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The Relationship Between the Coefficients and Roots of Polynomials of Degrees N

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by

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DEDICATION

I am dedicating this paper to my wife, Mrs. C. Anne Tave and two children, Robert and Cathy.

CHAPTER I

This paper is to develope and show some of the relations between coefficient and roots of polynomials of degree N.

The following definitions, facts and assumed theorems will be used throughout this paper.

Definition I: An integral domain D is a set of elements a,b,c,... having two operations, + and $\cdot$ and an equal relation, which satisfies the following postulates.

1. Closure: For each pair a,b of elements of integral domain D, $a + b$ and $a \cdot b$ are also elements of the integral domain D and are unique.
2. Commutative: For each pair a,b of elements of the domain D, $a + b = b + a$ and $a \cdot b = b \cdot a$
3. Associative: For each set of three elements a,b,c in D $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
4. Additive Identity (Zero): There exist an element Z in D such that for every element b, $b + z = z + b = b$ and $b \cdot z = z \cdot b = z$.
5. Multiplicative Identity (Unity): There exist an element U in D such that for every element b, $b \cdot u = u \cdot b = b$.
6. Additive Inverse: For each element b in D there exist an element b^* such that $b + b^* = b^* + b = z$ where z is the zero of postulate 4.
7. Cancellation: If a and b are elements in D, and if $c \neq z$

is an element such that $c \cdot a = c \cdot b$ then $a = b$.

8. Distributive: If a,b and c are elements in D, then $a \cdot (b + c) = a \cdot b + a \cdot c$, and $(a + b)\, c = a \cdot c + b \cdot c$.

Definition I.1: A field F has the same postulates as the integral domain with the addition of multiplicative inverse. Multiplicative Inverse: $b \neq z$ there exist a corresponding element b^{-1} such that $b \cdot b^{-1} = b^{-1} \cdot b = u$.

Definition I.2: A polynomial over a field C is denoted by $F(x) = a_0x^n + a_1x^{n-1} + \dots a_n$, $a_0 \neq 0$, can also be used to define a function of real or complex variables over a field C.

Definition I.3: A polynomial F(x) over D[x] is irreducible over D (or prime) if and only if it has no proper divisors in D[x]: F(x) is reducible over D if it has a proper divisor in D[x].

Definition I.4: Let F(x) be a polynomial of degree N over a field F. We say that the equation $F(x) = 0$ is an equation over the field F, and N is the degree of the equation.

Definition I.5: If $F(x) \mid g(x)$ and $g(x) \mid F(x)$, then F(x) and g(x) are associates.

Definition I.6: S denotes sum or product of roots depending on the subscript of S.

I.6 (a): S_1 = negative sum of the roots.

I.6 (b): S_2 = positive sum of the product of roots taken 2 at a time.

I.6 (c): S_3 = negative sum of the product of the roots taken 3 at a time.

I.6 (d): $S_i = (-1)^i$ (sum of the product of the roots taken i at a time).

I.6 (e): $S_n = (-1)^n$ (product of the roots).

Definition I.7: A polynomial F is divisible by polynomial g if $g \neq 0$ and there exists a polynomial h such that $F = gh$.

Fact I.8: Every polynomial of degree N over the field of complex numbers has all its roots in the field of complex numbers. That is the only irreducible, non constant polynomials over the field of complex numbers are those of degree 1.

Fact 1.9: The only irreducible polynomial over the field of real numbers are of degree 1 and 2.

Theorem 1: The factor theorem. If x_o is a root of the equation, $F(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots a_1 x + a_o = 0$, then $x - x_o$ divides F(x) and conversley.

Theorem 1.2: The number of positive roots of $F(x) = 0$ cannot exceed the number variations of signs in F(x).

Theorem 1.3: Every polynomial F(x) of degree n, with real coefficient denoted by $F(x) = a_o x^n + a_1 x^{n-1} + \ldots + a_n$, $a_o \neq 0$ can be factor in two n linear factor.

