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An Application To Astronomy

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AN APPLICATION TO ASTRONOMY

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A C K N O W L E D G E M E N T

The writer wishes to express his appreciation to Dr. A. D. Stewart for his assistance in the writing of this paper.

J.C.J.

D E D I C A T I O N

To my wife Peggy who has made my entire pursuits for an education a pleasant one.

J.C.J.

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CHAPTER I

INTRODUCTION AND NOTATIONS

From time immemorial, mankind has gazed out into the universe wondering about many things, one of them being whether a day might come when men could travel from earth to one or all of the distant planets. With the advent of modern technology this dream is not entirely out of reach, particularly because of the invention of rockets.

However, it is not the purpose of this paper to discuss whether man will reach another planet, though it does seem very likely he will, but to discuss the relation between two bodies separated by a distance r.

According to Newton's famous universal law of gravitation, any two bodies separated by a distance r and having masses M_1 and M_2 respectively, are attracted toward each other with a force having magnitude given by $F = GMM/r$, where $r > 0$, based on the assumption that r is taken to be the distance between their centers and where G is a universal constant having the same number for each pairs of particles.

Given any two bodies A and B in space many possibilities exist, some of which will be stated here.

1. Body A and Body B may be fixed, which would imply r is constant.

2. Body A may be fixed and Body B allowed to move in a circular path around body A, which would imply r is constant.

3. Both body A and body B may be in motion in such a way r is not constant.

4. Body A may be fixed and body B allowed to move in an elliptical path around body A, which would imply r is not constant.

The four possibilities stated above are not to be considered the only ones, but to make the reader aware of the fact the possibilities exist with respect to the distance r. It is because of these possibilities that the writer of this paper has selected a pair of bodies to write about, namely the sun and earth, instead of writing about any two bodies. According to astronomers observations the motion of these two bodies is described by possibility (4).

It will be assumed that any conclusions drawn with respect to the two selected bodies will hold for any two bodies described under similar conditions.

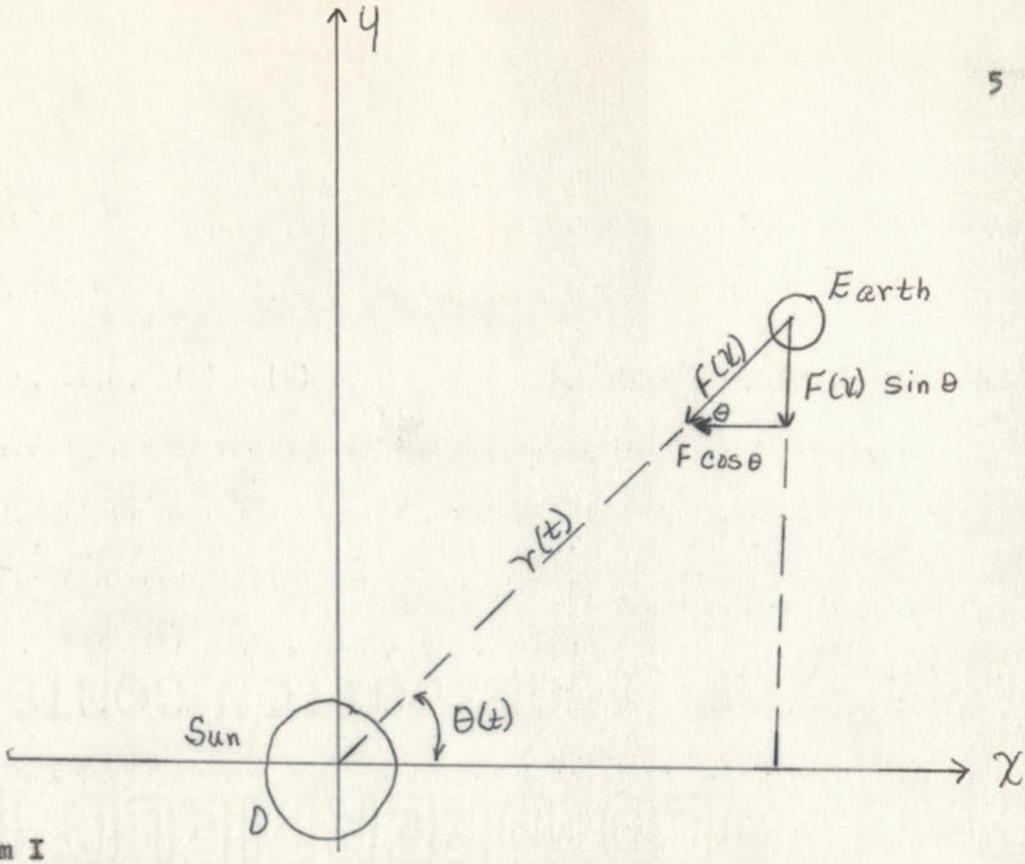
Notations	Meaning
1. $\{\dot{x}, \dot{y}, \dot{r}, \dot{e}, y'(t), x'(t)\}$	1. Differentiation with respect to <u>t</u> .
2. F_x , F_y	2. Mass of earth times acceleration in x and y directions, respectively.
3. m_s	3. Mass of sun
4. m_e	4. Mass of earth
5. G	5. Universal gravitation constant
6. $r(t)$	6. $\sqrt{x^2(t) + y^2(t)}$
7. $\{N, M, R, K, D, E\}$	7. Constants
8. $\{x(t), y(t), e(t), r(t)\}$	8. Functions of <u>t</u>
9. F_x	9. $-F(t) \cos e(t)$
10. F_y	10. $-F(t) \sin e(t)$
11. $F(t)$	11. $\frac{G\, m_e m_s}{r^2(t)}$

