Fuzzy Multi-context Systems

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***Abstract*—Multi-context systems provide an effective representation and reasoning framework for integrating heterogeneous knowledge obtained from different sources and have been applied in different fields. Because many application fields in real life have to deal with uncertain and fuzzy knowledge, the present work aims to combine the multi-context system and fuzzy logic theory effectively and systematically to deal with the representation and reasoning of uncertainty in heterogeneous contexts. The current research in this area is still relatively limited, especially in terms of systematic integration. Specifically, the present work proposes a class of heterogeneous non-monotonic fuzzy multi-context systems based on non-monotonic multi-context systems, in which an abstract logic is proposed to capture different types of logic and is used as a theoretical basis for fuzzy multi-context knowledge representation and setting up bridging rules to integrate heterogeneous knowledge. Fuzzy equilibria are used to describe the semantics of fuzzy multi-context systems. The syntactic and semantic framework of heterogeneous non-monotonic fuzzy multi-context systems is then systematically established. Finally, we show that the proposed fuzzy multi-context system not only extends the non-monotonic multi-context system to fuzzy settings, but also could expand the probabilistic multi-context system and the possibility multi-context system in the similar way.**

***Index Terms*—Knowledge integration, multi-context systems, abstract logics, fuzzy equilibria**

I. Introduction

DRIVEN by initiatives such as the Word Wide Web and the Internet of Things, there is a growing demand for heterogeneous knowledge sharing and reasoning, which poses various challenges to Knowledge Representation and Reasoning scheme in artificial intelligence [1]. Multi-context systems [2] can be deemed as a promising solution to address such challenges. The basic idea of multi-context systems is to capture different knowledge through distinct logical languages and establish the relationship between them through the bridge rules. Based on this, equilibria of multi-context systems are defined as acceptable global states which are "stable" with respect to information exchange.

Since multi-language systems (ML systems) were firstly developed to integrate multiple monotonic inference systems [3], [4], various multi-context systems have been proposed successively, such as multi-context systems based on default logic (ConDL) [5], [6], probabilistic multi-context systems (p-MCSs) with probabilistic reasoning [7], non-monotonic multi-context systems (MCSs) with non-monotonic reasoning [2], and possibilistic multi-context systems (poss-MCSs) with possibilistic reasoning [8]. Besides, multi-context systems have been applied to different domains, for instance, the Semantic Web [9], engineering executable agents [10], ambient intelligence [11], and optimization problems [12], to name just a few.

It is worth noting that MCSs [2] integrate monotonic and non-monotonic logics, have been extended into various forms, such as managed multi-context systems (mMCSs) with management capabilities (*e.g.*, removal or revision of information) [1], reactive multi-context systems (rMCSs) with reactive reasoning [13], and preferential multi-context systems (PM-CSs) with preference information [14]. Furthermore, several methods have been developed to address the inconsistency in MCSs that stems form information exchange [15]–[18].

However, the semantics of almost all existing multi-context systems are defined in terms of definite information states [3]–[6] or definite belief states [1], [2], [13], [14]. Therefore these systems are unsuitable for handling uncertainties in practical applications, with two exceptions: p-MCSs [7] and poss-MCSs [8]. The former is based on propositional probabilistic logics [19], and the latter is based on possibilistic logic programs [20]. Nevertheless, both of them are homogeneous multi-context reasoning frameworks because the methods of inference are the same in either probabilistic or possibilistic contexts.

As we all know, uncertainty in real-world applications contains fuzziness which arises when the boundary of information is not clear. Fuzzy knowledge indeed exists in practical applications, especially in the area of the Semantic Web [21], [22] and the Internet of Things [23], [24]. Thus, the issue of dealing with fuzziness is of growing importance in knowledge integration of different environments. That is, it is necessary to cope with fuzzy knowledge in the heterogeneous knowledge integration. Consider the following scenario as an illustration:

Example 1. Alice went to the hospital with a severe sore throat that lasted for two days. And she is a little allergic to antibiotics. Dr. John relies on his experience, interaction with Alice, and the results from some lab tests to treat sore throats.

(1) If sore throat is at least moderate, he will strongly suspect bacterial pharyngitis.

(2) It is quasi certain that if he suspects that the pharyngitis is caused by bacteria, he will recommend a blood test.

(3) He is almost certain that antibiotics can cure bacterial pharyngitis, and he is absolutely certain that lozenges can help relieve sore throats.

(4) When the blood test result from the laboratory is positive, he will prescribe antibiotics if no information indicates that Alice is highly allergic to antibiotics.

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(5) If the blood test result is negative, he will be very likely to prescribe a lozenge.

If John strongly recommends a blood test, Alice will accept John's suggestion, and then she will pay the fee. If Alice pays, the laboratory will conduct a blood test for Alice.

This medical scenario not only involves three heterogeneous contexts—the patient Alice, the doctor John, and the laboratory—but also contains a great deal of uncertainty caused by fuzzy knowledge such as the level of pain and the reaction of antibiotic allergy, and by uncertain knowledge such as the experience of the doctor.

As analyzed above, only p-MCSs and poss-MCSs provide uncertainty modeling frameworks in all existing multi-context systems. Nevertheless, (i) a probabilistic bridge rule in p-MCSs is incapable of expressing the local absent information from a context such as rule (4) in Example 1, since it contains no default negation; (ii) a possibilistic bridge rule in poss-MCSs is incapable of expressing the local uncertain information from a context such as rules (1) and (4) in Example 1, since a propositional atom in possibilistic bridge rules can not model uncertainty; and (iii) both of p-MCSs and poss-MCSs are less general since they are homogeneous multi-context reasoning frameworks. Besides, fuzzy sets [25], [26] and fuzzy logic theories [27]–[30] provide a solid theoretical foundation for the representation of, and the reasoning with fuzzy knowledge. However, all proposed multi-context systems neither incorporate fuzzy logic theory into them nor meet the need of handling fuzzy knowledge. In a nutshell, there is lack of a more general heterogeneous multi-context reasoning framework for dealing with uncertainties.

The present work aims to establish a generic heterogeneous multi-context reasoning framework for integrating heterogeneous knowledge under uncertainty. This reasoning framework unifies and generalizes other multi-context frameworks, including non-monotonic multi-context systems, probabilistic multi-context systems, possibilistic multi-context systems, and so forth. The main contributions of this work are simply summarized as follows:

- Our work is based on various logics. We thus define an abstract logic framework to capture different types of logic, for instance, classical propositional logic, fuzzy answer set programs [29], probabilistic logic under the propositional case [19], and possibilistic normal logic programs [20]. Moreover, we introduce notions of monotonicity and reducibility of abstract logics, which apply to any logic covered by this framework.
- We propose heterogeneous non-monotonic fuzzy multi-context systems called fuzzy multi-context systems (FMCSs in short), which consist of fuzzy contexts based on abstract logics. Specially, fuzzy equilibrium semantics for FMCSs is investigated in details, including the grounded fuzzy equilibrium and the well-founded fuzzy equilibrium, which are generated by fixpoint iteration. The proposed FMCSs contribute to the fusion of heterogeneous knowledge under uncertainty by using distinct logic theories. For instance, Fig. 1 illustrates an FMCS modeling the information exchange among three contexts C_1, C_2 and C_3 by fuzzy bridge rules. These contexts are associated with heterogeneous (uncertain) knowledge: a fuzzy answer set program K_1, a possibilistic logic program K_2, and a probabilistic logic theory K_3, respectively.

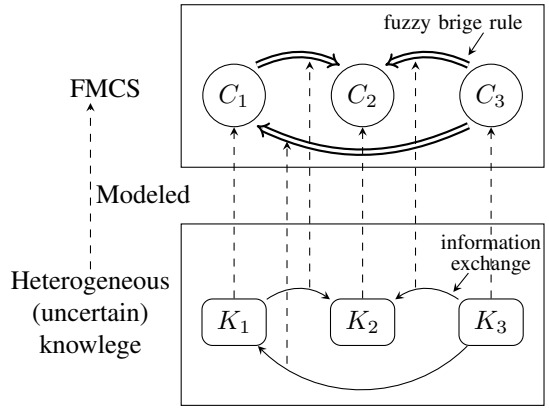

Fig. 1. Structure diagram of an FMCS

- We study the relationships between FMCSs and other multi-context systems: (i) FMCSs generalize non-monotonic multi-context systems; (ii) probabilistic multi-context systems and possibilistic multi-context systems can be embedded into FMCSs, but not vice versa.

The remainder of this paper is organized as follows. In Section II, some preliminaries, including logical operators and non-monotonic multi-context systems are briefly reviewed. In Section III, we introduce the formal concepts of abstract logics, monotonic abstract logics, and reducible abstract logics in our setting. Section IV presents a fuzzy multi-context systems framework including its syntax and semantics. The relationships between fuzzy multi-context systems and other multi-context systems are discussed in Section V. We conclude our work and point out future studies in Section VI.

II. Preliminaries

Four common logical connectives are negation, conjunction, disjunction, and implication, which are usually modeled in an algebraic structure by logical operators negator, t-norm, t-conorm, and implicator. In this section, we provide a brief review of logical operators and non-monotonic multi-context systems introduced by Brewka and Eiter [2].

A. Logical Operators

Complete lattice, $\mathcal{L}$-fuzzy sets, and logical operators are introduced in this subsection, more details can be found in [26], [27], [29].

Complete lattice A *complete lattice* is a partially ordered set $(\mathcal{L}, \leq_{\mathcal{L}})$ ($\mathcal{L}$ for short) such that each subset of $\mathcal{L}$ has the least upper bound (*supremum*) and the greatest lower bound (*infimum*) in $\mathcal{L}$. $1_{\mathcal{L}}$ and $0_{\mathcal{L}}$ are used to denote the greatest element and the least element of $\mathcal{L}$, respectively.

$\mathcal{L}$-fuzzy sets Let $\mathcal{U}$ be a universe and $\mathcal{L}$ a complete lattice. An *$\mathcal{L}$-fuzzy set* on $\mathcal{U}$ is a mapping A: $\mathcal{U} \longrightarrow \mathcal{L}$, which can be written as a set $\{(x, A(x)) \mid x \in \mathcal{U}, A(x) \in \mathcal{L}\}$ of ordered pairs. An *$\mathcal{L}$-fuzzy set* A is *crisp* if $A(x) \in \{1_{\mathcal{L}}, 0_{\mathcal{L}}\}$ for

all $x \in \mathcal{U}$. In this case A is the *characteristic function* of $\{x \in \mathcal{U} \mid A(x) = 1_{\mathcal{L}}\}$. $[0,1]$-fuzzy sets are referred to as fuzzy sets as usual. By $\mathcal{L}^{\mathcal{U}}$ we denote the *$\mathcal{L}$-fuzzy space* on $\mathcal{U}$, namely $\mathcal{L}^{\mathcal{U}} = \{A \mid A : \mathcal{U} \longrightarrow \mathcal{L}\}$. It is clear that $(\mathcal{L}^{\mathcal{U}}, \leq_{\mathcal{L}^{\mathcal{U}}})$ is a complete lattice, where $A \leq_{\mathcal{L}^{\mathcal{U}}} B$ *iff* $A(x) \leq_{\mathcal{L}} B(x)$ for any $x \in \mathcal{U}$.

Negator A unary operation $\mathcal{N}$ on a complete lattice $\mathcal{L}$ (*i.e.*, a mapping $\mathcal{L} \longrightarrow \mathcal{L}$) is a *negator* if it satisfies $\mathcal{N}(1_{\mathcal{L}}) = 0_{\mathcal{L}}$, $\mathcal{N}(0_{\mathcal{L}}) = 1_{\mathcal{L}}$, and for each $x, y \in \mathcal{L}$, $y \leq_{\mathcal{L}} x$ implies $\mathcal{N}(x) \leq_{\mathcal{L}} \mathcal{N}(y)$ (*decreasing*).

Triangular norm A binary operation $\mathcal{T}$ on a complete lattice $\mathcal{L}$ (*i.e.*, a mapping $\mathcal{L}^2 \longrightarrow \mathcal{L}$) is a *triangular norm* (*t-norm* for short) if it satisfies four conditions: for each $x, y, z \in \mathcal{L}$, $\mathcal{T}(x,y) = \mathcal{T}(y,x)$ (*commutativity*), $\mathcal{T}(\mathcal{T}(x,y),z) = \mathcal{T}(x,\mathcal{T}(y,z))$ (*associativity*), $x \leq_{\mathcal{L}} y$ implies $\mathcal{T}(x,z) \leq_{\mathcal{L}} \mathcal{T}(y,z)$ (*monotonicity*), and $\mathcal{T}(1_{\mathcal{L}}, x) = x$. A t-norm is *continuous* if it is a continuous function.

Triangular conorm A binary operation $\mathcal{S}$ on a complete lattice $\mathcal{L}$ (*i.e.*, a mapping $\mathcal{L}^2 \longrightarrow \mathcal{L}$) is a *triangular conorm* (*t-conorm* for short) if it satisfies four conditions: for each $x, y, z \in \mathcal{L}$, $\mathcal{S}(x,y) = \mathcal{S}(y,x)$ (*commutativity*), $\mathcal{S}(\mathcal{S}(x,y),z) = \mathcal{S}(x,\mathcal{S}(y,z))$ (*associativity*), $x \leq_{\mathcal{L}} y$ implies $\mathcal{S}(x,z) \leq_{\mathcal{L}} \mathcal{S}(y,z)$ (*monotonicity*), and $\mathcal{S}(0_{\mathcal{L}}, x) = x$.

