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An Alternative Approach To High School Geometry

James L. Williams

AN ALTERNATIVE APPROACH TO
HIGH SCHOOL GEOMETRY

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The emphasis throughout this paper was to give an understanding of the basic principles of approaching High School Geometry from an analytic geometry approach. Considerable care was taken with the proof of the main theorems so that we may develop an appreciation of the logical structure of a mathematical proof.

AN ALTERNATIVE APPROACH TO HIGH SCHOOL GEOMETRY

By

James L. Williams

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D E D I C A T I O N

This thesis is affectionately dedicated to my beloved wife, Mrs. Dorothy S. Williams, whose encouragements and inspirations have made this thesis possible.

T A B L E O F C O N T E N T S

CHAPTER I

Introduction and Terminology

The fundamental ideas of analytic geometry are usually attributed to the French mathematician and philosopher Descartes (1596-1650). The key to the expression of geometric facts in algebraic form lies in the representation of a point in the plane by means of a pair of real numbers called the coordinates of the point. This paper is devoted to some detailed proofs of fundamental theorems in High School Geometry based on an alternative approach.

The analytic geometry approach seems to be a more powerful attack upon many of the problems of High School Geometry than the methods which we have thus far employed. Analytic geometry not only simplifies the proofs of many of the propositions with which we are familiar, but enables us to attack successfully problems which we could handle in elementary geometry only with great difficulty, or not at all. With the tools already developed-the formulas for distance, point of division (midpoint), and slope- will aid in solving many problems of High School Geometry.

In analytic geometry the methods of algebra are combined with those of Euclidean geometry in the solution of geometry problems. The properties of a geometric figure depend upon the relations of the parts and not upon the particular position which the figure is drawn. Therefore, the properties of any geometric figure are independent of the way in which the axes are chosen. In the proof of geometric properties of figures it will, in general, be possible to choose the axes in more than one way. The axes will be chosen in the way which gives the simplest algebra.

The writer would like to point out that analytic geometry is not a different geometry but is a different approach to geometry. This approcah was used to prove theorems previously developed by the synthetic approach. In all such cases the analytic proof is not the only proof, but in many cases it is a far simpler proof.

The statement, symbol or notation on the left has meaning on the right.

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(1) Angle of inclination	(1) The statement that θ is the angle of inclination means that θ is the angle between the line ℓ and the x-axis on the positive side.
(2) Bisector of an angle	(2) The ray which divides the angle into two equal angles.
(3) Equilateral triangle	(3) A triangle having all congruent sides.
(4) Isosceles triangle	(4) A triangle with at least two congruent sides.
(5) Midpoint	(5) The point which divides the line segment into equal line segments.
(6) Parallelogram	(6) A quadrilateral in which both pairs of opposite sides are parallel.
(7) Perpendicular lines	(7) Two lines that meet to form congruent adjacent angles.
(8) Rectangle	(8) A parallelogram with four right angles.
(9) Reflexive Property of Equality	(9) Any quantity is equal to itself.
(10) Rhombus	(10) A parallelogram with a pair of adjacent sides equal.
(11) Right angle	(11) An angle of measure 90°.
(12) Slope of a line	(12) The statement that m is the slope of the line ℓ means that there exists two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ such that $m = \frac{y_2-y_1}{x_2-x_1}$, $x_2 > x_1$.

(13) Transitive Property of Equality	(13) Two numbers equal to the same or equal number are equal to each other.
(14) Trapezoid	(14) A quadrilateral with exactly two sides parallel.
(15) Trigonometric Cofunctions	(15) The statement that two functions are trigonometric cofunctions means their arguments are complementary.
(16) Trigonometric identities	(16) If f and g are trigonometric functions then the equation f = g is said to be an identity iff $f(x) = g(x)$ $\forall$ x domain of f $\cap$ g.
(17) $=$	(17) Is equal to
(18) $\neq$	(18) Is not equal to
(19) $>$	(19) Is greater than
(20) $\angle$	(20) Angle
(21) $\triangle$	(21) Triangle
(22) $\longleftrightarrow$	(22) Line
(23) $\overline{P_1P_2}$	(23) Line segment $\overline{P_1P_2}$
(24) $\perp$	(24) Is perpendicular to
(25) $\cong$	(25) Is congruent to
(26) $\parallel$	(26) Is parallel to
(27) $\sqrt{\ }$	(27) The square root of
(28) $\nparallel$	(28) Is not parallel to
(29) $\square$	(29) Parallelogram
(30) m	(30) Slope
(31) $\tan\ \theta$	(31) Tangent of angle θ

(32)	$\therefore$	(32)	Therefore
(33)	α	(33)	Alpha
(34)	β	(34)	Beta
(35)	δ	(35)	Delta
(36)	θ	(36)	Theta
(37)	S.A.S.	(37)	If two sides of one triangle are equal to two sides of a second triangle and the angles included by these sides are equal, then the triangles are congruent.
(38)	A.S.A.	(38)	If two angles of one triangle are equal to two angles of a second triangle and the side included by these angles are equal, then the triangles are congruent.
(39)	S.S.S.	(39)	If the three sides of one triangle are equal, respectively, to the three sides of a second triangle, then the triangles are congruent.
(40)	H.L	(40)	If the hypotenuse and a leg of one triangle are congruent to the hypotenuse and a leg of another right triangle, the triangles are congruent.
(41)	L.L.	(41)	If the legs of one right triangle are congruent to the legs of another right triangle, the triangles are congruent.
(42)	C.P.C.T.	(42)	Corresponding parts of congruent triangles are equal.

CHAPTER II

Theorems

In this chapter is a list of all basic theorems and their proofs to be used in proving High School Geometry from an analytic approach.

Theorem: 1.1

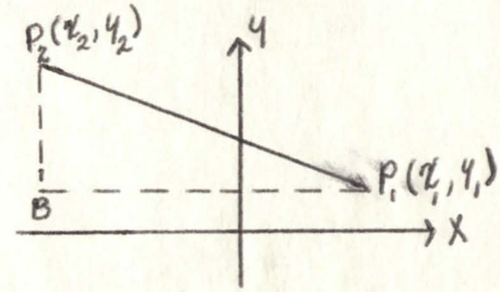
The Distance between Two Points

The distance between two points is given by the formula

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} \ .$$

Given: Points P_1 and P_2 with the coordinates (x_1,y_1) and (x_2,y_2) respectively $P_1P_2 = d$.

Prove: $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$.



Proof:

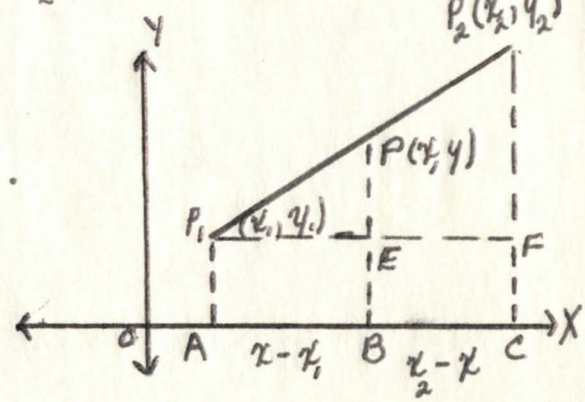
Statements	Reasons
1. Draw $P_1B \parallel$ to the x-axis and $P_2B \parallel$ to the y-axis .	1. Through a given point, a line can be constructed parallel to a given line.
2. $\angle P_2BP_1$ is a right angle .	2. Definition of perpendicular lines.
3. $BP_1 = x_2-x_1$ and $P_2B = y_2-y_1$.	3. The distance between two points having the same coordinates is the difference of their abscissas and the distance between two points having the same abscissas is the difference of their ordinates.
4. $\overline{P_2P_1}^2 = \overline{BP_1}^2 + \overline{P_2B}^2$.	4. Pythagorean theorem .
5. $d^2 = (x_2-x_1)^2 + (y_2-y_1)^2$.	5. Substitution .
6. $\therefore d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$.	6. Taking the square root of both sides of the equation.

Theorem 1.2

The Midpoint of a Line Segment

The coordinates of the midpoint of line segment are one-half the sums of the coordinate of the end points or

$$x = \frac{x_1 + x_2}{2} \quad \text{and} \quad y = \frac{y_1 + y_2}{2} \quad .$$



Given: P the midpoint of line segment P_1P_2.

Prove: $x = \frac{x_1+x_2}{2}$ and $y = \frac{y_1+y_2}{2}$.

Proof:

Statements	Reasons
1. Draw P_1A, PB, and $P_2C \perp$ the x-axis.	1. Through a given point, a line can be constructed perpendicular to a given line.
2. $P_1A \parallel PB \parallel P_2C$.	2. Two or more lines which are perpendicular to the same line are parallel.
3. $P_1P = \overline{PP_2}$.	3. Definition of midpoint .
4. $x - x_1 = x_2 - x$.	4. Substitution .
5. $2x = x_2 + x_1$.	5. Addition property .
6. $\therefore x = \frac{x_2 + x_1}{2}$.	6. Division property .

Similarly, a line through P_1,P, and P_2 perpendicular to the y-axis we can prove that

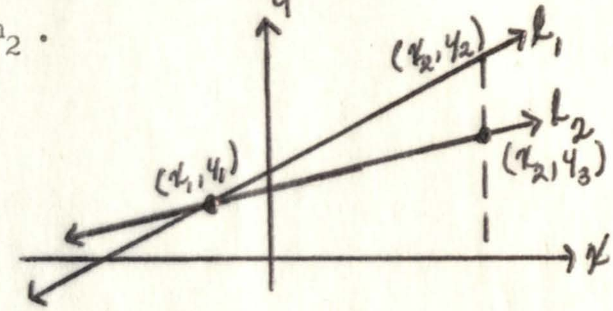
$$y = \frac{y_1 + y_2}{2} \quad .$$

Theorem 1.3

Two non-vertical lines are parallel if and only if they have the same slope.

Part I: Two non-vertical lines have the same slope, then they are parallel.

If $\ell_1 \nparallel \ell_2$ then $m_1 \neq m_2$.



Given: $\ell_1 \nparallel \ell_2$.

Prove: $m_1 \neq m_2$.

Proof:

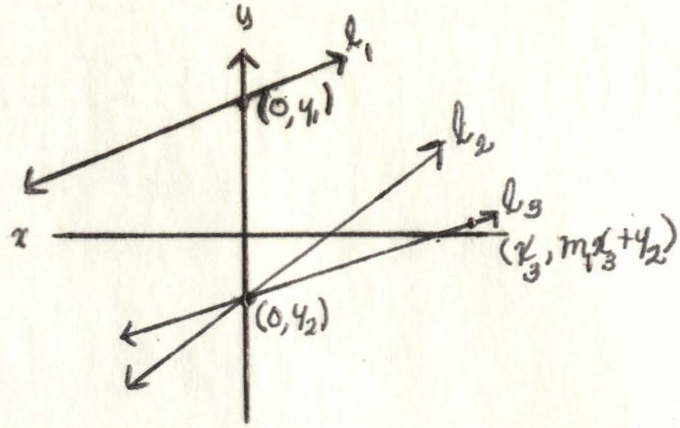
Statements	Reasons
1. ℓ_1 and ℓ_2 intersect at some common point (x_1, y_1) .	1. It was given that $\ell_1 \nparallel \ell_2$.
2. There exist point (x_2, y_2) and (x_2, y_3) on ℓ_1 and ℓ_2, respectively, $y_2 \neq y_3$.	2. Construction .
3. $m_1 = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $m_2 = \dfrac{y_3 - y_1}{x_2 - x_1}$.	3. Definition of slope .
4. $\therefore m_1 \neq m_2$.	4. From step 3 .

Part II: If two non-vertical lines are parallel, then they have the same slope.

Given: $\ell_2 \parallel \ell_1$, ℓ_1 and ℓ_2 are non-vertical;

Slope of $\ell_1 = m_1$; Slope of $\ell_2 = m_2$.

Prove: $m_1 = m_2$.



Proof:

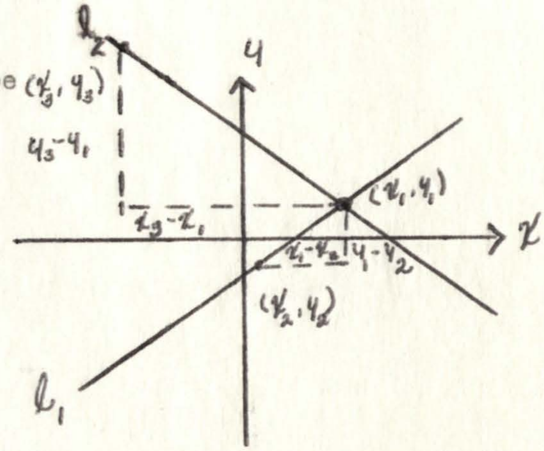
Statements	Reasons
1. Since ℓ_1 and ℓ_2 are non-vertical, they will intersect the y-axis at point $(0,y_1)$ and $(0,y_2)$ respectively. Assume $y_1 \neq 0$. (If $y_1 = 0$, interchange the role of y_1 in y_2 in the rest of the proof.)	
2. Suppose that $m_1 \neq m_2$.	
3. There exist some point having the coordinates $(x_3, m_1x_3+y_2)$, $(x_3 \neq 0)$.	3. There exist a one-to-one correspondence between points in a plane and ordered pairs of real numbers.
4. There exist a line ℓ_3 containing the point $(0,y_2)$ and $(x_3, m_1x_3+y_2)$.	4. Given any two points, there exists exactly one line containing them.
5. ℓ_3 has a slope of m_1 and is therefore parallel to ℓ_1.	5. Definition of slope (Theorem 1.3, part I).
6. But ℓ_3 and ℓ_2 pass through $(0,y_2)$ and are parallel to ℓ_1.	6. Construction.
7. $\therefore \ell_3 = \ell_2$ and it follows that $m_1 = m_2$.	7. Through a given point not on a line, there exists exactly one line parallel to the given line.

Lemma 1

Two non-vertical perpendicular lines ℓ_1 and ℓ_2 having slopes m_1 and m_2 then, one slope is positive and the other slope is negative.

Given: Two non-vertical $\perp$ lines ℓ_1 and ℓ_2 having slopes m_1 and m_2 respectively.

Prove: One slope is positive and one slope is negative.



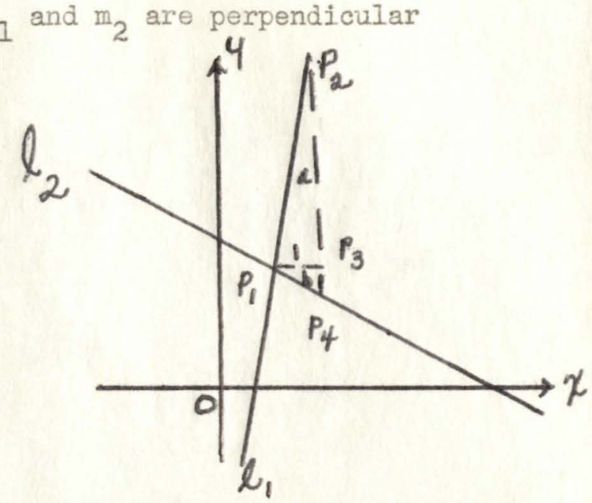
Proof:

Statements	Reasons
1. Let (x_1, y_1) be the point of intersection of lines ℓ_1 and ℓ_2.	1. Assumption .
2. $\ell_1 \perp \ell_2$.	2. Given .
3. Choose (x_2, y_2) on ℓ_1 and (x_3, y_3) on ℓ_2 .	3. Assume .
4. $m_1 = \dfrac{y_1 - y_2}{x_1 - x_2}$, $m_2 = \dfrac{y_3 - y_1}{x_2 - x_1}$.	4. Definition of slope .
5. $m_1 = \dfrac{y_1 - y_2}{x_1 - x_2} > 0$.	5. Ratio of two positive numbers is a positive number .

Statements	Reasons
6. $y_3 - y_1 > 0$ and $x_3 - x_1 < 0$ $$\Rightarrow \frac{y_3 - y_1}{x_3 - x_1} < 0$$ $$\Rightarrow m_2 < 0 \quad .$$	6. Ratio of a positive and negative number is negative.
7. $\therefore$ one slope is positive and one slope is negative.	7. From step 5 and step 6.

Theorem 1.4

If lines ℓ_1 and ℓ_2 having slopes m_1 and m_2 are perpendicular then $m_1m_2 = -1$.



Part I.

Given: Lines ℓ_1 and ℓ_2 with m_1 and m_2

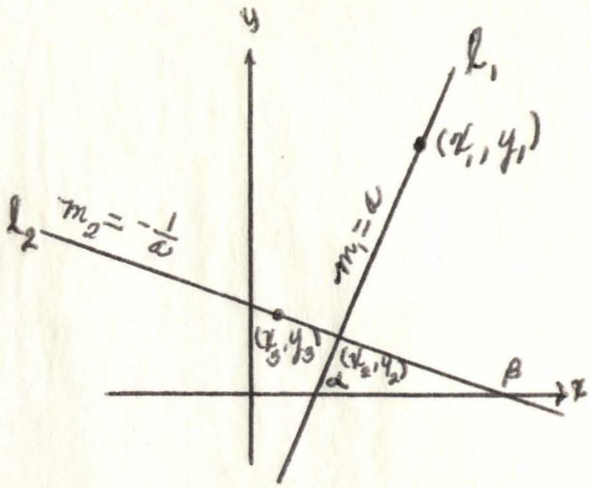
$\ell_1 \perp \ell_2$.

Prove: $m_1m_2 = -1$.

Proof:

Statements	Reasons
1. Let ℓ_1 and ℓ_2 intersect .	1. Assumption .
2. $\overline{P_1P_3}$ is parallel to the x-axis.	2. Construction .
3. $m_1 > 0$ and $m_2 < 0$.	3. Lemma 1 .
4. P_3 is 1 unit to the right of P_1.	4. Construction .
5. $\overline{P_3P_2}$ = a units vertical through P_3 to meet ℓ_1 at P_2 and ℓ_2 at P_4. $\overline{P_3P_4}$ = b units.	5. Construction .
6. $m_1 = a$ and $m_2 = -b$.	6. Lemma 1 .
7. $\overline{P_1P_3}$ is the altitude on the hypotenuse $\overline{P_2P_4}$ of the right $\triangle P_1P_2P_3$.	7. $\ell_1 \perp \ell_2$.
8. $\overline{P_1P_3}$ is the mean proportional between $\overline{P_3P_2}$ and $\overline{P_4P_3}$.	8. Definition of mean proportional .
9. $\overline{P_3P_2} \cdot \overline{P_4P_3} = \overline{P_1P_3}^2$.	9. Same as step 8 .
10. $a \cdot b = 1$.	10. Substitution .
11. $a \cdot b = -(m_1 \cdot m_2)$.	11. From step 7 .
12. $m_1m_2 = -1$.	12. Substitution .

Part II.



Given: Lines l_1 and l_2 with slopes m_1 and m_2; $m_1 m_2 = -1$.