CHAPTER II

STATEMENT OF PROBLEM

It is clear both from observations made over a long period of time, and studies made by astronomers that the earth does not take off in motion from the sun in a straight line, but moves around the sun in such a way that the distance between them is not fixed. This is further substantiated by the fact that we have four general seasons of the year, summer, fall, winter, and spring. Still more, each of these can be sub-divided. In fact, this could be thought of as a continuous process.

We shall in this chapter set up a system of differential equations which will completely describe the motion of the earth around the sun.



Theorem I

H_1: $F(x) = \dfrac{Gm_e m_s}{r^2(t)}$

H_2: $0 \leq \theta(t) \leq 360^\circ$

H_3: a is a constant

H_4: $r(t) \geq a > 0$

C: $\exists$ a system of differential equations

$$\begin{cases} m_e x''(t) = -\dfrac{m_e K x(t)}{(x^2(t)+y^2(t)^{3/2}} \\ \\ m_e y''(t) = -\dfrac{m_e K y(t)}{(x^2(t)+y^2(t)^{3/2}} \end{cases} ,$$

which precisely describe the motion of the system.

Proof:

1.
$$\begin{cases} m_e x''(t) = -F \cos \theta(t) \\ m_e y''(t) = -F \sin \theta(t) \end{cases}$$

2.
$$\begin{cases} m_e x''(t) = -\dfrac{Gm_e m_s}{r^2(t)} \cos \theta(t) \\ m_e y''(t) = -\dfrac{Gm_e m_s}{r^2(t)} \sin \theta(t) \end{cases}$$

Since $\sin \theta(t) = \dfrac{y(t)}{r(t)}$ and $\cos \theta(t) = \dfrac{x(t)}{r(t)}$,

(2) becomes

3.
$$\begin{cases} x''(t) = -\dfrac{Kx(t)}{r^3(t)} \\ y''(t) = -\dfrac{Ky(t)}{r^3(t)} \end{cases}$$

where $K = Gm_s$

4. $$\begin{cases} x''(t) = -\dfrac{Kx(t)}{\left(x^2(t)+y^2(t)\right)^{3/2}} \\ \\ y''(t) = -\dfrac{Ky(t)}{\left(x^2(t)+y^2(t)\right)} \end{cases}$$

As initial conditions we assume at $t = 0$, the earth is located on the x axis, a distance a from the sun, and is proceeding in the positive y direction with velocity V_0. Thus

5. $$t = 0 \quad \begin{cases} x(0) = a \\ y(0) = 0 \\ x'(0) = 0 \\ y'(0) = V_0 \end{cases}$$

If we can solve simultaneously equations (4) subject to conditions (5), we shall have the solution to our problem.