Implicator A binary operation $\mathcal{I}$ on a complete lattice $\mathcal{L}$ (*i.e.*, a mapping $\mathcal{L}^2 \longrightarrow \mathcal{L}$) is an *implicator* if it is increasing in the first argument and decreasing in the second (*i.e.*, hybrid monotonic) and satisfies $\mathcal{I}(0_{\mathcal{L}}, 0_{\mathcal{L}}) = 1_{\mathcal{L}}$ and $\mathcal{I}(1_{\mathcal{L}}, x) = x$ for any $x \in \mathcal{L}$. *Residual implicator* $\mathcal{I}$ based on a t-norm $\mathcal{T}$ is defined as $\mathcal{I}(x,y) = \sup\{z \in \mathcal{L} \mid \mathcal{T}(x,z) \leq_{\mathcal{L}} y\}$ for each $x, y, z \in \mathcal{L}$.

There are some important negators, t-norms, t-conorms, and residual implicators defined on $[0,1]$ [27], [29]:

(1) Łukasiewicz t-norm $\mathcal{T}_L$, t-conorm $\mathcal{S}_L$, negator $\mathcal{N}_L$, and residual implicator $\mathcal{I}_L$:

$$\mathcal{T}_L(x,y) = \max\{0, x+y-1\}, \quad \mathcal{S}_L(x,y) = \min(x+y, 1)$$
$$\mathcal{N}_L(x) = 1-x, \quad \mathcal{I}_L(x,y) = \min\{1-x+y, 1\}.$$

(2) Gödel t-norm $\mathcal{T}_G$, t-conorm $\mathcal{S}_G$, negator $\mathcal{N}_G$, and residual implicator $\mathcal{I}_G$:

$$\mathcal{T}_G(x,y) = \min\{x,y\}, \quad \mathcal{S}_G = \max(x,y)$$
$$\mathcal{N}_G(x) = \begin{cases} 0 & x > 0 \\ 1 & x = 0 \end{cases}, \quad \mathcal{I}_G = \begin{cases} y & x > y \\ 1 & x \leq y \end{cases}.$$

(3) Product t-norm $\mathcal{T}_P$, t-conorm $\mathcal{S}_P$, negator $\mathcal{N}_P$, and residual implicator $\mathcal{I}_P$:

$$\mathcal{T}_P = x \times y, \quad \mathcal{S}_P = x + y - x \times y$$
$$\mathcal{N}_P = \mathcal{N}_G, \quad \mathcal{I}_P(x,y) = \begin{cases} 1 & x \leq y \\ y/x & x > y \end{cases}.$$

B. Non-monotonic Multi-context Systems

Non-monotonic multi-context systems (MCSs) introduced by Brewka and Eiter is briefly reviewed in this subsection, more details can be found in [2].

Definition 1. [2] *A logic $\mathbb{L}$ is a tuple (KB, BS, ACC) consisting of three components:*

(1) *KB is the set of knowledge bases of $\mathbb{L}$. The element of a knowledge base is called a formula of $\mathbb{L}$;*
(2) *BS is the set of possible belief sets;*
(3) *$ACC : KB \longrightarrow 2^{BS}$ is a set-valued mapping describing the set of acceptable belief sets of each knowledge base in KB.*

Various logics, such as classical propositional logic, description logic, modal logic, answer set programs [31], and default logic [32], can be represented by $\mathbb{L}$ [2], [15], [16].

Definition 2. [2] *Assume $\mathbb{L} = \{\mathbb{L}_1, ..., \mathbb{L}_n\}$ is a set of logics, where $\mathbb{L}_k = (KB_k, BS_k, ACC_k)$ $(1 \leq k \leq n)$. An $\mathbb{L}_i$-bridge rule over the set $\mathbb{L}$, $1 \leq i \leq n$, is of the following form:*

$$a \leftarrow (r_1 : p_1), ..., (r_j : p_j), \textbf{\textit{not}}(r_{j+1} : p_{j+1}), ..., \textbf{\textit{not}}(r_m : p_m) \tag{1}$$

where for any $1 \leq k \leq m$, $1 \leq r_k \leq n$, $p_k \in \bigcup BS_{r_k}$, and $a \in \bigcup KB_i$, i.e., a formula of $\mathbb{L}_i$.

We denote $h(r) = a$ the *head* of a bridge rule r. Bridge rules describe definite information between contexts.

Definition 3. [2] *A multi-context system, or MCS for short, $M = \{C_1, ..., C_n\}$ is a set of contexts $C_i = (\mathbb{L}_i, kb_i, br_i)$ $(1 \leq i \leq n)$, where $\mathbb{L}_i = (KB_i, BS_i, ACC_i)$, $kb_i \in KB_i$, and br_i is a collection of $\mathbb{L}_i$-bridge rules over the set $\{\mathbb{L}_1, ..., \mathbb{L}_n\}$.*

Definition 4. [2] *A belief state of an MCS $M = \{C_1, ..., C_n\}$ is a sequence $\overline{S} = (S_1, ..., S_n)$ satisfying any $S_i \in BS_i$.*

Definition 5. [2] *Let $\overline{S} = (S_1, ..., S_n)$ be a belief state, a bridge rule of the form* (1) *is applicable w.r.t. $\overline{S}$ iff $p_i \in S_{r_i}$ for $1 \leq i \leq j$, and $p_k \notin S_{r_k}$ for $j+1 \leq k \leq m$.*

Definition 6. [2] *A belief state $\overline{S} = (S_1, ..., S_n)$ of an MCS M is an equilibrium iff for each $1 \leq i \leq n$,*

$$S_i \in ACC_i(kb_i \cup \{h(r) \mid r \in br_i \text{ is applicable in } \overline{S}\}).$$

The equilibrium semantics substantially defines the definite belief states that may be adopted by MCSs. However, MCSs focus on definite knowledge and ignore uncertainty in applications.

III. Abstract Logics

Logic theories play a central role in multi-context systems. In this section, we define a new abstract logic to cover different logics, including classical propositional logic, fuzzy answer set programs [29], possibilistic normal logic programs [20], and so on. Furthermore, we introduce the formal definitions of monotonic abstract logics and reducible abstract logics in our setting. A reducible abstract logic can be reduced to a monotonic abstract logic.

Definition 7. *An abstract logic $L = (\mathcal{KB}, \mathcal{U}, \mathcal{L}, \mathcal{FBS}, \mathcal{ACC})$ comprises five components as follows:*

(1) *$\mathcal{KB} = \{kb \mid kb$ is a set describing well-formed knowledge base of $L\}$, the element of kb is a formula of L;*
(2) *$\mathcal{U}$ is a universe;*
(3) *$\mathcal{L}$ is a complete lattice w.r.t. $\leq_{\mathcal{L}}$;*
(4) *$\mathcal{FBS} = \mathcal{L}^{\mathcal{U}} = \{\mathcal{S} \mid \mathcal{S} : \mathcal{U} \longrightarrow \mathcal{L}\}$ is a fuzzy belief space, where $\mathcal{S}$ is an $\mathcal{L}$-fuzzy set on $\mathcal{U}$, called a fuzzy belief set;*

(5) $\mathcal{ACC} : \mathcal{KB} \longrightarrow 2^{\mathcal{FBS}}$ *is a mapping describing the set of acceptable fuzzy belief sets of a knowledge base.*

Syntax and semantics are the fundamental components of any logic system. Intuitively, $\mathcal{ACC}$ captures the semantics of L—the relationship between the knowledge bases set $\mathcal{KB}$ and the fuzzy belief space $\mathcal{FBS}$, while the knowledge base of L—a set of formulas—captures the syntax of L. $\mathcal{U}$ could be a set of formula or a set of atoms. $\mathcal{L}$ may be the range of truth degrees, necessity degrees, or probabilities. The fuzzy belief space $\mathcal{FBS}$ is the $\mathcal{L}$-fuzzy space on $\mathcal{U}$. Obviously, $\mathcal{FBS}$ is a complete lattice *w.r.t.* $\leq_{\mathcal{FBS}}$, *i.e.*, $\leq_{\mathcal{L}^{\mathcal{U}}}$.

Remark 1. The logic $\mathbb{L} = (KB, BS, ACC)$ can be translated into an abstract logic $L_{\mathbb{L}} = (\mathcal{KB}_{\mathbb{L}}, \mathcal{U}_{\mathbb{L}}, \mathcal{L}_{\mathbb{L}}, \mathcal{FBS}_{\mathbb{L}}, \mathcal{ACC}_{\mathbb{L}})$ where

- $\mathcal{KB}_{\mathbb{L}} = KB$, $\mathcal{U}_{\mathbb{L}} = \bigcup BS$, $\mathcal{L}_{\mathbb{L}}$ is $\{0, 1\}$ with natural ordering, and $\mathcal{FBS}_{\mathbb{L}} = 2^{\mathcal{U}_{\mathbb{L}}}$;
- $\mathcal{ACC}_{\mathbb{L}}(kb)$ is the set of mappings $v : \mathcal{U}_{\mathbb{L}} \longrightarrow \{0, 1\}$ such that, for each $S \in ACC(kb)$,

$$v(s) = \begin{cases} 1, & s \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Example 2. Some logics captured by L.

(1) Classical propositional logic CPL

CPL can be expressed as an abstract logic $L_C = (\mathcal{KB}_C, \mathcal{U}_C, \mathcal{L}_C, \mathcal{FBS}_C, \mathcal{ACC}_C)$ comprising the following components:

- $\mathcal{KB}_C = 2^{F(\mathcal{A})}$, where $F(\mathcal{A})$ is the set of well-formed formulas generated by connectives $(\neg, \wedge, \vee, \rightarrow)$ over a propositional atoms set $\mathcal{A}$;
- $\mathcal{U}_C = F(\mathcal{A})$, $\mathcal{L}_C$ is $\{0, 1\}$ with natural ordering, and $\mathcal{FBS}_C = 2^{F(\mathcal{A})}$;
- $\mathcal{ACC}_C(kb) = \{Cn(kb)\}$, where $Cn(kb) = \{\varphi \in F(\mathcal{A}) \mid kb \models \varphi\}$ and $kb \models \varphi$ means that each model of kb is a model of φ. A model of a formula φ is an evaluation $v : \mathcal{A} \longrightarrow \{0, 1\}$ such that $v(\varphi) = 1$.

(2) Fuzzy answer set programs FASP [29]

An atom is an expression of the kind $P(x_1, ..., x_n)$ with an n-ary predicate P, where x_i $(1 \leq i \leq n)$, called a term, is either a constant or a variable.

A FASP P on a complete lattice $(\mathcal{L}, \leq_{\mathcal{L}})$ is a finite set of fuzzy rules of the form

$$a_0 \leftarrow f(a_1, ..., a_n; b_1, ..., b_m) \tag{2}$$

where each a_i $(0 \leq i \leq n)$ and $b_j (1 \leq j \leq m)$ is either an atom or an element of $\mathcal{L}$ and $f : \mathcal{L}^{n+m} \longrightarrow \mathcal{L}$ is a function that increases in its first n arguments and decreases in its last m arguments, such as a t-norm or a t-conorm. For convenience, any FASP rule of the form (2) will be abbreviated as $a_0 \leftarrow a_1$ if f is the identity function, $n = 1$, and $m = 0$. On this basis, if a_1 is the greatest element of $\mathcal{L}$, then $r : a_0 \leftarrow a_1$ will be further shortened as a_0.

A fuzzy interpretation I is a mapping from a atoms set $\mathcal{A}$ to $\mathcal{L}$. Furthermore, $I(a) = a$ for each $a \in \mathcal{L}$ and $I(f(a_1,..., a_n; b_1,..., b_m))$=$f(I(a_1),..., I(a_n);I(b_1),..., I(b_m))$. A fuzzy interpretation I satisfies a fuzzy rule r of the form (2) if $f(I(a_1), ..., I(a_n); I(b_1), ..., I(b_m)) \leq_{\mathcal{L}} I(a_0)$. It is a model of P if it satisfies every rule of P.

A fuzzy interpretation I is a fuzzy answer set of P iff it is a minimal model of the fuzzy reduct P^I which is obtained from P by replacing each b_j $(1 \leq j \leq m)$ in the rule of the form (2) with $I(b_j)$, *i.e.*,

$$P^I = \{a_0 \leftarrow f(a_1, ..., a_n; I(b_1), ..., I(b_m)) \mid a_0 \leftarrow f(a_1, ..., a_n; b_1, ..., b_m) \in P\}.$$

The minimal model of P^I coincides with the least fixpoint of the following immediate consequence operator

$$T_F(J)(a_0) = sup\{J(f(a_1, ..., a_n; I(b_1), ..., I(b_m))) \mid a_0 \leftarrow f(a_1, ..., a_n; I(b_1), ..., I(b_m)) \in P^I\}.$$

FASP can be represented as an abstract logic $L_F = (\mathcal{KB}_F, \mathcal{U}_F, \mathcal{L}_F, \mathcal{FBS}_F, \mathcal{ACC}_F)$ where

- $\mathcal{KB}_F = \{kb \mid kb$ is a fuzzy answer set program over a set $\mathcal{A}$ of atoms and complete lattice $\mathcal{L}\}$;
- $\mathcal{U}_F = \mathcal{A}$, $\mathcal{L}_F = \mathcal{L}$ (the complete lattice above), and $\mathcal{FBS}_F = \{\mathcal{S} \mid \mathcal{S} : \mathcal{A} \longrightarrow \mathcal{L}\}$;
- $\mathcal{ACC}_F(kb) = \{\mathcal{S} \mid \mathcal{S}$ is a fuzzy answer set of $kb\}$.