Prove: $l_1 \perp l_2$.

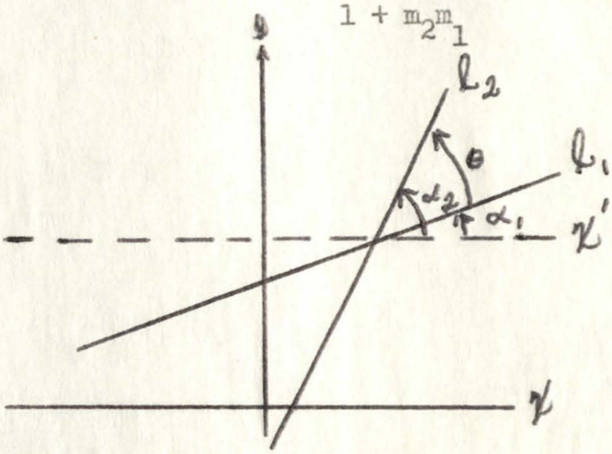
Proof:

Statements	Reasons
1. Let l_1 with slope m_1 and l_2 with slope m_2 be given .	1. Given .
2. $m_1 \cdot m_2 = -1 \Rightarrow m_1 = \frac{-1}{m2}$.	2. Given .
3. m_1 and m_2 are opposite each other.	3. Lemma 1 .
4. $m_1 = \frac{y_1 - y_2}{x_1 - x_2} = a, m_2 = \frac{y_3 - y_2}{x_3 - x_2} = -\frac{1}{a}$.	4. Definition of slope .
5. $m_1 = \tan \alpha = \frac{y_1 - y_2}{x_1 - x_2} = a$ $m_2 = \tan \beta = \frac{y_3 - y_2}{x_3 - x_2} = -\frac{1}{a}$.	5. Definition of trigonometric identity .
6. $\tan \alpha = -\cot \beta$.	6. Substitution .
7. $-\cot \beta = \tan(\beta + 90°)$.	7. Definition of cofunction .
8. $\beta = \alpha + 90°$.	8. Since tangent and cotangent are cofunctions .
9. $\therefore l_1 \perp l_2$.	9. From step 7 and step 8 .

Theorem 1.5

Let ℓ_1 and ℓ_2 be lines with slope m_1 and m_2 respectively, and let θ be the angle from ℓ_1 to ℓ_2. If $m_1 m_2 = -1$, $\theta = 90°$. Otherwise, θ is the angle such that

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}, \quad 0° \leq \theta < 180°.$$



Proof: Let m_1 be the slope of ℓ_1 and let m_2 be the slope of ℓ_2.

Then $m_1 = \tan\alpha_1$, $m_2 = \tan\alpha_2$.

θ is the angle between ℓ_2 and ℓ_1.

$$\tan\theta = \tan(\alpha_2 - \alpha_1)$$

$$= \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_2 \tan\alpha_1} \quad \text{(from trigonometry)}$$

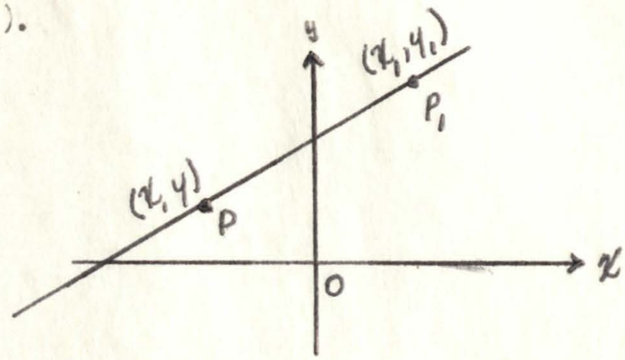
$$= \frac{m_2 - m_1}{1 + m_2 m_1}$$

Hence

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1}.$$

Theorem 1.6

The equation of the line passing through the point (x_1, y_1) and having slope m is $y - y_1 = m(x - x_1)$.



Given: Points P and P_1 with the coordinates (x, y) and (x_1, y_1) respectively.

Prove: $y - y_1 = m(x - x_1)$.

Proof: Let P be the point (x, y) and P_1 be the point (x_1, y_1).

The slope (m) of $\overline{PP_1}$ is expressed by $\frac{y-y_1}{x-x_1}$.

Since the slope of $\overleftrightarrow{PP_1}$ must be m, $\frac{y - y_1}{x - x_1} = m$.

Hence $y - y_1 = m(x - x_1)$.

Theorem 1.7

The circle with center (a,b) and radius r has the equation $(x - a)^2 + (y - b)^2 = r^2$.

CHAPTER III

Triangles

In this chapter, the two methods will be applied to several theorems related to triangles. First, the synthetic method will be applied to prove a theorem, then the analytic method will be used to prove the same theorem in sequence. Theorems proved by the synthetic method will be denoted by unprime numbers while the same theorems being proved by the analytic method will be denoted by prime numbers. Example, Theorem 1 and Theorem 1'.

Theorem 1

If two sides of a triangle are equal, the angles opposite these sides are equal.

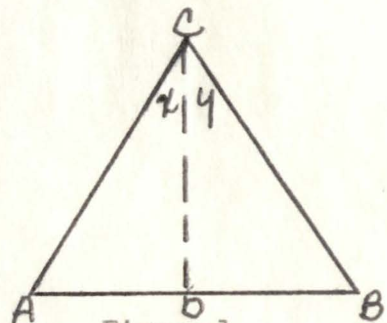


Figure 1

Given: $\triangle ABC$ with $\overline{AC} = \overline{BC}$.

Prove: $m\angle A = m\angle B$.

Proof: Construct $\overline{CD}$, the bisector of $\angle C$.

Statements	Reasons
1. $\overline{AC} = \overline{BC}$.	1. Given .
2. $\angle x = \angle y$.	2. Definition of bisector of an angle .
3. $\overline{DC} = \overline{DC}$.	3. Reflexive Property .
4. $\triangle ADC \cong \triangle BDC$.	4. S.A.S.
5. $\therefore m\angle A = m\angle B$.	5. C.P.C.T.

Theorem 1'

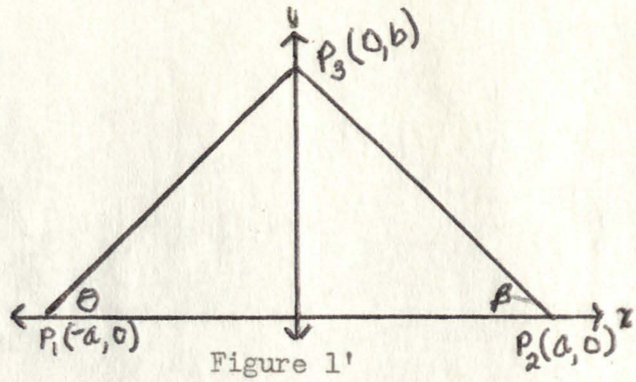


Figure 1'

Given: $\Delta P_1P_2P_3$ with $\overline{P_1P_3} = \overline{P_2P_3}$.

Prove: $m\angle\theta = m\angle\beta$.

Proof: Let P_1P_2 lie on the x-axis and the altitude from P_3 lie on the y-axis. Let the coordinates of P_1P_2 and P_3 be as shown. By the slope formula:

1. m_1 - slope of $\overline{P_1P_3}$, $\frac{b - 0}{0 - {}^-a} = \frac{b}{a}$

   m_2 - slope of $\overline{P_2P_3}$, $\frac{b - 0}{0 - a} = -\frac{b}{a}$

   m_3 - slope of $\overline{P_1P_2}$, $\frac{0 - 0}{a - {}^-a} = 0$.

2. $\tan\theta = \frac{m_1 - m_3}{1 + m_3 \cdot m_1} = \frac{\frac{b}{a} - 0}{1 + 0\ \frac{b}{a}} = \frac{b}{a}$.

3. $\tan\beta = \frac{m_3 - m_2}{1 + m_2 \cdot m_3} = \frac{0 - -\frac{b}{a}}{1 + 0\ -\frac{b}{a}} = \frac{b}{a}$.

4. $\tan\theta = \tan\beta$.

5. Hence $\theta = \beta$ since θ and $\beta < 180°$.

Theorem 2

If a point is on the perpendicular bisector of a line segment, it is equally distant from the ends of the segment.

Given: Line segment AB and perpendicular bisector ℓ and point P on line ℓ .

Prove : $\overline{AP} \cong \overline{BP}$.

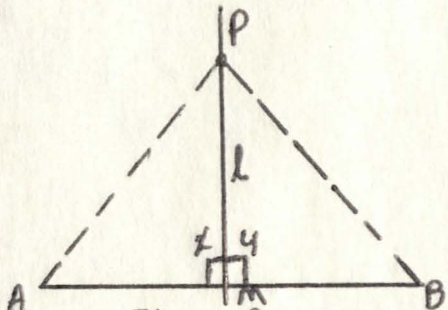


Figure 2

Proof:

Statements	Reasons
1. $\overline{AM} = \overline{BM}$.	1. Definition of bisector .
2. $\angle x = \angle y$.	2. Definition of perpendicular lines .
3. $\overline{PM} = \overline{PM}$.	3. Reflexive Property .
4. $\triangle AMP \cong \triangle BMP$.	4. S.A.S
5. $\therefore \overline{AP} \cong \overline{BP}$.	5. C.P.C.T.

Theorem 2'

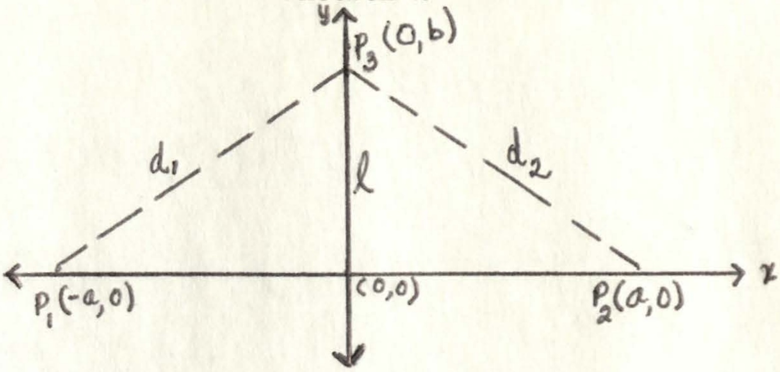


Figure 2'

Given: Line segment $\overline{P_1P_2}$ and perpendicular bisector ℓ and point P_3 on line ℓ.

Prove: $d_1 = d_2$.

Proof: Let P_1P_2 lie on the x-axis with line ℓ on the y-axis. Let the coordinates of P_1, P_2, and P_3 be as shown in the figure. $d_1 = \overline{P_1P_3}$ and $d_2 = \overline{P_2P_3}$. By the distance formula:

1. $d_1 = \sqrt{(-a - 0)^2 + (0 - b)^2} = \sqrt{a^2 + b^2}$.

2. $d_2 = \sqrt{(a - 0)^2 + (0 - b)^2} = \sqrt{a^2 + b^2}$.

3. Hence $d_1 = d_2$.

Theorem 3

If two parallel lines are crossed by a transversal, the alternate interior angles are equal.

Given: Line $\overleftrightarrow{XY} \parallel$ line $\overleftrightarrow{ZW}$. Both lines are cut by transversal TR at points A and B.

Prove: $\angle x = \angle y$.

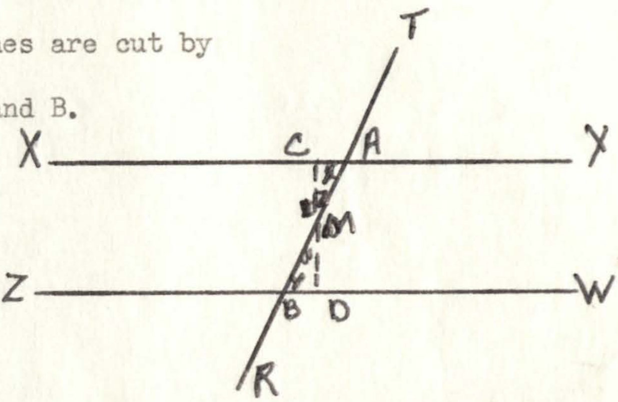


Figure 3

Proof: Construct a perpendicular to line ZW at DM, the midpoint of BA.

Statements	Reasons
1. $\overline{CD} \perp \overline{XY}$.	1. If one of two parallel lines is $\perp$ to a third line, the other is $\perp$ to it.
2. $\triangle BDM$ and $\triangle ACM$ are right triangles .	2. Definition of right triangles .
3. $\overline{BM} = \overline{AM}$.	3. Definition of a midpoint.
4. $\angle z = \angle w$.	4. Vertical angles are equal.
5. $\triangle BDM \cong \triangle ACM$.	5. Congruent hypotenuse and acute angle .
6. $\angle x = \angle y$.	6. C.P.C.T.

Theorem 3'

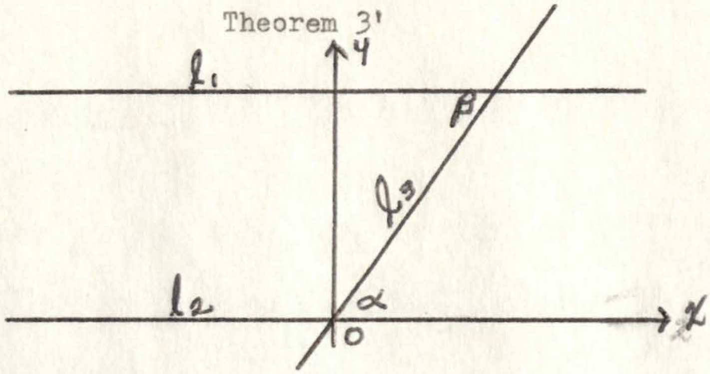


Figure 3'

Given: Line $l_1 \parallel$ line l_2. Both lines are cut by transversal l_3.

Prove: $\angle\alpha = \angle\beta$.

Proof: Let l_2 lie on the x-axis and $l_1 \parallel l_2$. Let l_3 intersect l_2 at the origin and l_3 at some point. By the slope formula:

1. m_1 - slope of l_2

   m_2 - slope of l_1

   m_3 - slope of l_3.

2. $$\tan\alpha = \frac{m_3 - m_1}{1 + m_3 m_1} = \frac{m_3 - 0}{1 + m_3(0)} = m_3$$

   $$\tan\beta = \frac{m_3 - m_2}{1 + m_3 m_1} = \frac{m_3 - 0}{1 + m_3(0)} = m_3 .$$

3. So $\tan\alpha = \tan\beta$ if α and $\beta < 180°$.

4. $\therefore \angle\alpha = \angle\beta$.

Theorem 4

The line that joins the midpoints of two sides of a triangle is parallel to the third sides.

Given: Line MN joining the midpoints of $\overline{AB}$ and $\overline{AC}$ of $\triangle ABC$. $\overline{CD}$ is drawn parallel to $\overline{AB}$, meeting $\overline{MN}$ extended at D.

Prove: $\overline{MN} \parallel \overline{BC}$.

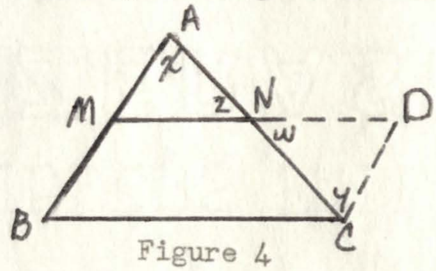


Figure 4

Proof:

Statements	Reasons
1. $\angle z = \angle w$.	1. Vertical angles are equal .
2. $\overline{AN} = \overline{CN}$.	2. Definition of midpoint .
3. $\angle x = \angle y$.	3. Alternate interior angles are equal .
4. $\triangle CND \cong \triangle AMN$.	4. A.S.A.
5. $\overline{CD} = \overline{AM}$.	5. C.P.C.T.
6. $\overline{BM} = \overline{AM}$.	6. Definition of midpoint.
7. $\overline{CD} = \overline{BM}$.	7. Transitive Property .
8. BMDC is a ▱ .	8. A pair of opposite sides of a quadrilateral are both equal and parallel.
9. $\therefore \overline{MN} \parallel \overline{BC}$.	9. Definition of a parallelogram .

Theorem 4'

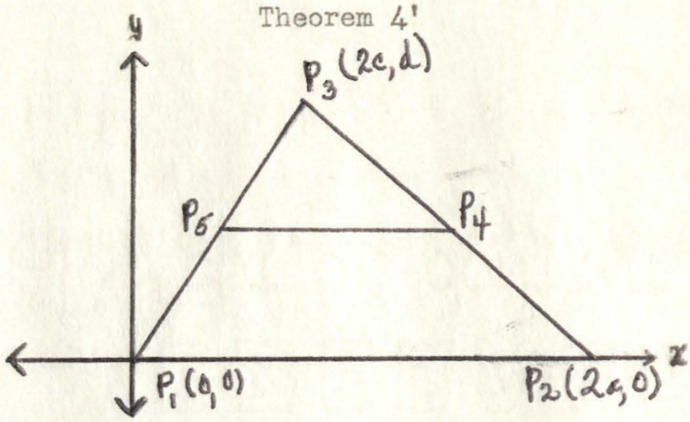


Figure 4'

Given: $\triangle P_1P_2P_3$ with P_5 and P_4 joining midpoints of $\overline{P_1P_3}$ and $\overline{P_2P_3}$.

Prove: $\overline{P_5P_4} \parallel \overline{P_1P_2}$.

Proof: Let P_1P_2 lie on the x-axis with P_1 at the origin and coordinates of P_2 and P_3 be as shown. By the midpoint formula P_5 is the point (c,d) and P_4 is the point (a+c,d).

1. m_1 - slope of $\overline{P_5P_4}$

   m_2 - slope of $\overline{P_1P_2}$.

2. $m_1 = \dfrac{d - d}{a+b-0} = 0$

   $m_2 = \dfrac{0 - 0}{a - 0} = 0$.

3. Hence $m_1 = m_2$.

4. $\therefore \overline{P_5P_4} \parallel \overline{P_1P_2}$.

Theorem 5

The line segment that joins the midpoint of two sides of a triangle is equal to one half of the third side.

Given: $\triangle ABC$ and line segment MN joining the midpoints of AB and AC.

Prove: $\overline{MN} = 1/2(\overline{BC})$.

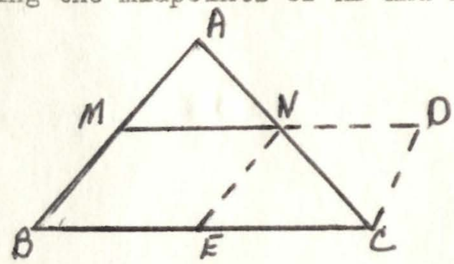


Figure 5

Proof: Draw $\overline{CD} \parallel \overline{AB}$ meeting MN extended to D. Find E, the midpoint.