Theorem 1.4: If $F(x) \neq 0$ has real coefficients of F[x] and a root $a + bi$ where a and b are real and $b \neq 0$ of multiplicity k, then $a - bi$ is a root of F(x) of multiplicity k.

Theorem 1.5: If $F(x) = a_o x^n + a x^{n-1} + \ldots a_{n-1} x + a_n$, where a_o, a_1, ..., a_{n-1}, a_n are integers and $a_o \neq 0$, $a_n \neq 0$, has a rational root $^p/_q$, where p and q are relatively prime integers, then p is a factor of

a_n and q a factor of a_0.

Theorem 1. 6: If F (x) is a polynomial of degree one or greater, with coefficients in field C of complex numbers, then the equation F (x) = 0 has at least one root in C.

Theorem 1. 7: If M (x) is a H.C.F. of F (x) and g (x), r is a common root of F (x) and g (x) if and only if r is a root of M (x).

CHAPTER II

ARITHMETIC IN POLYNOMIALS

Definition II. 1: An infinite sequence of elements of a set S is a function whose domain is the set of nonnegative integers and whose range is a subject of S. Such a sequence is represented by the symbol $(a_o, a_1, a_2, \ldots)$, where each $a_i \varepsilon S$ for every nonnegative integer i.

Definition II. 2: If F is a field, a polynomial over F is an infinite sequence of elements in F such that only a finite number of terms are different from zero. The set of all polynomials over F is referred to as the domain of polynomials over F and is denoted F[x].

Definition II. 3: If F is a field and if $x = (a_o, a_1, a_2, \ldots)$ and $y = (b_o, b_1, b_2, \ldots)$ are any two elements of F[x], then (1) x = y only if $a_i = b_i$ for every nonnegative integer i. (2) $xSy = (c_o, c_1, c_2, \ldots)$ were $c_i = a_i + b_i$ for every nonnegative integer i. (3) $xPy = (d_o, d_1, d_2, \ldots)$ where $d_i = a_oPb_iS\ a_1Pb_{i-1}S \ldots a_iPb_o$ for every nonnegative integer i.

Definition II. 4: The notation $\sum_{iSj=m} a_iPb_j$ means the sum of all terms a_iPb_j which can be formed with i and j nonnegative integers whose sum is m.

Example II. 4: The terms of $\sum_{iSj=4} a_iPb_j$ can be written out as follows:

$$\Sigma a_iPb_j = a_oPb_4Sa_1Pb_3Sa_2Pb_2Sa_3Pb_1Sa_4Pb_o$$

A comparison of definitions of addition and multiplication of polynomials written as sequences with familiar rules for operations with polynomials written in terms of x will reveal that these operations are identical.

The five basic laws of Algebra

(1) $U + V = V + U$

(2) $U \cdot V = V \cdot U$

(3) $U + (V + W) = (U + V) + W$

(4) $U \cdot (V \cdot W) = (U \cdot V) \cdot W$

(5) $U \cdot (V + W) = U \cdot V + U \cdot W$

Now we will establish 5 basic laws of algebra using the operations S and P.

(1.1) USV = VSU

(2.2) UPV = VPU

(3.3) US(VSW) = (USV)SW

(4.4) UP(VPW) = (UPV)PW

(5.5) UP(VSW) = (UPV) S (UPW)

It is necessary to use different signs for addition and multiplication. Otherwise, we find ourselves assuming things in algebra that have only been proved for arithmetic. There is every reason to believe that laws (1.1) through (5.5) will work for polynomials because the operations S and P are defined so that they give a formal statement of what is done in traditional algebra. The idea here is to show that an algebra can be built on any arithmetic. The arithmetic used here is over field F [x], and a set of symbols obeying the axioms of a field.

The following is a proof of laws (1.1) through (5.5) for polynomials over a field F [x]. In these proofs it will become apparent that many of the steps taken in algebraic calculations can be justified by appeal to to the commutative, associative and distributive laws, starting by assuming these laws for the arithmetic of the field F[x]. It will be shown by establishing (1.1) through (5.5) that these same principles also hold for polynomials over F[x].

