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Bhavneet Kaur
Lady Shri Ram College for Women

Sumit Kumar
University of Delhi

Rajiv Aggarwal
University of Delhi

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Effects of Viscosity, Oblateness, and Finite Straight Segment on the Stability of the Equilibrium Points in the RR3BP

[1]Bhavneet Kaur, [2*]Sumit Kumar and [3]Rajiv Aggarwal

[1]Department of Mathematics
Lady Shri Ram College for Women
University of Delhi
Delhi, India
bhavneet.lsr@gmail.com

[2]Department of Mathematics
University of Delhi
Delhi, India
sumit.ims85@gmail.com

[3]Department of Mathematics
Deshbandhu College
University of Delhi
Delhi, India
rajiv_agg1973@yahoo.com

*Corresponding Author

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Abstract

Associating the influences of viscosity and oblateness in the finite straight segment model of the Robe's problem, the linear stability of the collinear and non-collinear equilibrium points for a small solid sphere are analyzed. This small solid sphere is moving inside the first primary which is a homogeneous incompressible viscous fluid whose hydrostatic equilibrium figure is an oblate spheroid. The second primary is a finite straight segment. The existence of the equilibrium points is discussed after deriving the pertinent equations of motion of the small solid sphere. It is found that viscosity does not affect the location and number of equilibrium points but affects the stability as it converts the marginal stability to asymptotic stability. However, oblateness affects the locations of the equilibrium points. Applicability of the results of this study to an astrophysical problem is discussed, and we have calculated a lower bound on the ratio of the orbital radius and the total mass of the primaries of an astrophysical problem to which the results obtained may be applied. This ratio is called the scaling factor.

Keywords: Viscosity; Stability; Robe's Restricted Three-Body Problem; Oblate spheroid; Finite straight segment

MSC 2010 No.: 37N05, 70F07, 70F15

1. Introduction

Robe's restricted three-body problem (RR3BP) has a special importance and relevance in the problems of celestial mechanics. It was introduced by Robe (1977) to study the influences of the Moon's attraction on the Earth's core. He considered two primaries. The first primary was a rigid spherical shell filled with a homogeneous incompressible fluid, and the second primary was a point mass which lies outside the shell and revolves around the first primary in a circular orbit. By considering attraction of the fluid, attraction of the second primary, and the buoyancy force due to the fluid, he studied the motion of the third body which was a small solid sphere within the spherical shell. He discovered only one equilibrium point, namely the center of the spherical shell, and analyzed its linear stability. Shrivastav and Garain (1991) studied the effects of perturbations in the Coriolis and centrifugal forces on the location of this equilibrium point and proved that it shifted to the right side. Hallan and Rana (2001) analysed the same problem for the location and linear stability of the equilibrium points. They proved that the center of the spherical shell is not the only equilibrium point. They showed that, depending on the parameters, there may be infinite number of equilibrium points other than the center of shell. One lying on the line joining centers of the primaries, two lying outside the plane of motion of primaries, and an infinite number of equilibrium points lying on a circle in the plane of motion of primaries with center at second primary. They also deliberated on the linear stability of all these equilibrium points under the effects of perturbations in the Coriolis and centrifugal forces (Hallan and Rana (2003)).

Plastino and Plastino (1995) worked on the aforementioned problem with a modification. They took the first primary as a Roche's ellipsoid (Chandrashekhar (1987)), which affects the buoyancy force. Both the original Robe's model and the modified model (Plastino and Plastino (1995)) were studied by Giordano et al. (1996) under the effects of linear drag. Kaur and Aggarwal (2012) and Kaur and Aggarwal (2013) extended the original and the modified Robe's problem (Plastino and Plastino (1995)) to the $2 + 2$ body problem.

Over time, the RR3BP has been generalized by considering different shapes of the primaries. Hallan and Mangang (2007) probed the problem by taking first primary as an oblate spheroid. Singh and Sandah (2012) and Singh and Mohammed (2012) explored the problem by taking both the primaries as oblate spheroids and one oblate, another triaxial, respectively. Motivated from Jain and Sinha (2014) who explored the restricted three-body problem with a finite straight segment, Kumar et al. (2019) analyzed the effects of a finite straight segment in RR3BP. They assumed the shape of the second primary as a finite straight segment and the first primary as a rigid spherical shell filled with a fluid. Kaur et al. (2021) modified this finite straight segment model of the Robe's problem. Along with the finite straight segment shape of the second primary, they assumed the shape of the first primary as an oblate spheroid and probed the problem for locations and linear stability of the equilibrium points.

Apart from the shapes of the primaries, the RR3BP has been explored by considering different properties of the celestial bodies. Ghosh and Mishra (2001) and AbdulRaheem (2011) explored the RR3BP by considering the photo-gravitational effect of the celestial bodies. They considered the second primary as a radiating body and the first primary as an oblate body. They observed that the radiation and the oblateness affect the positions of the equilibrium points whereas in the general case the equilibrium points are unstable. Singh and Leke (2013a; 2013b; 2013c) explored the effects of variable masses on the Robe's problem. Singh and Leke (2013b) assumed that the primary m_1 is a fluid in the shape of a sphere and the masses of the bodies vary at same rate. Singh and Leke (2013a) considered the variable mass in accordance with the unified Meshcherskii law and determined the motion of the primaries by the Gylden-Meshcherskii law. Along with the effects of variable masses, Singh and Leke (2013c) explored the effect of perturbations in the Coriolis and centrifugal forces. Kaur et al. (2020a) explored the effect of perturbations in the Coriolis and centrifugal forces in the Robe's problem when second primary is a finite straight segment. They found that when the centrifugal force is constant, the Coriolis force acts as a stabilizing force and when the Coriolis force is constant, the centrifugal force act as a destabilizing force. Recently, Abouelmagd et al. (2021) explored the effect of the modified potential in the Robe's problem. They assumed that the second primary generates a modified Newtonian potential and the first primary is a spherical nebula. They evaluated the equilibrium points for the cases: (i) when there are perturbations in the Coriolis and centrifugal forces, and (ii) when there are no perturbations in these forces.

A fluid plays an important role in the RR3BP, and an interesting property of a fluid is viscosity. Ansari et al. (2019a) were the first to study the effects of viscosity in the Robe's problem with the first primary as a rigid spherical shell and the second primary as an oblate spheroid. They also explored the effects of viscosity in the perturbed RR3BP (Ansari et al. (2019b)). They concluded that the viscosity does not change the location of an equilibrium point but converts the marginal stability to asymptotic stability.

Motivated by Ansari et al. (2019a) and Ansari et al. (2019b), Kaur et al. (2020b) probed the effects of viscosity in the finite straight segment model (Kumar et al. (2019)) of the Robe's problem. Kaur and Kumar (2021) also analyzed the effects of viscosity in the perturbed circular RR3BP finite straight segment model. They proved that the viscosity and perturbation in the Coriolis force do not affect the location of the equilibrium points. However, viscosity affects the nature of stability. They also presented the applicability of the result obtained to an astrophysical problem.