A little experimenting with equations (4) shows that it is difficult if not impossible to eliminate $x(t)$ or $y(t)$. Upon noticing the combination $x^2(t)+y^2(t)$, a change to polar coordinates seems advisable. This is further evidenced by the fact that the position of the earth relative to the sun is perhaps better described by coordinates $[r(t), \theta(t)]$ then by $[x(t),y(t)]$. We therefore transform equations (4) to polar coordinates.

Theorem II

H_1: $x(t) \wedge y(t)$ are continuous functions

H_2: $x(t) = r(t) \cos \theta(t)$

H_3: $y(t) = r(t) \sin \theta(t)$

H_4: $x(t) \wedge y(t)$ have C^2

C: The system of differential equations in theorem I can be transform into an equivalent system of differential equations

$$\begin{cases} \ddot{r}(t) - r(t)\,\dot{\theta}^2 = -\dfrac{K}{r^2(t)} \\ \\ 2\dot{r}\dot{\theta} + r(t)\ddot{\theta} = 0 \end{cases} \qquad \begin{cases} r(0) = 0 \\ \theta(0) = 0 \\ \dot{r}(0) = 0 \\ \dot{\theta}(0) = \dfrac{V_0}{a} \end{cases}$$

in polar form.

Proof:

a. $x(t) = r(t)\cos\theta(t)$

b. $\dot{x}(t) = \left(-r(t)\sin\theta(t)\right)\dot{\theta} + \cos\theta(t)(\dot{r})$

c. $\ddot{x}(t) = (\ddot{r}-r\dot{\theta}^2)\cos\theta(t) - (2\dot{r}\dot{\theta} - r\ddot{\theta})\sin\theta(t)$

and

d. $y(t) = r(t)\sin\theta(t)$

e. $y''(t) = r(t)\cos\theta(t)\dot{\theta} + \sin\theta(t)\dot{r}$

f. $y''(t) = \left(\ddot{r} - r(t)\dot{\theta}^2\right)\sin\theta(t) + \left(2\dot{r} + r(t)\right)\cos\theta(t)$

Thus

$$6. \quad \begin{cases} x''(t) = \left(\ddot{r} - r(t)\dot{\theta}^2\right)\cos\theta(t) - \left(2\dot{r}\dot{\theta} + r(t)\dot{\theta}(t)\right)\sin\theta(t) \\ y''(t) = (\ddot{r} - r(t)\dot{\theta}^2)\sin\theta(t) + (2\dot{r}\dot{\theta} + r(t)\ddot{\theta})\cos\theta(t) \end{cases}$$

Equations (6) become upon making use of equations (4)

$$7. \quad (\ddot{r} - r(t)\dot{\theta}^2)\cos\theta(t) - (2\dot{r}\dot{\theta} + r(t)\ddot{\theta})\sin\theta(t) = -\frac{K\cos\theta(t)}{r^2(t)}$$

$$8. \quad (\ddot{r} - r(t)\dot{\theta}^2)\sin\theta(t) + (2\dot{r}\dot{\theta} + r(t)\ddot{\theta})\cos\theta(t) = -\frac{K\sin\theta(t)}{r^2(t)}$$

Multipling equation (7) by $\cos\theta(t)$, equation (8) by $\sin\theta(t)$ and add,

$$(\ddot{r} - r(t)\dot{\theta}^2)(\cos^2\theta(t) + \sin^2\theta(t) = -\frac{k}{r^2(t)} \quad (\cos^2\theta(t) + \sin^2\theta(t)$$

$$9. \quad (\ddot{r} - r(t)\dot{\theta}^2) = -\frac{k}{r^2(t)}$$

Also multiplying equation (7) by $\sin\theta(t)$ and equation (8) by $\cos\theta(t)$ and subtracting

$$\sin\theta(t) \quad \Big| \quad (\ddot{r} - r(t)\,\dot{\theta}^2)\cos\theta(t) - (2\,\dot{r}\,\dot{\theta} + r(t)\,\ddot{\theta})\sin\theta(t) = -\frac{K\cos\theta(t)}{r^2(t)}$$