FASP is an extension of answer set programs [31]. The consistency problem of FASP under Łukasiewicz semantics is NP-hard for normal logic programs and Σ_2^P-complete for disjunctive logic programs [33].

(3) Possibilistic normal logic programs PASP [20]

Let $\mathcal{A}$ be a set of atoms, a PASP P is a finite set of possibilistic rules of the form

$$r = (p_0 \leftarrow p_1, ..., p_j, \textbf{\textit{not}}\ p_{j+1}, ..., \textbf{\textit{not}}\ p_n., \ \alpha) \tag{3}$$

where $n \geq 0$, $\{p_0, p_1, ..., p_n\} \subseteq \mathcal{A}$, and $\alpha \in [0, 1]$, called necessity degree, represents the certainty degree of the information described by r. We denote r by $(h(r) \leftarrow B^+(r), B^-(r)., \ \alpha)$, where $h(r) = p_0$, $B^+(r) = \{p_1, ..., p_j\}$, and $B^-(r) = \{p_{j+1}, ..., p_n\}$. The possibilistic rule is positive if $B^-(r) = \emptyset$. For convenience, any possibilistic rule of the form (3) will be abbreviated as (p_0, α) if $n = 0$. On this basis, if $\alpha = 1$, then (p_0, α) will be further shortened as p_0.

A definite PASP consists of positive possibilistic rules. The possibilistic reduct of a PASP P *w.r.t.* a set A of atoms is the following definite PASP

$$P^A = \{(p_0 \leftarrow p_1, ..., p_j., \ \alpha) \mid r \in P \text{ is of the form (3) and } B^-(r) \cap A = \emptyset\}.$$

A possibilistic atom is of the form $(a, \alpha) \in \mathcal{A} \times [0, 1]$, where α is the necessity degree of a. A positive possibilistic rule: $(p_0 \leftarrow p_1, ..., p_n., \ \alpha)$ *w.r.t.* a set I of possibilistic atoms is β-applicable where $\beta = \min\{\alpha_1, ..., \alpha_n, \alpha\}$ if $\{(p_1, \alpha_1), ..., (p_n, \alpha_n)\} \subseteq I$, and 0-applicable otherwise.

A possibilistic interpretation I is a mapping: $\mathcal{A} \longrightarrow [0, 1]$, *i.e.*, a set of possibilistic atoms. It is a possibilistic answer set of a definite PASP P if it is the least fixpoint *lfp*(T_P) of the immediate possibilistic consequence operator T_P defined as

$$T_P(I) = \{(a,\alpha) \mid APP(P, I, a) \neq \emptyset \text{ and } \alpha = \max_{r \in APP(P,I,a)} \{\beta \mid r \ is\ \beta\text{-applicable in } I\}\}$$

where $APP(P, I, a) = \{r \mid r \in P, h(r) = a, r \text{ is } \beta\text{-applicable in } I, \text{and } \beta > 0\}$.

A possibilistic interpretation I is a possibilistic answer set of a PASP P iff it is a possibilistic answer set of P^{I^*}, where $I^* = \{a \mid (a, \alpha) \in I\}$.

PASP can be written as an abstract logic $L_P = (\mathcal{KB}_P, \mathcal{U}_P, \mathcal{L}_P, \mathcal{FBS}_P, \mathcal{ACC}_P)$ where

- $\mathcal{KB}_P = \{kb \mid kb$ is a possibilistic norm logic program over a set $\mathcal{A}$ of atoms$\}$;
- $\mathcal{U}_P = \mathcal{A}$, $\mathcal{L}_P$ is $[0, 1]$ with natural ordering, and $\mathcal{FBS}_P = \{\mathcal{S} \mid \mathcal{S} : \mathcal{A} \longrightarrow [0, 1]\}$;
- $\mathcal{ACC}_P(kb) = \{\mathcal{S} \mid \mathcal{S}$ is a possibilistic answer set of $kb\}$.

(4) Probabilistic logic: the propositional case [19], [34]

We only consider the probabilistic logic under the propositional case, called propositional probabilistic logic, which combines classical propositional logic and probability theory [19]. The related notions of classical propositional logic introduced above will continue to be adopted.

$F(\mathcal{A})$ is the set of propositional formulas on a finite set $\mathcal{A}$ of propositional atoms. A probability formula is a pair $(F, \mu) \in F(\mathcal{A}) \times [0, 1]$, where μ is the probability of F. We use kb to denote a finite set of probability formulas.

A subset w of $\mathcal{A}$ is called a possible world [34]. We use $W = \{w \mid w \subseteq \mathcal{A}\}$ to denote the set of all possible worlds. A probability distribution is a function $WP : W \longrightarrow [0, 1]$ satisfying $\Sigma_{w \in W} WP(w) = 1$. The probability μ_F of a formula F is determined by a probability distribution WP over W. That is $\mu_F = \Sigma_{w \models F} WP(w)$, which means that the probability of a formula F is the sum of probabilities of possible words w where F is true.

Let $kb = \{(F_1, \mu_{F_1}), ..., (F_m, \mu_{F_m})\}$ and $W = \{w \mid w \subseteq \mathcal{A}\}$ the set of possible words. The so-called probabilistic entailment of Nilsson's probability logic is to deduce a probability formula (φ, μ_φ) from kb, denoted by $kb \models_P (\varphi, \mu_\varphi)$, where $\varphi \in F(\mathcal{A})$, $\mu_\varphi = \Sigma_{w \models \varphi} WP(w)$, and WP is a probability distribution over W. It is worth pointing out that the above probability distribution WP is determined by the following equations with constraints:

(1) for each $(F_i, \mu_i) \in kb$, $\Sigma_{w \models F_i} WP(w) = \mu_{F_i}$,
(2) $WP(w_1) + \cdots + WP(w_n) = 1$,
(3) each $WP(w_i) \in [0, 1]$.

Propositional probabilistic logic can be expressed as an abstract logic $L'_P = (\mathcal{KB}'_P, \mathcal{U}'_P, \mathcal{L}'_P, \mathcal{FBS}'_P, \mathcal{ACC}'_P)$ where

- $\mathcal{KB}'_P = \{kb = \{(F_1, \mu_{F_1}), ..., (F_m, \mu_{F_m})\} \mid kb$ is a probability formulas set over a finite atoms set $\mathcal{A}\}$;
- $\mathcal{U}'_P = F(\mathcal{A})$, $\mathcal{L}'_P$ is $[0, 1]$ with natural ordering, and $\mathcal{FBS}'_P = \{\mathcal{S} \mid \mathcal{S} : F(\mathcal{A}) \longrightarrow [0, 1]\}$;
- $\mathcal{ACC}'_P(kb) = \{\{(\varphi, \mu_\varphi) \mid kb \models_P (\varphi, \mu_\varphi)\}\}$.

In addition, there are varieties of logics can be captured, such as basic fuzzy logic [27], monoidal t-norm based logic MTL [28], normal residuated logic programs NRLP [30], multi-adjoint normal logic programs [35], and logics covered by $\mathbb{L}$.

Definition 8. *An abstract logic $L = (\mathcal{KB}, \mathcal{U}, \mathcal{L}, \mathcal{FBS}, \mathcal{ACC})$ is monotonic iff*

(1) *for all $kb \in \mathcal{KB}$, $\mathcal{ACC}(kb)$ is a singleton set, and*
(2) *if $kb \subseteq kb'$ then $\mathcal{S} \leq_{\mathcal{FBS}} \mathcal{S}'$ where $\mathcal{ACC}(kb) = \{\mathcal{S}\}$ and $\mathcal{ACC}(kb') = \{\mathcal{S}'\}$.*

A monotonic abstract logic L guarantees that a knowledge base kb accepts a unique belief set. Abstract logic $L_C = (\mathcal{KB}_C, \mathcal{U}_C, \mathcal{L}_C, \mathcal{FBS}_C, \mathcal{ACC}_C)$ in Example 2 are monotonic, whereas $L_F = (\mathcal{KB}_F, \mathcal{U}_F, \mathcal{L}_F, \mathcal{FBS}_F, \mathcal{ACC}_F)$ and $L_P = (\mathcal{KB}_P, \mathcal{U}_P, \mathcal{L}_P, \mathcal{FBS}_P, \mathcal{ACC}_P)$ are non-monotonic.

Definition 9. *Let $L = (\mathcal{KB}, \mathcal{U}, \mathcal{L}, \mathcal{FBS}, \mathcal{ACC})$ be an abstract logic. If there is a nonempty subset $\mathcal{KB}'$ of $\mathcal{KB}$ and a function $R_L : \mathcal{KB} \times \mathcal{FBS} \longrightarrow \mathcal{KB}'$ such that*

(1) *$L' = (\mathcal{KB}', \mathcal{U}, \mathcal{L}, \mathcal{FBS}, \mathcal{ACC})$ is monotonic;*
(2) *for any $\mathcal{S} \in \mathcal{FBS}$, $R_L(kb, \mathcal{S}) = kb$ if $kb \in \mathcal{KB}'$;*
(3) *if $\mathcal{S}_1 \leq_{\mathcal{FBS}} \mathcal{S}_2$ and $\mathcal{ACC}(R_L(kb, \mathcal{S}_i)) = \{\mathcal{S}'_i\}$ $(i = 1, 2)$ then $\mathcal{S}'_2 \leq_{\mathcal{FBS}} \mathcal{S}'_1$;*
(4) *$\mathcal{ACC}(R_L(kb, \mathcal{S})) = \{\mathcal{S}\}$ iff $\mathcal{S} \in \mathcal{ACC}(kb)$.*

Then we say that L is reducible w.r.t. R_L, where R_L is called a reduction function of L.

From Definition 9, we know that (i) if $kb \in \mathcal{KB}'$ then it will not be reduced; (ii) condition (3) is weaker than the condition that R_L is decreasing in its second argument (*i.e.*, $R_L(kb, \mathcal{S}) \subseteq R_L(kb, \mathcal{S}')$ if $\mathcal{S}' \leq_{\mathcal{FBS}} \mathcal{S}$), since the latter implies the former by Definition 8, but not vice versa; and (iii) a fuzzy belief set $\mathcal{S}$ is accepted by a knowledge base kb if it is accepted by the reduced knowledge base $R_L(kb, \mathcal{S})$, in other words, we are able to use the reduction to verify whether $\mathcal{S}$ is accepted.

Example 3. There are some examples of reducible logics.

(1) Each monotonic abstract logic is reducible, its reduction function $R^m_L(kb, \mathcal{S}) = kb$.
(2) FASP is reducible, its reduction function $R^F_L(kb, \mathcal{S}) = kb^{\mathcal{S}}$, where kb is a fuzzy answer set programming, $\mathcal{S}$ is a fuzzy interpretation of kb, and $kb^{\mathcal{S}}$ is the fuzzy reduct of kb *w.r.t.* $\mathcal{S}$.
(3) PASP is reducible, its reduction function $R^P_L(kb, \mathcal{S}) = kb^{\mathcal{S}^*}$, where kb is a possibilistic normal logic programming, $\mathcal{S}$ is a set of possibilistic atoms, $\mathcal{S}^* = \{a \mid (a, \alpha) \in \mathcal{S}\}$ is a set of atoms, and $kb^{\mathcal{S}^*}$ is the possibilistic reduct of kb *w.r.t.* $\mathcal{S}^*$.

Moreover, normal residuated logic programming NRLP is also reducible, its reduction function is similar to FASP's (see [30] for more details).

IV. Fuzzy Multi-context Systems

In this section, we propose a fuzzy multi-context system framework including both of its syntax and semantics. Specially, we define the (grounded) fuzzy equilibrium and the well-founded fuzzy equilibrium to capture the semantics of fuzzy multi-context systems. Some examples are provided to illustrate different concepts of fuzzy multi-context systems.

A. Syntax of Fuzzy Multi-context Systems

A fuzzy multi-context system is composed of fuzzy contexts which are based on abstract logics and fuzzy bridge rules. Fuzzy bridge rules describe the flow of information between fuzzy contexts, which are defined below.

Definition 10. *Let $L_i = (\mathcal{KB}_i, \mathcal{U}_i, \mathcal{L}_i, \mathcal{FBS}_i, \mathcal{ACC}_i)$ $(1 \leq i \leq n)$ be abstract logics. A (fuzzy) L_i-bridge rule on the set of abstract logics $\{L_1, ..., L_n\}$ is of the form*

$$\begin{aligned} a \leftarrow & \langle c_1 : (a_1, \mu_1)\rangle, ..., \langle c_k : (a_k, \mu_k)\rangle, \\ & \textbf{\textit{not}}\langle c_{k+1} : (a_{k+1}, \mu_{k+1})\rangle, ..., \textbf{\textit{not}}\langle c_m : (a_m, \mu_m)\rangle \end{aligned} \tag{4}$$

where for any $1 \leq j \leq m$, $1 \leq c_j \leq n$, $a_j \in \mathcal{U}_{c_j}$, $\mu_j \in \mathcal{L}_{c_j}$, and $a \in \bigcup \mathcal{KB}_i$, i.e., a formula of L_i.

