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A Special Case of Rodriguez-Lallena and Ubeda-Flores Copula Based on Ruschendorf Method

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Abstract

Measure of dependence is a particular way of looking at the association between random variables, and one way to capture stochastic dependence is through the use of copula. In this study, a Rushendorf Method was applied to a bivariate function to obtain a copula through the use of a special case of Rodriguez-Lallena and Ubeda-Flores (RLUF) copula. Properties of the RLUF copula such as the density, measures of dependence, and lower and upper tail dependence were studied. In particular, measures of dependence such as Spearman's rho, Kendall's tau and Blomqvist's beta of RLUF copula are $\rho^{RLUF} \in$ [-0.1128, 0.3353], $\tau^{RLUF} \in$ [-0.0752, 0.2356] and $\beta^{RLUF} \in$ [-0.07902, 0.24769], respectively. Moreover, the Root-Mean-Square Error (RMSE), Sum-Square Error (SSE), Mean Absolute Error (MAE), Mean Square Error (MSE), Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) were used in deriving the best joint distribution between monthly precipitation and temperature in the Philippines from 1974 to 2013. The results showed that considering the monthly precipitation and temperature datas, RLUF copula outperformed the other existing bivariate copulas such as Ali-Mikhail-Haq (AHM), Farlie-Gumbel-Morgenstern (FGM), and Clayton copulas.

Keywords: RLUF copula; Bivariate copula; Ruschendorf method; Spearman's rho; Kendall's tau; Blomqvist's beta; Tail dependence; Model selection

MSC 2010 No.: 62H10, 62H12, 62E15

1. Introduction

Analyzing the correlation or dependence between random variables has remained a growing interest in the field of probability and statistics. Most of the commonly used correlation analysis of a stochastic variables methods are Pearson's Correlation, Spearman's Rank Rho and Kendall's tau, and these correlation measures the association between the variables in linear and non-linear relationship. But, these methods have their own limitations because they only quantify the degree of dependence and do not give information about the structure of the dependence. According to Schmidt (2007) and Fang et al. (2014), one way to give more information on the structure of the dependence was to construct a copula because it can describe the interconnection of two or more random variables.

For some of the reasons, the usual correlation analysis will show its restrictions that leads to closely ambiguous interpretations. Luckily, in recent years, Wen and Liu (2009) studied the improvement of copula technology bids a good application for the correlation analysis because it can better reveal the degree of correlation between two or more variables.

Sklar (1959) first proposed copula theory and was reviewed and summarized by Nelsen (2003). According to this theory, the joint distribution of two or more variables are big help in constructing the interconnection between the variables. According to the study of Nelsen (2007), the key benefits of the copula is that without the joint probability distribution, it can also perform correlation analysis using copula functions. Some of the distributions generated using copula, Macalos and Arcede (2015) studied Farlie-Gumbel-Morgenstern Copula which contain Bivariate Singh-Maddala and Bivariate Dagum distribition which is conducted by Pizon and Arcede (2019). Pathak and Vellaismy (2016) discuss on their paper a subclass of the Rodriguez-Lallena and Ubeda-Flores family of copulas, through order statistic. The study of Kumar (2010) discussed the statistical properties and tractable results Ali-Mikhail-Haq (AMH) copula and Yee et al. (2014) fitted the rainfall data using Ali-Mikhail-Haq (AMH), Clayton, Frank, Galambos, Gumbel-Hoogaurd (GH) and Plackett.

Due to these benefits, the interesting correlation of the copula perspective has been growing interest because of its usefulness and popularity from the researcher. This has led to enormous applications in different areas, such as the fields of finance and insurance which is studied by Frees and Valdez (1998), stock market which studied by Wen and Liu (2009), and floods which is studied by Shiau et al. (2006). Numerous studies have been shown to construct the properties of dependence and measures of correlation between two or more variables in terms of several copulas.

Construction of a copula becomes the main interest of most mathematicians. Rodriguez-Lallena and Ubeda-Flores (2004) presented a wide class of bivariate copulas depending on two univariate functions which generalizes many known family of copulas. According to the study of Djenndai et al. (2021), Arqub (2019), and Arqub and Rashaideh (2018), the advantages of having of copulas which are differentiable in parametric/semiparametric approach solve the problem of estimation and optimization problem. Various researches measure the goodness of fit such as; maximum likelihood and minimum distance estimators, which was conducted by Weiß (2019) and maximum pseudo-likelihood approach from the study of Genest and Favre (2007), which helps to identify the best criterion in copula selection. Serinaldi et al. (2012) applied the Sum-Square Error (SSE) to the performance of wood volume models and in the study of Fang et. al. (2013), AIC is superior to select the copula that provides the best fit. However, in the study of Shumway and Stoffer (2011), BIC performs better in large samples while AIC performs better in small samples. In this study, both criteria are used in selecting the best fit copula model and we also consider some computing indices accuracy such as Root-Mean Square Error (RMSE), Mean Absolute Error (MAE) and Mean-Square Error (MSE).