$$\cos\theta(t) \quad \Big| \quad (\ddot{r} - r(t)\,\dot{\theta}^2)\sin\theta(t) + (2\,\dot{r}\,\dot{\theta} + r(t)\ddot{\theta})\cos\theta(t) = -\frac{K\sin\theta(t)}{r^2(t)}$$

a) $$(\ddot{r} - r(t)\dot{\theta}^2)\sin\theta(t)\cos\theta(t) - (2\dot{r}\dot{\theta} + r(t)\ddot{\theta})\sin^2\theta(t) = -\frac{K\cos\theta(t)\sin\theta(t)}{r^2(t)}$$

b) $$(\ddot{r} - r(t)\dot{\theta}^2)\sin\theta(t)\cos\theta(t) + (2\dot{r}\dot{\theta} + r(t)\ddot{\theta})\cos^2\theta(t) = -\frac{K\cos\theta(t)\sin\theta(t)}{r^2(t)}$$

then (b) - (a) yields

c) $$-(2\dot{r}\dot{\theta} + r(t)\ddot{\theta})\sin^2\theta(t) - (2\dot{r}\dot{\theta} + r(t)\ddot{\theta})\cos^2\theta(t) = 0$$

d) $$-(2\dot{r}\dot{\theta} + r(t)\ddot{\theta}) = 0$$

10. $$2\dot{r}\dot{\theta} + r(t)\ddot{\theta} = 0$$

11. $$\begin{cases} \ddot{r} - r(t)\dot{\theta}^2 = -\dfrac{K}{r^2(t)} \\ 2\dot{r}\dot{\theta} + r(t)\ddot{\theta} = 0 \end{cases}$$

CHAPTER III

THE t-SOLUTION

Theorem III

H_1: $r(t) \geq a > 0$

H_2: $\ddot{r} - r(t)\dot{\theta}^2 = -\dfrac{K}{r^2(t)}$

H_3: $\dot{\theta} = \dfrac{a\,V_0}{r^2}$

H_4: $r(t)$ has C^2

C: $\exists$ a t-number associated with the differential equation in H_2 such that for any number $\underline{r}$, t will give the time the earth has been in motion in its orbital path around the sun:

$$t = \frac{1}{V_0^2 - \frac{2K}{a}} \sqrt{\left(V_0^2 - \frac{2K}{a}\right) r^2(t) + 2Kr(t) - a^2 V_0^2}$$

$$- \frac{2K}{2\left(V_0^2 - \frac{2K}{a}\right)} \operatorname{Log}_e \left[\frac{2\left(V_0^2 - \frac{2K}{a}\right)\sqrt{\left(V_0^2 - \frac{2K}{a}\right) r^2(t) + 8Kr(t) - a^2V_0^2} + 2\left(V_0^2 - \frac{2K}{a}\right) r(t) + 2K}{\sqrt{4\left(V_0^2 - \frac{2K}{a}\right)\left(-a^2 V_0^2\right) - 4K^2}} \right] +$$

$$\frac{2K}{2\left(V_0^2 - \frac{2K}{a}\right)} \operatorname{Log}_e \left[\frac{2\left(V_0^2 - \frac{2K}{a}\right) \cdot \sqrt{6Ka} + 2V_0 a}{\sqrt{4\left(V_0^2 - \frac{2K}{a}\right)\left(-a^2 V_0^2\right) - 4K^2}} \right]$$

Proof:

1. $$\ddot{r} = \frac{a^2 V_0^2}{r^3} - \frac{K}{r^2}$$

2. Let $\dot{r} = p$

3. $$p'(t) = p'(r) \cdot r'(t) = p \cdot p'(r) = \frac{a^2 V_0^2}{r^3(t)} - \frac{K}{r^2(t)}$$

4. $$p\, p'(r) = \frac{a^2 V_0^2}{r^3} - \frac{K}{r^2}$$

5. $$\int p\, p'(r) dr = a^2 V_0^2 \int \frac{dr}{r^3(t)} - K \int \frac{dr}{r^2} + C_1$$

6. $$\frac{p^2}{2} = \frac{K}{r(t)} - \frac{a^2 V_0^2}{2r^2(t)} + C_1 , \quad r(t) > 0$$