We denote the fuzzy bridge rule r by $h(r) \leftarrow \mathcal{B}(r)$, where $h(r) = a$ is the *head* of r and $\mathcal{B}(r)$ is called *body*. The body $\mathcal{B}(r)$ comprises $\mathcal{B}(r)^+$ and $\mathcal{B}(r)^-$ which stand for sets $\{\langle c_1 : (a_1, \mu_1)\rangle, ..., \langle c_k : (a_k, \mu_k)\rangle\}$ and $\{\langle c_{k+1} : (a_{k+1}, \mu_{k+1})\rangle, ..., \langle c_m : (a_m, \mu_m)\rangle\}$, respectively. The rule r is *positive* if $\mathcal{B}(r)^- = \emptyset$.

Intuitively, a fuzzy bridge rule r says that $h(r)$ should be added to some well-formed knowledge base of L_i if the degree of a_j $(1 \leq j \leq k)$ in some fuzzy belief set of L_{c_j} is at least μ_j and the degree of a_t $(k+1 \leq t \leq m)$ in some fuzzy belief set of L_{c_t} is less than μ_j.

Definition 11. *A fuzzy multi-context system, or FMC-S for short, M is a set $\{C_1, ..., C_n\}$ of fuzzy contexts $C_i = (L_i, kb_i, br_i)$ $(1 \leq i \leq n)$, where $L_i = (\mathcal{KB}_i, \mathcal{U}_i, \mathcal{L}_i, \mathcal{FBS}_i, \mathcal{ACC}_i)$ is an abstract logic, $kb_i \in \mathcal{KB}_i$, and br_i is a collection of fuzzy L_i-bridge rules over $\{L_1, ..., L_n\}$.*

In an FMCS M, the set $\{L_1, ..., L_n\}$ of logics specify knowledge types and reasoning characteristics in all fuzzy contexts and fuzzy bridge rules represent information flow between fuzzy contexts. Whether we add information delivered by fuzzy bridge rules to a fuzzy context depends on other fuzzy contexts involved in bodies of these fuzzy bridge rules.

Example 4. [Continued from Example 1] Example 1 describes three heterogeneous contexts—the patient Alice C_1, the doctor John C_2, and the laboratory C_3—and interactions among them. The level of pain, the experience of doctor, and the reaction of antibiotic allergy are full of uncertainty. We can use the following fuzzy multi-context system $M = \{C_1, C_2, C_3\}$ to represent this scenario, where C_i consists of L_i, kb_i, and br_i $(i = 1, 2, 3)$ as follows

(1) L_1 is the FASP over complete lattice $\mathcal{L} = [0, 1]$,
$kb_1 = \{throat\_pain \leftarrow 0.8, \quad allergy \leftarrow 0.3,$
$pay\_blood\_test \leftarrow accept\_blood\_test\}$,
$br_1 = \{(accept\_blood\_test, 1) \leftarrow$
$\langle 2 : (recommend\_blood\_test, 0.9)\rangle\}$.

(2) L_2 is the PASP,
$kb_2 = \{(cure \leftarrow antibiotics.,\ 0.9),$
$(relieve\_pain \leftarrow lozenges.,\ 1),$
$(recommend\_blood\_test \leftarrow$
$suspect\_bacteria.,\ 0.95)\}$,
$br_2 = \{(suspect\_bacteria, 0.85) \leftarrow$
$\langle 1 : (throat\_pain, 0.6)\rangle,$
$(lozenges, 0.8) \leftarrow \langle 3 : (negative, 1)\rangle,$
$(take\_antibiotics, 1) \leftarrow \langle 3 : (positive, 1)\rangle,$
$\textbf{\textit{not}}\langle 1 : (allergy, 0.7)\rangle\}$.

(3) L_3 is the CPL,
$kb_3 = \emptyset$,
$br_3 = \{(perform\_blood\_test, 1) \leftarrow$
$\langle 1 : (pay\_blood\_test, 1)\rangle\}$.

Definition 12. *A fuzzy context $C = (L, kb, br)$ is reducible w.r.t. R_L if*

(1) *R_L is a reduction function of L, and*
(2) *for any fuzzy belief set $\mathcal{S}$, $R_L(kb \cup H, \mathcal{S}) = R_L(kb, \mathcal{S}) \cup H$ whenever $H \subseteq \{h(r) \mid r \in br\}$.*

An FMCS $M = \{C_1, ..., C_n\}$ is reducible w.r.t. $(R_{L_1}, ..., R_{L_n})$ if for each i, C_i is reducible w.r.t. R_{L_i}.

In a reducible fuzzy context C, the abstract logic L is reducible and the head $h(r)$ of each fuzzy bridge rule r is no longer reduced. This means that each $B \subseteq \{h(r) \mid r$ is a bridge rule in $C\}$ belongs to the target class $(\mathcal{KB}')$. Note that a reducible FMCS based on monotonic logics may still be non-monotonic because its fuzzy bridge rules may be non-monotonic.

Example 5. [Continued from Example 4] The FMCS M presented in Example 4 is reducible.

From Example 3, we know that PASP, FASP, and CPL are reducible *w.r.t.* R_L^P, R_L^F, and R_L^m, respectively.

For each $\mathcal{S}$ in FASP, $R_L^F(kb_1 \cup H, \mathcal{S}) = kb_1 \cup H = R_L^F(kb_1, \mathcal{S}) \cup H$, where $H = \{(accept\_blood\_test, 1)\}$.

Therefore, C_1 is reducible *w.r.t.* R_L^F.

For any $\mathcal{S}$ in PASP, $R_L^P(kb_2 \cup H, \mathcal{S}) = kb_2 \cup H = R_L^P(kb_2, \mathcal{S}) \cup H$, where $H = \{(take\_antibiotics, 1), (lozenges, 0.8), (suspect\_bacteria, 0.85)\}$.

Hence, C_2 is reducible *w.r.t.* R_L^P.

For any $\mathcal{S}$ in CPL, $R_L^m(kb_3 \cup H, \mathcal{S}) = H = R_L^m(kb_3, \mathcal{S}) \cup H$, where $H = \{(perform\_blood\_test, 1)\}$.

Thus, C_3 is reducible *w.r.t.* R_L^m.

As a result, M is reducible *w.r.t.* (R_L^P, R_L^F, R_L^m).

Definition 13. *An FMCS $M = \{C_1, ..., C_n\}$ is definite iff*

(1) *M is reducible w.r.t. $(R_{L_1}, ..., R_{L_n})$,*
(2) *for each $C_i = (L_i, kb_i, br_i)$, $R_{L_i}(kb_i, \mathcal{S}) = kb_i$ whenever $\mathcal{S} \in \mathcal{FBS}_i$, and*
(3) *all fuzzy bridge rules in M are positive.*

A definite FMCS has desirable properties: it is reducible, each kb_i is the reduced form, and each fuzzy bridge rule is positive.

Example 6. Consider an FMCS $M = \{C_1, C_2, C_3\}$, where C_i consists of L_i, kb_i, and br_i $(i = 1, 2, 3)$ as follows:

(1) L_1 is the FASP over complete lattice $\mathcal{L} = [0, 1]$,
$kb_1 = \{like \leftarrow 0.95\}$,
$br_1 = \{(buy\_car, 0.9) \leftarrow \langle 1 : (like, 0.8)\rangle,$
$\langle 2 : (support, 0.7)\rangle, \langle 3 : (on\_sale\_car, 1)\rangle\}$.

(2) L_2 is the PASP,
$kb_2 = \{(support \leftarrow recommend.,\ 0.9),$
$(quality\_good \leftarrow .,\ 1)\}$.
$br_2 = \{(recommend, 0.85) \leftarrow \langle 2 : (quality\_good, 0.8)\rangle,$
$\langle 3 : (on\_sale\_car, 1)\rangle\}$.

(3) L_3 is the CPL, $kb_3 = \{on\_sale\_car\}$, and $br_3 = \emptyset$.

It is obvious that each fuzzy bridge rule in M is positive.

According to Example 3, FASP, PASP, and CPL are reducible *w.r.t.* R_L^F, R_L^P, and R_L^m, respectively.

For each $\mathcal{S}$ in FASP, $R_L^F(kb_1, \mathcal{S}) = kb_1$, which implies

$$R_L^F(kb_1 \cup \{(buy\_car, 0.9)\}, \mathcal{S}) = kb_1 \cup \{(buy\_car, 0.9)\} = R_L^F(kb_2, \mathcal{S}) \cup \{(buy\_car, 0.9)\}.$$

Hence, C_1 is reducible *w.r.t.* R_L^F.

For each $\mathcal{S}$ in PASP, $R_L^P(kb_2, \mathcal{S}) = kb_2$, and so

$$R_L^P(kb_2 \cup \{(recommend, 0.85)\}, \mathcal{S}) = kb_2 \cup \{(recommend, 0.85)\} = R_L^P(kb_2, \mathcal{S}) \cup \{(recommend, 0.85)\}.$$

Thus, C_2 is reducible *w.r.t.* R_L^P.

For each $\mathcal{S}$ in CPL, $R_L^m(kb_3, \mathcal{S}) = kb_3$, which implies $R_L^m(kb_3 \cup \emptyset, \mathcal{S}) = kb_3 = R_L^m(kb_3, \mathcal{S})$.

Therefore, C_3 is reducible *w.r.t.* R_L^m.

Consequently, M is reducible *w.r.t.* (R_L^F, R_L^P, R_L^m).

From the above, M is definite.

B. Semantics of Fuzzy Multi-context Systems

For an FMCS, its semantics is to discuss the acceptable fuzzy belief states it may adopt. In order to capture the semantics of FMCSs, the (grounded) fuzzy equilibrium and the well-founded fuzzy equilibrium are defined. Furthermore, the well-founded fuzzy equilibrium of a reducible FMCS is deemed as an approximation of its grounded fuzzy equilibria.

1) Fuzzy Equilibria:
A fuzzy equilibrium of an FMCS is essentially an acceptable fuzzy belief state, providing the basis of the semantics of fuzzy multi-context systems.

Definition 14. *Let $M = \{C_1, ..., C_n\}$ be a fuzzy multi-context system. $\mathbb{S} = \mathcal{FBS}_1 \times \cdots \times \mathcal{FBS}_n$ is called the fuzzy belief state space of M and $\mathrm{S} = (\mathcal{S}_1, ..., \mathcal{S}_n) \in \mathbb{S}$ a fuzzy belief state.*

Since, for each i, $\mathcal{FBS}_i$ is a complete lattice *w.r.t.* $\leq_{\mathcal{FBS}_i}$, $\mathbb{S}$ is the product of complete lattices. The *componentwise ordering* $\leq_{\mathbb{S}}$ on $\mathbb{S}$ is defined as: let $\mathrm{S}, \mathrm{S}' \in \mathbb{S}$,

$$\mathrm{S} \leq_{\mathbb{S}} \mathrm{S}' \textit{ iff } \mathcal{S}_i \leq_{\mathcal{FBS}_i} \mathcal{S}_i' \text{ for each } 1 \leq i \leq n.$$

According to [36], we know that the product of complete lattices is still a complete lattice *w.r.t. componentwise ordering*. For this reason, the following corollary holds.

Corollary 1. *The fuzzy belief state space $\mathbb{S}$ of an FMCS M is a complete lattice w.r.t. $\leq_{\mathbb{S}}$.*

Definition 15. *A fuzzy bridge rule r of the form* (4) *is applicable w.r.t. a fuzzy belief state $\mathrm{S} = (\mathcal{S}_1, ..., \mathcal{S}_n)$ iff*

(1) *for $1 \leq j \leq k$, $\mu_j \leq_{\mathcal{L}_{c_j}} \mathcal{S}_{c_j}(a_j)$, and*
(2) *for $k+1 \leq j \leq m$, $\mathcal{S}_{c_j}(a_j) <_{\mathcal{L}_{c_j}} \mu_j$.*

$H(br, \mathrm{S}) = \{h(r) \mid r \in br$ is applicable *w.r.t.* fuzzy belief state $\mathrm{S}\}$ denotes the set of heads of applicable fuzzy bridge rules. $H(br, \mathrm{S}) \neq \varnothing$ means that the information described by $H(br, \mathrm{S})$ will be added to kb.

Definition 16. *Let $\mathrm{S} = (\mathcal{S}_1, ..., \mathcal{S}_n)$ be a fuzzy belief state of an FMCS M. S is a fuzzy equilibrium for M iff for each $1 \leq i \leq n$, $\mathcal{S}_i \in \mathcal{ACC}_i(kb_i \cup H(br_i, \mathrm{S}))$.*

From Definition 16, we know that for a fuzzy context C_i, if $H(br_i, \mathrm{S}) \neq \emptyset$ then its acceptable fuzzy belief sets are affected by the given fuzzy belief sets of the other fuzzy contexts. A fuzzy equilibrium is a fuzzy belief state composed of an acceptable fuzzy belief set for each fuzzy context.