In the present article, we explore the effects of viscosity and oblateness in the finite straight segment model of the Robe's problem. We have taken the primary m_1 as an oblate spheroid and m_2 as a line segment. There are six sections in this article. Development of the Robe's problem and motivation is given in current section. Section 2 describes the configuration and the equations of motion. Section 3 is devoted to the location of the equilibrium points while the stability analysis is given in Section 4. Applicability of the results obtained are discussed in Section 5. Section 6 include some discussions on the findings. Finally, conclusions drawn are stated in Section 7.

2. Equations of motion

Let m_1, m_2, and m_3 be the three bodies, where

- m_1 is a homogeneous incompressible viscous fluid of density ρ_1 whose hydrostatic equilibrium figure is an oblate spheroid,
- m_2 is a finite straight segment of length $2l$,
- m_3 is a small solid sphere of density ρ_3.

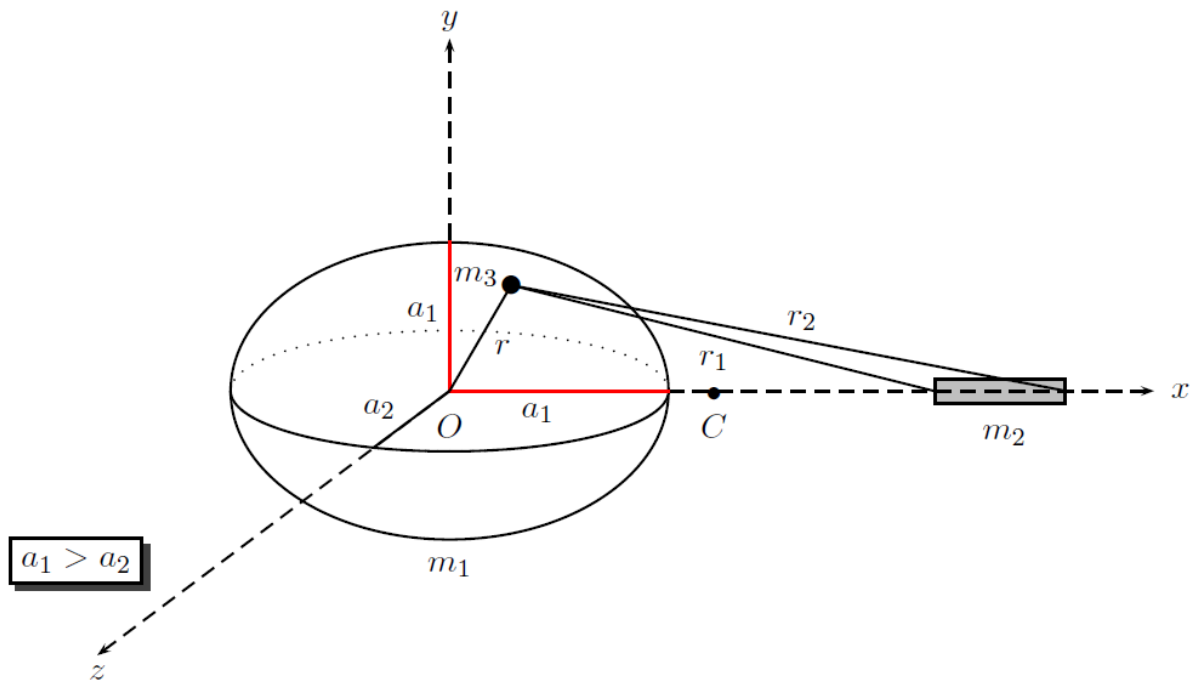

Figure 1. Geometric configuration of finite straight segment model of Robe's problem with an oblate body

The bodies m_1 and m_2, called primaries, describe a circular orbit about their common center of mass C. m_3 moves inside m_1 under the influences of the gravitation attraction by m_2, the attraction of the fluid, the buoyancy force due to fluid, and the viscous force of the fluid. Here, m_1, m_2, and m_3 also denote the masses of the bodies. Mass m_3 is assumed to be so small such that it does not affect the motion of m_1 and m_2. Kaur et al. (2021) studied the same model but without viscous force. Considering viscous force and following the procedure adopted by them and Ansari et al. (2019b), the equations of motion of m_3 in the uniformly rotating coordinate system $Oxyz$ with origin O at the center of mass of m_1 (see Figure 1) become

$$\begin{aligned} \ddot{x} - 2\omega\dot{y} &= -\alpha\dot{x} + W_x, \\ \ddot{y} + 2\omega\dot{x} &= -\alpha\dot{y} + W_y, \\ \ddot{z} &= -\alpha\dot{z} + W_z, \end{aligned} \tag{1}$$

where

$$\begin{aligned} W =& \rho\Bigg[\frac{1}{2}\omega^2\left((x-\mu)^2 + y^2\right) + \frac{\mu}{2l}\log\left(\frac{r_1 + r_2 + 2l}{r_1 + r_2 - 2l}\right) \\ &+ \pi\rho_1\left(I - A_1x^2 - A_1y^2 - A_2z^2\right)\Bigg], \text{ (Kaur et al. (2021))} \end{aligned}$$

$$
\begin{aligned}
\rho =& 1-\frac{\rho_1}{\rho_3},\\
\omega =& 1+\frac{3}{4}A+\frac{1}{2}l^2,\ l<<1,\ \text{(Kaur et al. (2021))}\\
A =& \frac{a_1^2-a_2^2}{5},\ 0<A<<1,\\
\mu =& \frac{m_2}{m_1+m_2},\ \ 0<\mu<1,\\
r_1^2 =& (x-1+l)^2+y^2+z^2,\\
r_2^2 =& (x-1-l)^2+y^2+z^2,\\
I =& 2a_1^2A_1+a_2^2A_2,\\
A_1 =& a_1^2a_2\int_0^\infty \frac{du}{\left(a_1^2+u\right)^2\left(a_2^2+u\right)^{1/2}},\\
A_2 =& a_1^2a_2\int_0^\infty \frac{du}{\left(a_1^2+u\right)\left(a_2^2+u\right)^{3/2}}.
\end{aligned}
$$

The coordinates x, y, z are dimensionless variables chosen such that the sum of the masses m_1, m_2 is 1 unit and the distance between m_1, m_2 is 1 unit and time t is chosen such that the value of gravitational constant G is 1 (McCuskey (1963)). Here, a_1 and a_2 denote the equatorial and polar radii of the oblate spheroid m_1. The terms $\alpha\dot{x}$, $\alpha\dot{y}$, and $\alpha\dot{z}$ (Ansari et al. (2019b)) are due to viscous force, where α is a positive constant. $W_i, i = x, y, z$ represent the partial derivatives of W with respect to x, y, z and dot denotes the derivative with respect to time t. The length parameter l and oblateness parameter A are assumed to be small such that $o(l^3) = o(A^2) \approx 0$.