7. From (6) since $p = \dot{r} = 0$ where $r = a$

8. $$C_1 = \frac{v_0^2}{2} - \frac{K}{a}$$

9. $$\frac{\dot{r}^2(t)}{2} = \frac{K}{r(t)} - \frac{a^2 v_0^2}{2r^2(t)} + \frac{v_0^2}{2} - \frac{K}{a} \ , \ r(t) > 0$$

10. $$r'(t) = \sqrt{\left(v_0^2 - \frac{2K}{a}\right) + \frac{2K}{r(t)} - \frac{a^2 v_0^2}{r^2(t)}}$$

11. $$r'(t) = \frac{1}{r(t)} \sqrt{\left(v_0^2 - \frac{2K}{a}\right) r^2(t) + 2Kr(t) - a^2 v_0^2} \ , \ r(t) \geq a$$

12. $$r(t) \cdot r'(t) = \sqrt{\left(v_0^2 - \frac{2K}{a}\right) r^2(t) + 2Kr(t) - a^2 v_0^2}$$

13. Let $N = v_0^2 - \frac{2K}{a}$, $M = 2K$ and $R = -a^2 v_0^2$

14. $$r(t)\, r'(t) = \sqrt{Nr^2(t) + Mr(t) + R}$$

15. $$1 = \frac{r(t)\, r'(t)}{\sqrt{Nr^2(t) + Mr(t) + R}}$$

16. $$\int 1dt = \int \frac{r(t)\ r'(t)dt}{\sqrt{Nr^2(t) + Mr(t) + R}} + C_2$$

17. $$t = \int \frac{r(t)\ r'(t)dt}{\sqrt{Nr^2(t) + Mr(t) + R}} + C_2$$

18. $$t = \frac{1}{2N} \int \frac{(2Nr(t) + M - M)\ r'(t)dt}{\sqrt{Nr^2(t) + Mr(t) + R}} + C_2$$

19. $$1 = \frac{1}{2N} \int \frac{(2Nr(t)+M)r'(t)dt}{\sqrt{Nr^2(t)+Mr(t)+R}} - \frac{M}{2N} \int \frac{r'(t)dt}{\sqrt{Nr^2(t)+Mr(t)+R}} + C_2$$

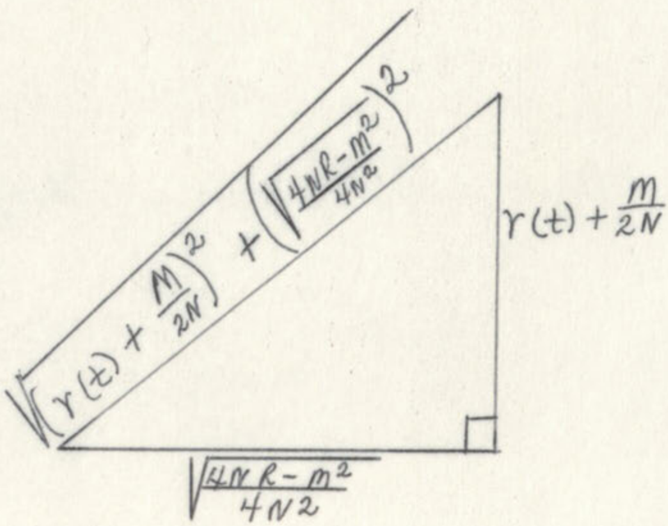
20. $$t = \frac{1}{2N}\left(\sqrt{Nr^2(t)+Nr(t)+R}\right) - \frac{M}{2N} \int \frac{r'(t)dt}{\sqrt{Nr^2(t)+Mr(t)+R}} + C_2$$

a) $$\int \frac{r'(t)dt}{\sqrt{r^2(t)+ \frac{M}{N} r(t)+ \frac{R}{N}}} = \frac{1}{\sqrt{N}} \int \frac{r'(t)dt}{\sqrt{r^2(t)+ \frac{M}{N} r(t)+ \frac{R}{N}}} + C_2$$