Example 7. [Continued from Example 6] The fuzzy belief state space of M in Example 6 is $\mathbb{S} = \mathcal{FBS}_P \times \mathcal{FBS}_F \times \mathcal{FBS}_C$. It is easy to verify that $\mathrm{S} = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$ is a fuzzy equilibrium of M, where

$\mathcal{S}_1 = \{(like, 0.95), (buy\_car, 0.9)\}$,
$\mathcal{S}_2 = \{(quality\_good, 1), (recommend, 0.85), (support, 0.85)\}$,
$\mathcal{S}_3 = Cn(\{on\_sale\_car\})$.

The fuzzy equilibrium S indicates that a red car is on sale in C_3, C_1 wants to buy the car very much (0.9), and C_2 is strongly (0.85) in favor of buying it.

Definition 17. *Let $\mathbb{S}$ be the fuzzy belief state space of an FMCS M. S is a minimal fuzzy equilibrium of M iff there exists no fuzzy equilibrium S' such that $\mathrm{S}' <_{\mathbb{S}} \mathrm{S}$.*

We will see later that S in Example 7 is the unique minimal fuzzy equilibrium of M in Example 6.

2) Grounded Fuzzy Equilibria:
A definite FMCS has a unique minimal fuzzy equilibrium, called the grounded fuzzy equilibrium, which can be calculated through the iteration of fixed point. A reducible FMCS can be transformed into a definite FMCS under a given fuzzy belief state. On this basis, grounded fuzzy equilibria for reducible FMCSs are defined.

Definition 18. *Let M be a definite FMCS and $\mathbb{S}$ the fuzzy belief state space of M. A fuzzy belief state S is the grounded fuzzy equilibrium of M if it is the unique minimal fuzzy equilibrium of M (under the ordering $\leq_{\mathbb{S}}$).*

How to calculate the grounded fuzzy equilibrium is crucial for a definite FMCS M. A method that depends on $\mathbb{T}_M$ is presented below.

Definition 19. *Let $M = \{C_1, ..., C_n\}$ be a definite FMCS and $\mathbb{S}$ the fuzzy belief state space of M. The immediate consequence operator $\mathbb{T}_M : \mathbb{S} \longrightarrow \mathbb{S}$ is defined as*

$$\mathbb{T}_M(\mathrm{S}) = \mathrm{S}^*, i.e., \mathbb{T}_M((\mathcal{S}_1, ..., \mathcal{S}_n)) = (\mathcal{S}_1^*, ..., \mathcal{S}_n^*)$$

where $\mathcal{ACC}_i(kb_i \cup H(br_i, \mathrm{S})) = \{\mathcal{S}_i^*\}$ $(1 \leq i \leq n)$.

Theorem 1. *Let $\mathbb{S}$ be the fuzzy belief state space of a definite FMCS $M = \{C_1, ..., C_n\}$. Then $\mathrm{S} \leq_{\mathbb{S}} \mathrm{S}'$ implies $\mathbb{T}_M(\mathrm{S}) \leq_{\mathbb{S}} \mathbb{T}_M(\mathrm{S}')$, i.e., $\mathbb{T}_M$ is monotonic.*

Proof: Suppose $\mathrm{S} = (\mathcal{S}_1, ..., \mathcal{S}_n)$, $\mathrm{S}' = (\mathcal{S}_1', ..., \mathcal{S}_n')$, $\mathbb{T}_M((\mathcal{S}_1, ..., \mathcal{S}_n)) = (\mathcal{S}_1^*, ..., \mathcal{S}_n^*)$, and $\mathbb{T}_M((\mathcal{S}_1', ..., \mathcal{S}_n')) = (\mathcal{S}_1'^*, ..., \mathcal{S}_n'^*)$, where $\mathcal{ACC}_i(kb_i \cup H(kb_i, \mathrm{S})) = \{\mathcal{S}_i^*\}$ and $\mathcal{ACC}_i(kb_i \cup H(kb_i, \mathrm{S}')) = \{\mathcal{S}_i'^*\}$ for all i.

Considering that M is definite, we conclude from Definitions 8, 9, 12 and 13 that, for each i,

(1) there exists a subset $\mathcal{KB}_i'$ of $\mathcal{KB}_i$ such that $(\mathcal{KB}_i', \mathcal{U}_i, \mathcal{L}_i, \mathcal{FBS}_i, \mathcal{ACC}_i)$ is monotonic,

(2) there is a reduction function R_{L_i} of L_i from $\mathcal{KB}_i \times \mathcal{FBS}_i$ to $\mathcal{KB}_i'$,

(3) each fuzzy bridge rule in M is positive,

(4) $R_{L_i}(kb_i \cup H(br_i, \mathrm{S}), \mathcal{S}_i) = R_{L_i}(kb_i, \mathcal{S}_i) \cup H(br_i, \mathrm{S}) = kb_i \cup H(br_i, \mathrm{S}) \in \mathcal{KB}'_i$,

(5) $R_{L_i}(kb_i \cup H(br_i, \mathrm{S}'_i), \mathcal{S}'_i) = R_{L_i}(kb_i, \mathcal{S}'_i) \cup H(br_i, \mathrm{S}') = kb_i \cup H(br_i, \mathrm{S}') \in \mathcal{KB}'_i$.

For each i, if $a \in H(br_i, \mathrm{S})$, then there is a positive fuzzy bridge rule r in br_i of the form

$$a \leftarrow \langle c_1 : (a_1, \mu_1)\rangle, ..., \langle c_k : (a_k, \mu_k)\rangle$$

such that $\mu_j \leq_{\mathcal{L}_{c_j}} \mathcal{S}_{c_j}(a_j)$ for each j by Definition 15. It follows from $\mathrm{S} \leq_{\mathbb{S}} \mathrm{S}'$ that $\mu_j \leq_{\mathcal{L}_{c_j}} \mathcal{S}'_{c_j}(a_j)$, and r is applicable *w.r.t.* S'. Therefore, $a \in H(br_i, \mathrm{S}')$, which gives $H(br_i, \mathrm{S}) \subseteq H(br_i, \mathrm{S}')$, so $kb_i \cup H(br_i, \mathrm{S}) \subseteq kb_i \cup H(br_i, \mathrm{S}')$. By Definition 8, we have $\mathcal{S}^*_i \leq_{\mathcal{FBS}_i} \mathcal{S}'^*_i$, and thus $\mathbb{T}_M(\mathrm{S}) \leq_{\mathbb{S}} \mathbb{T}_M(\mathrm{S}')$. ■

The theorem below indicates that each definite FMCS has a unique grounded fuzzy equilibrium, which can be computed by the following transfinite sequences.

Theorem 2. [Fixpoint Semantics] *Let $\mathbb{S}$ be the fuzzy belief state space of a definite FMCS $M = \{C_1, ..., C_n\}$. We define the transfinite sequences $\mathbb{T}^\alpha_M$ as follows:*

$\mathbb{T}^0_M = 0_{\mathbb{S}}$, *where $0_{\mathbb{S}}$ is the least element of $\mathbb{S}$;*

$\mathbb{T}^{\alpha+1}_M = \mathbb{T}_M(\mathbb{T}^\alpha_M)$, *if α is a successor ordinal;*

$\mathbb{T}^\alpha_M = \cup_{\beta<\alpha}\mathbb{T}^\beta_M$, *if α is a limit ordinal.*

Then,

(1) *there is an ordinal λ such that lfp$(\mathbb{T}_M) = \mathbb{T}^\lambda_M$, and*

(2) *the grounded fuzzy equilibrium of M is lfp$(\mathbb{T}_M)$.*

Proof: (1) It follows from Theorem 1 and the Tarski's theorem of fixpoint [37].

(2) It follows from Definition 18. ■

Example 8. [Continued from Example 6] The M in Example 6 is definite. We compute its grounded fuzzy equilibrium as follows.

$\mathbb{T}^0_M = (\mathcal{S}^0_1, \mathcal{S}^0_2, \mathcal{S}^0_3)$, where $\mathcal{S}^0_1 = \mathcal{S}^0_2 = \mathcal{S}^0_3 = \emptyset$.
$\mathbb{T}^1_M = \mathbb{T}_M(\mathbb{T}^0_M) = (\mathcal{S}^1_1, \mathcal{S}^1_2, \mathcal{S}^1_3)$, where
$\mathcal{S}^1_1 = \{(like, 0.95)\}$,
$\mathcal{S}^1_2 = \{(quality\_good, 1)\}$,
$\mathcal{S}^1_3 = Cn(\{on\_sale\_car\})$.
$\mathbb{T}^2_M = \mathbb{T}_M(\mathbb{T}^1_M) = (\mathcal{S}^2_1, \mathcal{S}^2_2, \mathcal{S}^2_3)$, where
$\mathcal{S}^1_2 = \{(like, 0.95)\}$,
$\mathcal{S}^2_2 = \{(quality\_good, 1), (recommend, 0.85), (support, 0.85)\}$,
$\mathcal{S}^2_3 = Cn(\{on\_sale\_car\})$.
$\mathbb{T}^3_M = \mathbb{T}_M(\mathbb{T}^2_M) = (\mathcal{S}^1_3, \mathcal{S}^2_3, \mathcal{S}^3_3)$, where
$\mathcal{S}^1_3 = \{(like, 0.95), (buy\_car, 0.9)\}$,
$\mathcal{S}^3_2 = \{(quality\_good, 1), (recommend, 0.85) (support, 0.85)\}$,
$\mathcal{S}^3_3 = Cn(\{on\_sale\_car\})$.
$\mathbb{T}^4_M = \mathbb{T}^3_M = lfp(\mathbb{T}_M)$.

As a consequence, *lfp*$(\mathbb{T}_M)$ is the grounded fuzzy equilibrium of M.

In order to define grounded fuzzy equilibria for reducible FMCSs, we first introduce the following reduction.

Definition 20. *Let an FMCS $M = \{C_1, ..., C_n\}$ be reducible w.r.t. $(R_{L_1}, ..., R_{L_n})$ and $\mathrm{S} = (\mathcal{S}_1, ..., \mathcal{S}_n)$ a fuzzy belief state. The S-reduct of M is*

$$M^{\mathrm{S}} = \{C^{\mathrm{S}}_1, ..., C^{\mathrm{S}}_n\}$$

where for each $1 \leq i \leq n$, $C^{\mathrm{S}}_i = (L_i, R_{L_i}(kb_i, \mathcal{S}_i), br^{\mathrm{S}}_i)$ and br^{S}_i is obtained from br_i by

- *removing rules with **not** $\langle k : (a, \mu)\rangle$ in the body such that $\mu \leq_{\mathcal{L}_k} \mathcal{S}_k(a)$, and*
- *removing **not** $\langle k : (a, \mu)\rangle$ from all other rules.*

For each reducible FMCS M and each fuzzy belief state S, M^{S} is a definite FMCS.

Example 9. Consider the FMCS $M = \{C_1, C_2\}$ consisting of fuzzy contexts $C_i = (L_i, kb_i, br_i)$ defined as follows

(1) L_1 is the PASP,
$kb_1 = \{(a \leftarrow b, c., 0.9),\ (c \leftarrow ., 0.8)\}$,
$br_1 = \{(b, 0.7) \leftarrow \langle 1 : (c, 0.65)\rangle, \textbf{\textit{not}}\langle 2 : (d, 0.5)\rangle\}$.

(2) L_2 is the FASP over $[0, 1]$,
$kb_2 = \{e \leftarrow 0.3,\ d \leftarrow \mathcal{N}_L(m)\}$,
$br_2 = \{(m, 0.8) \leftarrow \langle 2 : (e, 0.2)\rangle, \textbf{\textit{not}}\langle 1 : (a, 0.9)\rangle\}$.

$\mathbb{S} = \mathcal{FBS}_P \times \mathcal{FBS}_F$ is the fuzzy belief state space of M.

Let $\mathrm{S} = (\mathcal{S}_1, \mathcal{S}_2)$ where $\mathcal{S}_1 = \{(a, 0.7), (b, 0.7), (c, 0.8)\}$ and $\mathcal{S}_2 = \{(d, 0.2), (e, 0.3), (m, 0.8)\}$, the reduction of M *w.r.t.* S is a definite FMCS $M^{\mathrm{S}} = \{C^{\mathrm{S}}_1, C^{\mathrm{S}}_2\}$ where

(1) $C^{\mathrm{S}}_1 = (L_1, kb'_1, br'_i)$,
$kb'_1 = kb_1$,
$br^{\mathrm{S}}_1 = \{(b, 0.7) \leftarrow \langle 1 : (c, 0.65)\rangle\}$.

(2) $C^{\mathrm{S}}_2 = (L_2, kb'_2, br'_i)$,
$kb'_2 = kb^{\mathcal{S}_2}_2 = \{e \leftarrow 0.3,\ d \leftarrow 0.2\}$,
$br^{\mathrm{S}}_2 = \{(m, 0.8) \leftarrow \langle 2 : (e, 0.2)\rangle\}$.

Definition 21. *Let S be a fuzzy belief state of a reducible FMCS M. S is a grounded fuzzy equilibrium of M if S is the grounded fuzzy equilibrium of M^{S}, i.e., S = lfp$(\mathbb{T}_{M^{\mathrm{S}}})$.*

A grounded fuzzy equilibrium S of a reducible FMCS M is substantially the grounded fuzzy equilibrium of the definite FMCS M^{S}.

