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A Fundamental Theorem in Linear Programming

George E. Higgs

A FUNDAMENTAL THEOREM IN LINEAR PROGRAMMING

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A Thesis

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by

George E. Higgs

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DEDICATION

This paper is dedicated to my wife Olivette, and my two children, Cherie and Marcus who so willingly and untiringly have made my entire pursuits for an education possible.

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INTRODUCTION

Linear Programming is a recent empiral tool made available to the market world. It has already found many important applications and offers exciting opportunities in the future. It can serve as an important management aid to marketing firms.

Linear Programming originated during World War II as a method for specifying routes that would minimize travel distance for the limited shipping facilities available to the Allies; for determining the best method of allocating scarce labor, machine tools, and plant facilities to produce war goods; and for similar purposes. Since the end of the war, the method has become highly refined and is being used by a large number of private firms and research organizations.

Linear Programming techniques involve the maximization or minimization of linear function, subject to linear inequalities. The linear function ordinarily is a profit or cost function.

Linear Programming can be applied as to procedure for selecting a beginning plan which is consistent with sets of two or more inequalities defining the upper and lower limit to real activities or unused resources.

Since linear programming is a fairly new concept in the solution of certain types of problems, a thorough understanding about these properties must be comprehended by the teacher of high school algebra.

In the teaching of high school algebra the method that is used to find the maximum or minimum of a function $f(X)=AX+b$ over a polygonal convex set is as follows: Find the corner points of the set, there will be a finite number of them; substitute the coordinates of each into the function; the largest of the value so obtained will be a maximum of the function and the smallest value will be the minimum of the function. Knowing that this method always brings about a practical solution in all cases, the question brought to the writer's mind is why does the maximum or minimum always occur at a corner point?

The primary objective of this paper is to show that given a linear function f defined over a polygonal convex set, its maximum and minimum will occur at the corner points.

In Chapter I we have basic definitions and theorems that will be used throughout this paper.

In Chapter II we show conditions under which a system of inequalities in R^2 has a solution and in fact we show that if a solution exists it is a convex set. Furthermore we assume that a system of inequalities in R^n has a solution and show this solution is convex.

In Chapter III we prove two lemmas that are to be used in the proof to the main theorem. We also show in the main theorem that if given a linear function f define over a polygonal set, its maximum or minimum will occur at a corner point.

In Chapter IV we show two examples that are applicable to the main theorem. Although these applications will give the reader some idea of the adaptability of linear-programming methods, it is not the writer's purpose to describe the complete range of linear-programming applications. These applications were selected mainly for their ability to illustrate the techniques of formulating a linear-programming model.

CHAPTER I

DEFINITIONS AND TERMINOLOGY

1.0 Definition: Any ordered pair of real numbers (x,y) is called a vector.

1.1 Definition: $R^n = \{x \mid (x^1, x^2 \quad , x^n)$ where $x^i \in R$ for $i=1, 2, \cdots, \overline{n}$

1.2 Definition: If V=(x,y), then the norm of V is designated by $\|V\|$ and is defined by $\|V\| = \sqrt{x^2+y^2}$.

1.3 Definition: The intersection of closed half planes is a polygonal convex set.

1.4 Definition: Let K denote a polygonal convex set then P is a corner point of K if P is on the boundaries of K and is the intersection of two lines.

1.5 Definition: If each of x and y is an element of R^n, then

1) the sum of x and y is an element of R^n, denoted by $x+y=(x^1+y^1,\ x^2+y^2, \cdots x^n+y^n)$

2) the dot product of x and y is an element of R denoted by

$$xy = \left[x^1,\ x^2,\ \cdots\ x^n\right]\begin{bmatrix} y^1 \\ y^2 \\ \vdots \\ y^n \end{bmatrix} = x^1y^1+x^2y^2+\ldots+x^ny^n$$

1.6 Definition: An (m,n) matrix A is a rectangular array of real numbers denoted symbolically by

$$A= \begin{bmatrix} a_{11} & a_{12}--- & a_{1n} \\ a_{21} & a_{22}--- & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2}--- & a_{mn} \end{bmatrix}$$

having n rows and m columns. If n=1, A is a column vector and m=1, A is a row vector.