In this study, one of the popular class bivariate copula developed by Rodriguez-Lallena and Ubeda-Flores (2004) is considered. Though there are some methods on copula construction, Ruschendorf (1985) has developed a general method on construction of multivariate distributions having uniform marginal. Given this helpfulness, Mah and Shitan (2014) construct a copula based on Ruschendorf Method that contains an exponential function. However, in this paper, we consider a general form of quadratic-exponential constructed copula based on the Ruschendorf Method. Hence, this study, considered two parameters having quadratic-exponential function and obtained the properties, i.e., density, Spearman's rho, Kendall's tau, Blomqvist's beta, and lower and upper tail dependence. We also present the goodness of fit of this copula where our only focus is to apply on a real data sets.

2. Bivariate Copula Construction

In this section, Ruschendorf method of constructing copulas are in the immediately following Equation (3).

This paper apply some special functions and this includes the error function

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2} dt, \tag{1}$$

where $\pi \approx 3.1415...$

Lemma 2.1.

For any $x > 0$, $a > 0$, and $k \in N$,

$$\int_0^x \mathrm{e}^{kat^2} dt = \frac{\sqrt{\pi}\operatorname{erf}(\sqrt{-ka}x)}{2\sqrt{-ka}}. \tag{2}$$

This time, we will be able to choose an arbitrary function to construct new copulas based on the Ruschendorf method. Here, we choose a function that contain a parameter in the component of quadratic-exponential, which is defined to be,

$$f(x, y) = x^3 y^3 \mathrm{e}^{ax^2 + by^2}. \tag{3}$$

The procedure of construction of copula in Equation (3) were given in the following steps.

Step 1. Let $f(x,y) = x^3y^3\mathrm{e}^{ax^2+by^2}$ for $x,y \in [0,1]$. We then compute,

$$f(x) = \int_0^1 f(x,y)dy, f(y) = \int_0^1 f(x,y)dx, A = \int_0^1\int_0^1 f(x,y)dxdy,$$

as follows,

$$f(x) = \int_0^1 x^3y^3\mathrm{e}^{ax^2+by^2}dy = x^3\mathrm{e}^{ax^2}\Big(\frac{b\mathrm{e}^b - e^b + 1}{2b^2}\Big),$$

and

$$f(y) = \int_0^1 x^3y^3\mathrm{e}^{ax^2+by^2}dx = y^3\mathrm{e}^{by^2}\Big(\frac{a\mathrm{e}^a - e^a + 1}{2a^2}\Big).$$

Then,

$$A = \int_0^1\int_0^1 x^3y^3\mathrm{e}^{ax^2+by^2}dydx = \Big(\frac{b\mathrm{e}^b - e^b + 1}{2b^2}\Big)\Big(\frac{a\mathrm{e}^a - e^a + 1}{2a^2}\Big).$$

Step 2. Construct $f^*(x,y) = f(x,y) - f(x) - f(y) + A$. Hence, we have

$$f^1(x,y) = \Big[x^3\mathrm{e}^{ax^2} - \Big(\frac{a\mathrm{e}^a - e^a + 1}{2a^2}\Big)\Big]\Big[y^3\mathrm{e}^{by^2} - \Big(\frac{b\mathrm{e}^b - e^b + 1}{2b^2}\Big)\Big]. \tag{4}$$

at this point, we let $h(x,y) = 1 + \theta\Big[x^3\mathrm{e}^{ax^2} - \Big(\frac{a\mathrm{e}^a - e^a + 1}{2a^2}\Big)\Big]\Big[y^3\mathrm{e}^{by^2} - \Big(\frac{b\mathrm{e}^b - e^b + 1}{2b^2}\Big)\Big]$ and transform it into Rodriguez-Lallena and Ubeda-Flores Copula form by performing the double integral of the function $h(x,y)$, which is given below,

$$\begin{aligned} C(u,v) &= \int_0^v\int_0^u h(x,y)dxdy \\ &= uv + \theta\Big[\Big(\frac{au^2\mathrm{e}^{au^2} - e^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\Big) \\ &\qquad \Big(\frac{bv^2\mathrm{e}^{bv^2} - e^{bv^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)v}{2b^2}\Big)\Big]. \end{aligned}$$

Here, we derive a copula based on the arbitrary function and satisfies the condition of Rodriguez-Lallena and Ubeda-Flores copula immediately before Theorem 2.1.

Theorem 2.1.

Let $0 < a \leq b$ and $\Big(\mathrm{e}^a - \frac{(a\mathrm{e}^a - e^a + 1)}{2a^2}\Big) \leq \theta \leq \Big(\mathrm{e}^b - \frac{(b\mathrm{e}^b - e^b + 1)}{2b^2}\Big)$. Then,

$$\begin{aligned} C^{RLUF}(u,v) = uv + \theta\Big[&\Big(\frac{au^2\mathrm{e}^{au^2} - e^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\Big) \\ &\Big(\frac{bv^2\mathrm{e}^{bv^2} - e^{bv^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)v}{2b^2}\Big)\Big], \end{aligned} \tag{5}$$

is a copula.