3. Equilibrium points

Equilibrium points of the problem are the constant solutions of the system of differential equations (1). Therefore, at an equilibrium point all the derivatives of x, y, z are zero. Hence, a point (x, y, z) is an equilibrium point of the system provided

$$W_x(x,y,z) = W_y(x,y,z) = W_z(x,y,z) = 0.$$

The above system has been solved for equilibrium points by Kaur et al. (2021). For $\rho > 0$, they found the equilibrium points which are stated as follows:

Collinear equilibrium points: $L_1(p_1, 0, 0)$ and $L_2(x_1 + p_2, 0, 0)$ are two collinear equilibrium points, where

$$
\begin{aligned}
x_1 &= 1+\frac{\mu+\left(\mu^2+8\pi\rho_1A_1\mu^2-4\mu\right)^{1/2}}{2\left(1-2\pi\rho_1A_1\right)}\ \text{(Hallan and Mangang (2007))},\\
p_1 &= \frac{3A\mu}{2\left(1+2\mu-2\pi\rho_1A_1\right)},
\end{aligned}
$$

$$p_2 = \frac{(x_1-\mu)(x_1-1)(l^2+3A/2)-\mu l^2(1-x_1)^{-3}}{2\mu+(1-2\pi\rho_1 A_1)(1-3x_1)},$$

with $o(p_1^2) = o(p_2^2) \approx 0$. The equilibrium point $L_2(x_1+p_2, 0, 0)$ exists only if $1-2\pi\rho_1 A_1 + 3\mu/4 < 0$ and $-a_1 < x_1 < a_1$.

Non-collinear equilibrium points: A point $L_3(x, y, 0)(x \neq 0, y \neq 0)$ is a non-collinear equilibrium point if it lies within the oblate spheroid m_1 and at the circle given by

$$(1-x)^2 + y^2 = 1 - A - \frac{2l^2}{3}. \tag{2}$$

The non-collinear equilibrium points are infinite in number and exist provided

$$2\pi\rho_1 A_1 = (1-\mu)\left(1+l^2+3A/2\right) \text{ (Kaur et al. (2021)).}$$

The nomenclature collinear and non-collinear equilibrium point are justified as L_1, L_2 lie on x-axis and L_3 does not lie on this line.

4. Stability

For a dynamical system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}),\ \mathbf{x}(\mathbf{t_0}) = \mathbf{x_0}$, the equilibrium points are the steady state solutions. Therefore, the solutions which start close to an equilibrium point are of special interest for us. This is important to us because when we give a small deviation, we would like to know whether the trajectory also remains in the vicinity of the equilibrium point or not. If all the trajectories remain in the vicinity of the equilibrium point, then the equilibrium point is a called stable equilibrium point, otherwise it is called an unstable equilibrium point. Moreover, if all the trajectories converge to the equilibrium point, then the equilibrium point is called asymptotically stable and if they remain in the vicinity but do not converge to the equilibrium point, then the equilibrium point is called marginally stable. Such analysis is termed as stability analysis.

Here, we perform the linear stability analysis and we take help of Routh–Hurwitz stability criterion (Clark (1996)) for this analysis. To perform this, let $\mathbf{r_0} = x_0\hat{i} + y_0\hat{j} + z_0\hat{k}$ be the position vector of the equilibrium point (x_0, y_0, z_0) and let $\mathbf{r} = \xi\hat{i} + \eta\hat{j} + \zeta\hat{k}$ be a small vector. Consider a solution which starts from an arbitrary point (x, y, z) with position vector $\mathbf{r} + \mathbf{r_0}$. Therefore, we have

$$x = x_0 + \xi,\ y = y_0 + \eta,\ z = z_0 + \zeta.$$

Substituting these values of x, y, z in the system of differential equations (1) and retaining only the linear terms in $\xi, \eta, \zeta, \dot{\xi}, \dot{\eta}$, and $\dot{\zeta}$, we get the variational equations as

$$\begin{aligned} \ddot{\xi} - 2\omega\dot{\eta} &= W^0_{xx}\xi + W^0_{xy}\eta + W^0_{xz}\zeta - \alpha\dot{\xi}, \\ \ddot{\eta} + 2\omega\dot{\xi} &= W^0_{xx}\xi + W^0_{xy}\eta + W^0_{xz}\zeta - \alpha\dot{\eta}, \\ \ddot{\zeta} &= W^0_{xx}\xi + W^0_{xy}\eta + W^0_{xz}\zeta - \alpha\dot{\zeta}, \end{aligned} \tag{3}$$

where W^0_{pq}, p, q=x, y, z represent the second order partial derivatives of W with respect to p and q at (x_0, y_0, z_0).

4.1. Stability of the collinear equilibrium points

4.1.1. *Stability of the equilibrium point L_1*

Coordinates of the equilibrium point L_1 are $(p_1, 0, 0)$, i.e., $x_0 = p_1, y_0 = z_0 = 0$. Therefore, we have

$$\begin{aligned}
W^0_{xx} &= \frac{3\rho\mu A}{2p_1}, \\
W^0_{yy} &= \frac{3\rho\mu A}{2p_1} - 3\rho\mu, \\
W^0_{zz} &= -\rho\left(2\pi\rho_1 A_2 + \mu + 3p_1\mu + 2\mu l^2\right), \\
W^0_{xy} &= W^0_{yz} = W^0_{zx} = 0.
\end{aligned}$$

Thus, the system of variational equations (3) reduce to

$$\begin{aligned}
\ddot{\xi} - 2\omega\dot{\eta} &= W^0_{xx}\xi - \alpha\dot{\xi}, \\
\ddot{\eta} + 2\omega\dot{\xi} &= W^0_{yy}\eta - \alpha\dot{\eta}, \\
\ddot{\zeta} &= W^0_{zz}\zeta - \alpha\dot{\zeta}.
\end{aligned} \tag{4}$$

It is clear from system (4) that a change in ζ does not affect the motion in $\xi\eta$-plane. Hence, motion parallel to z-axis and parallel to xy-plane are independent to each other. This motivates us to study the stability parallel to z-axis and parallel to xy-plane independently.

Theorem 4.1. (Stability of L_1 parallel to z-axis)

Let

$$p_0 = -\frac{2\pi\rho_1 A_2 + \mu + 2\mu l^2}{3\mu}.$$

The equilibrium point L_1 of the system (1) is

(a) marginally stable parallel to z-axis, provided $p_1 = p_0$.
(b) asymptotically stable parallel to z-axis, provided $p_1 > p_0$.