1.7 Definition: <u>Greater</u> <u>Than</u> ($>$); For every pair x,y of real numbers, $x>y$ if, and only if, $x - y$ is a positive number.

1.8 Definition: Less Than ($\leq$); For every pair x,y of real numbers $x\leq y$ if, and only if $x - y$ is a negative number.

1.9 Definition: Let $f: R^n \longrightarrow R$ and $x_o \in R^n$, then $f(x_o)$ is a maximum of f if for each $y \in R^n$, $f(y) \leq f(x_o)$.

2.0 Definition: Let $f: R^n \longrightarrow R$ and $x_o \in R^n$, then $f(x_o)$ is a minimum of f if for each $y \in R^n$, $f(y) \geq f(x_o)$.

2.1 Definition: The statement that $[\overline{A};\overline{B}]$ is a linear path from the point A to the point B in the R^n means; there is a continuous function f from $[0,1]$ into R^n such that if t is an element of $[0,1]$, then $f(t)=(1 - t)A+tB$. $[A;B]'=\{f(t) \mid 0 \leq t \leq 1\}$ is called the carrier of the linear path from A to B.

2.2 Let C denote a subset of R^n, then the set C is said to be convex provided that if A and B are points in C, the $[A;B]'$ is a subset of C.

Example:

H_1: C is a closed circular disc with point P as its center.

H_2: $C=\{x \mid \|X - P\| \leq r$

C: C is a convex set.

Proof:

Let A and B be points in C and $[A;B]$ be the linear path from A to B and X a point in $[A;B]'$, then there exist $t \in [0,1]$ such that $X=(1 - t)$ $A+tB$.

$$\begin{aligned} d(P,X) &= \|X - P\|, \quad X=(1 - t)A+tB \text{ and } P=(1 - t)P+tP \\ &= \|(1-t)A+tB - [(1-t)P+tP]\| \\ &= \|(1-t)\ (A-P)+t(B-P)\| \\ &\leq \|(1-t)\ (A-P)\| + \|t(B-P)\| \\ &= \|(1-t)\| \|A-P\| + \|t\|\ \|B-P\| \end{aligned}$$

Since $A-P \leq r$ and $B-P \leq r$ implies that

$$(1-t)r+tr=r$$

Therefore C is a convex set.

SYMBOLS USED

1. $\{\ \}$ ----------------------Set
2. R ----------------- $\{x \mid x \text{ is a real number}\}$
3. d(x,y)---------The distance between the points x and y.
4. $\Longrightarrow$ -----------------------Implies
5. iff-----------------------If and only if

CHAPTER II

PROPERTIES OF LINEAR INEQUALITIES

We shall not prove that under certain conditions a system of inequalities has a solution, but we will observe the following:

Consider the system of inequalities:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n \geq c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n \geq c_2 \\ \cdots\cdots + \cdots\cdots\cdots\cdots\cdots\cdots \quad . \\ \cdots\cdots + \cdots\cdots\cdots\cdots\cdots\cdots \quad . \\ a_{m1}x_1 + a_{m2}x_2 \cdots\cdots + a_{mn}x_n \geq c_n \end{cases}$$

If the associated system of equalities,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n = c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n = c_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad . \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \quad . \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n = c_n \end{cases}$$

has a solution, then so does the system of inequalities.

It is interesting to note that given the system

$$a_{11}x_1 + a_{12}x_2 \geq c_1$$
$$a_{21}x_1 + a_{22}x_2 \geq c_2$$

in case $\dfrac{a_{11}}{a_{21}} = \dfrac{a_{12}}{a_{22}}$

the associated system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = c_1 \\ a_{21}x_1 + a_{22}x_2 = c_2 \end{cases}$$

has no solution, but the system of inequalities may have a solution.