Theorem (1.1): USV = VSU

Proof:

Let $U = (a_o, \ldots a_n, \ldots)$ and $V = (b_o, \ldots b_n, \ldots)$

$$USV = (a_oSb_o, \ldots, a_nSb_n, \ldots)$$
$$= (b_oSa_o, \ldots, b_nSa_n, \ldots) = VSU$$

Theorem 2.2: UPV = VPU

Proof:

$$UPV = (a_o, \ldots a_n, \ldots) P (b_o, \ldots b_n, \ldots)$$
$$= a_oPb_o, \ldots, \sum_{iSj=n} a_iPb_j \ldots)$$
$$= (b_oPa_o, \ldots, \Sigma\, b_jPa_i, \ldots)$$
$$= (b_o, \ldots, b_n, \ldots) P (a_o, \ldots, a_n, \ldots)$$
$$= VPU$$

Theorem 3.3: US(VSW) = (USV)SW

Proof:

Let $W = (c_o, \ldots, c_n, \ldots)$

$$US(VSW) = (a_o, \ldots, a_n, \ldots) S (b_oSc_o, \ldots, b_nSc_n, \ldots)$$
$$= (a_oS\,[b_oSc_o], \ldots, a_nS\,[b_nSc_n], \ldots)$$
$$= ([a_oSb_o]\,Sc_o, \ldots, [a_nSb_n]\,Sc_n, \ldots)$$
$$= (a_oSb_o, \ldots, a_nSb_n, \ldots) S (c_o, \ldots c_n, \ldots)$$
$$= (USV)\,SW$$

Theorem 4.4: $UP(VPW) = (UPV)PW$

Proof:

$$UP(VPW) = (a_o, \ldots, a_n, \ldots)\, P[b_o, \ldots, b_n, \ldots)\, P\, (c_o, \ldots, c_n, \ldots)]$$

$$= (a_o, \ldots, a_n, \ldots)\, P\, (b_oPc_o, \ldots, \sum_{jSk=n} b_jPc_k \ldots)$$

$$= (a_oPb_oPc_o, \ldots, \sum_{iSjSk=n} a_iPb_jPc_k, \ldots)$$

Similarly,

$$(UPV)PW = [(a_o, \ldots, a_n, \ldots)\, P\, (b_o, \ldots, b_n, \ldots)]\, P\, (c_o, \ldots c_n, \ldots)$$

$$= (a_oPb_o, \ldots, \sum_{iSj=n} a_iPb_j, \ldots)\, P\, (c_o, \ldots, c_n, \ldots)$$

$$= (a_oPb_oPc_o, \ldots, \sum_{iSjSk=n} a_iPb_jPc_k, \ldots)$$

Theorem 5.5: $UP(VSW) = (UPV)\, S\, (UPW)$

Proof:

$$UP(VSW) = (a_o, \ldots, a_n, \ldots)\, P[(b_o, \ldots, b_n, \ldots)\, S\, (c_o, \ldots c_n, \ldots)]$$

$$= (a_o, \ldots, a_n, \ldots)\, P\, (b_oSc_o, \ldots, b_nSc_n, \ldots)$$

$$= a_oP\, [b_oSc_o], \ldots, \sum_{iSj=n}(a_jP\, (b_jSc_j), \ldots)$$

$$= (a_oPb_oSa_oPc_o, \ldots, \sum_{iSj=n}(a_iPb_jSa_iPc_j), \ldots)$$

$$= (a_oPb_oSa_oPc_o, \ldots, \sum_{iSj=n} a_iPb_jS \sum_{iSj=n} a_iPc_j, \ldots)$$

$$= (a_o, \ldots, a_n, \ldots)\, P\, (b_o, \ldots, b_n, \ldots)\, S\, (a_o, \ldots, a_n, \ldots)\, P\, (c_o, \ldots, c_n, \ldots)$$

$$= (UPV)\, S\, (UPW)$$

Theorem II. 5 $x^m\, Px^n = x^{mSn}$

Proof:

Let x^m be $(a_o, a_1, a_2, \ldots a_m)$ where $a_m = 1$, and all other terms are zero's.