Proof:

The characteristic equation for the last equation of the system (4) is given by

$$\lambda^2 + \alpha\lambda - W^0_{zz} = 0. \tag{5}$$

If $p_1 = p_0$, then $W^0_{zz} = 0$. Therefore, zero and $-\alpha$ shall be the roots of Equation (5). Consequently, the equilibrium point is marginally stable which proves (a).

If $p_1 > p_0$, then $-W^0_{zz} > 0$. So there is no sign change in Routh sequence $\{1, \alpha, -W^0_{zz}\}$, i.e., the first column of the Routh-Hurwitz array

1	$-W^0_{zz}$
α	0
$-W^0_{zz}$	

of Equation (5). Consequently, both the roots of Equation (5) lie in the left half plane, i.e., both the roots have negative real parts. Hence, the equilibrium point is asymptotically stable which proves (b). ■

Theorem 4.1 is represented graphically in Figure 2.

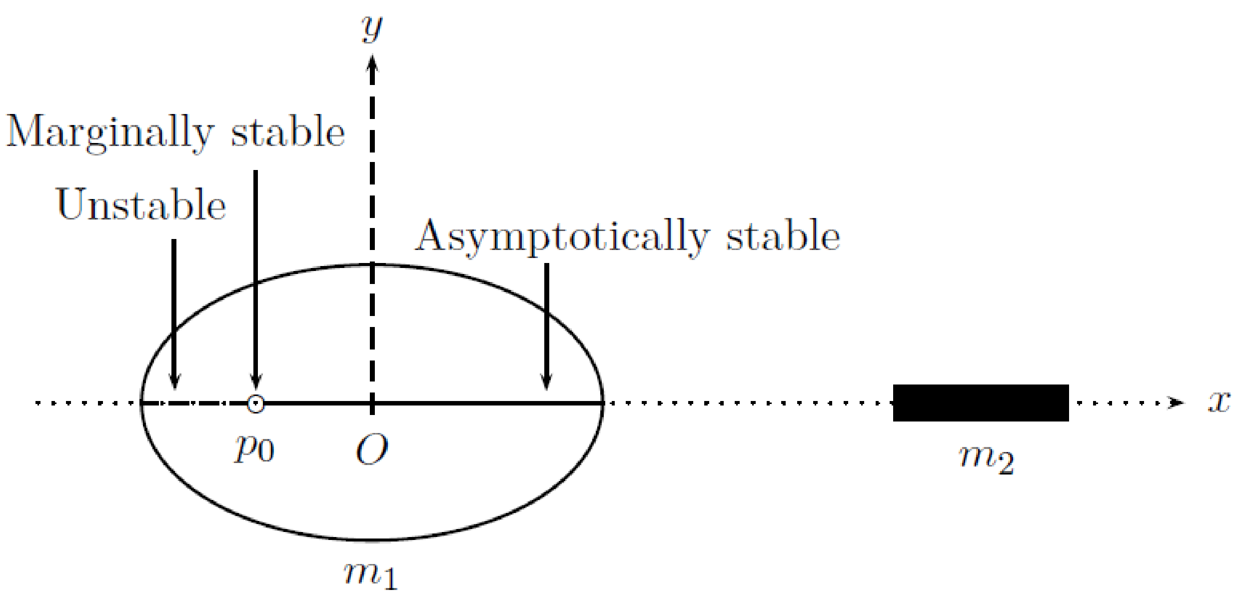

Figure 2. Bifurcation diagram for stability (in direction parallel to z-axis) of the equilibrium point $L_1(p_1, 0, 0)$

Theorem 4.2. (Stability of L_1 parallel to xy-plane)

If $A \neq 0$, then the equilibrium point $L_1(p_1, 0, 0)$ of the system (1) is asymptotically stable parallel to xy-plane only if $p_1 < 0$.

Proof:

The characteristic equation for the first two equations of the system of equations (4) is

$$\lambda^4 + a'_1\lambda^3 + a'_2\lambda^2 + a'_3\lambda + a'_4 = 0, \tag{6}$$

where

$$\begin{aligned} a'_1 =& 2\alpha, \\ a'_2 =& \alpha^2 + 4\omega^2 - \left(W^0_{xx} + W^0_{yy}\right), \\ a'_3 =& -\alpha\left(W^0_{xx} + W^0_{yy}\right), \\ a'_4 =& W^0_{xx}W^0_{yy}. \end{aligned}$$

Case 1. $p_1 < 0$.
Since $p_1 < 0$ and ρ, μ, A are positive, so $W^0_{xx} = \frac{3\rho\mu A}{2p_1} < 0$ and $W^0_{yy} = 3\rho\mu\left(\frac{A}{2p_1} - 1\right) < 0$. Hence, all the coefficients $a_i, i = 1, 2, 3, 4$ of the characteristic equation (6) are positive and the corresponding Routh-Hurwitz array becomes

1	a'_2	a'_4
a'_1	a'_3	0
Δ_1	a'_4	
Δ_2	0	
a'_4		

where

$$\Delta_1 = 4\omega^2 + \alpha^2 - \frac{1}{2}(W^0_{xx} + W^0_{yy}) > 0,$$

and

$$\Delta_2 = \frac{\alpha}{2\Delta_1}\left[(W^0_{xx} - W^0_{yy})^2 - 2(W^0_{xx} + W^0_{yy})(4\omega^2 + \alpha^2)\right] > 0.$$

Hence, signs of all the terms in the Routh sequence $1, a'_1, \Delta_1, \Delta_2, a'_4$ are positive. Therefore, real parts of all the roots of characteristic Equation (6) are negative which implies that the equilibrium point is asymptotically stable.

Case 2. $0 < p_1 < A/2$.
In this case both W^0_{xx} and W^0_{yy} are positive. Therefore, the coefficient a'_3 is negative and a'_4 is positive. At least one root is positive and hence, the equilibrium point is unstable.

Case 3. $p_1 = A/2$.
$p_1 = A/2$ implies $W^0_{xx} > 0$ and $W^0_{yy} = 0$. Therefore, the coefficient a'_4 is zero and a'_3 is negative. Since a'_1 is positive, so at least one root is positive. Hence, the equilibrium point is unstable.

Case 4. $p_1 > A/2$.
In this case $W^0_{xx} > 0$ and $W^0_{yy} < 0$. Therefore, the coefficient a'_4 is negative. Since a'_1 is positive, so at least one root is positive. Hence, the equilibrium point is unstable.

Case 5. $p_1 = 0$.
$p_1 = 0$ implies $A = 0$, a contradiction. ∎

Theorem 4.2 is represented graphically in Figure 3.

Theorem 4.3. (Stability criterion for L_1)

Let $A \neq 0$. The equilibrium point L_1 of the system (1) is

(a) marginally stable, provided $p_1 = p_0$,
(b) asymptotically stable, provided $p_0 < p_1 < 0$.