We shall state in the following theorem the necessary and sufficient conditions for such a system to have a solution.

Theorem I

H_1: $\begin{cases} a_1x_1 + b_1x_2 \geq c_2 \\ a_1x_1 + b_1x_2 \geq c_1 \end{cases}$ $\quad c_1 \neq c_2$

C: The system in H_1 has a solution.

Proof:

Since $c_1 \neq c_2$, then either $c_2 > c_1$ or $c_1 > c_2$

Suppose $c_2 > c_1$

Let $K = \{(x_1, x_2) | a_1x_1 + b_1x_2 = c_2 + d\}$ where $d \geq 0$

We will show that K is a solution set in H_1:

$a_1x_1 + b_1x_2 = c_2 + d \geq c_2 \Longrightarrow a_1x_1 + b_1x_2 \geq c_2$

$a_1x_1 + b_1x_2 = c_2 + d > c_1 + d \geq c_1$

Therefore K is a solution set to H_1

<u>Theorem</u> <u>II</u>

$$H_1: \begin{cases} a_1x_1 + b_1x_1 \geq c_1 \\ a_1x_1 + b_1x_1 \leq c_2 \end{cases}$$

C: There exist a solution set to H_1 iff $c_2 \geq c_1$.

Assume a solution set exists

Prove: $c_1 \leq c_2$

Proof:

$$1. \quad \begin{cases} a_1x_1 + b_1x_1 - c_1 \geq 0 \\ a_1x_1 + b_1x_1 - c_2 \leq 0 \end{cases}$$

$$2. \quad \begin{cases} a_1x_1 + b_1x_1 - c_1 \geq 0 \\ -a_1x_1 - b_1x_1 + c_2 \geq 0 \end{cases}$$ Divide second inequality by -1.

3. $c_2 - c_1 \geq 0$

Therefore $c_2 \geq c_1$

Assume $c_1 \leq c_2$, show there exist a solution.

1. Since $c_1 \leq c_2$, $c_1 \leq \frac{c_1 + c_2}{2} \leq c_2$

2. Let $H = \left\{(x_1, x_2) \mid a_1x_1 + b_1x_2 = \frac{c_1 + c_2}{2}\right\}$

3. Claim H is a solution set to H_1.

4. Let $K = \frac{c_1 + c_2}{2}$, Then $a_1x_1 + b_1x_2 = K$

5. $x_1 = \frac{K - b_1x_2}{a_1}$ $a_1 \neq 0$ and $x_2 = \frac{K - a_1x_1}{b_1}$ $b_1 \neq 0$

$$6. \quad \cancel{a}_1\left(\frac{K - b_1x_2}{\cancel{a}_1}\right) + \cancel{b}_1\left(\frac{K - a_1x_1}{\cancel{b}_1}\right) \geq c_1$$

$$\cancel{a}_1\left(\frac{K - b_1x_1}{\cancel{a}_1}\right) + \cancel{b}_1\left(\frac{K - a_1x_2}{\cancel{b}_1}\right) \leq c_2$$

$$7. \quad \begin{cases} K - b_1x_2 + K - a_1x_1 \geq c_1 \\ K - b_1x_2 + K - a_1x_1 \leq c_2 \end{cases}$$

$$8. \quad \begin{cases} 2K - b_1x_2 - a_1x_1 \geq c_1 \\ 2K - b_1x_2 - a_1x_1 \leq c_2 \end{cases}$$

$$9. \quad \begin{cases} \cancel{2}\left(\frac{c_1 + c_2}{\cancel{2}}\right) - b_1x_2 - a_1x_1 \geq c_1 \\ \cancel{2}\left(\frac{c_1 + c_2}{\cancel{2}}\right) - b_1x_2 - a_1x_1 \leq c_2 \end{cases}$$

$$10. \quad \begin{cases} c_2 - b_1x_1 - a_1x_1 \geq 0 \\ c_1 - b_1x_2 - a_1x_1 \leq 0 \end{cases}$$