Let x^n be $(b_o, b_1, b_2, \ldots b_n)$ where $b_n = 1$, and all other terms are zero's.

Let $x^m\,Px^n$ be $(c_o, c_1, c_2, \ldots, c_k)$ where $c_k = 1$, and all other terms are zero's. Then $c_k = \sum_{i=0}^{k} a_i P b_{k-1}$, c_k is zero if $a_i = 0$, or b_{k-1} is zero, but $a_i \neq 0$ if $i = m$ and $b_{k-1} \neq 0$. Therefore the sum of $(1 = m)$ and $(k - 1 = n)$ is $(k = nSm)$. Hence $c_k = c_{msn}$, then $c_{msn} = 1$. Therefore $x^m\,Px^n = x^{msn}$.

CHAPTER III

ROOTS AND COEFFICIENTS OF POLYNOMIALS

Theorem III. If $f(x)$ and $g(x)$ are in $F[x]$, and the $f(x) \neq 0$, $g(x)$ irreducible over F, and $f(x)$ and $g(x)$ have a common root, then $f(x) = g^n(x)\, h(x)$ where n is a positive integer, $h(x)$ is in $F[x]$, and $g(x)$ and $h(x)$ have no common roots.

Proof: Let $M(x)$ be HCF of $f(x)$ and $g(x)$ in $F[x]$. Every common root of $f(x)$ and $g(x)$ is a root of $M(x)$ by theorem 1.7. Since $f(x)$ and $g(x)$ have a common root, $M(x)$ is of degree at least one. But $g(x)$ is irreducible over F, so that its only factor in $F[x]$ are constants and associates by Definition 1.3 and 1.5. Since $M(x)$ is a factor of $f(x)$, $g(x)$ which is associate $M(x)$ is also a factor of $f(x)$.

Let h be the highest power of $g(x)$ which divides $f(x)$, then $f(x) = g^h(x)\, h(x)$ where $h(x)$ in $F[x]$ by Definition 1.8.

$g(x)$ and $h(x)$ have no common root, for if they did have, then the same argument as above would show that $h(x)$ is divisible by $g(x)$, but by definition of n this is impossible.

Theorem III. 1: If $f(x)$ is of degree $n > 2$ and has real coefficients, then $f(x)$ is reducible over the real numbers.

Proof:

By theorem 1.6 $f(x)$ has a root r. By theorem 1, $x - r$ is a factor $f(x)$.

If r is real, then $f(x)$ has factor with real coefficients. If $r = a + bi$ is complex, then by theorem 1.4 $[x - (a + bi)]\,[x - (a - bi)]$, which has real coefficients is a factor with real coefficient and of lower degree than $f(x)$. Hence $f(x)$ is reducible over $F[x]$, and the theorem follows.

Theorem III. 2: A quadratic equation cannot have more than two roots.

Proof by contradiction:

Consider the quadratic equation $ax^2 + bx + c = 0$. Let α, β, γ be three different roots. Now since each root will satisfy the equation we have, (1) $a\alpha^2 + b\alpha + c = o$, (2) $a\beta^2 + b\beta + c = 0$ and (3) $a\gamma^2 + b\gamma + c = 0$.