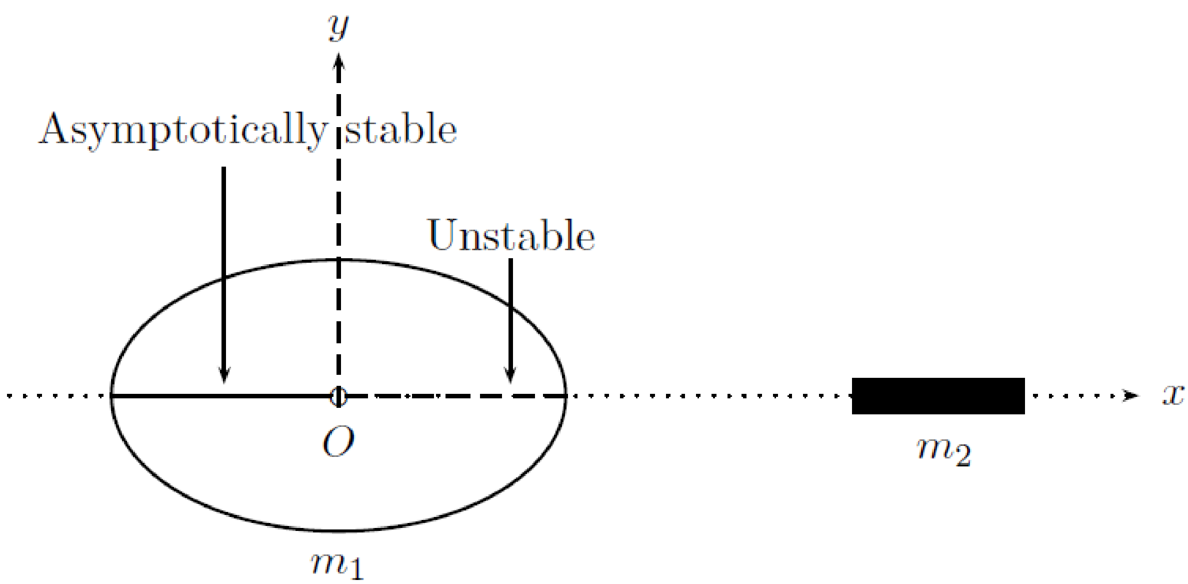

Figure 3. Bifurcation diagram for stability (in direction parallel to xy-plane) of the equilibrium point $L_1(p_1, 0, 0)$

Proof:

This follows directly from Theorems 4.1 and 4.2. ■

Theorem 4.3 is represented graphically in Figure 4.

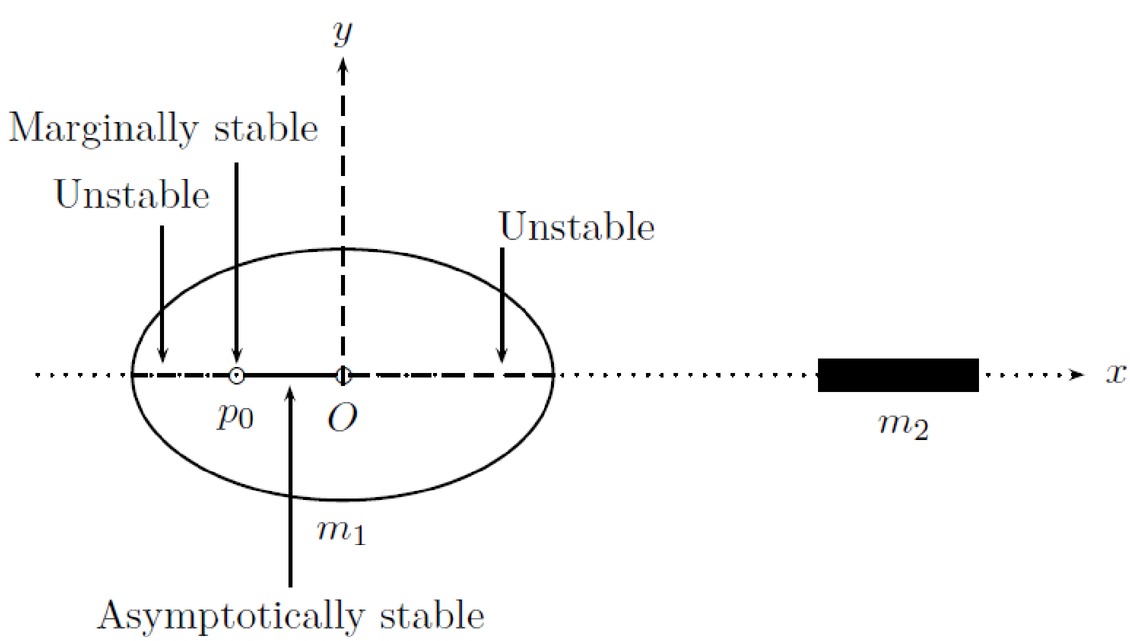

Figure 4. Bifurcation diagram for stability of the equilibrium point $L_1(p_1, 0, 0)$

4.1.2. *Stability of the equilibrium point L_2*

Coordinates of the equilibrium point L_2 are $(x_1 + p_2, 0, 0)$, i.e., $x_0 = x_1 + p_2 = x_{11}(\text{say}), y_0 = z_0 = 0$. Therefore, we have

$$\begin{aligned}
W^0_{xx} =& \rho\Big[1 + 3A/2 + l^2 - 2\pi\rho_1 A_1 + 2\mu a^3 \left(1 + 2l^2a^2\right)\Big] = a_{11},\\
W^0_{yy} =& \rho\Big[1 + 3A/2 + l^2 - 2\pi\rho_1 A_1 - \mu a^3 \left(1 + 2l^2a^2\right)\Big] = a_{22},\\
W^0_{zz} =& -\rho\left[2\pi\rho_1 A_2 + \mu a^3 \left(1 + 2l^2a^2\right)\right] = a_{33},\\
W^0_{xy} =& W^0_{yz} = W^0_{zx} = 0,
\end{aligned}$$

where

$$a = \frac{1}{1 - x_{11}}.$$

Thus, the system of variational equations (3) reduce to

$$\begin{aligned}
\ddot{\xi} - 2\omega\dot{\eta} &= a_{11}\xi - \alpha\dot{\xi},\\
\ddot{\eta} + 2\omega\dot{\xi} &= a_{22}\eta - \alpha\dot{\eta},\\
\ddot{\zeta} &= a_{33}\zeta - \alpha\dot{\zeta}.
\end{aligned} \tag{7}$$

Similar to the results about the stability of L_1, we have the following results about the stability of L_2.

Theorem 4.4. (Stability of L_2 parallel to z-axis)

The equilibrium point L_2 is always asymptotically stable parallel to z-axis.