Proof:

Observe that the function $C^{RLUF}(u,v)$ has the form of $C_\theta(u,v) = uv + \theta f(u)g(v)$ and satisfies the boundary condition from the fact that

$$f(0) = f(1) = g(0) = g(1) = 0.$$

Let f, g: [0,1] $\to \mathbb{R}$ where $f(u) = \left(\frac{au^2\mathrm{e}^{au^2} - e^{au^2}+1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\right)$ and $g(v) = \left(\frac{bv^2\mathrm{e}^{bv^2} - e^{bv^2}+1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)v}{2b^2}\right)$, provided that $0 < a \leq b$. The functions f and g are differentiable functions since $f'(u) = u^3\mathrm{e}^{au^2} - \frac{(a\mathrm{e}^a - e^a + 1)}{2a^2}$ and $g'(v) = v^3\mathrm{e}^{au^2} - \frac{(a\mathrm{e}^a - e^a + 1)}{2a^2}$ exist for every $0 < a \leq b$ and $u, v \in$ [0, 1]. Using basic concepts in calculus, the product and sum of polynomial and exponential functions are continuous on [0, 1], so f and g are continuously differentiable functions. Consequently, f and g are absolutely continuous functions.

Notice the following variables in the definition of Rodriguez-Lallena and Ubeda-Flores copula:

$$\begin{aligned}
\alpha &= \inf\{f'(u) : u \in A\} < 0,\\
\beta &= \sup\{f'(u) : u \in A\} > 0,\\
\gamma &= \inf\{g'(v) : v \in B\} < 0,\\
\delta &= \sup\{g'(v) : v \in B\} > 0,
\end{aligned}$$

where $A = \{u \in [0,1] : f'(u) \text{ exists}\}$ and $B = \{v \in [0,1] : g'(v) \text{ exists}\}$. We see that

$$\max\{f'(u)g'(v)\} = \max\{\alpha\gamma, \beta\delta\},$$

and

$$\max\{-f'(u)g'(v)\} = -\min\{f'(u)g'(v)\} = -\min\{\alpha\delta, \beta\gamma\}.$$

So, the admissible range of Lallena Flores copula parameter is

$$-\frac{1}{\max\{\alpha\gamma, \beta\delta\}} \leq \theta \leq -\frac{1}{\min\{\alpha\delta, \beta\gamma\}}.$$

Therefore, the admissible range of θ for $0 < a \leq b$ is given by

$$\frac{-1}{\left(\mathrm{e}^a - \frac{(a\mathrm{e}^a - e^a + 1)}{2a^2}\right)\left(\mathrm{e}^b - \frac{(b\mathrm{e}^b - e^b + 1)}{2b^2}\right)} \leq \theta \leq \frac{1}{\frac{(a\mathrm{e}^a - e^a + 1)}{2a^2}\left(\mathrm{e}^b - \frac{(b\mathrm{e}^b - e^b + 1)}{2b^2}\right)}. \qquad \blacksquare$$

Theorem 2.2.

Let $0 < a \leq b$ and $\frac{-1}{\left(\mathrm{e}^a - \frac{(a\mathrm{e}^a - e^a + 1)}{2a^2}\right)\left(\mathrm{e}^b - \frac{(b\mathrm{e}^b - e^b + 1)}{2b^2}\right)} \leq \theta \leq \frac{1}{\frac{(a\mathrm{e}^a - e^a + 1)}{2a^2}\left(\mathrm{e}^b - \frac{(b\mathrm{e}^b - e^b + 1)}{2b^2}\right)}$. Then,

$$c^{RLUF}(u,v) = 1 + \theta\left[\left(u^3\mathrm{e}^{au^2} - \frac{(a\mathrm{e}^a - e^a + 1)}{2a^2}\right)\left(v^3\mathrm{e}^{bv^2} - \frac{(b\mathrm{e}^b - e^b + 1)}{2b^2}\right)\right], \tag{6}$$

is density copula.

Proof:

With the copula given by Equation (5), the results follow directly by solving $\dfrac{\partial C^2(u,v)}{\partial u \partial v}$. ■

3. Measures of Dependence of Copula

Theorem 3.1.