Therefore $\quad c_2 \geq b_1x_2 + a_1x_1$

$$c_1 \leq b_1x_2 + a_1x_1$$

Theorem III

$$H_1: \begin{cases} a_{11}x_1 \quad a_{12}x_2 \cdots a_{1n}x_n \geq c_1 \\ a_{21}x_1 \quad a_{22}x_2 \qquad a_{2n}x_n \geq c_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \quad . \\ \cdots\cdots\cdots\cdots\cdots\cdots \quad . \\ \cdots\cdots\cdots\cdots\cdots\cdots \quad . \\ a_{m1}x_1 \quad a_{n2}x_2 \qquad a_{mn}x_n \geq c_n \end{cases}$$

H_2: $AX \geq C$

H_3: K is solution set for the inequality H_2.

C: K is a convex set

Proof:

Let R and S be points in K and $[R;S]$ be the linear path R to S and let X be a point in $[R:S]'$, then there exist a $t \in [0;1]$ such that $X=(1-t)R+tS$. Since the dimensions in H_1 are known it can be written in the following form:

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}} \underset{X}{\begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}} \underset{C}{\begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ c_n \end{bmatrix}}$$

1. A is called the coefficient matrix of the system of inequality.
2. C is called the constant vector of the system.

1. $A[(1 - t)R+tS] =$

2. $AR(1 - t)+ASt \leq$

3. $C(1 - t)+Ct = C$

Therefore K is a convex set.

Theorem IV

H_1:

$$a_1x_1 + b_1x_2 \geq c_1$$
$$a_2x_1 + b_2x_2 \geq c_2$$
$$x_1 \geq 0$$
$$x_2 \geq 0$$

H_2: S is a positive Solution to H_1.

C: S is a convex set.

Proof:

Let L and M be points in S and $[L;S]$ is a linear path from L to M and if K is a point in $[L;M]'$ there exist a $t \in [0,1]$ such that $K=(1 - t)L+Mt$ is a positive number.

Let $K=(x_1,x_2)$

1. $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (1 - t) \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} + t \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$$

2. $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (1 - t)1_1 \\ (1 - t)1_2 \end{pmatrix} + \begin{pmatrix} tm_1 \\ tm_2 \end{pmatrix}$$

3. $$x_1 = (1 - t)1_1 + tm_1$$
$$x_2 = (1 - t)1_2 + tm_2$$

4. Substituting in inequality #1 in H_1, we have

$$a_1[(1 - t)1_1 + tm_1] = a_1 1_1(1 - t) + a_1 m_1 t \geq c(1 - t) + ct = c_1.$$

5. Substituting in inequality #2 in H_1, we have

$$a_2[(1 - t)1_2 + tm_2] = a_2 1_2(1 - t) + a_2 m_2 t \geq c(1 - t) + ct = c_2.$$

6. Substituting in inequality #3, in H_1 we have

$$(1 - t)1_1 + tm_1 \geq 0(1 - t) + t0 = 0$$

7. Substituting in inequality #4, in H_1, we have

$$(1 - t)1_2 + tm_2 \geq 0(1 - t) + t0 = 0$$

Therefore S is a convex set.

CHAPTER III

MAIN THEOREM

Lemma I:

H_1: $[A,B]$ is an interval in R

H_2: $f(x)=ax+b$ is defined on $[A,B]$, where A and B are in R

H_3: $f(A) \leq f(B)$

C: If $Z \in [A,B]$, then $f(A) \leq f(Z) \leq f(B)$

Proof:

Let A and B be an interval and Z is a point in $[A,B]$ such that $Z=(1 - t)A+tB$, and $t \in [0,1]$, there exist a $f(Z) \in [f(A); f(B)]$ such that $f(Z)=aZ+b$.