Example 10. [Continued from Example 9] We further verify that S given in Example 9 is a grounded fuzzy equilibrium of M. That is to verify S = *lfp*$(\mathbb{T}_{M^{\mathrm{S}}})$, where M^{S} has been given in Example 9 .

$\mathbb{T}^0_{M^{\mathrm{S}}} = (\mathcal{S}^0_1, \mathcal{S}^0_2)$, where $\mathcal{S}^0_1 = \mathcal{S}^0_2 = \emptyset$.
$\mathbb{T}^1_{M^{\mathrm{S}}} = \mathbb{T}_M(\mathbb{T}^0_{M^{\mathrm{S}}}) = (\mathcal{S}^1_1, \mathcal{S}^1_2)$, where
$\mathcal{S}^1_1 = \{(c, 0.8)\}$, $\mathcal{S}^1_2 = \{(d, 0.2), (e, 0.3)\}$.
$\mathbb{T}^2_{M^{\mathrm{S}}} = \mathbb{T}_M(\mathbb{T}^1_{M^{\mathrm{S}}}) = (\mathcal{S}^2_1, \mathcal{S}^2_2)$, where
$\mathcal{S}^2_1 = \{(a, 0.7), (b, 0.7), (c, 0.8)\}$,
$\mathcal{S}^1_2 = \{(d, 0.2), (e, 0.3), (m, 0.8)\}$.
$\mathbb{T}^3_{M^{\mathrm{S}}} = \mathbb{T}^2_{M^{\mathrm{S}}} = lfp(\mathbb{T}_{M^{\mathrm{S}}}) = \mathrm{S}$.

The following theorem provides a method to check whether a fuzzy belief state S is a minimal fuzzy equilibrium of a reducible FMCS M. Namely, S is a minimal fuzzy equilibrium of M if S = *lfp*$(\mathbb{T}_{M^{\mathrm{S}}})$.

Theorem 3. *Let M be a reducible FMCS. A grounded fuzzy equilibrium of M is a minimal fuzzy equilibrium of M.*

Proof: Let $M = \{C_1, ..., C_n\}$ be reducible *w.r.t.* $(R_{L_1}, ..., R_{L_n})$, $\mathbb{S}$ the fuzzy belief state space of M, and $\mathrm{S} = (\mathcal{S}_1, ..., \mathcal{S}_n)$ a grounded fuzzy equilibrium of M. The proof will be divided into three steps.

Step 1: we show that $H(br_i, \mathrm{S}) = H(br_i^{\mathrm{S}}, \mathrm{S})$.

For each $C_i = (L_i, kb_i, br_i)$, let $a \in H(br_i^{\mathrm{S}}, \mathrm{S})$, by Definition 15 there is a positive fuzzy bridge rule $r^{\mathrm{S}} \in br_i^{\mathrm{S}}$ of the form $a \leftarrow \langle c_1 : (a_1, \mu_1)\rangle, ..., \langle c_k : (a_k, \mu_k)\rangle$, such that for all j, $\mu_j \leq_{\mathcal{L}_{c_j}} \mathcal{S}_{c_j}(a_j)$.

By Definition 20, there exists a $r \in br_i$ corresponding to r^{S}, which is one of the following situations

(i) $r = r^{\mathrm{S}}$, where for all j, $\mu_j \leq_{\mathcal{L}_{c_j}} \mathcal{S}_{c_j}(a_j)$,

(ii) r is of the following form

$a \leftarrow \langle c_1 : (a_1, \mu_1)\rangle, ..., \langle c_k : (a_k, \mu_k)\rangle,$
$\textbf{\textit{not}}\langle c_{k+1} : (a_{k+1}, \mu_{k+1})\rangle, ..., \textbf{\textit{not}}\langle c_m : (a_m, \mu_m)\rangle$

where $\mu_j \leq_{\mathcal{L}_{c_j}} \mathcal{S}_{c_j}(a_j)$ for $1 \leq j \leq k$ and $\mathcal{S}_{c_j}(a_j) <_{\mathcal{L}_{c_j}} \mu_j$ for $k+1 \leq j \leq m$.

In either case, that r is applicable *w.r.t.* S follows from Definition 15. Therefore, $a \in H(br_i, \mathrm{S})$, which implies $H(br_i^{\mathrm{S}}, \mathrm{S}) \subseteq H(br_i, \mathrm{S})$. In a similar way, $H(br_i, \mathrm{S}) \subseteq H(br_i^{\mathrm{S}}, \mathrm{S})$ can be proved. Consequently, $H(br_i, \mathrm{S}) = H(br_i^{\mathrm{S}}, \mathrm{S})$.

Step 2: we prove that S is a fuzzy equilibrium of M.

As $\mathrm{S} = (\mathcal{S}_1, ..., \mathcal{S}_n)$ is a grounded fuzzy equilibrium of M, we have $\mathrm{S} = \mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}}})$ by Definition 21. Hence, for each i, by Definitions 9, 12, 19, and 20,

$$\{\mathcal{S}_i\} = \mathcal{ACC}_i(R_{L_i}(kb_i, \mathcal{S}_i) \cup H(br_i^{\mathrm{S}}, \mathrm{S}))$$
$$\{\mathcal{S}_i\} = \mathcal{ACC}_i(R_{L_i}(kb_i \cup H(br_i^{\mathrm{S}}, \mathrm{S}), \mathcal{S}_i))$$
$$\mathcal{S}_i \in \mathcal{ACC}_i(kb_i \cup H(br_i^{\mathrm{S}}, \mathrm{S}))$$
$$\mathcal{S}_i \in \mathcal{ACC}_i(kb_i \cup H(br_i, \mathrm{S}))$$

Hence, S is a fuzzy equilibrium of M by Definition 16.

Step 3: we show that S is a minimal fuzzy equilibrium of M by contradiction.

Let $\mathrm{S}' = (\mathcal{S}'_1, ..., \mathcal{S}'_n)$ be a fuzzy equilibrium of M and $\mathrm{S}' <_{\mathbb{S}} \mathrm{S}$. By Definitions 9, 12, and 16, we have that for any i,

$$\mathcal{S}'_i \in \mathcal{ACC}_i(kb_i \cup H(br_i, \mathrm{S}'))$$
$$\{\mathcal{S}'_i\} = \mathcal{ACC}_i(R_{L_i}(kb_i \cup H(br_i, \mathrm{S}'), \mathcal{S}'_i))$$
$$\{\mathcal{S}'_i\} = \mathcal{ACC}_i(R_{L_i}(kb_i, \mathcal{S}'_i) \cup H(br_i, \mathrm{S}'))$$
$$\mathcal{S}'_i \in \mathcal{ACC}_i(R_{L_i}(kb_i, \mathcal{S}'_i) \cup H(br_i^{\mathrm{S}'}, \mathrm{S}'))$$

Hence, S' is a fuzzy equilibrium of $M^{\mathrm{S}'}$ by Definitions 16 and 20, which implies $\mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}'}}) \leq_{\mathbb{S}} \mathrm{S}'$.

Since S is a grounded fuzzy equilibrium of M, we have $\mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}}}) = \mathrm{S}$.

Thus $\mathbb{W}_M(\mathrm{S}') \leq_{\mathbb{S}} \mathrm{S}'$ and $\mathbb{W}_M(\mathrm{S}) = \mathrm{S}$ by Definition 22.

$\mathrm{S}' <_{\mathbb{S}} \mathrm{S}$ implies $\mathbb{W}_M(\mathrm{S}) <_{\mathbb{S}} \mathbb{W}_M(\mathrm{S}')$ by Proposition 1.

Therefore, $\mathrm{S} = \mathbb{W}_M(\mathrm{S}) <_{\mathbb{S}} \mathbb{W}_M(\mathrm{S}') \leq_{\mathbb{S}} \mathrm{S}'$, which contradicts with $\mathrm{S}' <_{\mathbb{S}} \mathrm{S}$. ■

3) Well-founded Fuzzy Equilibrium:
Motivated by the stable class semantics of logic programs and default logic [38], the well-founded fuzzy equilibrium of a reducible FMCS M is defined based on the operator $\mathbb{W}_M(\mathrm{S}) = \mathit{lfp}\ (\mathbb{T}_{M^{\mathrm{S}}})$.

Definition 22. *Let $M = \{C_1, ..., C_n\}$ be a reducible FMCS and $\mathbb{S}$ the fuzzy belief state space of M. The operator $\mathbb{W}_M : \mathbb{S} \longrightarrow \mathbb{S}$ is defined as $\mathbb{W}_M(\mathrm{S}) = \mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}}})$.*

Proposition 1. *Let $\mathbb{S}$ be the fuzzy belief state space of a reducible FMCS M. Then $\mathbb{W}_M$ is antimonotone, i.e., $\mathrm{S} \leq_{\mathbb{S}} \mathrm{S}'$ implies $\mathbb{W}_M(\mathrm{S}') \leq_{\mathbb{S}} \mathbb{W}_M(\mathrm{S})$.*

Proof: Let $\mathrm{S} = (\mathcal{S}_1, ..., \mathcal{S}_n)$ and $\mathrm{S}' = (\mathcal{S}'_1, ..., \mathcal{S}'_n)$.

It follows that $\mathbb{W}_M(\mathrm{S}) = \mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}}}) = \mathbb{T}^{\lambda}_{M^{\mathrm{S}}}$ and $\mathbb{W}_M(\mathrm{S}') = \mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}'}}) = \mathbb{T}^{\lambda}_{M^{\mathrm{S}'}}$ from Definition 22 and Theorem 2.

Next, we show that $\mathbb{W}_M(\mathrm{S}') \leq_{\mathbb{S}} \mathbb{W}_M(\mathrm{S})$, *i.e.*, $\mathbb{T}^{\lambda}_{M^{\mathrm{S}'}} \leq_{\mathbb{S}} \mathbb{T}^{\lambda}_{M^{\mathrm{S}}}$ by transfinite induction.

For $\lambda = 0, \mathbb{T}^0_{M^{\mathrm{S}}} = \mathbb{T}^0_{M^{\mathrm{S}'}} = 0_{\mathbb{S}}$.

For each $\beta \leq \lambda$, assume that $\mathbb{T}^{\beta}_{M^{\mathrm{S}'}} \leq_{\mathbb{S}} \mathbb{T}^{\beta}_{M^{\mathrm{S}}}$ by induction hypothesis.

If $\lambda = \alpha + 1$ is a successor ordinal then $\mathbb{T}^{\alpha}_{M^{\mathrm{S}'}} \leq_{\mathbb{S}} \mathbb{T}^{\alpha}_{M^{\mathrm{S}}}$.

Hence, by monotony of $\mathbb{T}_M$ and induction hypothesis, $\mathbb{T}^{\lambda}_{M^{\mathrm{S}'}} = \mathbb{T}_{M^{\mathrm{S}'}}(\mathbb{T}^{\alpha}_{M^{\mathrm{S}'}}) \leq_{\mathbb{S}} \mathbb{T}_{M^{\mathrm{S}'}}(\mathbb{T}^{\alpha}_{M^{\mathrm{S}}}) \leq_{\mathbb{S}} \mathbb{T}_{M^{\mathrm{S}}}(\mathbb{T}^{\alpha}_{M^{\mathrm{S}}}) = \mathbb{T}^{\lambda}_{M^{\mathrm{S}}}$.

If λ is a limit ordinal, then

$$\mathbb{T}^{\lambda}_{M^{\mathrm{S}'}} = \cup_{\beta<\lambda}\mathbb{T}^{\beta}_{M^{\mathrm{S}'}} \leq_{\mathbb{S}} \cup_{\beta<\lambda}\mathbb{T}^{\beta}_{M^{\mathrm{S}}} = \mathbb{T}^{\lambda}_{M^{\mathrm{S}}}.$$ ■

$\mathbb{W}^2_M = \mathbb{W}_M \cdot \mathbb{W}_M$, the mapping that applies $\mathbb{W}_M$ twice, is monotonic on $\mathbb{S}$.

Corollary 2. *$\mathbb{W}^2_M$ is monotone on $\mathbb{S}$.*

Owing to the Tarski's fixpoint theorem [37], $\mathbb{W}^2_M$ has the least fixpoint and the greatest fixpoint, denoted by $\mathit{lfp}(\mathbb{W}^2_M)$ and $\mathit{gfp}(\mathbb{W}^2_M)$, respectively. Based on this, we define the well-founded fuzzy equilibrium of a reducible FMCS M.

Definition 23. *Let M be a reducible FMCS. The well-founded fuzzy equilibrium of M is WFE$(M) = (\mathit{lfp}\ (\mathbb{W}^2_M), \mathit{gfp}(\mathbb{W}^2_M))$.*

The $\mathit{lfp}(\mathbb{W}^2_M)$ can be computed through the upward iteration of $\mathbb{W}^2_M$ from the least element $0_{\mathbb{S}}$ of $\mathbb{S}$, while the $\mathit{gfp}(\mathbb{W}^2_M)$ can be obtained by the downward iteration of $\mathbb{W}^2_M$ from the greatest element $1_{\mathbb{S}}$ of $\mathbb{S}$.

Example 11. [Continued from Example 9] Let us calculate the well-founded fuzzy equilibrium of M in Example 9. $\mathbb{S} = \mathcal{FBS}_P \times \mathcal{FBS}_F$ is the fuzzy belief state space of M.