Subtracting (2) from (1) gives $a\,(\alpha^2 - \beta^2) + b\,(\alpha - \beta) = 0$

or $(\alpha - \beta)\,[a(\alpha + \beta) + b) = 0$

but $\alpha - \beta \neq 0$, hence $a\,(\alpha - \beta) + b = 0$

Subtracting (3) from (1) gives $a\,(\alpha^2 - \gamma^2) + b\,(\alpha - \gamma) = 0$

or $(\alpha - \gamma)\,[a(\alpha + \gamma) + b] = 0$

but $\alpha - \gamma \neq 0$, hence $a(\alpha + \gamma) + b = 0$

Therefore, $a(\alpha + \beta) + b - a\,(\alpha + \gamma) = 0$.

$$a(\alpha + \beta - \alpha - \gamma] = 0$$

$a(\beta - \gamma) = 0$

either $a = 0$ or $\beta - \gamma = 0$,

but $a \neq 0$ if so then the quadratic becomes linear, therefore $\beta = \gamma$, then our assumption is wrong that all three are distinct. Hence a quadratic can have only two roots.

Statement: $\alpha + \beta = -\frac{b}{a}$ and $\alpha \cdot \beta = \frac{c}{a}$

Theorem III. 3: Every integral rational equation of degree n, $f(x) = a_o x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0$, a_o has at most n roots.

Proof:

1) $f(x) = c(x - r_1)(x - r_2) \ldots (x - r_n)$ conversely, if

2) $(x - r_1)$, $(x - r_2)$, ..., $(x - r_n)$ are divisors of $f(x)$, then each r_i for $i = 1, 2, \ldots, n$ is a zero of $f(x)$.

3) The second statement in proof follows immediately from the factor theorem. That is if $x - r_i$ is a divisor of $f(x)$, then $f(r_i) = o$.

4) The first statement of the proof will be proven by induction on n. If $f(x)$ has degree one, $f(x) = cx + d$, $c + d \in F$. If r_1 is a zero of $f(x)$, $f(r_1) = 0$ and $cr_1 + d = 0$ or $d = -cr_1$, then $f(x) = cx + d = cx - cr_1 = c(x-r_1)$.

5) Suppose that the theorem holds for all polynomials of degree k and let $f(x)$ be of degree $k + 1$, with leading coefficient

$c \neq 0$ and distinct zeros, $r_1, r_2, \ldots, r_{k+1}$. Since r_{k+1} is a zero of $f(x) = (x - r_{k+1})\, g(x)$ for some $g(x) \in F$. By theorem 1. the degree of $g(x)$ must be k and that $g(x)$ has leading coefficient $c \neq 0$. For any zero r_i of $f(x)$, with $f(r_i) = (r_i - r_{k+1})\, g(r_i)$.

6) Since the zero of $f(x)$ are distinct, $r_1 - r_{k+1} \neq 0$ therefore, $g(r_i) = 0$ so that r_i is also a zero of $g(x)$.

7) Thus $g(x)$ has k distinct zero's $r_1, r_2, \ldots, r_k$, by induction hypothesis, $g(x) = c(x - r_1)(x - r_2) \ldots (x - r_k)$ substitute this in the expression for $f(x)$, gives

8) $f(x) = c(x - r_1)\,(x - r_2) \ldots (x - r_k)\,(x - r_{k+1})$

Relation between roots and coefficients:

Let n = 1

$f(x) = x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n = (x - r_1)\,(x - r_2) \ldots (x - r_n)$

By multiplying the linear factors and comparing the resulting coefficients with $a_1, a_2, \ldots, a_n$, we obtain a relation among the roots and the coefficiants of $f(x)$.

If $n = 1$ then $f(x) = x + a_1$, so that $r_1 = -a_1$

If $n = 2$ then $f(x) = x^2 + a_1x + a_2 = (x - r_1)\,(x - r_2) = x^2 - (r_1 + r_2)\,x + r_1r_2$, so that $r_1 + r_2 = -a_1$, $r_1r_2 = a_2$

If $n = 3$ then

$$
\begin{aligned}
f(x) &= x^3 + a_1x^2 + a_2x + a_3 \\
&= (x - r_1)\,(x - r_2)\,(x - r_3) \\
&= x^3 - (r_1 + r_2 + r_3)\,x^2 + (r_1r_2 + r_1r_3 + r_2r_3)\,x - r_1r_2r_3
\end{aligned}
$$

so that $r_1 + r_2 + r_3$

$= - a_1$, $r_1r_2 + r_1r_3 + r_2r_3 = a_2$, $r_1r_2r_3 = - a_3$.