1. $f(A)=aA+b$ and $f(B)=aB+b$
2. $Z=(1 - t)A+tB, \quad 0 \leq t \leq 1$
3. $f(Z)=a[(1 - t)A+tB]+b$
4. $= aA(1 - t)+aBt+b \leq aB+b=f(B)$
5. $f(Z)=a[(1 - t)A+tB]+b =$
6. $=aA(1 - t)+tB+b \geq$
7. $aA+b = f(A)$

Therefore $f(A) \leq f(Z) \leq f(B)$

Lemma II

H_1: $[A;B]$ is a linear path in R^n

H_2: $f:[A;B]' \to R$ defined by

$f(X)=CX+b$, where $X=(x_1 x_2 \ldots x_n)$ and $C=(c_1 c_2 \ldots c_n)$

$b \in R$, then $f(X)=c_1x_1+c_2x_2 \ldots c_nx_n+b$

H_3: $f(A) \leq f(B)$

C: If $Z \in [A;B]'$, then $f(A) \leq f(Z) \leq f(B)$

Proof:

1. Let A and B be points in R^n and Z is a point in $[A;B]'$ such $Z=(1-t)A+tB$ and $t \in [0,1]$, then $f(Z)=CZ+b$. We must show that $f(Z) \in [f(A);(B)]$.

2. $A = (a_1, a_2, \ldots a_n)$

   $B = (b_1, b_2, \ldots b_n)$

   $Z = (z_1, z_2, \ldots b_n)$

3. $f(A)=CA+b=c_1a_1+c_2a_2+ \ldots + c_na_n+b$

   $f(B)=CB+b=c_1b_1+c_2b_2+ \ldots + c_nb_n+b$

   $f(Z)=CZ+b=c_1z_1+c_2z_2+ \ldots + c_nz_n+b$

4. $f(Z)=c_1[(1-t)a_1+tb_1]+b+c_2[(1-t)a_2+tb]+b+\ldots$
   $+c_n[(1-t)a_n+tb_n]+b$

   $= c_1a_1(1-t)+c_1b_1t+b+c_2a_2(1-t)+c_2b_2t+b+c_na_n(1-t)+c_nb_nt+b$

   $\geq c_1a_1+c_2a_2+ \ldots + c_na_n+b=f(A)$

5. $f(Z)=c_1[(1-t)a_1+tb_1]+b+c_2[(1-t)a_2+tb_2]+b+\ldots$
$+c_n[(1-t)a_n+tb_n]+b$
$=c_1a_1(1-t)+c_1b_1t+b+c_2a_2(1-t)+c_2b_2t+b+c_na_n(1-t)$
$+c_nb_nt+b$
$\leq c_1b_1+c_2b_2+ \ldots\ldots +c_nb_n+b=f(B)$

Therefore $f(A)\leq f(Z)\leq f(B)$

Theorem

H_1: K is a convex set

H_2: X is a point in K

H_3: f is non-constant linear function defined on K.
$f(X)=AX+b$ for X in K. $A=(a_1,\ a_2\ \ldots\ a_n)$
$X=(x_1,\ x_2,\ \ldots\ x_n)$ and $b\in R$

H_4: $f(X_o)$ is a minimum for some $X_o\in K$
X_o is a corner point of K

Proof:

1. Suppose X_o is not a corner point in K, since K is a convex set, there exist a point X_o in K such that $X_o\in[X_o;X_o']'$. Extend $[X_o;X_o']$ so that it intersects the boundaries of K at points P and Q either $f(P)\geq f(Q)$ or $f(Q)\geq f(P)$. Assume $f(P)\leq f(Q)$, then $f(P)\leq f(X_o)\leq f(Q)\Rightarrow f(X_o)=f(P)$ since $f(X_o)$ is a minimum.

2. If P is a corner point we know $P \neq X_o$, but $f(P)=f(X_o)$ Therefore since f is linear it must be constant, but this leads to a contradiction of H_1. Hence the theorem holds.