Proof:

The characteristic equation corresponding to the last equation of the system (7) is

$$\lambda^2 + \alpha\lambda - a_{33} = 0. \tag{8}$$

Since a_{33} is always negative, so real parts of both the roots of equation (8) are negative. Hence, the equilibrium point is asymptotically stable. ■

Theorem 4.5. (Stability of L_2 parallel to xy-plane)

The equilibrium point L_2 (if it exist) of the system (1) is

(a) marginally stable parallel to xy-plane, provided $a_{11} = 0$ and $a_{22} < 0$ or $a_{11} < 0$ and $a_{22} = 0$.
(b) asymptotically stable parallel to xy-plane, provided $a_{11} < 0$ and $a_{22} < 0$.

Proof:

Similar to the proof of Theorem 4.2. ■

Combining Theorems 4.4 and 4.5, we can state the stability criterion for L_2 as follows.

Theorem 4.6. (Stability criterion of L_2)

The equilibrium point L_2 (if it exist) of the system (1) is

(a) marginally stable, provided $a_{11} = 0$ and $a_{22} < 0$ or $a_{11} < 0$ and $a_{22} = 0$.
(b) asymptotically stable, provided $a_{11} < 0$ and $a_{22} < 0$.

4.2. Stability of the non-collinear equilibrium points

The non-collinear equilibrium points lie on the circle given by Equation (2) and all the points on the circle are of the form

$$\left(1-\left(1-\frac{A}{2}-\frac{1}{3}l^2\right)\cos\theta, \left(1-\frac{A}{2}-\frac{1}{3}l^2\right)\sin\theta, 0\right).$$

Therefore, for an arbitrary non-collinear equilibrium point we have

$$\begin{aligned}
x_0 =& 1-\left(1-\frac{A}{2}-\frac{1}{3}l^2\right)\cos\theta,\\
y_0 =& \left(1-\frac{A}{2}-\frac{1}{3}l^2\right)\sin\theta,\\
z_0 =& 0.
\end{aligned}$$

At these points, the second order partial derivatives of W are

$$\begin{aligned}
W^0_{xx} =& \frac{\rho\mu}{2}\Big[3\left(2+3A\right)\cos^2\theta + \left(3-24\cos^2\theta+35\cos^4\theta\right)l^2\Big], \qquad (9)\\
W^0_{yy} =& \frac{\rho\mu}{2}\Big[3\left(2+3A\right)\sin^2\theta + \left(1+\sin^2\theta-5\cos^2\theta+35\cos^2\theta\sin^2\theta\right)l^2\Big],\\
W^0_{xy} =& \frac{\rho\mu}{2}\Big[3\left(2+3A\right) + \left(35\cos^2\theta-9\right)l^2\Big]\sin\theta\cos\theta,\\
W^0_{zz} =& -\frac{\rho}{2}\left[2\mu+4\pi\rho_1A_2+3\mu A+\mu\left(1+5\cos^2\theta\right)l^2\right],\\
W^0_{xz} =& 0 \text{ and } W^0_{yz} = 0.
\end{aligned}$$

Hence, the system of variational Equations (3) becomes

$$\begin{aligned}
\ddot{\xi} - 2\omega\dot{\eta} =& -\alpha\dot{\xi} + W^0_{xx}\xi + W^0_{xy}\eta,\\
\ddot{\eta} + 2\omega\dot{\xi} =& -\alpha\dot{\eta} + W^0_{xy}\xi + W^0_{yy}\eta, \qquad (10)\\
\ddot{\zeta} =& -\alpha\dot{\zeta} + W^0_{zz}\zeta.
\end{aligned}$$

Theorem 4.7. (Stability of non-collinear equilibrium point parallel to z-axis)

The non-collinear equilibrium points of the system (1) are asymptotically stable parallel to z-axis.

Proof:

The characteristic equation for the last equation of the system of variational equations (10) is

$$\lambda^2 + \alpha\lambda - W_{zz}^0 = 0. \tag{11}$$

Since W_{zz}^0 is always negative, the equilibrium points are asymptotically stable. ■

Theorem 4.8. (Stability of non-collinear equilibrium point parallel to xy-plane)

The non-collinear equilibrium points of the system (1) are unstable parallel to xy-plane.

Proof:

The characteristic equation for the first two equations of the system of variational equations (10) is

$$\lambda^4 + b_1\lambda^3 + b_2\lambda^2 + b_3\lambda + b_4 = 0, \tag{12}$$

where

$$\begin{aligned} b_1 =& 2\alpha, \\ b_2 =& \alpha^2 + 4\omega^2 - \left(W_{xx}^0 + W_{yy}^0\right), \\ b_3 =& -\alpha\left(W_{xx}^0 + W_{yy}^0\right), \\ b_4 =& W_{xx}^0 W_{yy}^0 - W_{xy}^2. \end{aligned}$$

Using values of W_{xx}^0 and W_{yy}^0 from (9), the coefficient b_3 becomes

$$-\frac{\alpha\rho\mu}{2}\left[3(2+3A) + 5\left(1+\cos^2\theta\right)l^2\right],$$

which is less than zero. Hence, the equilibrium points are unstable parallel to xy-plane. ■

Combining **Theorems 4.7 and 4.8**, we can state following theorem about the stability of non-collinear equilibrium points.

Theorem 4.9. (Stability of non-collinear equilibrium points)

The non-collinear equilibrium points of the system (1) are unstable.

5. Applications

From **Theorem 4.3** it is clear that L_1 is stable only if it lies to the left of center of m_1, i.e., $p_1 < 0$, and $1 + 2\mu - 2\pi\rho_1 A_1 < 0 \implies p_1 < 0$. Hence, $\max\limits_{0<\mu<1}(1+2\mu) < 2\pi\rho_1 A_1 \implies p_1 < 0$.