The Spearman's rho of the bivariate case of Rodriguez-Lallena and Ubeda-Flores copula given in Theorem 2.1 is given by

$$\rho^{RLUF} = 12\theta\Big(\frac{2\mathrm{e}^a}{4a^2} - \frac{3\sqrt{\pi}\,\mathrm{erf}(\sqrt{-a})}{8a^2\sqrt{-a}} - \frac{a\mathrm{e}^a}{4a^2} + \frac{1}{4a^2}\Big)\Big(\frac{2\mathrm{e}^b}{4b^2} - \frac{3\sqrt{\pi}\,\mathrm{erf}(\sqrt{-b})}{8b^2\sqrt{-b}} - \frac{b\mathrm{e}^b}{4b^2} + \frac{1}{4b^2}\Big). \tag{7}$$

Proof:

The Spearman's rho can be obtained using the formula,

$$\begin{aligned}\rho^{RLUF} &= 12\int_0^1\int_0^1 C^{RLUF}(u,v)dudv - 3 \\ &= 12\theta\int_0^1 \Big(\frac{au^2\mathrm{e}^{au^2} - e^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\Big)du \\ &\qquad \int_0^1 \Big(\frac{bv^2\mathrm{e}^{bv^2} - e^{bv^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)v}{2b^2}\Big)dv.\end{aligned} \tag{8}$$

Consider,

$$\rho_u = \int_0^1 \Big(\frac{au^2\mathrm{e}^{au^2} - e^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\Big)du, \tag{9}$$

and

$$\rho_v = \int_0^1 \Big(\frac{bv^2\mathrm{e}^{bv^2} - e^{bv^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)v}{2b^2}\Big)dv. \tag{10}$$

Observe that

$$\begin{aligned}\rho_m &= \int_0^1 \Big(\frac{am^2\mathrm{e}^{am^2} - e^{am^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)m}{2a^2}\Big)dm \\ &= \int_0^1 \frac{am^2\mathrm{e}^{am^2}}{2a^2}dm - \int_0^1 \frac{\mathrm{e}^{am^2}}{2a^2}dm + \int_0^1 \frac{1}{2a^2}dm - \int_0^1 \frac{(a\mathrm{e}^a - e^a + 1)m}{2a^2}dm.\end{aligned}$$

Using Lemma 2.1 and simplifying the results, we can integrate each terms of ρ_m as follows,

$$\rho_1 = \int_0^1 \frac{au^2\mathrm{e}^{au^2}}{2a^2}du = \frac{\mathrm{e}^a}{4a^2} - \frac{\sqrt{\pi}\,\mathrm{erf}(\sqrt{-a})}{8a^2\sqrt{-a}},$$

$$\rho_2 = \int_0^1 \frac{\mathrm{e}^{au^2}}{2a^2}du = \frac{\sqrt{\pi}\,\mathrm{erf}(\sqrt{-a})}{4a^2\sqrt{-a}},$$

$$\rho_3 = \int_0^1 \frac{1}{2a^2}du = \frac{1}{2a^2}u\Big|_0^1 = \frac{1}{2a^2}(1) - \tfrac{1}{2a^2}(0) = \tfrac{1}{2a^2},$$

$$\rho_4 = \int_0^1 \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}du = \frac{(a\mathrm{e}^a - e^a + 1)}{4a^2}.$$

The results follow directly by plugging ρ_1 - ρ_4 to ρ_u and ρ_v. ■

Given the specific values for a, b, and θ in the equation of Spearman's rho, this helps us to identify which possible values of the parameters that could give us the desired dependence measure we want including the maximum and the minimum Spearman's rho. Further, these numerical values provide an interpretation of the dependence using the copula.

In this note, a and b are arbitrary values as long as they satisfy the admissible range of copula parameter and choosing a and b very close to 0, the maximum value of θ give the largest Spearman's rho of the copula which is 0.3533. On the other hand, the minimum value of θ give the lowest Spearman's rho of the copula which is -0.1128. This implies that the copula can be best fitted with data having a Spearman's rho within -0.1128 and 0.3353.

Theorem 3.2.

The Kendall's tau of the bivariate case of Rodriguez-Lallena and Ubeda-Flores copula given in Theorem 2.1 is

$$\tau^{RLUF} = 8\theta\Big(\frac{2\mathrm{e}^a}{4a^2} - \frac{3\sqrt{\pi}\,\mathrm{erf}(\sqrt{-a})}{8a^2\sqrt{-a}} - \frac{a\mathrm{e}^a}{4a^2} + \frac{1}{4a^2}\Big)\Big(\frac{2\mathrm{e}^b}{4b^2} - \frac{3\sqrt{\pi}\,\mathrm{erf}(\sqrt{-b})}{8b^2\sqrt{-b}} - \frac{b\mathrm{e}^b}{4b^2} + \frac{1}{4b^2}\Big). \quad (11)$$

Proof:

The Kendall's tau can be obtained using the formula,

$$\begin{aligned}\tau^{RLUF} &= 4\int_0^1\int_0^1 c^{RLUF}(u,v)C^{RLUF}(u,v)dudv - 1\\ &= 4\int_0^1\int_0^1 \Bigg[uv + \theta\Big(\frac{au^2\mathrm{e}^{au^2} - e^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\Big)\times\\ &\qquad \Big(\frac{bv^2\mathrm{e}^{bv^2} - e^{bv^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)v}{2b^2}\Big)\\ &\qquad + \theta\Big(u^4\mathrm{e}^{au^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\Big)\Big(v^4\mathrm{e}^{bv^2} - \frac{(b\mathrm{e}^b - e^b + 1)v}{2b^2}\Big)\end{aligned} \quad (12)$$

$$+ \theta^2\Big(\frac{au^2\mathrm{e}^{au^2} - e^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\Big)\Big(u^3\mathrm{e}^{au^2} - \frac{(a\mathrm{e}^a - e^a + 1)}{2a^2}\Big)$$
$$\Big(\frac{bv^2\mathrm{e}^{bv^2} - e^{bv^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)v}{2b^2}\Big)\Big(v^3\mathrm{e}^{bv^2} - \frac{(b\mathrm{e}^b - e^b + 1)}{2b^2}\Big)\Bigg]dudv - 1.$$

To integrate Equation (12), we assign every terms as follows,

$$
\begin{aligned}
\tau_1 &= \int_0^1 \int_0^1 uvdudv,\\
\tau_2 &= \int_0^1 \int_0^1 \theta\Big(\frac{au^2\mathrm{e}^{au^2} - e^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\Big)\times\\
&\qquad\qquad \Big(\frac{bv^2\mathrm{e}^{bv^2} - e^{bv^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)v}{2b^2}\Big)dudv,\\
\tau_3 &= \int_0^1 \int_0^1 \theta\Big(u^4\mathrm{e}^{au^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\Big)\Big(v^4\mathrm{e}^{bv^2} - \frac{(b\mathrm{e}^b - e^b + 1)v}{2b^2}\Big)dudv,\\
\tau_4 &= \int_0^1 \int_0^1 \theta^2\Big(\frac{au^2\mathrm{e}^{au^2} - e^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\Big)\Big(u^3\mathrm{e}^{au^2} - \frac{(a\mathrm{e}^a - e^a + 1)}{2a^2}\Big)\times\\
&\qquad\qquad \Big(\frac{bv^2\mathrm{e}^{bv^2} - e^{bv^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)v}{2b^2}\Big)\Big(v^3\mathrm{e}^{bv^2} - \frac{(b\mathrm{e}^b - e^b + 1)}{2b^2}\Big)dudv.
\end{aligned}
$$

Using Lemma 2.1 and simplifying the results, we can integrate each term as follows,

$$
\begin{aligned}
\tau_1 &= \tfrac{1}{4},\\
\tau_2 &= \theta\Big(\tfrac{2\mathrm{e}^a}{4a^2} - \tfrac{3\sqrt{\pi}\,\mathrm{erf}(\sqrt{-a})}{8a^2\sqrt{-a}} - \tfrac{a\mathrm{e}^a}{4a^2} + \tfrac{1}{4a^2}\Big)\Big(\tfrac{2\mathrm{e}^b}{4b^2} - \tfrac{3\sqrt{\pi}\,\mathrm{erf}(\sqrt{-b})}{8b^2\sqrt{-b}} - \tfrac{b\mathrm{e}^b}{4b^2} + \tfrac{1}{4b^2}\Big),\\
\tau_3 &= \theta\Big(\tfrac{a\mathrm{e}^a - 2\mathrm{e}^a - 1}{4a^2} + \tfrac{3\sqrt{\pi}\,\mathrm{erf}(\sqrt{-a})}{8a^2\sqrt{-a}}\Big)\Big(\tfrac{b\mathrm{e}^b - 2\mathrm{e}^b - 1}{4b^2} + \tfrac{3\sqrt{\pi}\,\mathrm{erf}(\sqrt{-b})}{8b^2\sqrt{-b}}\Big),\\
\tau_4 &= 0.
\end{aligned}
$$

The results follow directly by plugging the results. ■

Choosing a and b very close to 0, the maximum value of θ give the largest Kendall's tau of the copula which is 0.2356. On the other hand, the minimum value of θ give the lowest Kendall's tau of the copula which is -0.0752. This implies that copula can be best fitted with data having a Kendall's tau within -0.0752 and 0.2356.

Theorem 3.3.

The Blomqvist's beta of the bivariate case of Rodriguez-Lallena and Ubeda-Flores copula given in Theorem 2.1,

$$
\begin{aligned}
\beta^{RLUF} = 4\theta\Big[\Big(&\frac{a(\frac{1}{2})^2\mathrm{e}^{a(\frac{1}{2})^2} - e^{a(\frac{1}{2})^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)(\frac{1}{2})}{2a^2}\Big)\times\\
&\Big(\frac{b(\frac{1}{2})^2\mathrm{e}^{b(\frac{1}{2})^2} - e^{b(\frac{1}{2})^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)(\frac{1}{2})}{2b^2}\Big)\Big].
\end{aligned}
\tag{13}
$$

Proof:

The Blomqvist's beta can be obtained using the formula,

$$\beta^{RLUF} = -1 + 4C^{RLUF}\left(\frac{1}{2}, \frac{1}{2}\right).$$

The results follow directly by plugging $C^{RLUF}(\frac{1}{2}, \frac{1}{2})$ to the Blomqvist's beta formula. ∎

Choosing a and b very close to 0, the maximum value of θ give the largest Blomqvist's beta of the copula which is 0.24769. On the other hand, the minimum value of θ give the lowest Blomqvist's beta of the copula which is -0.07902. This implies that copula can be best fitted with data having a Blomqvist's beta within -0.07902 and 0.24769.