3. If P is not a corner point, then $p \in [R;S]'$ where R and S are corner points in K.

4. Assume $f(R) \leq f(S)$, then $f(R) \leq f(P) \leq f(S)$. According to statement 1 $f(X_o)=f(P)$, then we have $f(R) \leq f(X_o) \leq f(S)$

5. $f(R) \leq f(X_o) \Rightarrow f(X_o)$ is not a minimum which leads to a contradiction.

Therefore X_o is a corner point.

CHAPTER IV

APPLICATIONS

Example #1

To fill his vitamin needs, a man is to buy 100 pills which must contain at least 750 units of B_1, 600 units of B_2, and 280 units of B_6. At 5 cents per pill, he can buy pill I, which contains 10 units of B_1, 5 of B_2 and 3 of B_6. At 6 cents per pill, he can buy pill II which has 12 units of B_1, 2 of B_2 and 11 of B_6. Also, pill III, which contains 6 units of B_1, 7 of B_2, and 2 of B_6, is available at 4 cents per pill. How many of each kind of pill should he use in order to fulfill his needs at the lowest cost.

Solution

Let x = number of pills at 5¢
y = number of pills at 6¢
100-x-y = number of pills at 4¢

Total cost to be minimized is $5x+6y+400 - 4x - 4y$ or $x\ 2y\ 400$.

	pill I (5¢)	pill II (6¢)	pill III (4¢)	Amount Needed
B_1	10	12	6	750
B_2	5	2	7	600
B_6	3	11	2	280

a. $10x+12y+600 - 6x-6y \geq 750$; $4x+6y \geq 150$; $2x+3y \geq 75$

b. $5x+2y+700 - 7x-7y \geq 600$; $-2x-5y \geq -100$; $2x+5y \leq 100$

c. $3x+11y+200 - 2x-2y \geq 280$; $x+9y \geq 80$

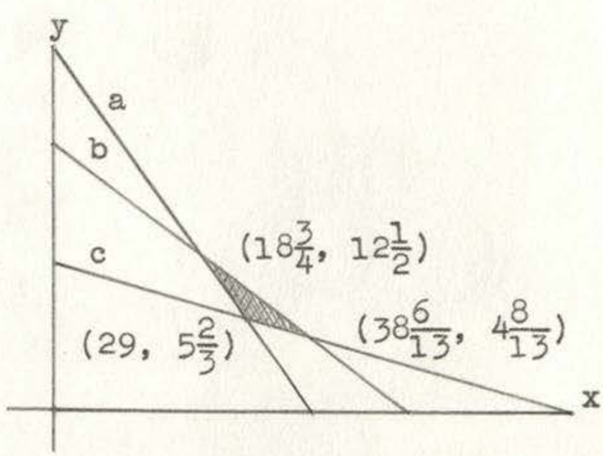
d. $x \geq 0$

e. $y \geq 0$

f. x and y must be integers

Solving the system of inequalities, we find that the vertices as listed on the chart below.

Vertex	x+2y+400	Cost
$(18\frac{3}{4}, 12\frac{1}{2})$	$18\frac{3}{4}+25+400$	$443\frac{3}{4}$ or \$4.44
$(38\frac{6}{13}, 4\frac{8}{13})$	$38\frac{6}{13}+9\frac{3}{13}+400$	$447\frac{9}{13}$ or \$4.48
$(29, 5\frac{2}{3})$	$29+11\frac{1}{3}+400$	$440\frac{1}{3}$ or \$4.40



$(29, 5\frac{2}{3})$ is vertex of minimum value

Therefore $x \doteq 29$ and $y \doteq 6$

29 pills can be purchased at 5¢

6 pills can be purchased at 6¢

65 pills can be purchased at 4¢

The 100 pills can be purchased for \$4.40.