Statements	Reasons
1. $\overline{MN} \parallel \overline{BC}$.	1. The line that joins the midpoints of two sides of a triangle is parallel to the third side.
2. BCDM is a ▱ .	2. Definition of parallelogram.
3. $\overline{NE} \parallel \overline{AB}$.	3. Same as step 1 .
4. BENM is a ▱ .	4. Definition of parallelogram .
5. $\overline{MN} = \overline{BE}$.	5. Opposite sides of a parallelogram are equal .
6. $\overline{BE} = 1/2(\overline{BC})$.	6. Definition of a midpoint .
7. $\overline{MN} = 1/2(\overline{BC})$.	7. Transitive Property .

Theorem 5'

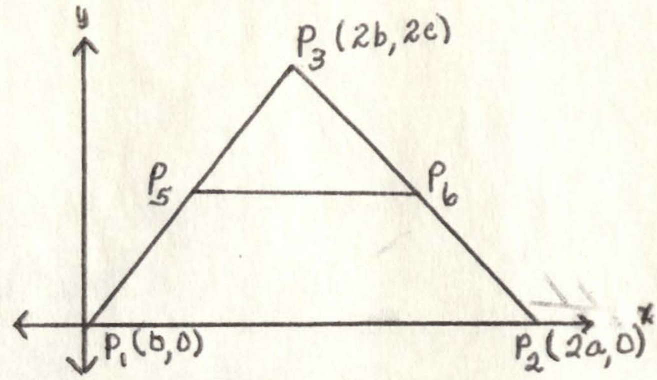


Figure 5'

Given: $\Delta P_1P_2P_3$; P_5 is the midpoint of $\overline{P_1P_3}$ and P_6 is the midpoint of $\overline{P_2P_3}$.

Prove: $\overline{P_5P_6} = 1/2(\overline{P_1P_2})$.

Proof: Let the line containing P_1P_2 be the x-axis and let the coordinates of P_1, P_2 and P_3 be as shown. By the midpoint formula:

1. $P_5 = \left(\frac{0 + 2b}{2}, \frac{0 + 2c}{2}\right) = (b,c)$.

2. $P_6 = \left(\frac{2a + 2b}{2}, \frac{0 + 2c}{2}\right) = \frac{2(a + b, c)}{2} = (a + b, c)$.

3. $\overline{P_1P_2} = \sqrt{(2a - 0)^2 + (0 - 0)^2} = \sqrt{4a^2} = 2a$.

4. $\overline{P_5P_6} = \sqrt{(a + b - b)^2 + (c - c)^2} = \sqrt{a^2} = a$.

5. $\therefore \overline{P_5P_6} = 1/2(\overline{P_1P_2})$.

Theorem 6

In an isosceles triangle, two medians are congruent.

Given: Isosceles $\triangle ABC$; $\overline{AD} = \overline{BE}$; $\angle BAC = \angle ABC$.

Prove: $\overline{AE} = \overline{BD}$.

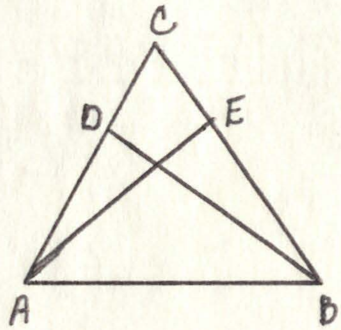


Figure 6

Proof:

Statements	Reasons
1. $\overline{AD} = \overline{BE}$; $\angle BAC = \angle ABC$.	1. Given .
2. $\overline{AB} = \overline{AB}$.	2. Reflexive Property.
3. $\triangle ABD \cong \triangle ABE$.	3. S.A.S.
4. $\overline{AE} = \overline{BD}$.	4. C.P.C.T.

Theorem 6'

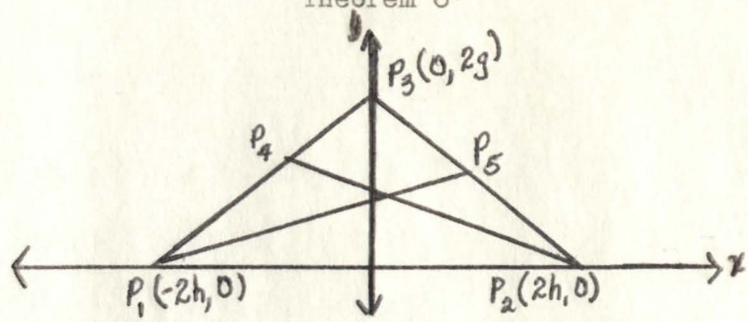


Figure 6'

Given: Isosceles $\triangle P_1P_2P_3$ with $\overline{P_1P_4} = \overline{P_2P_5}$.

Prove: $\overline{P_1P_5} = \overline{P_2P_4}$.

Proof: Let P_1P_2 lie on the x-axis and the altitude from P_3 lie on the y-axis. Let the coordinates of P_1, P_2 and P_3 be as shown in the figure. By the midpoint formula:

1. $P_4 = \left(\frac{0 + {}^-2h}{2}, \frac{2g + 0}{2}\right) = (-h, g)$

$P_5 = \left(\frac{0 + 2h}{2}, \frac{2g + 0}{2}\right) = (h, g)$.

2. $\overline{P_2P_4} = \sqrt{(2h - {}^-h)^2 + (0-g)^2} = \sqrt{(3h)^2 + g^2} = \sqrt{9h^2 + g^2}$

$\overline{P_1P_5} = \sqrt{(-2h-h)^2 + (g-0)^2} = \sqrt{(-3h)^2 + g^2} = \sqrt{9h^2 + g^2}$.

3. Hence $\overline{P_1P_5} = \overline{P_2P_4}$.

Theorem 7

In an equilateral triangle, the three medians are congruent.

Given: Equilateral $\triangle ABC$ with midpoints E, F, and G of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively.

Prove: $\overline{AF} = \overline{BG} = \overline{CE}$.

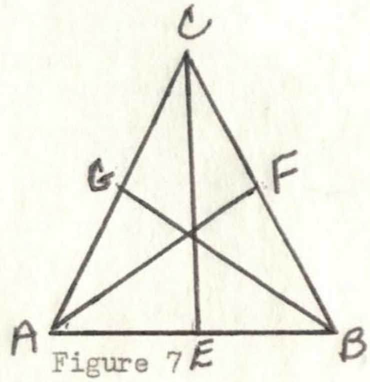


Figure 7

Proof:

Statements	Reasons
1. $\overline{AC} = \overline{BC}$.	1. Definition of equilateral triangle .
2. $\overline{AG} = \overline{BF}$.	2. Halves of equals are equal .
3. $\angle GAE = \angle FBE$.	3. Angles of equilateral triangles are equal .
4. $\overline{AB} = \overline{AB}$.	4. Reflexive Property
5. $\triangle ABG \cong \triangle AFB$.	5. S.A.S.
6. $\overline{AF} = \overline{BG}$.	6. C.P.C.T.
7. $\overline{CB} = \overline{AB}$.	7. Same as 1 .
8. $\overline{EB} = \overline{AG}$.	8. Same as 2 .
9. $\angle B = \angle A$.	9. Same as 3 .
10. $\triangle CBE \cong \triangle BAG$.	10. S.A.S.
11. $\overline{CE} = \overline{BG}$.	11. C.P.C.T.
12. $\overline{AF} = \overline{BG} = \overline{CE}$.	12. Transitive Property .

Theorem 7'

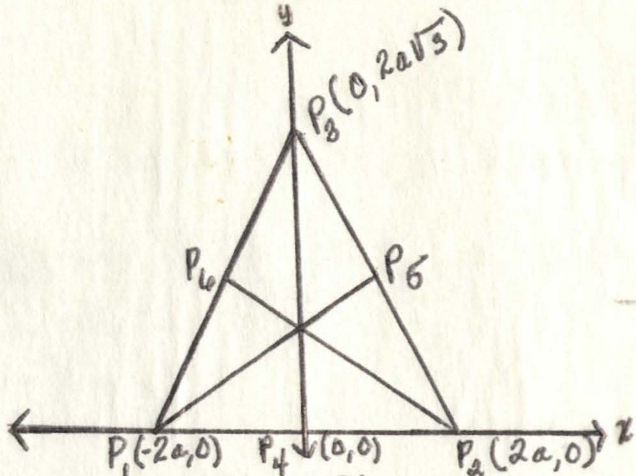


Figure 7'

Given: Equilateral $\Delta P_1P_2P_3$ with midpoints P_4, P_5, and P_6 of $\overline{P_1P_2}$, $\overline{P_2P_3}$ and $\overline{P_1P_3}$ respectively.

Prove: $\overline{P_1P_5} = \overline{P_2P_6} = \overline{P_3P_4}$.

Proof: Let $\overline{P_1P_2}$ lie on the x-axis with P_4 at the origin. Let P_1 be point (-2a,0) and P_2 be point (2a,0). Let P_3 be point (0,y). Since $\overline{P_1P_2} = \overline{P_2P_3}$, $\sqrt{4a^2 + y^2} = 4a$; $4a^2 + y^2 = 16a^2$; $y^2 = 12a^2$; $y = 2a\sqrt{3}$. P_3 is point $(0, 2a\sqrt{3})$. By the midpoint formula:

1. $P_6 = \left(\frac{-2a + 0}{2}, \frac{0 + 2a\sqrt{3}}{2}\right) = (-a, a\sqrt{3})$.

2. $P_5 = \left(\frac{2a + 0}{2}, \frac{0 + 2a\sqrt{3}}{2}\right) = (a, a\sqrt{3})$.

3. $\overline{P_2P_6} = \sqrt{(-a-2a)^2+(a\ \ 3-0)^2} = \sqrt{9a^2+3a^2} = \sqrt{12a^2} = 2a\sqrt{3}$.

4. $\overline{P_1P_5} = \sqrt{(a-{-2a})^2+(a\ \ 3-0)^2} = \sqrt{9a^2+3a^2} = \sqrt{12a^2} = 2a\sqrt{3}$.

5. $\overline{P_3P_4} = \sqrt{(0 - 0)^2 + (2a\ \ 3 - 0)^2} = 2a\sqrt{3}$.

6. $\therefore \overline{P_1P_5} = \overline{P_2P_6} = \overline{P_3P_4}$.

Theorem 8

The union of the three segments joining, in pairs, the midpoints of the sides of an isosceles triangle is an isosceles triangle.

Given: Isosceles $\triangle ABC$ with $\overline{AC} = \overline{BC}$ and midpoints D, F and E of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively.

Prove: $\triangle DEF$ is isosceles.

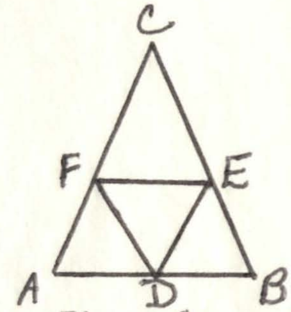


Figure 8

Proof:

Statements	Reasons
1. D, F and E are midpoints of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively .	1. Given .
2. $\overline{AC} = \overline{BC}$.	2. Given .
3. $\overline{AD} = \overline{BC}$.	3. Definition of midpoint .
4. $\angle FAD = \angle EBC$.	4. If two sides of an isosceles triangle are equal, the angles opposite the two sides are equal.
5. $\overline{AF} = \overline{BE}$.	5. Halves of equals are equal .
6. $\triangle AFD \cong \triangle BED$.	6. S.A.S.
7. $\overline{FD} = \overline{ED}$.	7. C.P.C.T.
8. $\therefore$ DFE is isosceles .	8. Definition of isosceles triangle .

Theorem 8'

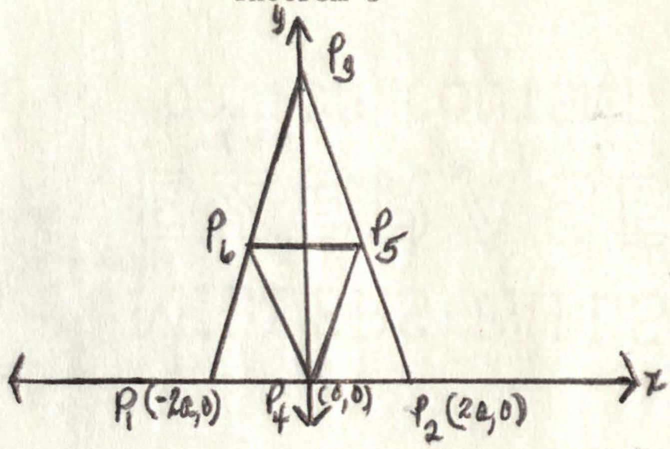


Figure 8'

Given: Isosceles $\triangle P_1P_2P_3$ with $\overline{P_1P_3} = \overline{P_2P_3}$ and midpoints P_4, P_5 and P_6 of $\overline{P_1P_2}$, $\overline{P_2P_3}$ and $\overline{P_1P_3}$ respectively.

Prove: $\triangle P_4P_5P_6$ is isosceles .

Proof: Let P_1P_2 be on the x-axis with P_4 on the origin and P_1 and P_2 having coordinates (-2a,0) and (2a,0) respectively. Let P_3 be the point (0,2b). By the midpoint formula:

1. $P_5 = \left(\frac{2a + 0}{2}, \frac{0 + 2b}{2}\right) = (a,b)$.

2. $P_6 = \left(\frac{-2a + 0}{2}, \frac{0 + 2b}{2}\right) = (-a,b)$.

3. By the distance formula:

$$\overline{P_4P_6} = \sqrt{(-a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2}$$

$$\overline{P_4P_5} = \sqrt{(a-0)^2 + (b-0)^2} = \sqrt{a^2 + b^2} .$$

4. $\therefore \overline{P_4P_6} = \overline{P_4P_5}$ and $P_4P_5P_6$ is isosceles.

Theorem 9

The line segments joining the midpoints of the side of an equilateral triangle form another equilateral triangle.

Given: The equilateral $\triangle ABC$ with $\overline{EF}$, $\overline{FG}$ and $\overline{GE}$ joining the midpoints of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$.

Prove: $\triangle GEF$ is equilateral.

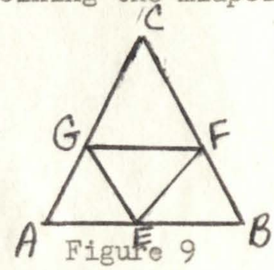


Figure 9

Proof:

Statements	Reasons
1. E, F and G are midpoints of $\overline{AB}$, BC and AC respectively.	1. Given .
2. $\overline{AB} = \overline{BC} = \overline{CA}$.	2. Given .
3. $\overline{AE} = \overline{EB}$, $\overline{BF} = \overline{FC}$ and $\overline{CG} = \overline{GA}$.	3. Definition of bisector .
4. $2\overline{AE} = 2\overline{BF} = 2\overline{GC}$.	4. A quantity may be substituted for its equal .
5. $\overline{AE} = \overline{BF} = \overline{CG}$.	5. If equals are divided by equals, the quotients are equal.
6. $\overline{AG} = \overline{BE} = \overline{CF}$.	6. From steps 4 and 5 .
7. $\angle A = \angle B = \angle C$.	7. An equilateral triangle is equiangular .
8. $\triangle AEG \cong \triangle BFE \cong \triangle CGF$.	8. S.A.S.
9. $\overline{GE} = \overline{EF} = \overline{FG}$.	9. C.P.C.T.
10. $\therefore \triangle GEF$ is equilateral .	10. Definition of equilateral triangle .

Theorem 9'

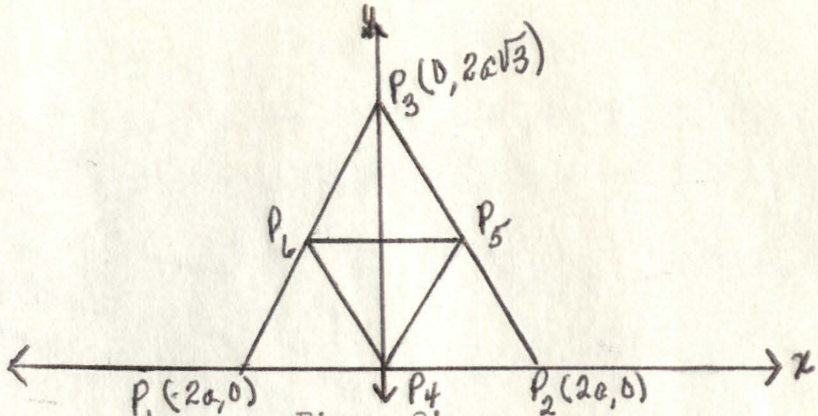


Figure 9'

Given: $\triangle P_1P_2P_3$ is equilateral with $\overline{P_4P_5}$, $\overline{P_6P_4}$ and $\overline{P_5P_6}$ joining the midpoint of $\overline{P_1P_2}$, $\overline{P_2P_3}$ and $\overline{P_3P_1}$.

Prove: $P_6P_4P_5$ is equilateral .

Proof: Let $\overline{P_1P_2}$ lie on the x-axis with P_4 at the origin and P_1, the point $(-2a,0)$, P_2 the point $(2a,0)$ and P_3 the point $(0,2a\sqrt{3})$.

By the midpoint formula:

1. $P_4 = \left(\frac{2a+0}{2}, \frac{0+2a\sqrt{3}}{2}\right) = (a, a\sqrt{3})$.

2. $P_5 = \left(\frac{-2a+0}{2}, \frac{0+2a\sqrt{3}}{2}\right) = (-a, a\sqrt{3})$.

3. $\overline{P_4P_5} = \sqrt{(a-0)^2 + (a\sqrt{3})^2} = \sqrt{a^2 + 3a^2} = \sqrt{4a^2} = 2a$.

4. $\overline{P_5P_6} = \sqrt{(-a-a)^2 + (a\sqrt{3} - a\sqrt{3})^2} = \sqrt{4a^2 + 0} = 2a$.

5. $\overline{P_6P_4} = \sqrt{(0 - -a)^2 + (a\sqrt{3} - 0)^2} = \sqrt{a^2 + 3a^2} = \sqrt{4a^2} = 2a$.

6. $\therefore \overline{P_4P_5} = \overline{P_5P_6} = \overline{P_6P_4}$ and $\triangle P_6P_4P_5$ is equilateral.

Theorem 10

The altitudes of a triangle are concurrent.

Given: $\triangle ABC$ with the altitudes $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$.

Prove: $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ are concurrent .