Compute the $\mathit{gfp}(\mathbb{W}^2_M)$ through the downward iteration of $\mathbb{W}^2_M$ from the greatest element $1_{\mathbb{S}}$ of $\mathbb{S}$.

$\mathbb{W}_M(1_{\mathbb{S}}) = \mathit{lfp}(\mathbb{T}_{M^{1_{\mathbb{S}}}}) = \mathrm{S}_1 = (\mathcal{S}^1_1, \mathcal{S}^1_2)$, where
$\mathcal{S}^1_1 = \{(c, 0.8)\}$, $\mathcal{S}^1_2 = \{(e, 0.3)\}$.
$\mathbb{W}^2_M(1_{\mathbb{S}}) = \mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}_1}}) = \mathrm{S}_2 = (\mathcal{S}^2_1, \mathcal{S}^2_2)$, where
$\mathcal{S}^2_1 = \{(a, 0.7), (b, 0.7), (c, 0.8)\}$,
$\mathcal{S}^2_2 = \{(d, 1), (e, 0.3), (m, 0.8)\}$.
$\mathbb{W}_M(\mathrm{S}_2) = \mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}_2}}) = \mathrm{S}_3 = (\mathcal{S}^3_1, \mathcal{S}^3_2)$, where
$\mathcal{S}^3_1 = \{(c, 0.8)\}$, $\mathcal{S}^3_2 = \{(d, 0.2), (e, 0.3), (m, 0.8)\}$.
$\mathbb{W}^2_M(\mathrm{S}_2) = \mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}_3}}) = \mathrm{S}_4 = (\mathcal{S}^4_1, \mathcal{S}^4_2)$, where
$\mathcal{S}^4_1 = \{(a, 0.7), (b, 0.7), (c, 0.8)\}$,
$\mathcal{S}^4_2 = \{(d, 0.2), (e, 0.3), (m, 0.8)\}$.
$\mathbb{W}_M(\mathrm{S}_4) = \mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}_4}}) = \mathrm{S}_4$.
$\mathbb{W}^2_M(\mathrm{S}_4) = \mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}_4}}) = \mathrm{S}_4 = \mathit{gfp}(\mathbb{W}^2_M)$.

We can also compute the $\mathit{lfp}(\mathbb{W}^2_M) = \mathrm{S}_4$ through the upward iteration of $\mathbb{W}^2_M$ from the least element $0_{\mathbb{S}}$ of $\mathbb{S}$.

As a result, WFE$(M) = (\mathrm{S}_4, \mathrm{S}_4)$.

As shown below, the well-founded fuzzy equilibrium can be deemed as an approximation of grounded fuzzy equilibria for a reducible FMCS.

Theorem 4. *Let $\mathbb{S}$ be the fuzzy belief state space of a reducible FMCS M. Then, $\mathit{lfp}\ (\mathbb{W}^2_M) \leq_{\mathbb{S}} \mathrm{S} \leq_{\mathbb{S}} \mathit{gfp}(\mathbb{W}^2_M)$ for any grounded fuzzy equilibrium S of M.*

Proof: Since S is the grounded fuzzy equilibrium of M, $\mathrm{S} = \mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}}})$, *i.e.*, $\mathrm{S} = \mathbb{W}_M(\mathrm{S})$ which implies $\mathbb{W}^2_M(\mathrm{S}) = \mathrm{S}$.

Hence, $\mathit{lfp}(\mathbb{W}_M^2) \leq_{\mathbb{S}} \mathrm{S} \leq_{\mathbb{S}} \mathit{gfp}(\mathbb{W}_M^2)$ by the Tarski's fixpoint theorem [37]. ■

Particularly, if $\mathit{lfp}(\mathbb{W}_M^2) = \mathit{gfp}(\mathbb{W}_M^2) = \mathrm{S}$, we denote the well-founded fuzzy equilibrium of M by $\mathit{WFE}(M) = \mathrm{S}$. In such a case, $\mathit{WFE}(M)$ captures the unique grounded fuzzy equilibrium of M.

Proposition 2. *Let M be a definite FMCS. If S is the grounded fuzzy equilibrium of M, then $\mathit{WFE}(M) = \mathrm{S}$.*

Proof: Let $\mathbb{S}$ be the fuzzy belief state space of M. $0_{\mathbb{S}}$ and $1_{\mathbb{S}}$ are the least element and the greatest element of $\mathbb{S}$, respectively.

Since M is definite, for each $\mathrm{S}' \in \mathbb{S}$, $M^{\mathrm{S}'} = M$ by Definitions 13 and 20.

According to Definition 22, we have that

$\mathbb{W}_M(0_{\mathbb{S}}) = \mathit{lfp}(\mathbb{T}_{M^{0_{\mathbb{S}}}}) = \mathit{lfp}(\mathbb{T}_M) = \mathrm{S}$

$\mathbb{W}_M^2(0_{\mathbb{S}}) = \mathbb{W}_M(\mathrm{S}) = \mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}}}) = \mathit{lfp}(\mathbb{T}_M) = \mathrm{S}$

$\mathbb{W}_M(1_{\mathbb{S}}) = \mathit{lfp}(\mathbb{T}_{M^{1_{\mathbb{S}}}}) = \mathit{lfp}(\mathbb{T}_M) = \mathrm{S}$

$\mathbb{W}_M^2(1_{\mathbb{S}}) = \mathbb{W}_M(\mathrm{S}) = \mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}}}) = \mathrm{S}$

$\mathbb{W}_M^2(\mathrm{S}) = \mathbb{W}_M(\mathbb{W}_M(\mathrm{S})) = \mathbb{W}_M(\mathrm{S}) = \mathrm{S}$

Hence, $\mathit{WFE}(M) = \mathrm{S}$. ■

The above proposition shows that the grounded fuzzy equilibrium coincides with the well-founded fuzzy equilibrium for a definite FMCS.

C. A Scenario Example Illustration

So far, we have only used the reducible fuzzy multi-context system M (see Examples 4 and 5) to represent the medical scenario described in Example 1. Next, we compute the minimal fuzzy equilibrium of M to predict which treatment the doctor John will give.

Example 12. [Continued from Example 4] Compute the minimal fuzzy equilibrium of M presented in Example 4. Considering that well-founded fuzzy equilibrium is an approximation of the minimal fuzzy equilibrium, we first calculate its well-founded fuzzy equilibrium.

$\mathbb{S} = \mathcal{FBS}_F \times \mathcal{FBS}_P \times \mathcal{FBS}_C$ is the fuzzy belief state space of the reducible FMCS M.

We first compute the $\mathit{lfp}(\mathbb{W}_M^2)$ through the upward iteration of $\mathbb{W}_M^2$ from the least element $0_{\mathbb{S}}$ of $\mathbb{S}$.

$\mathbb{W}_M(0_{\mathbb{S}}) = \mathit{lfp}(\mathbb{T}_{M^{0_{\mathbb{S}}}}) = \mathrm{S} = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3)$, where

$\mathcal{S}_1 = \{(throat\_pain, 0.8), (allergy, 0.3),$
$(accept\_blood\_test, 1), (pay\_blood\_test, 1)\}$,

$\mathcal{S}_2 = \{(suspect\_bacteria, 0.85),$
$(recommend\_blood\_test, 0.85)\}$,

$\mathcal{S}_3 = Cn(\{perform\_blood\_test\})$.

$\mathbb{W}_M^2(0_{\mathbb{S}}) = \mathbb{W}_M(\mathrm{S}) = \mathit{lfp}(\mathbb{T}_{M^{\mathrm{S}}}) = \mathrm{S}$.

$\mathbb{W}_M^2(\mathrm{S}) = \mathbb{W}_M(\mathrm{S}) = \mathrm{S} = \mathit{lfp}(\mathbb{W}_M^2)$.

We can also calculate $\mathit{gfp}(\mathbb{W}_M^2) = \mathrm{S}$ by the downward iteration of $\mathbb{W}_M^2$ from the greatest element $1_{\mathbb{S}}$ of $\mathbb{S}$.

Therefore, $\mathrm{WFE}(M) = \mathrm{S}$, which is also the minimal fuzzy equilibrium of M. It indicates that John will strongly (0.85) advise Alice to take a blood test, Alice will absolutely accept this advice, and the laboratory will perform the test for Alice.

Since John strongly suspects the throat pain is caused by bacterial pharyngitis, the initial treatment for Alice is further examination. The treatment did not end before the test results were given. Furthermore, suppose that the test result is positive, an FMCS M' can be obtained by adding the formula *perform_blood_test* $\rightarrow$ *positive* into kb_3 in M. Its minimal fuzzy equilibrium is $\mathrm{S}' = (\mathcal{S}'_1, \mathcal{S}'_2, \mathcal{S}'_3)$ where

$\mathcal{S}'_1 = \{(throat\_pain, 0.8), (allergy, 0.3),$
$(accept\_blood\_test, 1), (pay\_blood\_test, 1)\}$,

$\mathcal{S}'_2 = \{(cure, 0.9), (suspect\_bacteria, 0.85),$
$(recommend\_blood\_test, 0.85), (take\_antibiotics, 1)\}$,

$\mathcal{S}'_3 = Cn(\{perform\_blood\_test, positive\})$.

In this case, the test result shows that the throat pain is caused by bacterial pharyngitis, John will prescribe antibiotics, and the bacterial pharyngitis will be almost (0.9) cured.

In contrast, suppose that the test result is negative, an FMCS M'' can be obtained by adding the formula *perform_blood_test* $\rightarrow$ *negative* into kb_3 in M. Its minimal fuzzy equilibrium is $\mathrm{S}'' = (\mathcal{S}''_1, \mathcal{S}''_2, \mathcal{S}''_3)$ where

$\mathcal{S}''_1 = \{(throat\_pain, 0.8), (allergy, 0.3),$
$(accept\_blood\_test, 1), (pay\_blood\_test, 1)\}$,

$\mathcal{S}''_2 = \{(suspect\_bacteria, 0.85), (relieve, 0.8)$
$(recommend\_blood\_test, 0.85), (lozenges, 0.8)\}$,

$\mathcal{S}''_3 = Cn(\{perform\_blood\_test, negative\})$.

In such a case, the test result shows that the throat pain is not caused by bacteria, John is very (0.8) certain to prescribe lozenges, and Alice's pain will be greatly (0.8) relieved.

V. Comparison with other Multi-Context Systems

Representation and reasoning for multi-context systems have been extensively investigated from ML systems [4] to ConDL [5], MCSs [2], p-MCSs [7], poss-MCSs [8], and so forth.

From the literatures above, we easily acquire the following relationships among these multi-context systems:

- p-MCSs extend ML systems [7], and
- MCSs generalize ML systems and ConDL [2].

MCSs, poss-MCSs, and p-MCS relate to our work. We thus compare FMCSs with them and summarize the results in the following subsections.

A. Relationship to MCSs

In this subsection, we show how to translate an MCS M to an FMCS $\tau(M)$ while "preserving" the equilibria of M, but not vice versa.

Note that, MCSs are based on a set $\{\mathbb{L}_1, ..., \mathbb{L}_n\}$ of logics. The logic $\mathbb{L} = (KB, BS, ACC)$ corresponds to the abstract logic $L_{\mathbb{L}} = (\mathcal{KB}_{\mathbb{L}}, \mathcal{U}_{\mathbb{L}}, \mathcal{L}_{\mathbb{L}}, \mathcal{FBS}_{\mathbb{L}}, \mathcal{ACC}_{\mathbb{L}})$ (see Remark 1). We can also translate a bridge rule to a fuzzy bridge rule.

Remark 2. A bridge rule r of the form (1) can be translated into the following fuzzy bridge rule $\tau(r)$

$$a \leftarrow \langle r_1 : (p_1, 1)\rangle, ..., \langle r_j : (p_j, 1)\rangle, \\ \textbf{\textit{not}}\langle r_{j+1} : (p_{j+1}, 1)\rangle, ..., \textbf{\textit{not}}\langle r_m : (p_m, 1)\rangle. \tag{5}$$

Definition 24. *Let $M = \{C_1, ..., C_n\}$ be an MCS, where $C_i = (\mathbb{L}_i, kb_i, br_i)$ for each i. The corresponding FMCS is $\tau(M) = \{\tau(C_1), ..., \tau(C_n)\}$ where $\tau(C_i) = (L_{\mathbb{L}_i}, kb_i, \tau(br_i))$ and $\tau(br_i) = \{\tau(r) \mid r \in br_i\}$.*

The following theorem manifests that the equilibria of any MCS M can be precisely captured by the fuzzy equilibria of the corresponding FMCS $\tau(M)$.

Theorem 5. *A belief state $\overline{S} = (S_1, \ldots, S_n)$ is an equilibrium of an MCS M iff $\mathrm{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is a fuzzy equilibrium of $\tau(M)$, where $\mathcal{S}_i$ is the mapping $g : \bigcup BS_i \longrightarrow \{0, 1\}$ such that $g(s) = 1$ if $s \in S_i$, and 0 otherwise.*

Proof Sketch: For an MCS $M = \{C_1, ..., C_n\}$ with $C_i = (L_i, kb_i, br_i)$, $1 \leq i \leq n$, by Definition 24 we can construct an FMCS $\tau(M) = \{\tau(C_1), ..., \tau(C_n)\}$, $1 \leq i \leq n$, where $\tau(C_i) = (L_{\mathbb{L}_i}, kb_i, \tau(br_i))$, $\tau(br_i) = \{\tau(r) \mid r \in br_i\}$. Firstly, by Definitions 5 and 15 we can prove

$$\{h(r) \mid r \in br_i \text{ is applicable in } \overline{S}\} = H(\tau(br_i), \mathrm{S}).$$

Then, it follows from Definitions 6 and 16 that $\overline{S} = (S_1, \ldots, S_n)$ is an equilibrium of M iff $\mathrm{S} = (\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is a fuzzy equilibrium of $\tau(M)$. ■

We next show that FMCSs are unable to be converted into MCSs.