b) $$\int \frac{r'(t)dt}{\sqrt{r^2(t) + \frac{M}{N} r(t) + \frac{R}{N}}} = \frac{1}{\sqrt{N}} \int \frac{r'(t)dt}{\sqrt{r^2(t) + \frac{M}{N} r(t) + \frac{M}{4N} + \frac{R}{M} - \frac{M}{4N}}} + C_2$$

c) $$\int \frac{r'(t)dt}{\sqrt{r^2(t) + \frac{M}{N} r(t) + \frac{R}{N}}} = \frac{1}{N} \int \frac{r'(t)dt}{\sqrt{\left(r(t) + \frac{M}{2N}\right)^2 + \left(\sqrt{\frac{4NR - M^2}{4N^2}}\right)^2}} + C_2$$

d) Triangle of reference



e) $$\tan\alpha = \frac{2Nr(t) + M}{\sqrt{4NR - M^2}}$$

f) $$\sec\alpha = \frac{\sqrt{\left(r(t) + \frac{M}{2N}\right)^2 + \frac{4NR - M^2}{2N}}}{\sqrt{\frac{4NR - M^2}{2N}}}$$

g) $$r'(t)dt = \sqrt{\frac{4NR - M^2}{2N}}\ \mathrm{Sec}^2\alpha\ d\alpha$$

h) $$\sqrt{\frac{4NR - M^2}{2N}}\ \sec\alpha = \sqrt{\left(r(t) + \frac{M}{2N}\right)^2 + \frac{4NR - M^2}{4N}}$$

i) $$\frac{1}{\sqrt{N}}\int\frac{dr(t)}{\sqrt{\left(r(t)+\frac{M}{2N}\right)^2+\frac{4NR-M^2}{4N^2}}} = \frac{\sqrt{\frac{4N^2-M^2}{2N}}}{\sqrt{\frac{4N^2-M^2}{2N}}}\cdot\frac{1}{\sqrt{N}}\int\frac{\sec^2\alpha\, d\alpha}{\sec\alpha} + C_2$$

j) $$\frac{1}{\sqrt{N}}\int\frac{dr(t)}{\sqrt{\left(r(t)+\frac{M}{2N}\right)^2+\frac{4NR-M^2}{4N^2}}} = \frac{1}{\sqrt{N}}\int\frac{\sec^2\alpha}{\sec\alpha} + C_2$$

k) $$\frac{1}{\sqrt{N}}\int\frac{dr(t)}{\sqrt{\left(r(t)+\frac{M}{2N}\right)^2+\frac{4NR-M^2}{4N^2}}} = \frac{1}{\sqrt{N}}\int\sec\alpha\, d\alpha + C_2$$

l) $$\frac{1}{\sqrt{N}}\int\frac{dr(t)}{\sqrt{\left(r(t)+\frac{M}{2N}\right)^2+\frac{4NR-M^2}{4N^2}}} = \frac{1}{\sqrt{N}}\operatorname{Log}_e\left[\sec\alpha+\tan\alpha\right]+C_2$$

Using (1) in (20) we have

21. $$t = \frac{1}{N}\left(\sqrt{Nr^2(t)+Mr(t)+R}\right) - \frac{M}{2N^{3/2}}\,\mathrm{Log}_e\left[\frac{2\sqrt{N}\sqrt{Nr^2(t)+4Mr(t)+R}+2Nr(t)+M}{\sqrt{4NR-M^2}}\right] + C_2$$

$\therefore$

22. $$t = \frac{1}{V_0^2 - \frac{2K}{a}}\sqrt{\left(V_0^2 - \frac{2K}{a}\right) r^2(t) + 2Kr(t) - a^2 V_0^2}$$

$$- \frac{2K}{2\left(V_0^2 - \frac{2K}{a}\right)}\,\mathrm{Log}_e\left[\frac{2\left(V_0^2 - \frac{2K}{a}\right)\sqrt{\left(V_0^2 - \frac{2K}{a}\right) r^2(t) + 8Kr(t) - a^2V_0^2} + 2\left(V_0^2 - \frac{2K}{a}\right) r(t) + 2K}{\sqrt{4\left(V_0^2 - \frac{2K}{a}\right)\left(-a^2V_0^2\right) - 4K^2}}\right]$$

$$+ \frac{2K}{2\left(V_0^2 - \frac{2K}{a}\right)}\,\mathrm{Log}_e\left[\frac{2\left(V_0^2 - \frac{2K}{a}\right)\cdot\sqrt{6Ka} + 2V_0 a}{\sqrt{4\left(V_0^2 - \frac{2K}{a}\right)\left(-a^2V_0^2\right) - 4K^2}}\right]$$