4. Tail Dependence of a Copula

Theorem 4.1.

The bivariate case of Rodriguez-Lallena and Ubeda-Flores copula given in Theorem 2.1 is upper and lower tail independent.

Proof:

We shall show that $\lambda_U = \lambda_L = 0$. We begin solving the upper tail dependence and evaluate the limit of Step 6 using L'Hopital's rule.

$$\begin{aligned}
\lambda_U &= \lim_{u\to 1^-} \frac{1 - 2u + C^{RLUF}(u,u)}{1-u} \\
&= \lim_{u\to 1^-} \frac{1 - 2u + u^2 + \theta\left[\left(\frac{au^2\mathrm{e}^{au^2} - e^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\right)\left(\frac{bu^2\mathrm{e}^{bu^2} - e^{bu^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)u}{2b^2}\right)\right]}{1-u} \\
&= \lim_{u\to 1^-} \frac{1 - 2u + u^2}{1-u} + \theta \lim_{u\to 1^-} \frac{\left[\left(\frac{au^2\mathrm{e}^{au^2} - e^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\right)\left(\frac{bu^2\mathrm{e}^{bu^2} - e^{bu^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)u}{2b^2}\right)\right]}{1-u} \\
&= \lim_{u\to 1^-} \frac{(1-u)^2}{1-u} + \theta \lim_{u\to 1^-} \frac{\left[\left(\frac{au^2\mathrm{e}^{au^2} - e^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\right)\left(\frac{bu^2\mathrm{e}^{bu^2} - e^{bu^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)u}{2b^2}\right)\right]}{1-u} \\
&= \lim_{u\to 1^-} 1 - u + \theta \lim_{u\to 1^-} \frac{\left[\left(\frac{au^2\mathrm{e}^{au^2} - e^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\right)\left(\frac{bu^2\mathrm{e}^{bu^2} - e^{bu^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)u}{2b^2}\right)\right]}{1-u} \\
&= \theta \lim_{u\to 1^-} \frac{\left[\left(\frac{au^2\mathrm{e}^{au^2} - e^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - e^a + 1)u}{2a^2}\right)\left(\frac{bu^2\mathrm{e}^{bu^2} - e^{bu^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - e^b + 1)u}{2b^2}\right)\right]}{1-u} \\
&= -\theta \lim_{u\to 1^-} \Big[\Big(\frac{2a^2u^3\mathrm{e}^{au^2} - a\mathrm{e}^a - \mathrm{e}^a + 1}{a^2}\Big)\Big(\frac{bu^2\mathrm{e}^{bu^2} - \mathrm{e}^{bu^2} + 1}{2b^2} - \frac{(b\mathrm{e}^b - \mathrm{e}^b + 1)u}{2b^2}\Big)\times \\
&\qquad \Big(\frac{2b^2u^3\mathrm{e}^{bu^2} - b\mathrm{e}^b - \mathrm{e}^b + 1}{b^2}\Big)\Big(\frac{au^2\mathrm{e}^{au^2} - \mathrm{e}^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^a - \mathrm{e}^a + 1)u}{2a^2}\Big)\Big] = 0.
\end{aligned}$$

On the other hand, the lower tail dependence is computed as follows and we evaluate the limit of Step 4 using L'Hopital's rule.