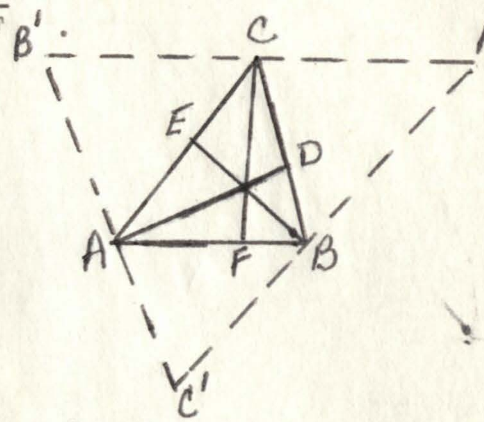


Figure 10

Proof:

Statements	Reasons
1. Draw B'A' through C $\parallel$ $\overline{AB}$; C'A' through B $\parallel$ $\overline{AC}$; C'B' through A $\parallel$ $\overline{BC}$.	1. Through a given point only one line can be constructed parallel to a given line.
2. ABCB' and ABA'C are ▱s .	2. Opposite sides are parallel .
3. $\therefore$ B'C = $\overline{AB}$ and CA' = AB .	3. Opposite sides of a parallelogram are equal .
4. $\therefore$ B'C = CA' .	4. Quantities equal to the same quantity are equal to each other.
5. $\overline{CF}$ $\perp$ AB .	5. Given CF altitude of AB .
6. CF $\perp$ B'A' .	6. If a line is $\perp$ to one of two parallel lines, it is $\perp$ to the other also .
7. $\therefore$ $\overline{CF}$ is the $\perp$ bisector of B'A'.	7. CF bisects B'A' and is $\perp$ to B'A'.
8. In like manner, BE and AD are the perpendicular bisectors of C'A' and B'C' respectively .	8. Same as 8 .
9. $\therefore$ AD, BE and CF are concurrent .	9. The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices.

Theorem 10'

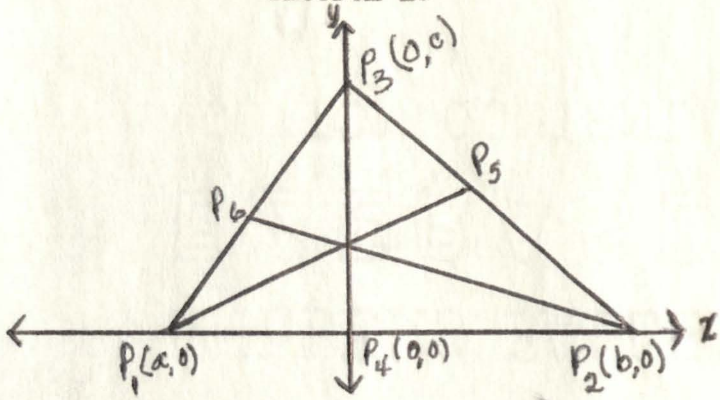


Figure 10'

Given: Any $\triangle ABC$ with P_4, P_5 and P_6 the points where the altitudes intersect $\overleftrightarrow{P_1P_2}$, $\overleftrightarrow{P_2P_3}$, and $\overleftrightarrow{P_1P_3}$ respectively.

Prove: $\overline{P_1P_5}$, $\overline{P_2P_6}$ and $\overline{P_3P_4}$ intersect at a common point.

Proof: Let P_1P_2 lie on the x-axis, with the altitude P_3P_4 lying on the y-axis. Let the coordinates of P_1, P_2 and P_3 be as shown in the figure. By the slope formula:

1. m_1 - slope of $P_2P_3 = \frac{c-0}{0-b} = \frac{c}{-b}$, slope of $P_1P_5 = \frac{b}{c}$.

2. m_2 - slope of $P_1P_2 = \frac{0-0}{b-a} = 0$.

3. m_3 - slope of $P_1P_3 = \frac{c-0}{0-a} = \frac{c}{-a}$, slope of $P_2P_6 = \frac{a}{c}$.

4. Equations of the line containing altitudes

$$P_1P_5 = y - 0 = m(x - a)$$

$$y = \frac{b}{c}(x - a) \; .$$

$$P_2P_6 = y - 0 = m(x - b)$$

$$y = \frac{a}{c}(x - b) \; .$$

5. P_2P_6 and P_1P_5 intersect at the point where

$$\frac{b}{c}(x - a) = \frac{a}{c}(x - b)$$

$$\frac{bx - ab}{c} = \frac{ax - ab}{c}$$

$$bx - ab = ax - ab$$

$$bx - ax = -ab + ab$$

$$x(b - a) = 0$$

$$x = 0$$

$$y = \frac{-ab}{c} \quad .$$

6. $\therefore P_1P_5$, P_2P_6 and P_3P_4 intersect at a common point .

Theorem 11

The midpoint of the hypotenuse of a right triangle is equally distant from all three vertices.

Given: Right triangle ABC, M is the midpoint of $\overline{BC}$.

Prove: $\overline{CM} = \overline{BM} = \overline{AM}$.

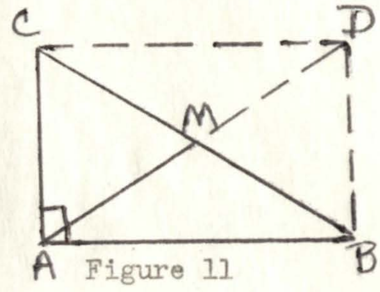


Figure 11

Proof: Draw $\overline{CD} \parallel \overline{AB}$ and $\overline{BD} \parallel \overline{AC}$.

Statements	Reasons
1. $\overline{CM} = \overline{BM}$.	1. Definition of a midpoint .
2. ABDC is a parallelogram .	2. Definition of a parallelogram .
3. ABDC is a rectangle .	3. A parallelogram with a right angle is a rectangle .
4. M bisects $\overline{AD}$.	4. Diagonals of a parallelogram bisect each other .
5. $\overline{AM} = 1/2(\overline{AD})$.	5. Definition of a midpoint .
6. $\overline{AD} = \overline{BC}$.	6. The diagonals of a rectangle are equal .
7. $\overline{AM} = 1/2(\overline{BC})$.	7. Substitution Property .
8. $\overline{BM} = 1/2(\overline{BC})$.	8. Definition of a midpoint .
9. $\therefore \overline{AM} = \overline{BM} = \overline{CM}$.	9. Transitive Property .

Theorem 11'

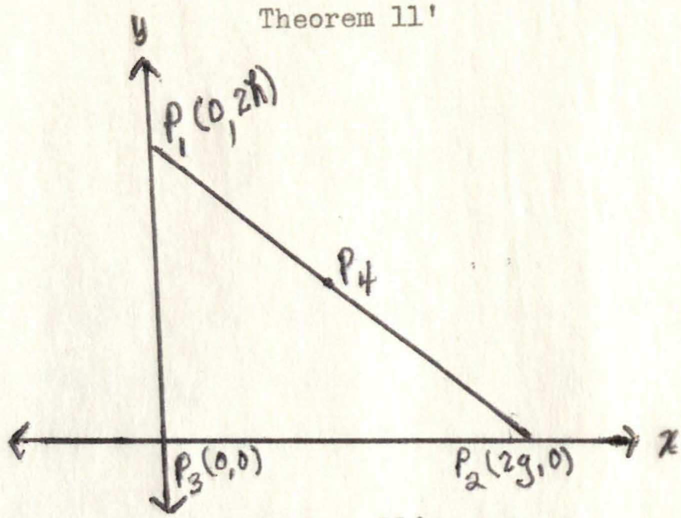


Figure 11'

Given: Right $\Delta P_1P_2P_3$ with P_4 the midpoint of $\overline{P_1P_2}$.

Prove: $\overline{P_1P_4} = \overline{P_2P_4} = \overline{P_3P_4}$.

Proof: Let P_2P_3 lie on the x-axis and P_1P_3 lie on the y-axis. Let the coordinates of P_1 and P_2 be as shown in the figure. By the midpoint formula, P_4 is the point (g,h).

1. $\overline{P_1P_4} = \sqrt{(g - 0)^2 + (h - 2h)^2} = \sqrt{g^2 + h^2}$.

2. $\overline{P_2P_4} = \sqrt{(g - 2g)^2 + (h - 0)^2} = \sqrt{g^2 + h^2}$.

3. $\overline{P_3P_4} = \sqrt{(g - 0)^2 + (h - 0)^2} = \sqrt{g^2 + h^2}$.

4. $\overline{P_1P_4} = \overline{P_2P_4} = \overline{P_3P_4}$.

5. $\therefore P_4$ is equidistant from the three vertices .

CHAPTER IV

Quadrilaterals

In this chapter, the two methods will be applied to several theorems related to quadrilaterals. First, the synthetic method will be applied to prove a theorem, then the analytic method will be used to prove the same theorem in sequence. Theorems proved by the synthetic method will be denoted by unprime numbers while the same theorems being proved by the analytic method will be denoted by prime numbers. Example, Theorem 12 and Theorem 12'.

Theorem 12

In a parallelogram, the opposite sides are congruent.

Given: ▱ ABCD, diagonal AC .

Prove: $\overline{AB} = \overline{DC}$; $\overline{AD} = \overline{BC}$.

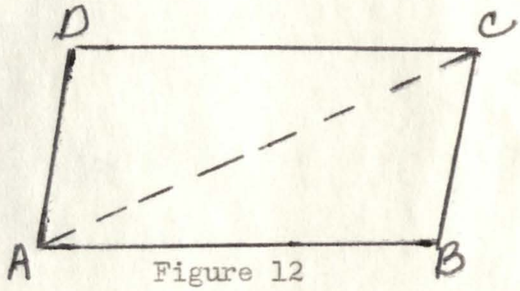


Figure 12

Proof:

Statements	Reasons
1. $\angle BAC \cong \angle DCA$.	1. If two parallel lines are intersected by a transversal, then the pairs of alternate interior angles are equal.
2. $\angle BCA \cong \angle DAC$.	2. Same as 1 .
3. $\overline{AC} \cong \overline{AC}$.	3. Reflexive Property.
4. $\triangle ABC \cong \triangle ADC$.	4. A.S.A.
5. $\therefore \overline{AB} = \overline{DC}$; $\overline{AD} = \overline{BC}$.	5. C.P.C.T.

Theorem 12'

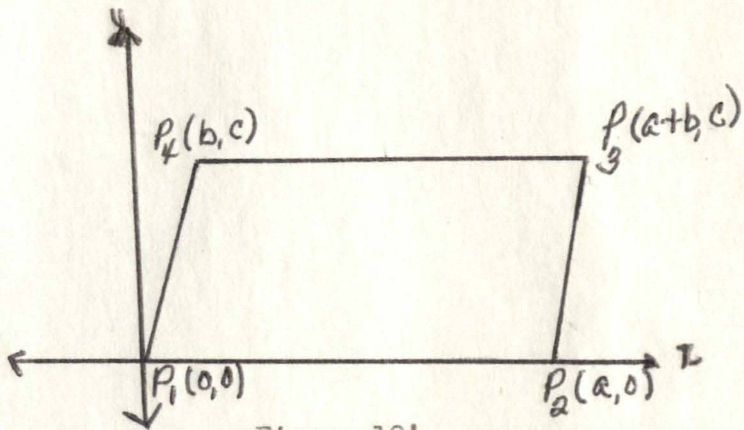


Figure 12'

Given: ▱ $P_1P_2P_3P_4$.

Prove: $\overline{P_1P_2} = \overline{P_4P_3}$; $\overline{P_1P_4} = \overline{P_2P_3}$.

Proof: Let P_1P_2 lie on the x-axis with P_1 at the origin and coordinates of P_2 and P_4 as shown in the figure. Then P_3 is point (a+b,c). By the distance formula:

1. $\overline{P_1P_2} = \sqrt{(a-0)^2 + (0-0)^2} = \sqrt{a^2} = a$.

2. $\overline{P_4P_3} = \sqrt{(a+b-b)^2 + (c-c)^2} = \sqrt{a^2} = a$.

3. $\overline{P_1P_4} = \sqrt{(c-0)^2 + (b-0)^2} = \sqrt{b^2 + c^2}$.

4. $\overline{P_2P_3} = \sqrt{(c-0)^2 + (a+b-a)^2} = \sqrt{b^2 + c^2}$.

5. $\therefore \overline{P_1P_2} = \overline{P_4P_3}$; $\overline{P_1P_4} = \overline{P_2P_3}$.

Theorem 13

If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.

Given: ▱ ABCD with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at O, so that $\overline{AO} = \overline{OC}$ and $\overline{BO} = \overline{OD}$.

Prove: ABCD is a ▱ .

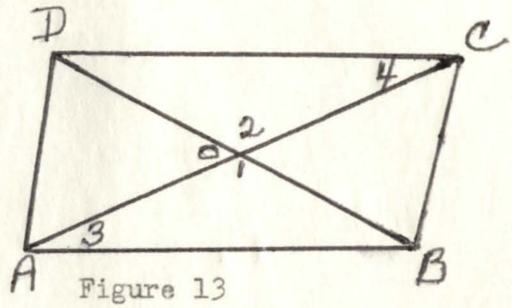


Figure 13

Proof:

Statements	Reasons
1. $\overline{AO} = \overline{OC}$.	1. Given .
2. $\overline{BO} = \overline{OD}$.	2. Given .
3. $\angle 1 = \angle 2$.	3. Vertical angles .
4. $\triangle AOB \cong \triangle COD$.	4. S.A.S.
5. $\angle 3 = \angle 4$.	5. C.P.C.T.
6. $\overline{AB} \parallel \overline{CD}$.	6. If two lines form equal alternate interior angles with a transversal, the lines are parallel.
7. $\overline{AB} = \overline{CD}$.	7. C.P.C.T.
8. ∴ ABCD is a ▱ .	8. If one side of a quadrilateral is equal and parallel to the opposite side, then the figure is a parallelogram.

Theorem 13'

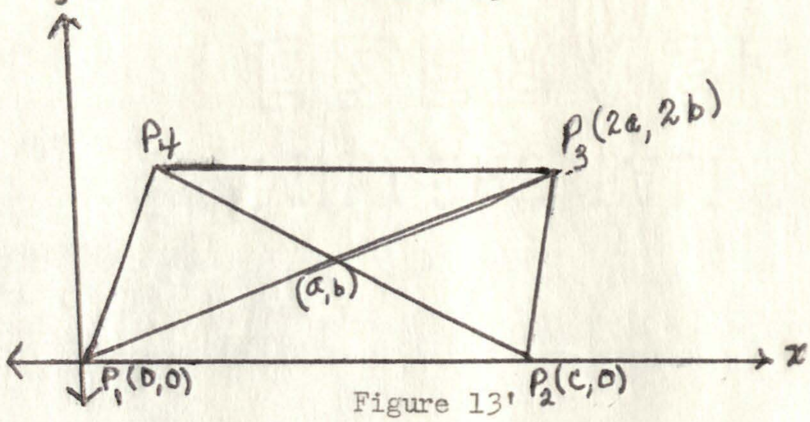


Figure 13'

Given: Quadrilateral $P_1P_2P_3P_4$ in which P_1P_3 bisect P_2P_4 .

Prove: $P_1P_2P_3P_4$ is a ▱ .

Proof: Let P_1P_2 lie on the x-axis with P_1 at origin; P_2 the point (c,0) and (a,b) the point of intersection of P_1P_3 and P_2P_4. By the midpoint formula, P_3 is point (2a,2b). Let P_4 be point (x,y). Then $\frac{x+c}{2} = a$; $x + c = 2a$; $x = 2a - c$; $\frac{0+y}{2} = b$; $y = 2b$. Hence P_4 is the point (2a - c, 2b).

1. The slope of $\overline{P_1P_4} = \frac{2b - 0}{2a-c-0} = \frac{2b}{2a - c}$.
2. The slope of $\overline{P_1P_2} = \frac{0 - 0}{c - 0} = 0$.
3. The slope of $\overline{P_2P_3} = \frac{2b - 0}{2a - c} = \frac{2b}{2a - c}$.
4. The slope of $\overline{P_3P_4} = \frac{2b - 2b}{2a - 2a - c} = 0$.
5. Since the slopes are equal, $\overline{P_1P_2} \parallel \overline{P_4P_3}$ and $\overline{P_1P_4} \parallel \overline{P_2P_3}$.
6. Hence $P_1P_2P_3P_4$ is a ▱ .

Theorem 14

If two sides of a quadrilateral are congruent and parallel, the quadrilateral is a parallelogram.

Given: Quadrilateral ABCD with $\overline{AB}$ equal and parallel to $\overline{CD}$.

Prove: ABCD is a ▱ .

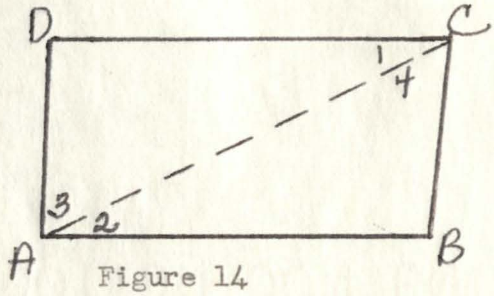


Figure 14

Proof: Draw diagonal AC .

Statements	Reasons
1. $\overline{AC} = \overline{AC}$.	1. Reflexive Property
2. $\angle 1 = \angle 2$.	2. Alternate interior angles of parallel lines AB and CD .
3. $\overline{AB} = \overline{CD}$.	3. Given .
4. $\triangle ABC \cong \triangle CDA$.	4. S.A.S.
5. $\angle 3 = \angle 4$.	5. C.P.C.T.
6. $\overline{AD} \parallel \overline{BC}$.	6. If two lines form equal alternate interior angles with a transversal, the lines are parallel .
7. $\therefore$ ABCD is a ▱ .	7. Opposite sides are parallel .

Theorem 14'

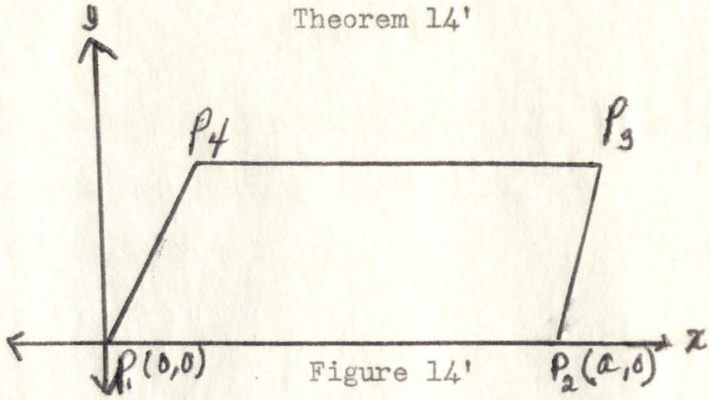


Figure 14'

Given: Quadrilateral $P_1P_2P_3P_4$ with $\overline{P_1P_2} = \overline{P_3P_4}$ and $P_1P_2 \parallel P_3P_4$.

Prove: $P_1P_2P_3P_4$ is a ▱ .

Proof: Let P_1P_2 lie on the x-axis with P_1 at the origin and P_2 the point (a,0).

1. The slope of $\overline{P_1P_2}$ is 0.

2. Since $\overline{P_4P_3} \parallel \overline{P_1P_2}$, the slope of $\overline{P_4P_3} = 0$.

3. Let the coordinates of P_4 be (b,d) and the coordinates of P_3, (c,d) .

4. Since $\overline{P_1P_2} = \overline{P_3P_4} = a$, $\sqrt{(c - b)^2}$, $c - b = a$, $c = a + b$.

5. The coordinates of P_3 are (a + b, d) .

6. The slope of $\overline{P_1P_4} = \frac{d - 0}{b - 0} = \frac{d}{b}$.

   The slope of $\overline{P_2P_3} = \frac{d - 0}{a + b - a} = \frac{d}{b}$.