Example #2

A poultry farmer raises chickens, ducks, and turkeys and has room for 500 birds on his farm. While he is willing to have a total of 500 birds, he does not want more than 300 ducks on his farm at any one time. Suppose that a chicken costs $1.50, a duck $1.00, and a turkey $4.00 to raise to maturity. Assume that the farmer can sell chickens for $3.00, ducks for $2.00, and turkeys for T dollars each. He wants to decide which kind of poultry to raise in order to maximize his profit.

a. Let x be the number of chickens and y be the number of ducks he will raise, then 500 -x - y is the number of turkeys to be raised. What is the convex of possible values of x and y which satisfy the restrictions.

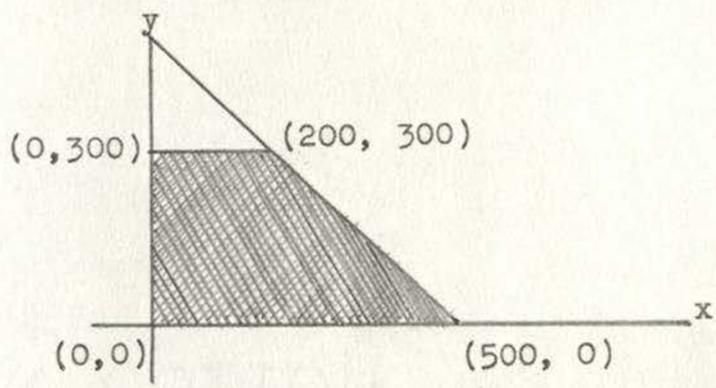
Solution:

$y \leqq 300$

$x+y \leqq 500$

$x \geqq 0$

$y \geqq 0$



b. Find the expression for the cost of raising x chickens, y ducks, and (500 - x - y) turkeys. Find the total amount he gets for these birds. Compute the profit which he would make under the circumstances.

Solution:

$S(x,y)=3.00x+2.00y+T(500 - x - y)$

$C(x,y)=1.50x+1.00y+4(500 - x - y)$

$P(x,y)=1.50x+1.00y+(T - 4)(500 - x - y)$

c. If T=\$6.00, show that to obtain the maximum profit the farmer should raise only turkeys.

Solution:

$P(0,0)=1.50(0)+1.00(0)+(6 - 4)(500 - 0 - 0)=\$1000.00$

d. If T=\$5.00, show that he should raise only chickens and find a maximum profit.

Solution:

$P(500,0)=1.50(500)+1.00(0)+(5 - 4)(500-500-0)=\$750.00$

e. If T=\$5.50, show that he can raise any combination of chickens and turkeys and find a maximum profit.

Solution:

$P(500,0)=1.50(500)+1.00(0)+(5.5 - 4)(500-500-0)=\$750.00$

$P(0,0)=1.50(0)+1.00(0)+(5.5 - 4)(500-0-0)=\$750.00$

f. Show if the price of turkeys drops below \$5.50, the farmer should raise only chickens.

$P(0,300)=1.50(0)+1.00(300)+(5.49-4.00)(500-0-300)=\$598.00$

$P(200,300)=1.50(200)+1.00(300)+(5.49-4.00)(500-200-300)$
$=\$650.00$

$P(0,0)=1.50(0)+1.00(0)+(5.49-4.00)(500-0-0)=\$745.00$

$P(500,0)=1.50(500)+1.00(0)+(5.49-4.00)(500-500-0)$
$=\$750.00$

g. Show that if the price is above \$5.50 he should raise only turkeys.

P(0,0)=1.50(0)+1.00(0)+(5.51-4.00)(500-0-0)=\$755.00

P(500,0)=1.50(500)+1.00(0)+(5.51-4.00)(500-500-0)=\$750.00

P(200,300)=1.50(200)+1.00(300)+(5.51-4.00)(500-200-300)
=\$650.00

P(0,300)=1.50(0)+1.00(300)+(5.51-4.00)(500-0-300)=\$602.00

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