The generalization is stated as follows:

Theorem III. 4: If the coefficient of the highest-degree term in an equation is unity, the coefficient of the second-highest-degree term is the negative of the sum of the roots. The coefficient of the third-highest-degree term is the sum of the roots multiplied two at a time, etc., and finally the constant term is plus or minus the product of the roots according as the number is even or odd.

Proof:

We have already proved this for $n = 1, 2, 3$. Proceeding by mathematical induction, suppose it true for $n = k$, let $n = k + 1$.

If $g(x) = (x - r_1)(x - r_2) \ldots (x - r_k) = x^k + b_1x^{k-1} + \ldots + b_k$, then

$$\begin{aligned} f(x) = x^{k+1} + a_1x^k + \ldots + a_{k+1} &= (x - r_1)(x - r_2)\ldots(x - r_{k+1}) \\ &= (x^k + b_1x^{k-1} + \ldots + b_k)(x - r_{k+1}) \\ &= x^{K+1} + (b_1 - r_{k+1})x^k + (b_2 - \\ &\quad r_{k+1}b_1)x^{k-1} + (b_3 - r_{k+1}b_2)x^{k-2} + \\ &\quad \ldots + (b_k - r_{k+1}b_{k-1})x - r_{k+1}b_k. \end{aligned}$$

Hence, $a_1 = b_1 - r_{k+1}$, $a_i = b_i - r_{k+1}b_{i-1}$ for $i = 2, 3, \ldots, k$, $a_{k+1} = r_{k+1}b_k$.

By the hypothesis of the induction applied to $g(x)$, $b_i = (-1)^iS^1$ $(i = 1, 2, \ldots, k)$ where $S^1{}_i$ is the sum of the products of $r_1, r_2, \ldots, r_k$ taken i at a time.

Therefore,

$a_1 = b_1 - r_{k+1} = - (r_1 + r_2 + \ldots r_k) - r_{k+1} = S_1$

$$a_{k+1} = - r_{k+1}b_k = - r_{k+1}[(-1)^k r_1 r_2 \ldots r_k] = (-1)^{k+1} s_{k+1}$$

$$a_i = b_i - r_{k+1}b_{i-1} \quad (i = 2, 3, \ldots, k)$$
$$= (-1)^i s^1_{i-1} = (-1)^i (s^1_i + r_{k+1}s^1_{i-1})$$

s^1_{i-1} contains all the products of $r_1, r_2, \ldots, r_k$ taken i-1 at a time. Therefore $r_{k+1}s^1_{i-1}$ contains all those products of $r_1, r_2, \ldots, r_k, r_{k+1}$ taken i at a time which have r_{k+1} as a factor. All the products of $r_1, r_2, \ldots, r_k, r_{k+1}$ taken i at a time which do not have r_{k+1} as a factor are in s^1_i. Thus $s^1 + r_{k+1}s^1_{i-1}$ is the sum of all the products of $r_1, r_2, \ldots, r_k, r_{k+1}$ taken i at a time. Hence $s^1_i + r_{k+1}s^1_{i-1} = s_i$, which proves the theorem for $n = k+1$. By the principle of mathematical induction, the theorem follows.

Example 1. $3x^4 - 5x^3 + 4x^2 + 12x - 15 = 0$. Reducing the coefficient of x^4 to 1, we have $x^4 - \frac{5}{3}x^3 + \frac{4}{3}x^2 + 4x - 5 = 0$

Therefore: $s_1 = -(-\frac{5}{3}) = \frac{5}{3}$

$s_2 = \frac{4}{3}$

$s_3 = -(4) = -4$

$s_4 = -5$

Example 2. Find a root of the equation when all the roots are given except one.