7. $\therefore \overline{P_1P_4} \parallel \overline{P_2P_3}$ and $P_1P_2P_3P_4$ is a ▱ .

Theorem 15

In a parallelogram, opposite angles are equal.

Given: Parallelogram ABCD, AB || CD, AC || BD .

Prove: ∠A = ∠D , ∠B = ∠C .

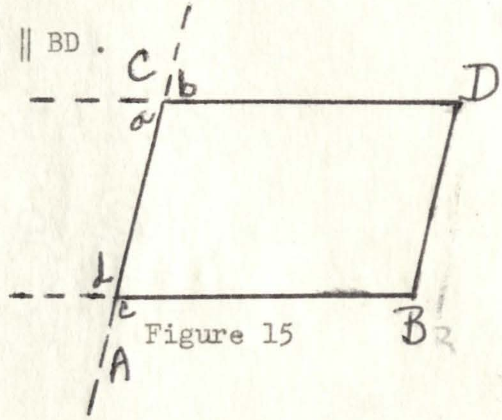


Figure 15

Proof:

Statements	Reasons
1. ∠a = ∠A .	1. Alternate interior angles are equal .
2. ∠a = ∠b .	2. Vertical angles are equal .
3. ∠A = ∠b .	3. Transitive Property .
4. ∠b = ∠D .	4. Alternate interior angles are equal .
5. ∴ ∠A = ∠D .	5. Transitive Property .
6. ∠d = ∠C .	6. Alternate interior angles are equal .
7. ∠d = ∠c .	7. Vertical angles are equal .
8. ∠C = ∠c .	8. Transitive Property .
9. ∠d = ∠B .	9. Alternate interior angles are equal .
10. ∴ ∠B = ∠C .	10. Transitive Property .

Theorem 15'

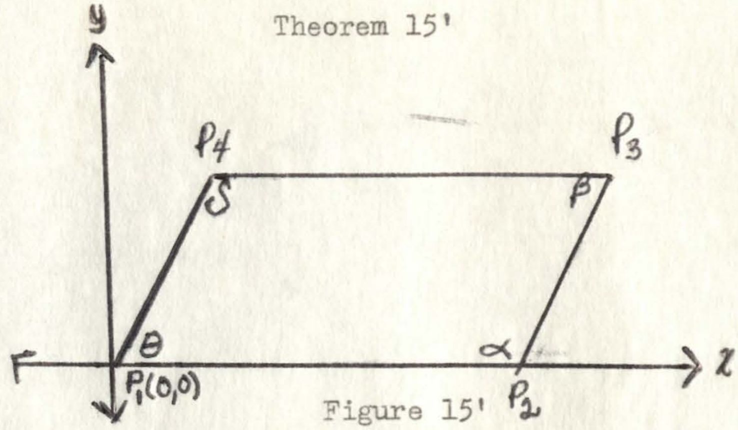


Figure 15'

Given: ▱ $P_1P_2P_3P_4$, $P_1P_2 \parallel P_3P_4$; $P_1P_4 \parallel P_2P_3$.

Prove: $\angle\theta = \angle\beta$, $\angle\alpha = \angle\delta$.

Proof: Let P_1P_2 lie on the x-axis with P_1 at the origin; P_1 is the angle θ, P_2 is the angle α , P_3 is the angle β and P_4 is the angle δ. By the slope formula:

1. m_1 - slope of P_1P_4 and P_2P_3

   m_2 - slope of P_1P_2 and P_4P_3

   $m_2 = 0$.

2. $\tan\theta = \dfrac{m_1 - m_2}{1 + m_1 \cdot m_2} = \dfrac{m_1 - 0}{1 + m_1 \cdot 0} = m_1$

   $\tan\beta = \dfrac{m_1 - m_2}{1 + m_1 \cdot m_2} = \dfrac{m_1 - 0}{1 + m_1 \cdot 0} = m_1$.

3. Hence $\tan\theta = \tan\beta$.

4. Hence $\theta = \beta$ since θ and $\beta < 180°$.

5. $\tan\alpha = \dfrac{m_2 - m_1}{1 + m_2 \cdot m_1} = \dfrac{0 - m_1}{1 + 0 \cdot m_1} = -m_1$

   $\tan\delta = \dfrac{m_2 - m_1}{1 + m_2 \cdot m_1} = \dfrac{0 - m_1}{1 + 0 \cdot m_1} = -m_1$.

6. Hence $\tan\alpha = \tan\delta$, if α and $\delta < 180°$.

7. $\therefore \angle\alpha = \angle\delta$.

Theorem 16

In any parallelogram the diagonals bisect each other.

Given: ▱ ABCD, diagonals $\overline{AC}$ and $\overline{BD}$.

Prove: E is the midpoint of $\overline{AC}$ and $\overline{BD}$.

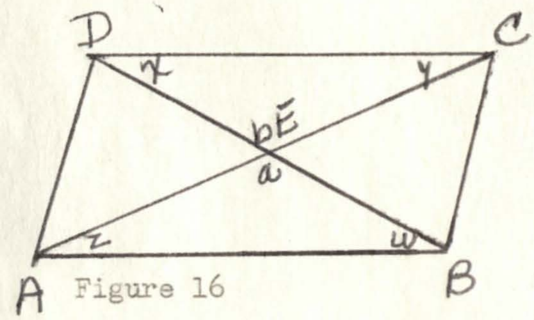


Figure 16

Proof:

Statements	Reasons
1. $\triangle ABC \cong \triangle ADC$.	1. In a parallelogram a diagonal forms two congruent triangles.
2. $\angle z = \angle y$.	2. C.P.C.T.
3. $\angle a = \angle b$.	3. Vertical angles are equal
4. $\triangle ABC \cong \triangle BDC$.	4. Same as 1.
5. $\angle w = \angle x$.	5. C.P.C.T.
6. $\overline{AB} = \overline{DC}$.	6. Opposite sides of a parallelogram are equal.
7. $\triangle ABE \cong \triangle DCE$.	7. A.S.A.
8. $\overline{DE} = \overline{BE}$. $\overline{AE} = \overline{CE}$.	8. C.P.C.T.
9. E is the midpoint of $\overline{AC}$ and $\overline{BD}$.	9. Definition of midpoint.

Theorem 16'

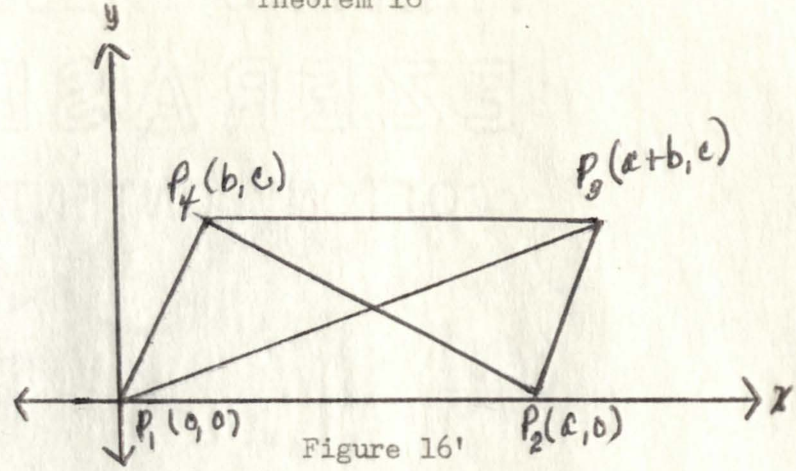


Figure 16'

Given: ▱ $P_1P_2P_3P_4$, diagonals $\overline{P_1P_2}$ and $\overline{P_2P_4}$.

Prove: $\overline{P_1P_3}$ bisect $\overline{P_2P_4}$.

Proof: Let P_1P_2 lie on the x-axis with P_1 at the origin and coordinates of P_2 and P_4 as shown in the figure. Then P_3 is point (a + b, c). By the midpoint formula:

1. $\overline{P_1P_3} = \left(\frac{0 + a + b}{2}, \frac{0 + c}{2}\right) = \left(\frac{a + b}{2}, \frac{c}{2}\right)$.

2. $\overline{P_2P_4} = \left(\frac{a + b}{2}, \frac{0 + c}{2}\right) = \left(\frac{a + b}{2}, \frac{c}{2}\right)$.

3. ∴ The midpoints of the two segments are the same point, the diagonals bisect each other.

Theorem 17

In a parallelogram, a diagonal forms two congruent triangles.

Given: ▱ ABCD with diagonal BD .

Prove: $\triangle ABD \cong \triangle BDC$.

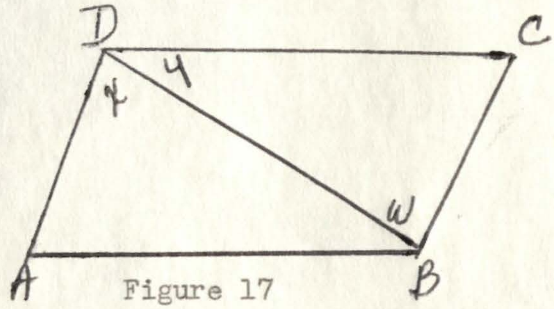


Figure 17

Proof:

Statements	Reasons
1. $\overline{AB} = \overline{DC}$.	1. Opposite sides of a parallelogram are equal .
2. $\overline{AD} = \overline{BC}$.	2. Same as 1 .
3. $\overline{BD} = \overline{BD}$.	3. Reflexive Property
4. $\triangle ABD \cong \triangle BDC$.	4. S.S.S.

Theorem 17'

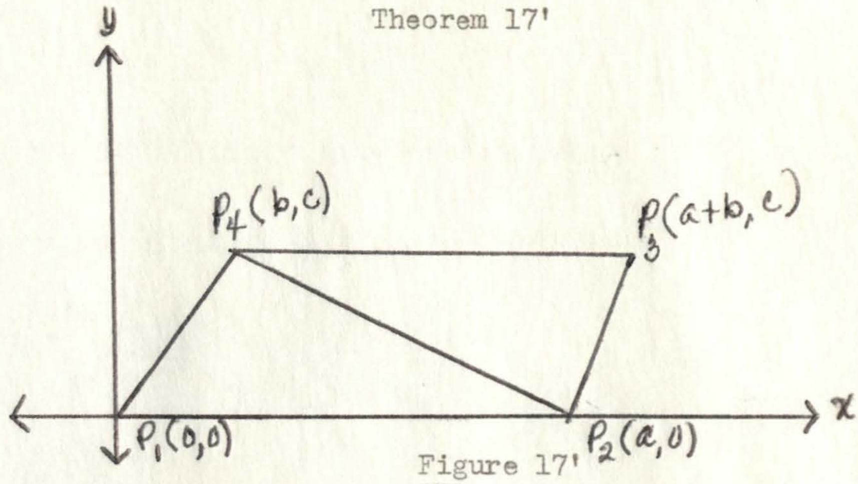


Figure 17'

Given: ▱ $P_1P_2P_3P_4$ with diagonal $\overline{P_2P_4}$.

Prove: $\triangle P_1P_2P_4 \cong \triangle P_2P_4P_3$.

Proof: Let P_1P_2 lie on the x-axis with P_1 at the origin and coordinates of P_2 and P_4 as shown in the figure. Then P_3 is point (a + b, c). By the distance formula:

1. $\overline{P_1P_2} = \sqrt{(a - 0)^2 + (0 - 0)^2} = \sqrt{a^2} = a.$

2. $\overline{P_4P_3} = \sqrt{(a + b - b)^2 + (c - c)^2} = \sqrt{a^2} = a .$

3. $\overline{P_1P_4} = \sqrt{(c - 0)^2 + (b - 0)^2} = \sqrt{c^2 + b^2} .$

4. $\overline{P_2P_3} = \sqrt{(c - 0)^2 + (a + b - a)^2} = \sqrt{c^2 + b^2} .$

5. $\overline{P_2P_4} = \sqrt{(c - 0)^2 + (b - a)^2} = \sqrt{c^2 + (b - a)^2} .$

6. $\overline{P_2P_4} = \overline{P_2P_4}$.

7. $\therefore \triangle P_1P_2P_4 \cong \triangle P_2P_4P_3$.

Theorem 18

The diagonals of a rectangle are equal.

Given: Rectangle ABCD, diagonals AC and BD .

Prove: $\overline{AC} = \overline{BD}$.

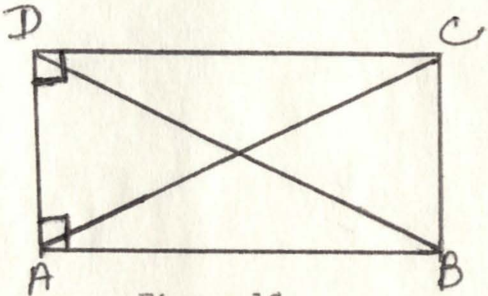


Figure 18

Proof:

Statements	Reasons
1. $\angle$ DAB and $\angle$ ADC are right $\angle$s .	1. Definition of rectangle .
2. $\triangle$DAB and $\triangle$ADC are right triangles .	2. Definition of right triangle .
3. $\overline{AB} = \overline{DC}$, $\overline{AD} = \overline{BC}$.	3. Opposite side of a parallelogram are equal .
4. $\triangle DAB \cong \triangle ADC$.	4. LL .
5. $\overline{AC} = \overline{BD}$.	5. C.P.C.T.

Theorem 18'

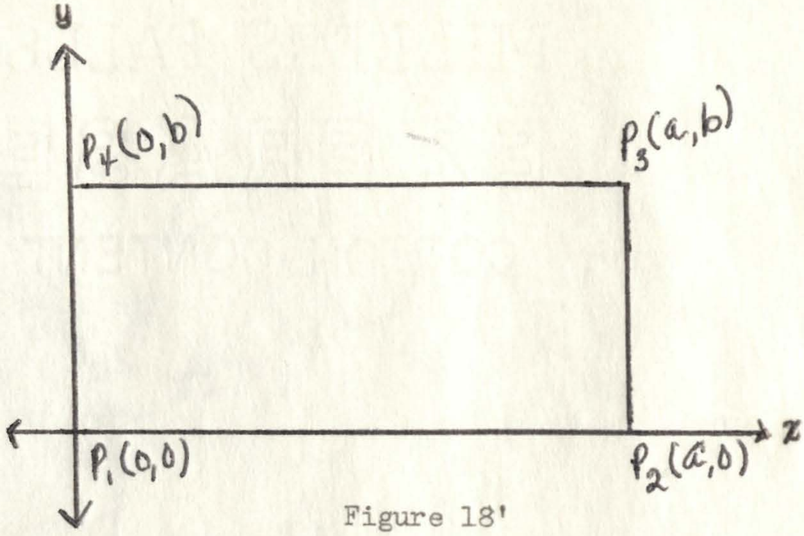


Figure 18'

Given: Rectangle $P_1P_2P_3P_4$, diagonals $\overline{P_1P_3}$ and $\overline{P_2P_4}$.

Prove: $\overline{P_1P_3} = \overline{P_2P_4}$.

Proof: Let $\overline{P_1P_2}$ lie on the x-axis and $\overline{P_1P_4}$ lie on the y-axis with coordinates of P_1, P_2, P_3 and P_4 as shown in the figure.

1. $\overline{P_1P_3} = \sqrt{(a - 0)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$.

2. $\overline{P_2P_4} = \sqrt{(0 - a)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$.

3. $\overline{P_1P_3} = \overline{P_2P_4}$, hence the diagonals are equal.

Theorem 19

The diagonals of a rhombus are perpendicular.

Given: Rhombus RSTQ .

Prove: $\overline{RT} \perp \overline{SQ}$.

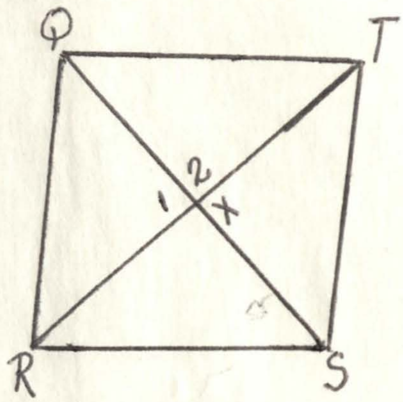


Figure 19

Proof:

Statements	Reasons
1. $\overline{RX} = \overline{TX}$.	1. The diagonals of a parallelogram bisect each other .
2. $\overline{QX} = \overline{QX}$.	2. Reflexive Property
3. $\overline{RQ} = \overline{TQ}$.	3. Definition of a rhombus .
4. $\triangle RXQ \cong \triangle TXQ$.	4. S.S.S.
5. $\angle 1 = \angle 2$.	5. C.P.C.T.
6. $\overline{RT} \perp \overline{SQ}$.	6. Two lines that meet to form congruent adjacent angles are perpendicular .

Theorem 19'

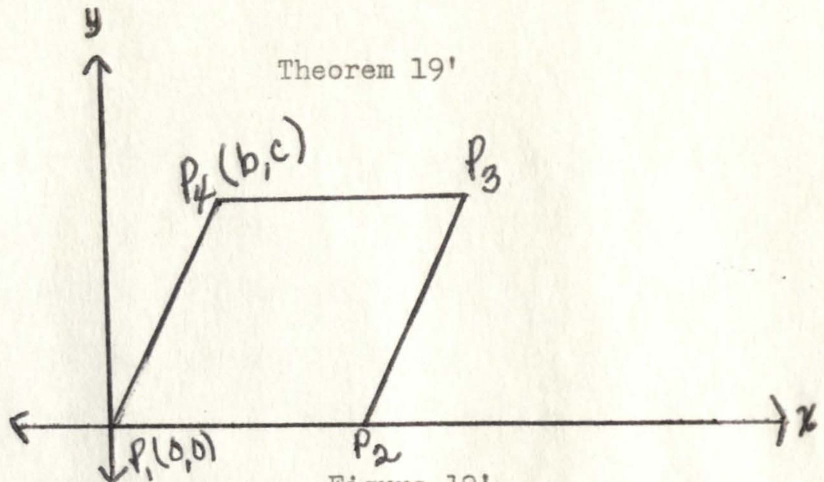


Figure 19'

Given: Rhombus $P_1P_2P_3P_4$.

Prove: $\overline{P_1P_3} \perp \overline{P_4P_2}$.

Proof: Let $\overline{P_1P_2}$ lie on the x-axis with P_1 at the origin and coordinates of P_4 (b,c). Since $\overline{P_1P_4} = \sqrt{b^2 + c^2} = \overline{P_1P_2}$. P_2 has coordinates $(\sqrt{b^2 + c^2}, 0)$. P_3 has coordinates $(b + \sqrt{b^2 + c^2}, c)$.

1. m_1 slope of P_1P_3

   m_2 slope of P_4P_2 .

2. $m_1 = \dfrac{c - 0}{b + \sqrt{b^2+c^2} - 0} = \dfrac{c}{b + \sqrt{b^2 + c^2}}$

   $m_2 = \dfrac{c - 0}{b - \sqrt{b^2 + c^2}} = \dfrac{c}{b - \sqrt{b^2 + c^2}}$.