$$\begin{aligned}
\lambda_L &= \lim_{u\to 0^+} \frac{C^{RLUF}(u,u)}{u}\\
&= \lim_{u\to 0^+} \frac{u^2 + \theta\Big[\Big(\frac{au^2\mathrm{e}^{au^2}-e^{au^2}+1}{2a^2} - \frac{(a\mathrm{e}^{a}-e^{a}+1)u}{2a^2}\Big)\Big(\frac{bu^2\mathrm{e}^{bu^2}-e^{bu^2}+1}{2b^2} - \frac{(b\mathrm{e}^{b}-e^{b}+1)u}{2b^2}\Big)\Big]}{u}\\
&= \lim_{u\to 0^+} \frac{u^2}{u} + \theta \lim_{u\to 0^+} \frac{\Big[\Big(\frac{au^2\mathrm{e}^{au^2}-e^{au^2}+1}{2a^2} - \frac{(a\mathrm{e}^{a}-e^{a}+1)u}{2a^2}\Big)\Big(\frac{bu^2\mathrm{e}^{bu^2}-e^{bu^2}+1}{2b^2} - \frac{(b\mathrm{e}^{b}-e^{b}+1)u}{2b^2}\Big)\Big]}{u}\\
&= \theta \lim_{u\to 0^+} \frac{\Big[\Big(\frac{au^2\mathrm{e}^{au^2}-e^{au^2}+1}{2a^2} - \frac{(a\mathrm{e}^{a}-e^{a}+1)u}{2a^2}\Big)\Big(\frac{bu^2\mathrm{e}^{bu^2}-e^{bu^2}+1}{2b^2} - \frac{(b\mathrm{e}^{b}-e^{b}+1)u}{2b^2}\Big)\Big]}{u}\\
&= \theta \lim_{u\to 0^+} \Big[\Big(\frac{2a^2u^3\mathrm{e}^{au^2} - a\mathrm{e}^{a} - \mathrm{e}^{a} + 1}{a^2}\Big)\Big(\frac{bu^2\mathrm{e}^{bu^2} - \mathrm{e}^{bu^2} + 1}{2b^2} - \frac{(b\mathrm{e}^{b} - \mathrm{e}^{b} + 1)u}{2b^2}\Big)\\
&\qquad \Big(\frac{2b^2u^3\mathrm{e}^{bu^2} - b\mathrm{e}^{b} - \mathrm{e}^{b} + 1}{b^2}\Big)\Big(\frac{au^2\mathrm{e}^{au^2} - \mathrm{e}^{au^2} + 1}{2a^2} - \frac{(a\mathrm{e}^{a} - \mathrm{e}^{a} + 1)u}{2a^2}\Big)\Big] = 0.
\end{aligned}$$

This completes the proof. ■

The variables are said to be asymptotically independent, since $\lambda_U = \lambda_L = 0$. This means that there is a probability in the limit that one variable takes a very low value, given that the other also takes a very low value. Similarly, there is a probability in the limit that one variable takes a very high value, given that the other also takes a very high value.

5. The Data

In this section, we will compare the RLUF copula to the existing bivariate copulas such as Ali-Mikhail-Haq (AHM), Farlie-Gumbel-Morgenstern (FGM), and Clayton copula. The procedures of the analysis are given by in which RMSE, SSE, MAE, MSE, AIC and BIC were used determine the best model.

This study used secondary data obtained from https://power.larc.nasa.gov/data-access-viewer/, which provide the data sets containing average monthly precipitation (mm) and temperature ($°C$) in the province of Iloilo, Philippines, specifically in the Municipality of Conception. The data consists of 468 (months) starting from January 1974 to December 2013 and it was used fit the copula model.

Table 1 display the descriptive statistics of monthly precipitation and temperature. The table revealed that Philippines has minimum monthly precipitation of 0.36 and maximum monthly precipitation of 24.77, and it happened on March 2013 and December 2013, respectively. On the hand, Philippines has minimum monthly temperature of 24.77 and maximum monthly temperature of 30.09, and it happened on January 1974 and May 2013, respectively.

Table 1. Descriptive Statistics

	N	Minimum	Maximum	Mean	Std. Deviation
Precipitation	468	0.36	629.82	169.63	121.18
Temperature	468	24.77	30.09	27.54	1.09

Figure 1 presents the scatter plot between monthly precipitation (Y) and monthly temperature (X). Figure 2 presents the scatter plot empploying the rescaled versions of empirical margins given as $u = F_n(x) = \dfrac{R_i}{n+1}$ and $v = G_n(y) = \dfrac{S_i}{n+1}$.

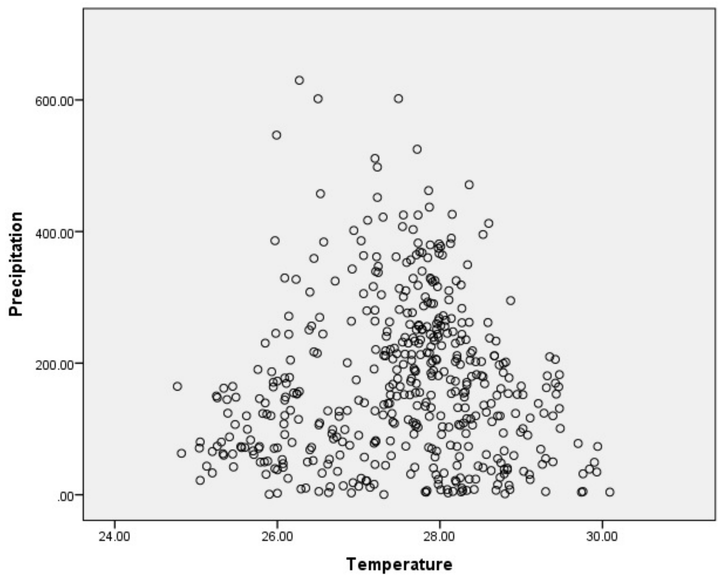

Figure 1. Scatter plot between monthly precipitation and temperarure

6. Copula Model Selection

In this section, the procedures for analysis and results for model fitting with the data will be given. The maple software will be used in estimating the parameters in order to obtain the RMSE, SSE, MAE, MSE, AIC and BIC values. The steps are as follows.