Two roots of $2x^3 - 3x^2 - 23x - 12 = 0$ are 3 and -4 the remaining root is $\frac{-3}{2} - [3 + (-4)] = -\frac{1}{2}$ or $6 \div 3(-4) = -\frac{1}{2}$

Example 3. Find the roots of the equation $x^3 + 8x^2 + 5x - 50 = 0$ having given that it has a double root:

Represent the roots a, a, b. Then $2a + b = -8$, $a^2 + 2ab = 5$, and $a^2b = 50$, solving the second of these equations for a and we have $a = -5$, $b = 2$ and $a = -\frac{1}{3}$, $b = \frac{-22}{3}$, the numbers $a = -5$, $b = 2$ satisfy the equation $a^2b = 50$, but the numbers $a = \frac{-1}{3}$, $b = \frac{-22}{3}$ do not satisfy this equation hence the required roots are -5, -5 and 2.

Example 4. Find the roots of the equation $x^3 + 6x^2 + 7x - 2 = 0$ in arithmetic progression.

Let the roots of the equation $x^3 + 6x^2 + 7x - 2 = 0$ be $a - d$, a, $a + d$. The sum of the roots is $3a$, hence $3a = -6$ and $a = -2$. The product of the roots gives $(a^2 - d^2)\,a = 2$. Substituting $a = -2$ and solving for d we find $d = \pm\sqrt{5}$ the roots are then $-2 - \sqrt{5}$, -2, $-2 + \sqrt{5}$.

Theorem III. 5: If n is an integer greater than 2, and k is a non zero integer, prove that $\sum\limits_{r=0}^{n} (r_{k+1})x^{n-r}$ has no integral zeros.

Proof: By contradiction.

Suppose that for $n > 2$ and $k \neq 0$, the polynomial $\sum\limits_{r=0}^{n}$ $(r_{k+1})x^{n-r}$ has an integral zero $x = a$, then

1) $a^n + (k+1)\,a^{n-1} + (2k+1)\,a^{n-2} + \ldots + nk+1 = 0$

2) $a^n + ka^{n-1} + a^{n-1} + 2ka^{n-2} + a^{n-2} + \ldots + nk+1 = 0$

3) $a^n + a^{n-1} + a^{n-2} + \ldots + 1 + ka^{n-1} + ka^{n-2} + \ldots n = 0$

4) $a^n + a^{n-1} + a^{n-2} + \ldots + 1 + k(a^{n-1} + 2a^{n-2} + \ldots n) = 0$

5) $a^n + a^{n-1} + a^{n-2} + \ldots + 1 = -k\,(a^{n-1} + 2a^{n-2} + \ldots + n)$

since $k \neq 0$, it follows that $a \neq -1$

6) $1 = -k\ (a^{n-1} + 2a^{n-2} + \ldots + n) + (-a - a^{n-1} - a^{n-2} - \ldots)$

7) $1 = -k\ (a^{n-1} + 2a^{n-2} + \ldots + n) + (a^n - a^n + 2a^{n-1} - a^{n-1} +$

$3a^{n-2} - a^{n-2} + \ldots n) + (-a^n - 2a^{n-1} - 3a^{n-2} \ldots n)$

8) $n + 1 = (-k + 1 - a)\ (a^{n-1} + 2a^{n-2} + \ldots + n)$

9) $(a^{n-1} + 2a^{n-2} + \ldots + n)$ divides $(n + 1)$. By inspection $a \leq -2$.

10) Statement 9) hold except for $a = -2$, $n = 3$ and $a = -2$, $n = 4$ when $n \geq 3$ and $a \leq -2$.

CHAPTER IV

SUMMARY

This paper was presented in order that someone might be given an insight on the solution of equations of various degrees.

Abel, a mathematician who has proved that no real solution exists between roots and coefficients of equations of degrees higher than four.

This paper can be used to solve various equations of degrees with certain information given about the equation.

This paper also gives means and ideas about how mathematical systems are built in using axioms and properties.

It can be utilized to serve as check on roots of equations.

The main Theorem is proven by mathematical induction.

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