Therefore,

$$3 < 2\pi\rho_1 A_1 \implies p_1 < 0. \tag{13}$$

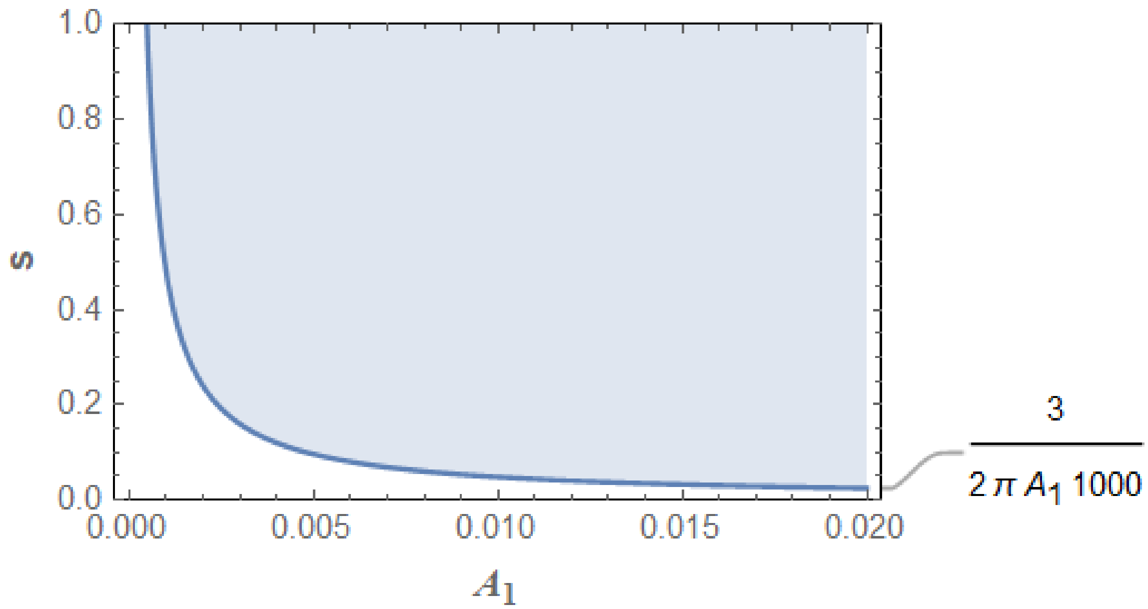

Figure 5. Lower bound on scaling factor s for the stability of the equilibrium point $L_1(p_1, 0, 0)$ when $\rho_1 = 1000$ Kg/m^3

If ρ_1, A_1 are dimensional quantity, then $\rho_1 A_1 s$ is dimensionless, where $s = R^3/M$. Here, R is orbital radius and M is total mass of primaries m_1, m_2. Hence, s act as scaling factor for $\rho_1 A_1$. Therefore,

$$s > 3/2\pi\rho_1 A_1 \implies p_1 < 0.$$

So, $3/2\pi\rho_1 A_1$ is a lower bound for scaling factor s. In Figure 5, lower bound for s has been plotted against A_1 when $\rho_1 = 1000$ Kg/m^3 (density of water). If (A_1, s) lies in blue region, then the equilibrium point L_1 is asymptotically stable. In other words, we can say that the blue region (region above the lower bound curve) is a stability region in sA_1-plane. Hence, the binary system for which the (A_1, s) lies in the blue region may be approached with this mathematical model and results. In Figure 6, we have plotted the lower bound curve of s for four different values of ρ_1.

6. Discussion

In the present study, the finite straight segment model of the Robe's problem is studied by considering the effects of oblateness of m_1 and the viscosity of the fluid. In the model considered, the oblate body m_1 and the finite straight segment m_2 are moving in a circular orbit around their common center of mass. The equations of motion of the third body are evaluated under the effects of:

- Viscous force due to fluid,
- Buoyancy force due to fluid,
- Attraction force due to fluid,
- Attraction force due to m_2.

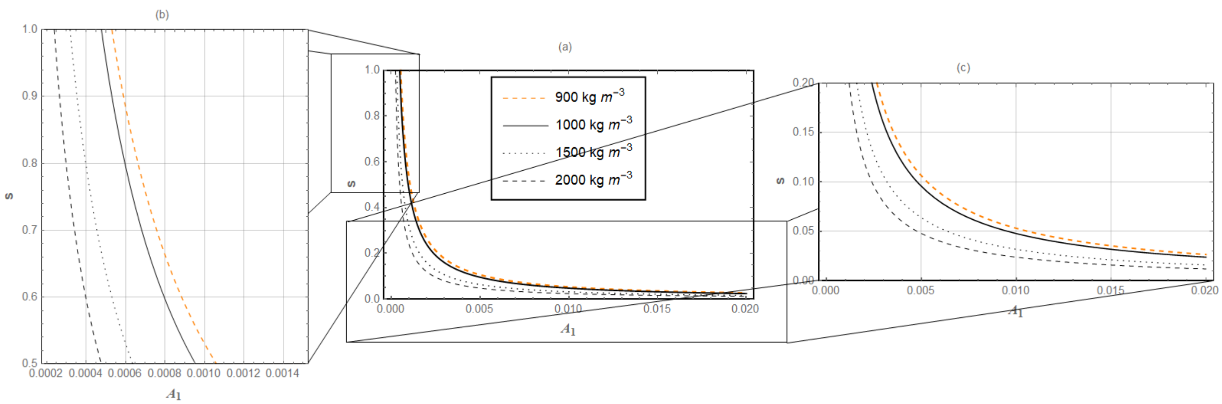

Figure 6. **(a)** Lower bound on scaling factor s for the stability of the equilibrium point $L_1(p_1, 0, 0)$ when $\rho_1 = 900$ Kg/m^3, 1000 Kg/m^3, 1500 Kg/m^3, 2000 Kg/m^3 **(b)** Show the zoomed portion of the upper part of graph (a) and **(c)** show the zoomed portion of the lower part of the graph (a)

The equilibrium points of two different nature are stated, namely collinear and non-collinear equilibrium points. There are two collinear $\{L_1, L_2\}$ and infinite non-collinear equilibrium points. Novelty of the work lies in the rigorous linear stability analysis of these equilibrium points. We have analyzed the one-dimensional (parallel to z-axis), two-dimensional (planer, parallel to xy-plane), and three-dimensional linear stability of the equilibrium points.

It is seen that the non-collinear equilibrium points are always asymptotically stable parallel to $z-$axis. However, they are unstable parallel to xy-plane and consequently unstable. Thus, viscous force converts the marginal stability to asymptotic stability but does not affect the unstable nature of the equilibrium points.

The collinear equilibrium point L_1 is found to be one dimensional marginally stable if it lies at critical point $L_0(p_0, 0, 0)$ and asymptotically stable if it lies on the right side of L_0. Similarly, in direction parallel to xy-plane it is asymptotically stable if it lies to the left of the center of m_1. Hence, it is asymptotically stable if it lies between L_0 and the center of m_1. A clear picture of these bifurcations are depicted in Figures 2, 3, and 4. Similarly, the equilibrium point L_2 is always asymptotically stable parallel to z-axis and parallel to xy-plane it is marginally or asymptotically stable depending on the values of parameters involved. Thus, in general L_2 is marginally stable or asymptotically stable depending on the values of parameters involved. Hence, again it is observed that the viscous force converts the marginal stability to asymptotic stability but does not affect the unstable nature of the equilibrium points.

Apart from stability, we have provided the scope of applicability of the model and results to an astrophysical problem. We have provided a lower bound on the scaling factor $s = R^3/M$, i.e., ratio of the orbital radius and the total mass of the primaries m_1, m_2 and proved that a binary system with $s > 3/2\pi\rho_1 A_1$ may be approached with this study. We have plotted this lower bound against A_1 for different values of ρ_1 in Figure 6.