3. $(\overline{P_1P_2})(\overline{P_4P_2}) = \dfrac{c}{b + \sqrt{b^2 + c^2}} \; \dfrac{c}{b - \sqrt{b^2+c^2}} = \dfrac{c^2}{b^2 - (b^2+c^2)} = -1$.

4. $\overline{P_1P_3} \perp \overline{P_4P_2}$.

Theorem 20

If the diagonals of a parallelogram are perpendicular, the parallelogram is a rhombus.

Given: ▱ ABCD; $AC \perp BC$.

Prove: ABCD is a rhombus .

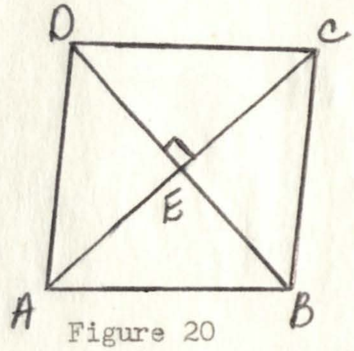


Figure 20

Proof:

Statements	Reasons
1. $\overline{DE} = \overline{BE}$.	1. The diagonals of a parallelogram bisect each other .
2. $\overline{CE} = \overline{CE}$.	2. Reflexive Property .
3. ∠ CED and ∠ CEB are right angles .	3. Perpendicular meet to form right angles .
4. $\triangle CED \cong \triangle CEB$.	4. S.A.S.
5. $\overline{DC} = \overline{BC}$.	5. C.P.C.T.
6. ∴ ABCD is a rhombus .	6. A parallelogram with two consecutive sides congruent is a rhombus .

Theorem 20'

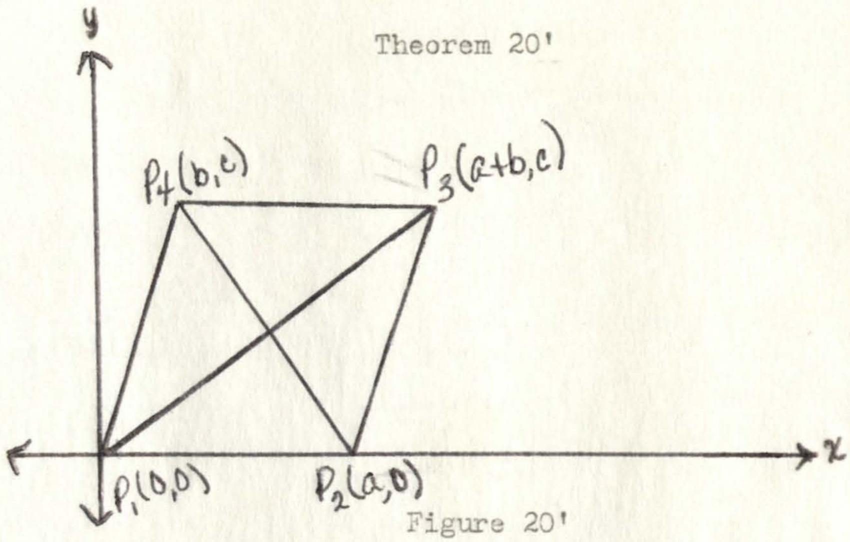


Figure 20'

Given: ▱ $P_1P_2P_3P_4$ with $\overline{P_1P_3} \perp \overline{P_2P_4}$.

Prove: $P_1P_2P_3P_4$ is a rhombus .

Proof: Let P_1 be point (0,0) with P_1P_2 lying on the x-axis and P_2, P_3P_4 as shown in the figure.

1. m_1 - slope of P_1P_3
   m_2 - slope of P_2P_4 .

2. $m_1 = \frac{c - 0}{a + b - 0} = \frac{c}{a + b}$
   $m_2 = \frac{c - 0}{b - a} = \frac{c}{b - a}$.

3. Since $\overline{P_1P_3} \perp \overline{P_2P_4}$, $\frac{c}{a + b} = \frac{a - b}{c}$; $c^2 = a^2 + b^2$.

4. By the distance formula, $P_4P_1 = \sqrt{b^2+c^2} = \sqrt{b^2+a^2-b^2} = \sqrt{a^2} = a$.

5. Since $\overline{P_1P_2} = a$, $\overline{P_1P_2} = \overline{P_4P_1}$.

6. Hence $P_1P_2P_3P_4$ is a rhombus .

Theorem 21

The segments joining the midpoints of the opposite sides of a quadrilateral bisect each other.

Given: Quadrilateral ABCD with midpoints Q, R, S, and T of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{AD}$ respectively.

Prove: $\overline{TR}$ and $\overline{QS}$ bisect each other.

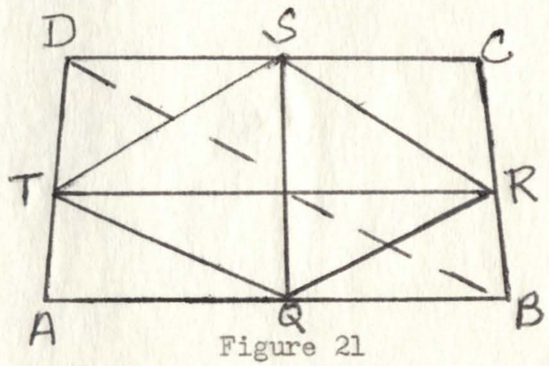


Figure 21

Proof:

Statements	Reasons
1. $\overline{RS} \parallel \overline{BD}$ and $\overline{RS} = 1/2(\overline{BD})$.	1. The line segment joining the midpoints of two sides of a $\triangle$ is parallel to the third and equal to one half of it.
2. $\overline{TQ} \parallel \overline{BD}$ and $\overline{TQ} = 1/2(\overline{BD})$.	2. Same as 1 .
3. $\overline{TQ} = \overline{RS}$.	3. Transitive Property .
4. $\overline{TQ} \parallel \overline{RS}$.	4. If two lines are parallel to a third line, they are parallel to each other .
5. QRST is a parallelogram .	5. If a pair of opposite sides of a quadrilateral are both parallel and equal, the quadrilateral is a parallelogram.
6. $\therefore$ $\overline{TR}$ and $\overline{QS}$ bisect each other .	6. The diagonals of a parallelogram bisect each other.

Theorem 21'

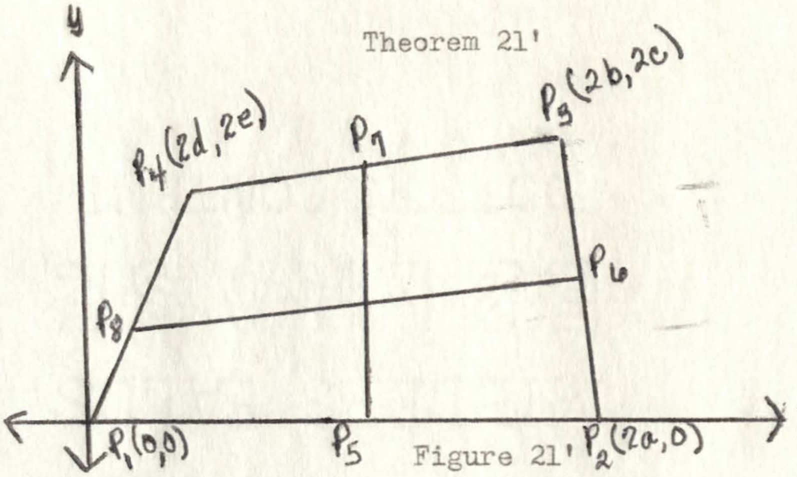


Figure 21'

Given: Quadrilateral $P_1P_2P_3P_4$ with midpoints P_5, P_6, P_7 and P_8 of $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$, $\overline{P_1P_4}$ respectively.

Prove: $\overline{P_8P_6}$ and $\overline{P_5P_7}$ bisect each other.

Proof: Let coordinate axes and coordinates of P_1, P_2, P_3 and P_4 be as shown in the figure. By the midpoint formula:

1. $P_5 = \left(\frac{0 + 2a}{2}, \frac{0 + 0}{2}\right) = (a,0)$.
2. $P_6 = \left(\frac{2a + 2b}{2}, \frac{0 + 2c}{2}\right) = (a + b,c)$.
3. $P_7 = \left(\frac{2b + 2b}{2}, \frac{2c + 2e}{2}\right) = (b + d, c + e)$.
4. $P_8 = \left(0 + 2d , \frac{0 + 2e}{2}\right) = (d,e)$.
5. The midpoint of $\overline{P_8P_6} = \left(\frac{a + b + d}{2}, \frac{c + e}{2}\right)$.
6. The midpoint of $\overline{P_7P_5} = \left(\frac{a + b + d}{2}, \frac{c + e}{2}\right)$.
7. The midpoints lie on the same point. Hence P_8P_6 and P_5P_7 bisect each other.

Theorem 22

The median of a trapezoid is parallel to the bases.

Given: Trapezoid ABCD, M is the midpoint of $\overline{AD}$.

N is the midpoint of $\overline{BC}$.

Prove: $\overline{MN} \parallel \overline{AB}$ and $\overline{MN} \parallel \overline{DC}$.

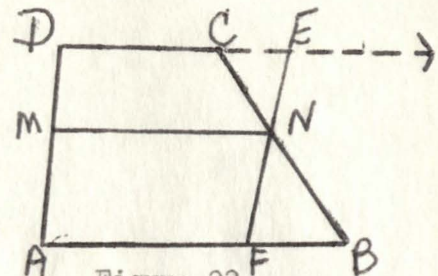


Figure 22

Proof: Draw $\overline{FE} \parallel \overline{AD}$ through N. AFED is a ▱ .

Statements	Reasons
1. $\overline{CN} = \overline{BN}$.	1. Definition of a midpoint .
2. $\angle CNE = \angle FNB$.	2. Vertical angles are equal .
3. $\angle NCE = \angle NBF$.	3. Alternate interior angles are equal .
4. $\triangle FNB \cong \triangle CNE$.	4. A.S.A.
5. $\overline{FN} = \overline{EN}$.	5. C.P.C.T.
6. $\overline{AD} = \overline{FE}$.	6. Opposite sides of a parallelogram are equal .
7. $\overline{DM} = 1/2(\overline{AD})$.	7. Definition of midpoint .
8. $\overline{DM} = \overline{EN}$.	8. Halves of equals are equal.
9. DMNE is a ▱ .	9. A quadrilateral with one pair of sides both equal and parallel is a parallelogram .
10. $\overline{MN} \parallel \overline{DC}$.	10. Opposite sides of a parallelogram are parallel .
11. $\overline{AM} = 1/2(\overline{AD})$.	11. Definition of midpoint .
12. $\overline{AM} = \overline{FN}$.	12. Halves of equals are equal .
13. AMNF is a ▱ .	13. A quadrilateral with one pair of sides both parallel and equal is a parallelogram .
14. $\overline{MN} \parallel \overline{AF}$.	14. Definition of a parallelogram .

Theorem 22'

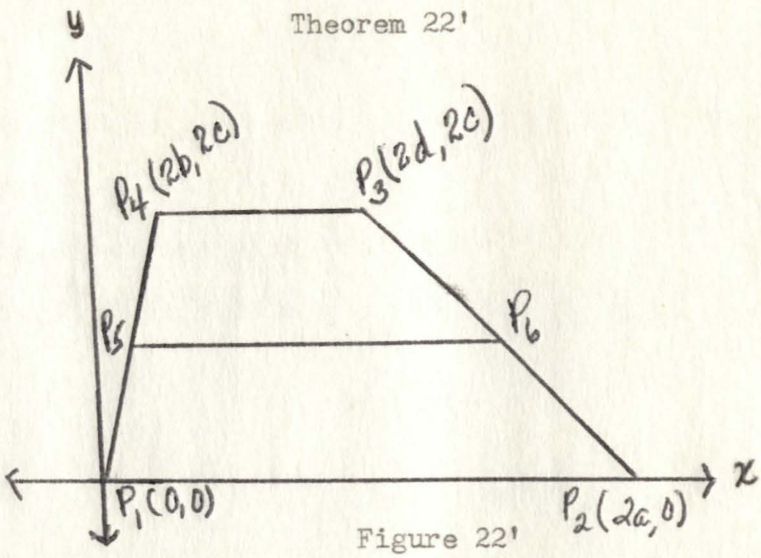


Figure 22'

Given: Trapezoid $P_1P_2P_3P_4$ with median $\overline{P_5P_6}$.

Prove: $\overline{P_5P_6} \parallel \overline{P_1P_2}$ and $\overline{P_5P_6} \parallel \overline{P_3P_4}$.

Proof: Let the axes and coordinates be as shown in the figure. By the midpoint formula P_5 is the point (b,c) and P_6 is the point (a + d, c).

1. m_1 - slope of P_1P_2 and P_3P_4

   m_2 - slope of P_5P_6 .

2. $m_1 = \frac{0 - 0}{2a - 0} = 0$, $\frac{2c - 2c}{2d - 2d} = 0$

   $m_2 = \frac{c - c}{a + d - b} = 0$.

3. Hence $m_1 = m_2$.

4. Since the three slopes are equal, $\overline{P_5P_6} \parallel \overline{P_1P_2}$ and $\overline{P_5P_6} \parallel \overline{P_3P_4}$.

Theorem 23

The median of a trapezoid is parallel to the bases and equal to half their sums.

Given: Trapezoid ABCD with the median $\overline{EF}$.

Prove: $\overline{EF} \parallel \overline{AB}$ and $\overline{DC}$ and $\overline{EF} = 1/2(\overline{AB} + \overline{DC})$.

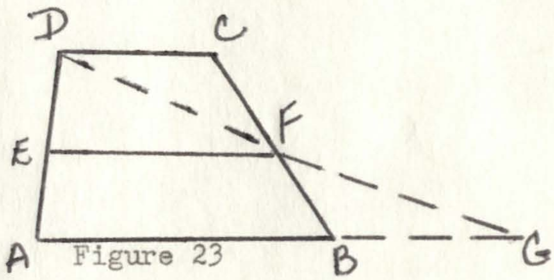


Figure 23

Proof:

Statements	Reasons
1. Draw $\overline{DF}$.	1. Through two points, one and only one straight line can be drawn.
2. Extend $\overline{DF}$ to meet $\overline{AB}$ produced at G .	2. A straight line may be extended to any required length .
3. $\triangle FCD \cong \triangle FBG$.	3. A.S.A.
4. $\overline{DF} = \overline{FG}$ and DC = BG .	4. C.P.C.T.
5. $\overline{EF} \parallel \overline{AG}$.	5. The line segment joining the midpoints of two sides of a triangle is parallel to the third side and equal to one half of it.
6. $\overline{EF} \parallel \overline{DC}$.	6. Two lines parallel to a third line are parallel to each other .
7. $\overline{EF} = 1/2(\overline{AG})$ or $1/2(\overline{AB} + \overline{BG})$.	7. Same as 5 .
8. $\therefore \overline{EF} = 1/2(\overline{AB} + \overline{DC})$.	8. Substitution .

Theorem 23'

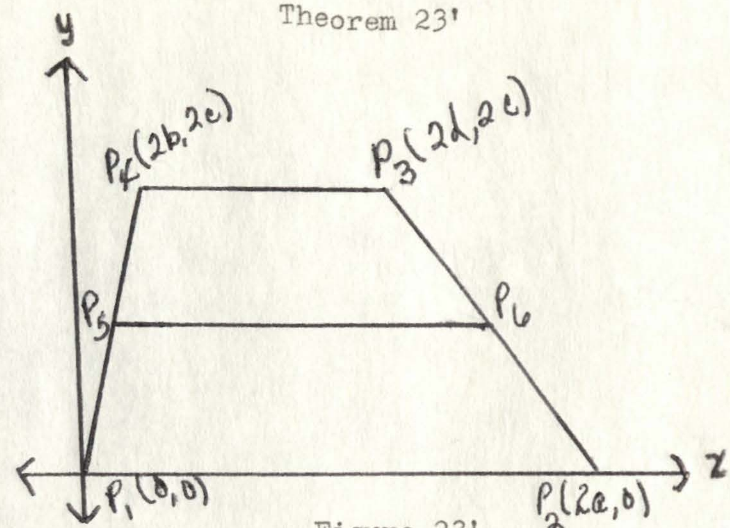


Figure 23'

Given: Trapezoid $P_1P_2P_3P_4$ with median $\overline{P_5P_6}$.

Prove: $\overline{P_5P_6} = 1/2(\overline{P_1P_2} + \overline{P_4P_3})$.

Proof: Let the axes and coordinates be as shown in the figure. By the midpoint formula, P_5 is point (b,c) and P_6 is point (a + d,c). By the distance formula:

1. $\overline{P_4P_3} = \sqrt{(2d - 2b)^2 + (2c - 2c)^2} = \sqrt{(2d-2b)^2+0} = 2d - 2b$.

2. $\overline{P_1P_2} = \sqrt{(2a - 0)^2+(0 - 0)^2} = \sqrt{(2a)^2+ 0} = \sqrt{(2a)^2} = 2a$.

3. $\overline{P_5P_6} = \sqrt{(a + d - b)^2+(c-c)^2} = \sqrt{(a+d-b)^2+0} = a + d - b$.

4. $\overline{P_1P_2} + \overline{P_4P_3} = 2a + 2d - 2b = 2(a + d - b)$.

5. Hence $\overline{P_5P_6} = 1/2(\overline{P_1P_2} + \overline{P_4P_3})$.

Theorem 24

Base angles of an isosceles trapezoid are congruent.

Given: Trapezoid ABCD with $\overline{DC} \parallel \overline{AB}$ and $\overline{AD} = \overline{BC}$.

Prove: $\angle A = \angle B$.

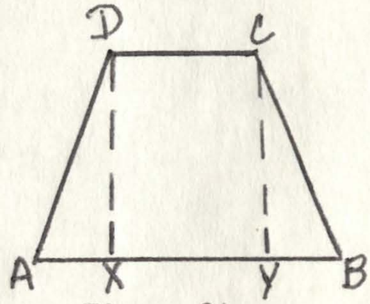


Figure 24

Proof:

Statements	Reasons
1. Draw $\overline{DX} \perp \overline{AB}$ and $\overline{CY} \perp \overline{AB}$.	1. Through a point not on a line exactly one line can be drawn $\perp$ to the given line.
2. $\overline{DX} \parallel \overline{CY}$.	2. In a plane, lines $\perp$ to the same line are parallel .
3. $\overline{DC} \parallel \overline{AB}$.	3. Given .
4. XYCD is a ▱ .	4. Definition of a ▱ .
5. $\overline{DX} = \overline{CY}$.	5. Opposite sides of a are equal .
6. $\overline{AD} = \overline{BC}$.	6. Given .
7. $\triangle AXD \cong \triangle BYC$.	7. HL .
8. $\angle A = \angle B$.	8. C.P.C.T.