CHAPTER IV

THE r-SOLUTION

Theorem IV

H_1: $r(t) \geq a$

H_2: $r(t)$ has C^2

H_3:
$$\begin{cases} \ddot{r} - r(t)\dot{\theta}^2 = -\dfrac{K}{r^2(t)} \\[2ex] \dot{\theta} = \dfrac{a V_0}{r^2(t)} \end{cases}$$

C: $\exists$ a r-number associated with the system of differential equations in H_3:

$$r(\theta) = \frac{\dfrac{a V_0}{K}}{1 + \cos \theta}$$

such that for any $0 \leq \theta \leq 360$, $\underline{r}$ will give the distance between the centers of the earth and sun.

Proof:

1.
$$r'(t) = \sqrt{\left(V_0^2 - \frac{2K}{a}\right) + \frac{2K}{r(t)} - \frac{a^2 V_0^2}{r^2(t)}}\ ;\ \theta'(t) = \frac{a V_0^2}{r^2(t)}$$

2.
$$r'(\theta) = \sqrt{\left(V_0^2 - \frac{2K}{a}\right) \frac{r^4(t)}{a^2 V_0^2} + \frac{2Kr^3(t)}{a^2 V_0^2} - \frac{r^4(t)}{a^2 V_0^2} \frac{a^2 V_0^2}{r^2(t)}}$$

3.
$$r'(\theta) = \sqrt{r^2(\theta) \left\{\left(\frac{1}{a^2} - \frac{2K}{a^3 V_0^2}\right) r^2(\theta) + \frac{2Kr(\theta)}{a^2 V_0^2} - 1\right\}}$$

4.
$$r'(\theta) = r(\theta) \sqrt{\left(\frac{1}{a^2} - \frac{2K}{a^3 V_0^2}\right) r^2(t) + \frac{2Kr(t)}{a^2 V_0^2} - 1}$$

5. Let $D = \frac{1}{a^2} - \frac{2K}{a^3 V_0^2}$, $E = \frac{K}{a^2 V_0^2}$

6.
$$r'(\theta) = r(\theta) \sqrt{Dr^2(\theta) - 2Er(\theta) - 1}$$

7. $$\int \frac{dr(\theta)}{r(\theta)\sqrt{Dr^2(\theta)-2Er(\theta)-1}} = \int d\theta + C_3$$

8. Let

$$r(\theta) = \frac{1}{u}$$

$$u = \frac{1}{r(\theta)}$$

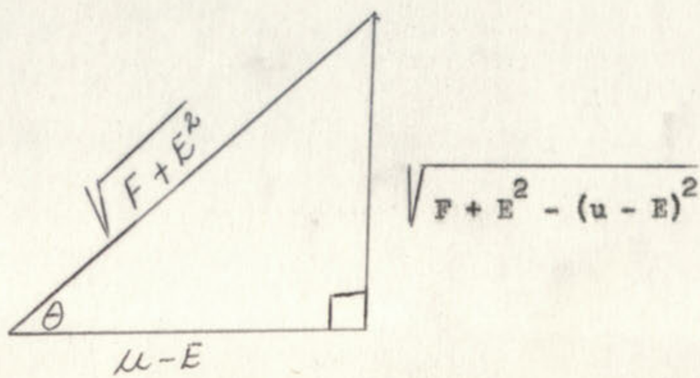
$$du = -\frac{1}{r^2(\theta)}\, dr(\theta)$$

$$-\frac{du}{u^2} = dr(\theta)$$

9. $$\int \frac{du}{\frac{u^2}{u}\sqrt{\frac{D}{u^2} - \frac{2E}{u} - 1}} = \int \frac{du}{\sqrt{D - 2Eu - u^2}} + C_3$$

10. $$\int \frac{du}{\frac{u^2}{u}\sqrt{\frac{D}{u^2} - \frac{2E}{u} - 1}} = \int \frac{du}{\sqrt{D + E^2 - (u - E)^2}} + C_3$$

11.