Step 1: Obtain the empirical frequency of each data points points $x_1, x_2, x_3, \ldots, x_n$ using the empirical formula of the copula

$$C_n(x_i, y_i) = \frac{1}{n}\sum_{k=1}^{n} 1\left(\frac{R_i}{n+1} \leq x_i, \frac{S_i}{n+1} \leq y_i\right).$$

Step 2: Obtain the sum of the square error

$$SSE = \sum_{i=1}^{n} \left((C(u_i, v_i) - C^*(u_i, v_i)\right)^2,$$

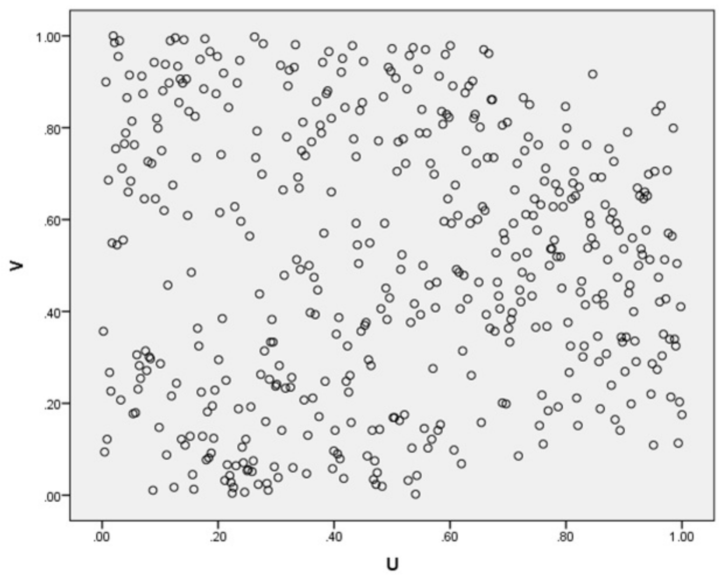

Figure 2. Scatter plot between monthly precipitation and temperarure (rescaled version)

where C is the fitted copula, and C^* is the empirical copula.

Step 3: Minimize the SSE by applying the optimization method from the calculus. We obtain the partial derivatives of SSE with respect to each parameter, set the partial derivatives to zero and solve for the parameters. The calculated values of the parameters will minimize the SSE.

Step 4: Substitute the value of the parameters to the following model selection:

$$RMSE = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\Big((C(u_i,v_i) - C^*(u_i,v_i)\Big)^2},$$

$$MAE = \frac{1}{n}\sum_{i=1}^{n}\Big|(C(u_i,v_i) - C^*(u_i,v_i)\Big|,$$

$$MSE = \frac{1}{n}\sum_{i=1}^{n}\Big(C(u_i,v_i) - C^*(u_i,v_i)\Big)^2.$$

Step 5: Substitute the value of the parameters to the copula to get maximum value of the log-likelihood $l(\theta)$ and apply to the formula of

$$AIC = -2l(\theta) + 2p,$$

and

$$BIC = -2l(\theta) + p\log(n).$$

where p is the number of free parameters and n is the sample size.

Step 6: The copula with the smallest RMSE, SSE, MAE, SSE, AIC or BIC value is the best fit model.

Table 2. Copula Fitting of Temperature and Relative Humidity

COPULA	Parameters	RMSE	SSE	MAE	MSE	AIC	BIC
RLUF	$\theta = -0.1234$ $a = 1.0545$ $b = 1.2657$	**0.0215**	**0.2164**	**0.0164**	**0.00046**	**-11.1191**	**-10.4488**
AMH	$\theta = 0.0415$	0.0222	0.2318	0.0177	0.00049	2.0970	2.7672
FGM	$\theta = 0.0360$	0.0222	0.2319	0.0177	0.00049	2.0999	2.7702
Clayton	$\theta = 0.0750$	0.0216	0.2200	0.0174	0.00047	0.4100	1.0802

Table 2 presents the copula fitting among the RLUF, Ali-Mikhail-Haq, FGM and Clayton copula. It can be seen from the table that the generated copula has the smallest value of RMSE = 0.0215, SSE = 0.2164, MAE = 0.0164, MSE = 0.00046, AIC $= -11.119124$ and BIC $= -10.448878$. The results implies that RLUF copula has the best fit.

7. Conclusion

The construction of a new copula approach has been interesting over time. In this article, we choose a quadratic-exponential having two parameters and apply the Rushendorf method to obtained a function similar to the form of Rodriguez-Lallena and Ubeda-Flores copula. We have also studied measures of dependence properties such as Spearman's Rho, Kendall's tau, Blomqvist beta, and the tail dependence. Moreover, the RLUF copula outperformed the other existing bivariate copulas such as Ali-Mikhail-Haq (AHM), Farlie-Gumbel-Morgenstern (FGM), and Clayton copulas.

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