Theorem 24'

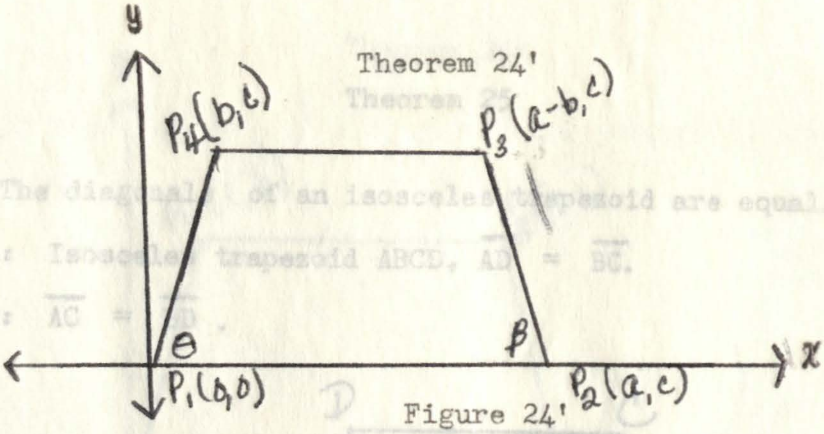


Figure 24'

Given: Isosceles trapezoid $P_1P_2P_3P_4$ with $P_4P_3 \parallel P_1P_2$ and $\overline{P_1P_4} = \overline{P_2P_3}$.

Prove: $\angle\theta = \angle\beta$.

Proof: Let P_1P_2 be on the x-axis with P_1 at the origin and coordinates of P_2 and P_4 as shown in the figure. Then P_3 is the point (a-b,c).

1. m_1 - slope of P_1P_2 and P_4P_3 , $\dfrac{0-0}{a-0} = 0$, $\dfrac{c-c}{a-b-b} = 0$

   m_2 - slope of P_1P_4 , $\dfrac{c-0}{b-0} = \dfrac{c}{b}$

   m_3 - slope of P_2P_3, $\dfrac{c-0}{a-b-a} = -\dfrac{c}{b}$.

2. $$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{\frac{c}{b} - 0}{1 + 0\cdot\frac{c}{b}} = \frac{c}{b}$$

   $$\tan\beta = \frac{m_1 - m_3}{1 + m_1 m_3} = \frac{0 - \left(-\frac{c}{b}\right)}{1 + 0\left(-\frac{c}{b}\right)} = \frac{c}{b} .$$

3. $\tan\theta = \tan\beta$.

4. Hence $\theta = \beta$ since θ and $\beta < 180°$.

Theorem 25

The diagonals of an isosceles trapezoid are equal.

Given: Isosceles trapezoid ABCD, $\overline{AD} = \overline{BC}$.

Prove: $\overline{AC} = \overline{BD}$.

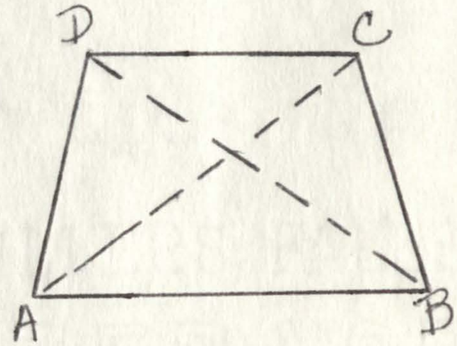


Figure 25

Proof:

Statements	Reasons
1. $\overline{AD} = \overline{BC}$.	1. Given .
2. $\overline{AB} = \overline{AB}$.	2. Reflexive Property .
3. $\angle A = \angle B$.	3. Base angles of an isosceles trapezoid are equal .
4. $\triangle ABC \cong \triangle ABD$.	4. S.A.S.
5. $\therefore \overline{AC} = \overline{BD}$.	5. C.P.C.T.

Theorem 25'

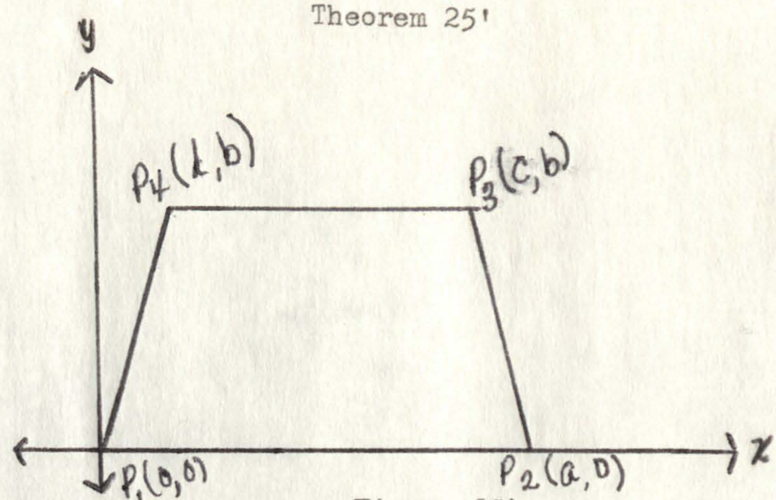


Figure 25'

Given: Isosceles trapezoid $P_1P_2P_3P_4$; $\overline{P_1P_4} = \overline{P_2P_3}$.

Prove: $\overline{P_1P_3} = \overline{P_2P_4}$.

Proof: Let the coordinate axes and coordinates be as shown in the figure. Let P_1P_4 and P_2P_3 be congruent legs in trapezoid $P_1P_2P_3P_4$. By the distance formula:

1. $\overline{P_1P_4} = \sqrt{(d - 0)^2 + (b - 0)^2} = \sqrt{d^2 + b^2}$.
2. $\overline{P_2P_3} = \sqrt{(a - c)^2 + (0 - b)^2} = \sqrt{(a - c)^2 + b^2}$.
3. Since $\overline{P_1P_4} = \overline{P_2P_3}$, $\sqrt{d^2 + b^2} = \sqrt{(a - c)^2 + b^2}$

   $d^2 + b^2 = (a - c)^2 + b^2$

   $d^2 = (a - c)^2$

   $d = a - c$

   Hence the coordinates of P_4 are $(a - c,b)$.
4. $\overline{P_1P_3} = \sqrt{(c - 0)^2 + (b - 0)^2} = \sqrt{c^2 + b^2}$.
5. $\overline{P_2P_4} = \sqrt{(a - c - a)^2 + (b - 0)^2} = \sqrt{c^2 + b^2}$.
6. $\therefore$ $\overline{P_1P_3} = \overline{P_2P_4}$.

Theorem 26

If the diagonals of a trapezoid are congruent, the trapezoid is isosceles.

Given: Trapezoid ABCD; $\overline{AC} = \overline{BD}$.

Prove: $\overline{AD} = \overline{BC}$.

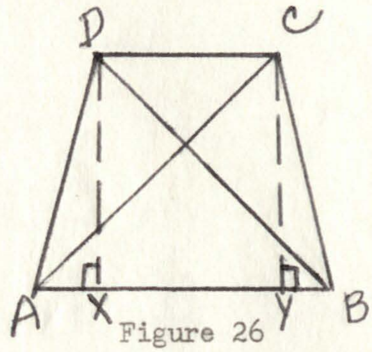


Figure 26

Proof:

Statements	Reasons
1. Draw $\overline{DX} \perp \overline{AB}$ and $\overline{CY} \perp \overline{AB}$.	1. Through a point not on a given line exactly one $\perp$ can be drawn to the line.
2. $\overline{DX} \parallel \overline{CY}$.	2. In a plane, two lines $\perp$ to the same line are parallel .
3. $\overline{DC} \parallel \overline{AB}$.	3. Definition of a trapezoid .
4. XYCD is a ▱ .	4. Definition of a parallelogram .
5. $\overline{DX} = \overline{CY}$.	5. Opposite sides of a parallelogram are congruent .
6. $\overline{AC} = \overline{BD}$.	6. Given .
7. $\triangle ACY \cong \triangle BDX$.	7. HL .
8. $\angle CAB = \angle DBA$.	8. C.P.C.T.
9. $\overline{AB} = \overline{AB}$.	9. Reflexive Property .
10. $\triangle CAB \cong \triangle DBA$.	10. S.A.S.
11. $\overline{AD} = \overline{BC}$.	11. C.P.C.T.

Theorem 26'

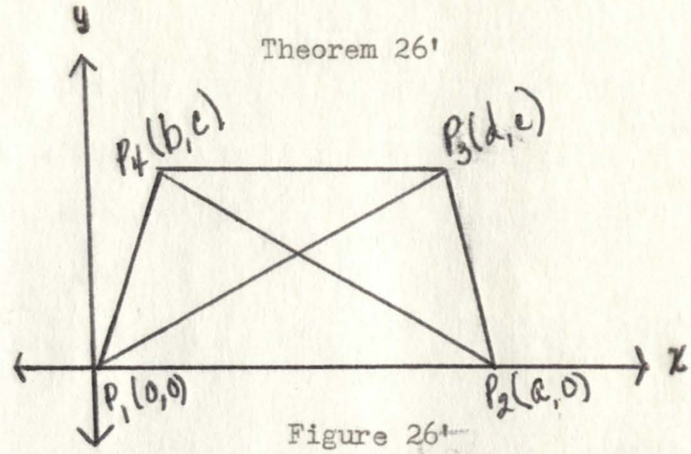


Figure 26'

Given: Trapezoid $P_1P_2P_3P_4$; with $\overline{P_1P_2} \parallel \overline{P_4P_3}$ and $\overline{P_1P_3} = \overline{P_2P_4}$.

Prove: $\overline{P_1P_4} = \overline{P_2P_3}$.

Proof: Let the axes and coordinates be as shown in the figure. By the distance formula:

1. $\overline{P_1P_3} = \sqrt{(d - 0)^2 + (c - 0)^2} = \sqrt{d^2 + c^2}$.

2. $\overline{P_2P_4} = \sqrt{(a - b)^2 + (0 - c)^2} = \sqrt{(a - b)^2 + c^2}$.

3. Since $\overline{P_1P_3} = \overline{P_2P_4}$, $\sqrt{d^2 + c^2} = \sqrt{(a - b)^2 + c^2}$

$$d^2 + c^2 = (a - b)^2 + c^2$$

$$d^2 = (a - b)^2$$

$$d = (a - b)$$

Hence the coordinates of P_3 are $(a - b,c)$.

4. $\overline{P_1P_4} = \sqrt{(b - 0)^2 + (c - 0)^2} = \sqrt{b^2 + c^2}$.

5. $\overline{P_2P_3} = \sqrt{(a - b - a)^2 + (c - 0)^2} = \sqrt{b^2 + c^2}$.

6. $\overline{P_1P_4} = \overline{P_2P_3}$.

7. Hence the trapezoid is isosceles .

Theorem 27

The quadrilateral formed by joining, in order, the midpoints of the sides of an isosceles trapezoid is a rhombus.

Given: Trapezoid ABCD; $\overline{AD} = \overline{BC}$; E, F, G and H are midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{AD}$.

Prove: EFGH is a rhombus .

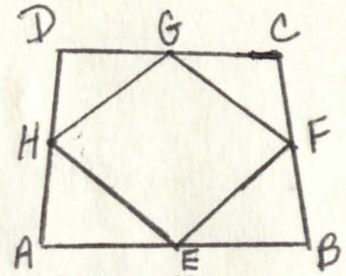


Figure 27

Proof:

Statements	Reasons
1. EFGH is a ▱ .	1. The figure formed by joining, in order, the midpoints of the sides of a quadrilateral is a ▱ .
2. $\overline{AH} = 1/2(\overline{AD})$, $\overline{BF} = 1/2(\overline{BC})$.	2. Definition of midpoint .
3. $\overline{AH} = \overline{BF}$.	3. Transitive Property .
4. $\angle A = \angle B$.	4. Base angles of an isosceles trapezoid are equal .
5. $\overline{AE} = \overline{BE}$.	5. Definition of a midpoint .
6. $\triangle AEH \cong \triangle BEF$.	6. S.A.S.
7. $\overline{HE} = \overline{FE}$.	7. C.P.C.T.
8. EFGH is a rhombus .	8. A ▱ with two consecutive sides equal is a rhombus .

Theorem 27'

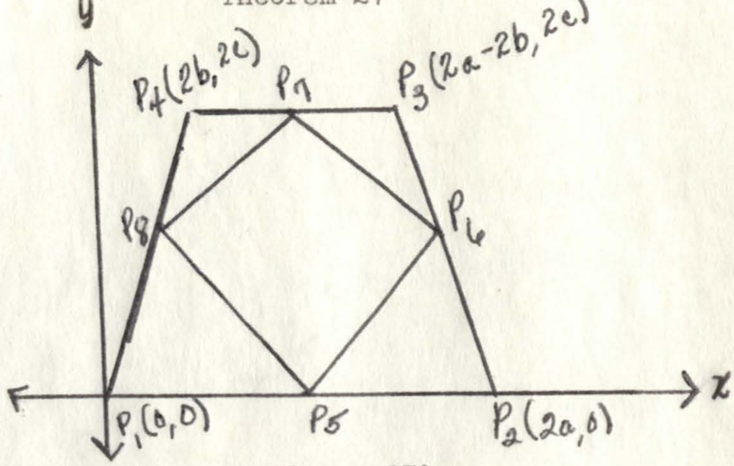


Figure 27'

Given: Isosceles trapezoid $P_1P_2P_3P_4$ with midpoints P_5, P_6, P_7 and P_8 of $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$ and $\overline{P_1P_4}$.

Prove: $P_5P_6P_7P_8$ is a rhombus.

Proof: Let the axes and coordinates be as shown in the figure. By the midpoint formula, P_5 is point (a,0), P_6 is point (2a - b,c), P_7 is point (a,2c) and P_8 is point (b,c).

1. m_1 - slope of P_5P_6, P_7P_8
   m_2 - slope of P_5P_8, P_6P_7.

2. $m_1 = \frac{c - 0}{2a - b - a} = \frac{c}{a - b}, \frac{2c - c}{a - b} = \frac{c}{a - b}$

   $m_2 = \frac{0 - c}{a - b} = \frac{-c}{a - b}, \frac{c - 2c}{2a - b - a} = \frac{-c}{a - b}$.

3. Since their slopes are equal, $P_5P_6 \parallel P_7P_8$ and $P_6P_7 \parallel P_8P_5$. Hence the figure is a ▱.

4. $\overline{P_5P_6} = \sqrt{(a - b)^2 + c^2}$ and $\overline{P_6P_7} = \sqrt{(a - b)^2 + c^2}$.

5. $\overline{P_5P_6} = \overline{P_6P_7}$, hence the ▱ is a rhombus.

Theorem 28

If a line parallel to the bases of a trapezoid bisects one leg, it bisects the other leg also.

Given: Trapezoid ABCD with $\overline{PQ} \parallel \overline{AB}$ and P the midpoint of $\overline{AD}$.

Prove: Q bisects $\overline{BC}$.

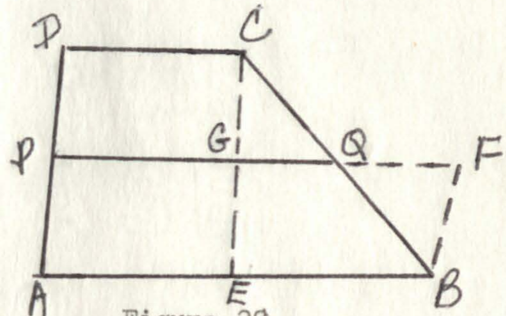


Figure 28

Proof: Draw $\overline{CE} \parallel \overline{AD}$ and $\overline{BF} \parallel \overline{AD}$. Extend $\overline{PQ}$ to F.

Statements	Reasons
1. $\overline{CE} \parallel \overline{BF}$.	1. Two lines parallel to the same line are parallel .
2. $\angle CGQ = \angle BFQ$.	2. Alternate interior angles .
3. $\angle GQC = \angle BQF$.	3. Vertical angles .
4. $\angle GCQ = \angle FBQ$.	4. Two angles of a triangle are equal, so third is equal .
5. $\overline{PA} = \overline{BF}$ and $\overline{PD} = \overline{GC}$.	5. Opposite sides of a parallelogram are equal .
6. $\overline{PA} = \overline{PD}$.	6. Definition of midpoint .
7. $\overline{BF} = \overline{GC}$.	7. Transitive Property .
8. $\triangle GQC \cong \triangle BQF$.	8. A.S.A.
9. $\overline{BQ} = \overline{QC}$.	9. C.P.C.T.
10. $\therefore$ Q bisects $\overline{BC}$.	10. Definition of bisector.

Theorem 28'

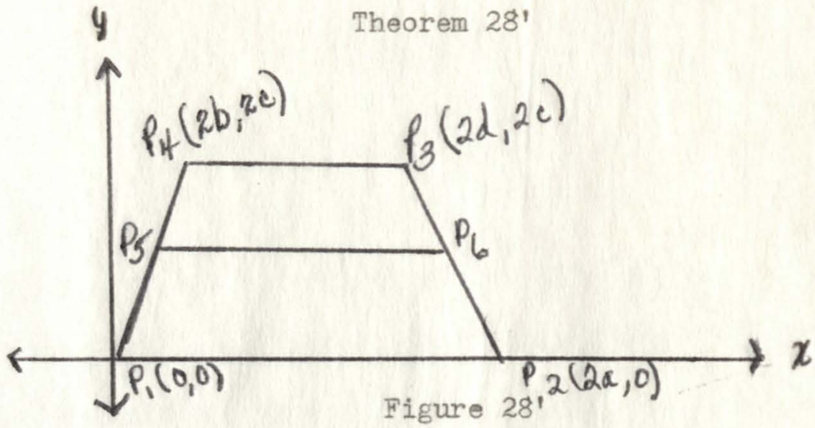


Figure 28'

Given: Trapezoid $P_1P_2P_3P_4$ with $\overline{P_5P_6} \parallel \overline{P_1P_2}$ and P_5 the midpoint of $\overline{P_1P_4}$.

Prove: P_6 bisects $\overline{P_2P_3}$.

Proof: Let coordinate axes and coordinates be as shown in the figure.

By the midpoint formula P_5 is point (b,c).

1. Slope of P_1P_2 is 0; since $\overline{P_5P_6} \parallel \overline{P_1P_2}$, $\overline{P_5P_6}$ slope is 0.

2. The equation of $\overleftrightarrow{P_5P_6}$ is $y - c = m(x - b) \qquad m = 0$

$$y - c = 0$$
$$y = c .$$

3. The slope of $\overleftrightarrow{P_2P_3} = \frac{2c - 0}{2d - 2a} = \frac{2c}{2d-2a} = \frac{2(c)}{2(d-a)} = \frac{c}{d - a}$.

4. The equation of $\overleftrightarrow{P_2P_3}$ is $y = \frac{c}{d - a}(x - 2a)$.

5. Intersection of $\overleftrightarrow{P_5P_6} \wedge \overleftrightarrow{P_2P_3}$; $c = \frac{c}{d - a}(x - 2a)$

$$d-a = x - 2a$$
$$x = d + a$$
$$y = c .$$

6. $\therefore P_6$ coordinates are (a + d,c).