12. $$\cos\theta = \frac{u - E}{\sqrt{D + E^2}}, \quad \sin\theta = \frac{\sqrt{D + E^2 - (u - E)^2}}{\sqrt{D + E^2}}$$

a) $$u = \sqrt{D + E^2}\cos\theta + E,$$

$$\sqrt{D + E^2 - (u - E)^2} = \sqrt{D + E^2}\sin\theta$$

b) $$du = -\sqrt{D + E^2}\sin\theta\, d\theta$$

13. $$-\int \frac{du}{\sqrt{D + E^2 - (u - E)^2}} = \frac{\sqrt{D + E^2}}{\sqrt{D + E^2}} \int \frac{\sin\theta\, d\theta}{\sin\theta} + C_3$$

14. $$-\int \frac{du}{\sqrt{D + E^2 - (u - E)^2}} = \int d\theta + C_3$$

15. $$-\int \frac{du}{\sqrt{D + E^2 - (u - E)^2}} = \theta + C_3$$

16. $$\text{arc cos} \frac{u - E}{\sqrt{D + E^2}} = \theta + C_3$$

17. $$(u - E) = \sqrt{D + E^2}\cos(\theta + C_3)$$

18. $$u = E\left[1 + \frac{\sqrt{D + E^2}}{E} \cos(\theta + C_3)\right]$$

19. Let $$e = \frac{\sqrt{D + E^2}}{E} = \frac{a V_0^2}{K} - 1$$

20. $$u = E\left\{1 + e\cos(\theta + C_3)\right\}$$

21. $$\frac{1}{r(\theta)} = \frac{K}{a^2 V_0^2}\left\{1 + \left(\frac{a V_0^2}{K} - 1\right)\cos(\theta + C_3)\right\}$$

22. $$\frac{1}{r(\theta)} = \frac{K}{a^2 V_0^2} + \left(\frac{1}{a} - \frac{K}{a^2 V_0^2}\right)\cos(\theta + C_3)$$

23. $$\frac{1}{r(\theta)} = \frac{K}{a^2 V_0^2} + \left(\frac{a V_0^2 - K}{a^2 V_0^2}\right)\cos(\theta + C_3)$$

24. $$r(\theta) = \frac{\frac{a^2 V_0^2}{K}}{1 + e\cos(\theta + C_3)}$$

25. $C_3 = 0$, From initial conditions

26. $$\therefore r(\theta) = \frac{\frac{a^2 V_0^2}{K}}{1 + e\cos\theta}$$

Since equation (26) represents the path of the earth around the sun, it must be an ellispe. Therefore, $0 < e < 1$. To show this consider the following:

1) $e = \frac{a{v_0}^2}{Gm_s} - 1$ according to appendixes A and B in Physcis for Students of Science and Engineering by David Holliday, the following are known:

2) $a = 9.344 \times 10^7$ miles

3) ${v_0}^2 = 342.25 \frac{mi.^2}{hrs.^2}$

4) $m_e = 5.983 \times 10^{24}$ kg.

5) $m_s = 329{,}390\ m_e$

6) $m_s = 19.8 \times 10^{29}$ kg.

7) $G = 6.673 \times 10^{-11}$ nt. $- \frac{m^2}{kg.^2}$

8) $$e = \frac{(9.344 \times 10^7 \text{ mi})(3.423 \times 10^2 \frac{mi^2}{sec.}\ 2}{(6.673 \times 10^{-11} \text{ nt.} - \frac{m^2}{kg.^2}\ (19.8 \times 10^{29} \text{ kg.})} - 1$$

9) $$e = \frac{91.62 \times 10^9 \text{ mi.}^3}{130 \times 10^{18}\ m^3} - 1$$

10) $$e = \frac{3.162 \times 10^{10}}{2.8 \times 10^{10}} - 1$$

11) $$e = \frac{3.162}{2.8} - 1$$

12) $e = 1.129 - 1$

13) $e = .129$

The path of the earth around the sun is elliptical because clearly $0 < e < 1$.

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