7. Conclusion

The Robe's restricted three-body problem is analyzed for equilibrium points and their stability. In the setting of the Robe's model, the first primary is assumed to be a homogeneous incompressible viscous fluid whose hydrostatic equilibrium figure is an oblate spheroid, the second primary as a finite straight segment, and the third body is a small solid sphere. When the density of the third body is greater than the density of fluid, there exist two collinear and infinite number of non-collinear equilibrium points. It is observed that location of these equilibrium points remain unaffected by the viscosity of the fluid but are substantially affected by the oblateness of the first primary and the length of the finite straight segment. Viscosity of the fluid affects the stability of these equilibrium points as it changes the nature of stability of the collinear equilibrium points from marginally stable to asymptotically stable. The stability is also affected by oblateness and length parameters. The non-collinear equilibrium points always remain unstable.

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REFERENCES

AbdulRaheem, A. (2011). Stability of collinear points in the generalized photogravitational Robes Restricted Three-Body Problem, International Journal of Astronomy and Astrophysics, Vol. 1, pp. 6-9. doi:10.4236/ijaa.2011.11002

Abouelmagd, E.I., Ansari, A.A. and Shehata, M.H. (2021). On Robe's restricted problem with a modified Newtonian potential, International Journal of Geometric Methods in Modern Physics, Vol. 18, No. 01, 2150005. https://doi.org/10.1142/S0219887821500055

Ansari, A.A., Singh, J., Alhussain, Z.A. and Belmabrouk, H. (2019a). Effect of oblateness and viscous force in the Robe's circular restricted three-body problem, New Astronomy, Vol. 73, 101280.

Ansari, A.A., Singh, J., Alhussain, Z.A. and Belmabrouk, H. (2019b). Perturbed Robe's CR3BP with viscous force, Astrophys. Space Sci., Vol. 364, No. 6, pp. 1-7.

Chandrashekhar, S. (1987). *Ellipsoidal Figures of Equilibrium*, Dover Publications Inc., New York.

Clark, R.N. (1996). *Control System Dynamics*, Cambridge University Press, New York.

Ghosh, R. N. and Mishra, B. N. (2001). Generalised photogravitational restricted three-body problem and the locations and stability of collinear equilibrium points, Indian J. Pure Applied Math., Vol. 32, No. 4, pp. 515–520.

Giordano, C.M., Plastino, A.R. and Plastino, A. (1996). Robe's restricted three-body problem with drag, Celes. Mech. and Dyn. Astron., Vol. 66, No. 2, pp. 229–242.

Hallan, P.P. and Mangang, K.B. (2007). Existence and linear stability of equilibrium points in the

Robe's restricted three body problem when the first primary is an oblate spheroid, Planetary and Space Science, Vol. 55, No. 4, pp. 512–516.

Hallan, P.P. and Rana, N. (2001). The existence and stability of equilibrium points in the Robe's restricted three-body problem, Celes. Mech. Dyn. Astron., Vol. 79, No. 2, pp. 145–155.

Hallan, P.P. and Rana, N. (2003). Effect of perturbations in the Coriolis and centrifugal forces on the locations and stability of the equilibrium points in Robe's circular problem with density parameter having arbitrary value, Ind. J. Appl. Math., Vol. 34, No. 7, pp. 1045–1059.

Jain, R. and Sinha, D. (2014). Stability and regions of motion in the restricted three-body problem when both the primaries are finite straight segments, Astrophys. Space Sci. Vol. 351, pp. 87–100.

Kaur, B. and Aggarwal, R. (2012). Robe's Problem: Its extension to 2+2 bodies, Astrophys. Space Sci., Vol. 339, pp. 283–294.

Kaur, B. and Aggarwal, R. (2013). Robe's restricted problem of 2+2 bodies when the bigger primary is a Roche ellipsoid, Acta Astronautica, Vol. 89, pp. 31–37.

Kaur, B., Chauhan, S. and Kumar, D. (2021). Outcomes of aspheric primaries in Robe's Circular Restricted Three-Body Problem, Applications and Applied Mathematics: An International Journal (AAM), Vol. 16, No. 1, pp. 463–480.

Kaur, B., Kumar, D. and Chauhan, S. (2020a). A study of small perturbations in the Coriolis and centrifugal forces in RR3BP with finite straight segment, Applications and Applied Mathematics: An International Journal (AAM), Vol. 15, No. 1, pp. 57–69.

Kaur, B. and Kumar, S. (2021). Stability analysis in the perturbed CRR3BP finite straight segment model under the effect of viscosity, Astrophys. Space Sci. Vol. 366, No. 5, pp. 1–12. https://doi.org/10.1007/s10509-021-03948-0

Kaur, B., Kumar, S., Chauhan, S. and Kumar, D. (2020b). Stability analysis of circular Robe's R3BP with finite straight segment and viscosity, Applications and Applied Mathematics: An International Journal (AAM), Vol. 15, No. 2, pp. 1072–1090.

Kumar, D., Kaur, B., Chauhan, S. and Kumar, V. (2019). Robe's restricted three-body problem when one of the primaries is a finite straight segment, Int. J. of Non-Linear Mech., Vol. 109, pp. 182–188.

McCuskey, S.W. (1963). *Introduction to Celestial Mechanics*, Addison-Wesely Publishing Company, Inc., New York.

Plastino, A.R. and Plastino, A. (1995). Robe's restricted three-body problem revisited, Celes. Mech. Dyn. Astron., Vol. 61, pp. 197–206.

Robe, H.A.G. (1977). A new kind of three-body problem, Celes. Mech. Dyn. Astron., Vol. 16, pp. 343–351.

Shrivastava, A. K. and Garain, D. (1991). Effect of perturbation on the location of libration point in the Robe's restricted problem of three bodies, Celes. Mech. Dyn. Astron., Vol. 51, pp. 67–73.

Singh, J. and Leke, O. (2013a). Existence and stability of equilibrium points in the Robe's Restricted Three-Body Problem with variable masses, International Journal of Astronomy and Astrophysics, Vol. 3, pp. 113–122.

Singh, J. and Leke, O. (2013b). On Robe's circular restricted problem of three variable mass bodies, Journal of Astrophysics, Vol. 2013, Article ID 898794, 11 pages. https://doi.org/10.1155/2013/898794

Singh, J. and Leke, O. (2013c). Robe's restricted three-body problem with variable masses and perturbing forces, ISRN Astron. and Astrophy, 8 pages.

Singh, J. and Mohammed, H.L. (2012). Robe's circular restricted three-body problem under oblate and triaxial primaries, Earth, Moon, and Planets, Vol. 109, pp. 1–11.

Singh, J. and Sandah, A.U. (2012). Existence and linear stability of equilibrium points in the Robe's restricted three-body problem with oblateness, Advances in Mathematical Physics, 18 pages.