7. By the midpoint formula the midpoint of P_2P_3 is $\left(\frac{2d+2a}{2}, c\right) = d+a,c = P_6$.

8. Hence P_6 bisects $\overline{P_2P_3}$.

CHAPTER V

Circles

In this chapter, the two methods will be applied to several theorems related to circles. First, the synthetic method will be applied to prove a theorem, then the analytic method will be used to prove the same theorem in sequence. Theorems proved by the synthetic method will be denoted by unprime numbers while the same theorems being proved by the analytic method will be denoted by prime numbers. Example, Theorem 29 and Theorem 29'.

Theorem 29

A line through the center of a circle perpendicular to a chord bisects the chord.

Given: Circle O with $\overline{AB}$ through center O $\perp$ to chord CD at E.

Prove: $\overline{CE} = \overline{ED}$.

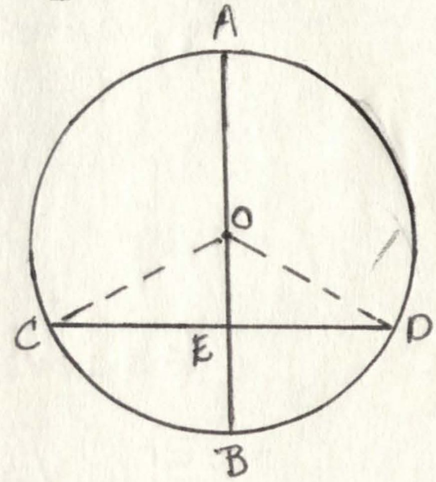


Figure 29

Proof:

Statements	Reasons
1. Draw radii OC and OD .	1. Construction .
2. $\overline{OC} = \overline{OD}$.	2. Radii of the same circle are equal .
3. $\overline{OE} = \overline{OE}$.	3. Reflexive Property .
4. $\overline{OE} \perp \overline{CD}$.	4. Given .
5. Right $\triangle OEC \cong$ Right $\triangle OED$.	5. HL .
6. $\therefore \overline{CE} = \overline{ED}$.	6. C.P.C.T.

Theorem 29'

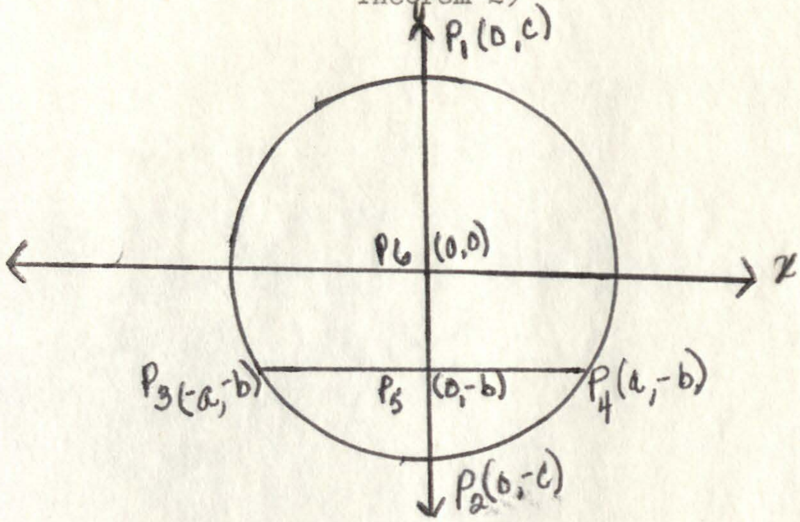


Figure 29'

Given: Circle O with $\overline{P_1P_2}$ through center $P_6 \perp$ to chord P_3P_4 at P_5.

Prove: $\overline{P_3P_5} = \overline{P_5P_4}$.

Proof: Let the origin be at the center of the circle. Let P_1 and P_2 intersect the y-axis. Let the coordinates be as shown in the figure. By the distance formula:

1. $\overline{P_3P_5} = \sqrt{(0 + a)^2 + (-b + b)^2} = \sqrt{a^2} = a.$

2. $\overline{P_5P_4} = \sqrt{(a - 0)^2 + (-b + b)^2} = \sqrt{a^2} = a .$

3. $\therefore \overline{P_3P_5} = \overline{P_5P_4}$.

Theorem 30

If a line is tangent to a circle, it is perpendicular to the radius drawn to the point of contact.

Given: $\overline{AB}$ tangent to circle O at C and OC a radius.

Prove: $\overline{AB} \perp \overline{OC}$.

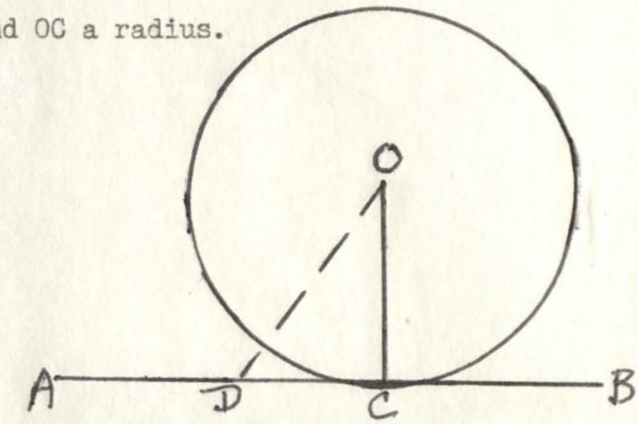


Figure 30

Proof:

Statements	Reasons
1. From D, any point on $\overline{AB}$ except C, draw DO.	1. Construction .
2. D is outside the circle O .	2. Definition of a tangent.
3. $\therefore \overline{OD} > \overline{OC}$, or $\overline{OC}$ is the shortest line segment from O to $\overline{AB}$.	3. Any point outside a circle is more that a radius distance from the center .
4. $\therefore \overline{OC} \perp \overline{AB}$ or $\overline{AB} \perp \overline{OC}$.	4. The shortest distance from a given exterior point to a line is the $\perp$ distance from the point to the line.

Theorem 30'

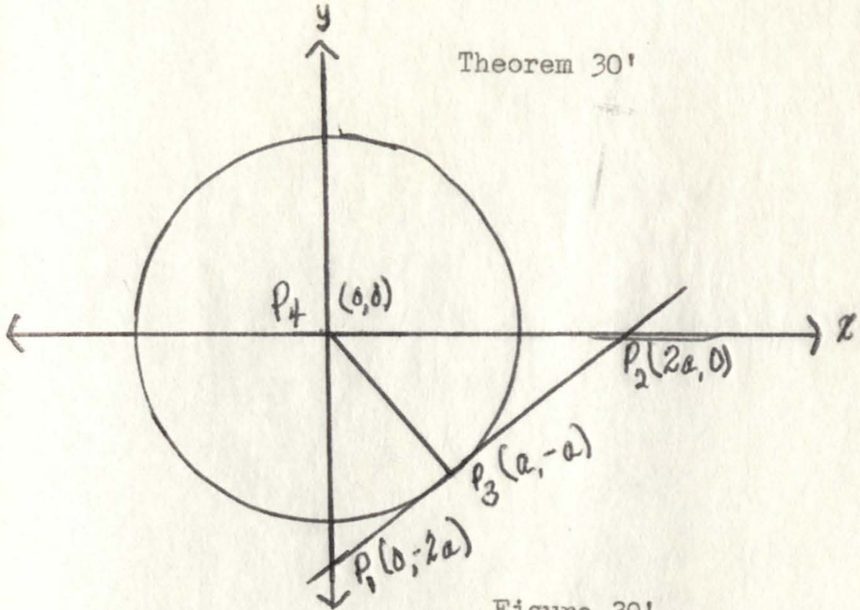


Figure 30'

Given: $\overline{P_1P_2}$ tangent to circle Q at P_3 and $\overline{P_4P_3}$ a radius.

Prove: $\overline{P_1P_2} \perp \overline{P_4P_3}$.

Proof: Let the origin be at the center of the circle. Let P_1 and P_2 intersect the x and y axes equal distance from P_4(the center). Let the coordinates of P_1, P_2 and P_4 be as shown in the figure. By the midpoint formula, P_3 is (a,-a). By the slope formula:

1. The slope of $\overline{P_1P_2} = m_1 = \frac{0 + 2a}{2a - 0} = \frac{2a}{2a} = 1$.

2. The slope of $\overline{P_4P_3} = m_2 = \frac{0 + a}{0 - a} = -\frac{a}{a} = -1$.

3. $m_1 \cdot m_2 = -1$.

4. $\therefore \overline{P_1P_2} \perp \overline{P_4P_3}$.

Theorem 31

In a circle or in equal circles, chords equidistant from the center are equal.

Given: Circle O and chords AB and CD with $\overline{OE} \perp \overline{AB}$ and $\overline{OF} \perp \overline{CD}$; distance $\overline{OE}$ = distance $\overline{OF}$.

Prove: $\overline{AB} = \overline{CD}$.

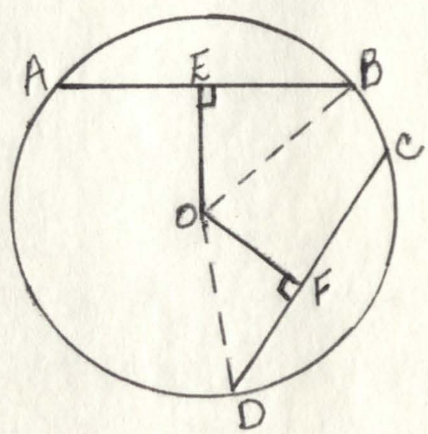


Figure 31

Proof:

Statements	Reasons
1. Draw radii OB and OD .	1. Construction .
2. $\overline{OB} = \overline{OD}$.	2. Radii of same circle are equal .
3. $\overline{OE} = \overline{OF}$.	3. Given .
4. $\overline{OE} \perp \overline{AB}$ and $\overline{OF} \perp \overline{CD}$.	4. Given .
5. Right $\triangle OEB \cong$ Right $\triangle OFD$.	5. HL .
6. $\therefore \overline{EB} = \overline{FD}$.	6. C.P.C.T.
7. $\overline{EB} = 1/2(\overline{AB})$ and $\overline{FD} = 1/2(\overline{CD})$.	7. A line through the center of a circle $\perp$ to a chord bisects the chord .
8. $\therefore \overline{AB} = \overline{CD}$.	8. Doubles of equals are equal .

Theorem 31'

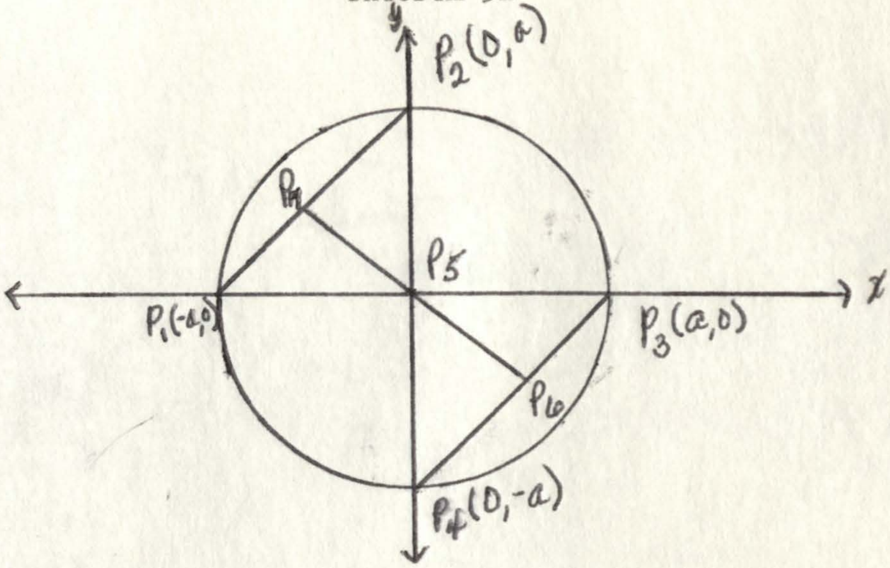


Figure 31'

Given: Circle O and chords P_1P_2 and P_3P_4 with $\overline{P_5P_7} \perp \overline{P_1P_2}$ and $\overline{P_5P_6} \perp \overline{P_3P_4}$; distance $\overline{P_5P_7}$ = distance $\overline{P_5P_6}$.

Prove: $\overline{P_1P_2} = \overline{P_3P_4}$.

Proof: Let the origin be at the center of the circle O. Let P_1 and P_3 intersect the x-axis and P_2 and P_4 intersect the y-axis. Let P_1, P_2, P_3 and P_4 coordinates be as shown in the figure. By the distance formula:

1. $\overline{P_1P_2} = \sqrt{(0 + a)^2 + (a - 0)^2} = \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}$.

2. $\overline{P_3P_4} = \sqrt{(a - 0)^2 + (0 + a)^2} = \sqrt{a^2 + a^2} = \sqrt{2a^2} = a\sqrt{2}$.

3. $\overline{P_1P_2} = \overline{P_3P_4}$.

Theorem 32

Tangents to a circle from an outside point are equal.

Given: Circle O with $\overline{PA}$ and $\overline{PB}$ tangent at A and B respectively; OP drawn.

Prove: $\overline{PA} = \overline{PB}$.

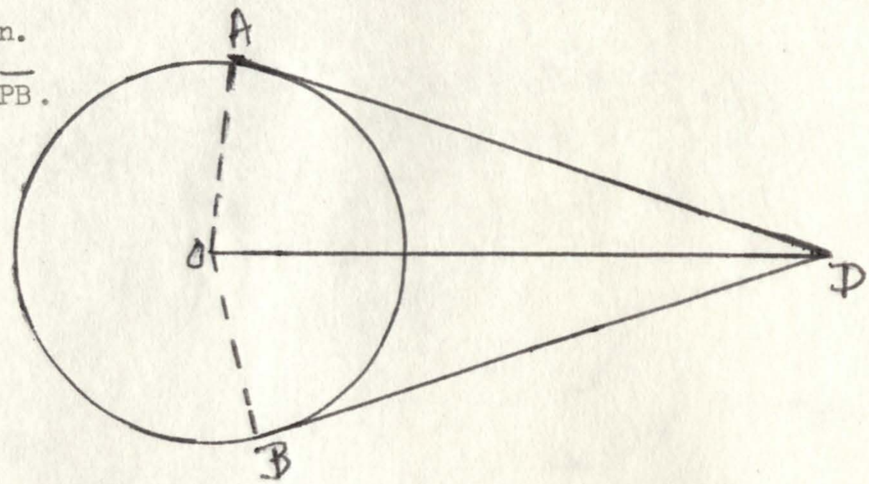


Figure 32

Proof:

Statements	Reasons
1. Draw OA and OB .	1. Construction .
2. $\overline{OA} = \overline{OB}$.	2. Radii of the same circle are equal .
3. $\angle$ A and $\angle$ B are right angles .	3. A tangent is $\perp$ to the radius drawn to the point of contact .
4. $\overline{OP} = \overline{OP}$.	4. Reflexive Property .
5. Right $\triangle OAP \cong$ Right $\triangle OBP$.	5. HL .
6. $\therefore \overline{PA} = \overline{PB}$.	6. C.P.C.T.

Theorem 32'

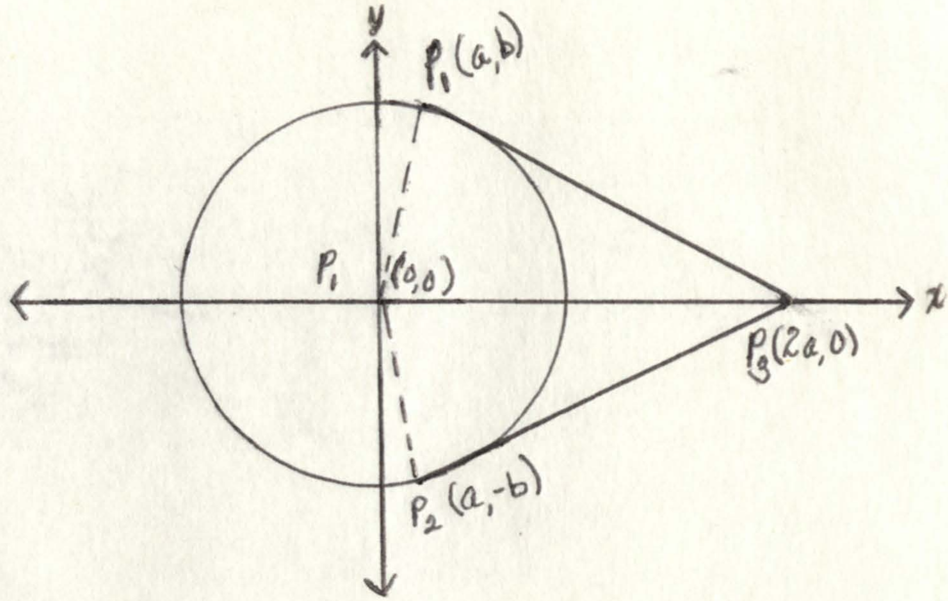


Figure 32'

Given: Circle O with $\overline{P_5P_4}$ and $\overline{P_5P_2}$ tangent at P_1 and P_2 respectively; $\overline{P_4P_3}$ drawn.

Prove: $\overline{P_3P_1} = \overline{P_3P_2}$.

Proof: Let the origin be at the center of the circle O. Let the coordinates of P_1, P_2, P_3 and P_4 be as shown in the figure. Let the point P_3 lie on the x-axis. By the distance formula:

1. $\overline{P_3P_1} = \sqrt{(a - 2a)^2 + (b - 0)^2} = \sqrt{a^2 + b^2}$.

2. $\overline{P_3P_2} = \sqrt{(2a - a)^2 + (0 + b)^2} = \sqrt{a^2 + b^2}$.

3. $\therefore \overline{P_3P_1} = \overline{P_3P_2}$.

SUMMARY

The emphasis throughout this paper was to give an understanding of the basic principles of approaching High School Geometry from an analytic geometry approach. Considerable care was taken with the proofs of the main theorems, so that we may develop an appreciation of the logical structure of a mathematical proof.

There are few subjects that afford a richer or more varied supply of interesting and thought-provoking problems than does analytic geometry. Yet many of us fail to reach a point where we can solve these problems. A major part of the difficulty arise from the fact the new subject matter and method is so abundant in the analytic geometry that there is that little time left to devote to problem solving.

An alternative view is that the most important outcomes from such an approach are (1) understanding of the essentials of developments (2) complete understanding of the results (3) ability to use the results in any problem situations. The teacher adopting such a view would make the first proofs completely in class and then do progressively less proving as the class advances.

The values to be derived from this approach may be divided into two groups, (1) intrinsic values and (2) preparation. Intrinsic values are the qualities of the subject that make its study result in increased problem-solving ability, increased appreciation of geometry as a useful and rigorous science and increased understanding of the relations among things mathematical. Analytic geometry as preparation for further mathematics and science.

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