Belief sets in MCSs are crisp sets in accordance with Definitions 2 and 4, whereas fuzzy belief sets in FMCSs are $\mathcal{L}$-fuzzy sets on a universe $\mathcal{U}$ by Definition 7.

To facilitate discussion, we first translate the abstract logic $L = (\mathcal{KB}, \mathcal{U}, \mathcal{L}, \mathcal{FBS}, \mathcal{ACC})$ to a logic

$$\mathbb{L}_L = (KB_L = \mathcal{KB}, BS_L, ACC_L = \mathcal{ACC}),$$

where $BS_L = \{S = \{(x, \mathcal{S}(x)) \mid x \in \mathcal{U}, \mathcal{S} \in \mathcal{FBS}\}\}$. It is worth pointing out that a belief set S in the logic $\mathbb{L}_L$ is not deemed as an $\mathcal{L}$-fuzzy sets on the universe $\mathcal{U}$, but as a crisp set on the universe $\mathcal{U} \times \mathcal{L}$.

Then, a fuzzy bridge rule r of the form (4) is translated into a bridge rule $Q(r)$ of the form (1) through treating (a_k, μ_k) $(1 \leq k \leq m)$ as an element of some belief set of the logic $\mathbb{L}_{L_{c_k}}$, *i.e.*, $(a_k, \mu_k) \in \bigcup BS_{L_{c_k}}$.

Furthermore, let $M = \{C_1, ..., C_n\}$ be an MCS, where $C_i = (L_i, kb_i, br_i)$ for each i. The corresponding MCS is $Q(M) = \{Q(C_1), ..., Q(C_n)\}$ where $Q(C_i) = (\mathbb{L}_{L_i}, kb_i, Q(br_i))$ and $Q(br_i) = \{Q(r) \mid r \in br_i\}$. Note that an FMCS M and the corresponding MCS $Q(M)$ have the same expression. The major difference between them is that fuzzy belief sets in M are $\mathcal{L}$-fuzzy sets on the universe $\mathcal{U}$, while belief sets in $Q(M)$ are viewed as crisp sets on the universe $\mathcal{U} \times \mathcal{L}$.

Finally, suppose that $\mathrm{S} = (\mathcal{S}_1, ..., \mathcal{S}_n)$ is a fuzzy equilibrium of an FMCS M, $\bar{S} = (S_1, ..., S_n)$ with $S_i = \{(x, \mathcal{S}_i(x)) \mid x \in \mathcal{U}\}$ is the corresponding belief state of S, and $Q(M)$ is the corresponding MCS of M. $\bar{S}$ may not be an equilibrium of $Q(M)$ since the notion of applicability of bridge rules (see Definition 5) differs from the notion of fuzzy bridge rules (see Definition 15).

For example, assume that M is the FMCS in Example 4 which encodes the medical scenario in Example 1. Then, according to the above analysis, $Q(M)$ is the MCS with the same expression as M. By Definition 6, it can be directly verified that the corresponding belief state $\bar{S}$ of the minimal fuzzy equilibrium S of M presented in Example 12, is not an equilibrium of $Q(M)$. Additionally, $\bar{S}' = (S'_1, S'_2, S'_3)$ with $S'_1 = \{(throat\_pain, 0.8), (allergy, 0.3)\}$, $S'_2 = \emptyset$, and $S'_3 = Cn(\emptyset)$ is the minimal equilibrium of the above $Q(M)$, which indicates that there is nothing the doctor John will do to help Alice. This does not match the fact that since Alice's throat pain is severe, John strongly suspects bacterial pharyngitis such that he strongly advises her to take a blood test.

B. Relationship to poss-MCSs

In this subsection, we show that poss-MCSs can be translated into FMCSs, but not vice versa.

1) poss-MCSs:

A poss-MCS [8] is based on possibilistic normal logic programs, comprising possibilistic contexts. The inference method in each possibilistic context is the same, since the inference is only rooted in possibilistic normal logic programs.

Definition 25. [8] *A poss-MCS $M = \{C_1, ..., C_n\}$ is a set of possibilistic contexts $C_i = (\mathcal{A}_i, K_i, B_i)$. For each $1 \leq i \leq n$, C_i consists of the following components:*

(1) *$\mathcal{A}_i$ is a set of atoms,*
(2) *K_i is a possibilistic logic program over $\mathcal{A}_i$, and*
(3) *B_i is a collection of possibilistic bridge rules defined as*

$$\begin{aligned} a \leftarrow (c_1 : p_1), ..., (c_j : p_j), \\ \textbf{\textit{not}}(c_{j+1} : p_{j+1}), ..., \textbf{\textit{not}}(c_m : p_m), \ [\alpha] \end{aligned} \tag{6}$$

where for each $1 \leq j \leq m$, $1 \leq c_j \leq n$, p_j is an atom in $\mathcal{A}_{c_j}$, $\alpha \in [0, 1]$, and a is an atom in $\mathcal{A}_i$.

*A poss-MCS is definite if all K_i and B_i do not contain **not**, otherwise it is normal.*

A possibilistic bridge rule of the form (6) is denoted by $h(r) \leftarrow \mathcal{B}(r)$, $[\alpha]$ where $h(r) = a$ is the *head* of r and $\mathcal{B}(r)$ is called *body*. The *body* $\mathcal{B}(r)$ comprises $\mathcal{B}(r)^+$ and $\mathcal{B}(r)^-$ which stand for sets $\{(c_1, a_1), ..., (c_j, a_j)\}$ and $\{(c_{j+1}, a_{j+1}), ..., (c_m, a_m)\}$, respectively.

Definition 26. [8] *A possibilistic belief state of a poss-MCS $M = \{C_1, ..., C_n\}$ with $C_i = (\mathcal{A}_i, K_i, B_i)$ is a sequence $\tilde{S} = (S_1, ..., S_n)$ where $S_i = \{(a, \alpha) \mid a \in \mathcal{A}_i$ and $\alpha \in [0, 1]\}$, i.e., a set of possibilistic atoms.*

Definition 27. [8] *Let $\tilde{S} = (S_1, ..., S_n)$ be a possibilistic belief state and r a possibilistic bridge rule of the form*

$$a \leftarrow (c_1 : p_1), ..., (c_m : p_m), \ [\alpha]$$

where $1 \leq c_i \leq n$ for each $1 \leq i \leq m$. The possibilistic bridge rule r is $\beta-$applicable in $\tilde{S}$ with $\beta = \min\{\alpha_1, ..., \alpha_m, \alpha\}$ if $(p_i, \alpha_i) \in S_{c_i}$ for each i, and 0-applicable otherwise.

Theorem 6. [8] *Let $M = \{C_1, ..., C_n\}$ with $C_i = (\mathcal{A}_i, K_i, B_i)$ be a definite poss-MCS and $\tilde{S} = (S_1, ..., S_n)$ a possibilistic belief state. $\tilde{S}$ is the possibilistic grounded equilibrium of M iff for each $1 \leq i \leq n$, S_i is the possibilistic answer set of P_i, i.e., $S_i = lfp(T_{P_i})$ where $P_i = K_i \cup \{(h(r), \beta) \mid r \in B_i, r$ is $\beta-$applicable in $\tilde{S}$, and $\beta > 0\}$ and T_{P_i} is the immediate possibilistic consequence operator.*

A normal poss-MCS can be reduced to a definite poss-MCS. The reduct of a normal poss-MCS is based on the possibilistic reduct of a PASP program.

Definition 28. [8] *Let $M = \{C_1, ..., C_n\}$ with $C_i = (\mathcal{A}_i, K_i, B_i)$ be a normal poss-MCS and $\tilde{S}^* = (S^*_1, ..., S^*_n)$ a*

sequence of atoms sets $S_i^* \subseteq \mathcal{A}_i$. The reduct of M w.r.t. $\tilde{S}^*$ is a definite poss-MCS $M^{\tilde{S}^*} = \{C_1^{\tilde{S}^*}, ..., C_n^{\tilde{S}^*}\}$ where, $1 \leq i \leq n$,

- $C_i^{\tilde{S}^*} = (\mathcal{A}_i, K_i^{S_i^*}, B_i^{\tilde{S}^*})$,
- $K_i^{S_i^*}$ *is the possibilistic reduct of* K_i *w.r.t.* S_i^*, *and*
- $B_i^{\tilde{S}^*} = \{h(r) \leftarrow \mathcal{B}(r)^+, [\alpha] \mid r \in B_i$ *and for each* $(c_k, p_k) \in \mathcal{B}(r)^-, p_k \notin S_{c_k}^*\}$.

Definition 29. [8] *Let $\tilde{S} = (S_1, ..., S_n)$ be a possibilistic belief state of a normal poss-MCS M. $\tilde{S}$ is a possibilistic grounded equilibrium of M if $\tilde{S}$ is the possibilistic grounded equilibrium of $M^{\tilde{S}^*}$, where $\tilde{S}^* = (S_1^*, ..., S_n^*)$ and for $1 \leq i \leq n$, $S_i^* = \{a \mid (a, \alpha) \in S_i\}$.*

2) FMCSs Generalize poss-MCSs:
A poss-MCS $M = \{C_1, ..., C_n\}$ with $C_i = (\mathcal{A}_i, K_i, B_i)$ is equivalent to a poss-MCS $M' = \{C_1', ..., C_n'\}$ based on pairwise disjoint atoms sets $\mathcal{A}_i'$, where for each $1 \leq i \leq n$, $C_i' = (\mathcal{A}_i', K_i', B_i')$, $\mathcal{A}_i' = \{a^i \mid a \in \mathcal{A}_i\}$, K_i' is obtained from K_i by replacing each atom a in K_i with a^i, and B_i' is obtained from B_i by replacing each atom a in $\mathcal{A}_j$ with a^j for each $1 \leq j \leq n$. We thus assume that atoms sets $\mathcal{A}_i$ in a poss-MCS M are pairwise disjoint in the rest of paper.

In Example 2, PASP logic corresponds to the abstract logic $L_P = (\mathcal{KB}_P, \mathcal{U}_P, \mathcal{L}_P, \mathcal{FBS}_P, \mathcal{ACC}_P)$. Any possibilistic bridge rule can be translated into a possibilistic rule.

Remark 3. A possibilistic bridge rule r of the form (6) can be translated into the following possibilistic rule $\psi(r)$

$$(a \leftarrow p_1, ..., p_j, \textbf{\textit{not}} p_{j+1}, ..., \textbf{\textit{not}} p_m,\ \alpha). \tag{7}$$

Definition 30. *Let $M = \{C_1, ..., C_n\}$ with $C_i = (\mathcal{A}_i, K_i, B_i)$ be a poss-MCS. The corresponding FMCS is $\psi(M) = \{(L_P, kb, br)\}$ where $kb = \bigcup(K_i \cup \psi(B_i))$, $\psi(B_i) = \{\psi(r) \mid r \in B_i\}$, and $br = \emptyset$.*

The above kb is essentially a possibilistic logic program over the set $\bigcup \mathcal{A}_i$ of atoms. The following theorem suggests that there is a one-to-one correspondence between the possibilistic grounded equilibrium of a definite poss-MCS M and the fuzzy equilibrium of the corresponding FMCS $\psi(M)$.

Theorem 7. *A possibilistic belief state $\tilde{S} = (S_1, ..., S_n)$ is the possibilistic grounded equilibrium of a definite poss-MCS M iff $\mathrm{S} = (\bigcup S_i)$ is the fuzzy equilibrium of $\psi(M)$.*

Proof Sketch: Let $M = \{C_1, ..., C_n\}$ with $C_i = (\mathcal{A}_i, K_i, B_i)$ be a definite poss-MCS, $\tilde{S} = (S_1, ..., S_n)$ a possibilistic belief state. Then, the corresponding FMCS is $\psi(M) = \{(L_P, kb, \emptyset)\}$ where $kb = \bigcup(K_i \cup \psi(B_i))$ and $\psi(B_i) = \{\psi(r) \mid r \in B_i\}$.

For each $1 \leq i \leq n$, let $P_i = K_i \cup H_i$ where $H_i = \{(h(r), \beta) \mid r \in B_i, r$ is $\beta-$applicable in $\tilde{S}$, and $\beta > 0\}$.

Firstly, we can prove that for $1 \leq i \leq n$, any r in B_i is $\beta-$applicable in $\tilde{S}$ iff $\psi(r)$ in $\psi(B_i)$ is $\beta-$applicable in $\bigcup S_i$, by Definition 27 and the definition of $\beta-$applicable of possibilistic rule (see Example 2).

Secondly, we can show that $\bigcup T_{P_i}(S_i) = T_{kb}(\bigcup S_i)$, by the definition of immediate possibilistic consequence operator (see Example 2).

Finally, we can prove that $\tilde{S} = (S_1, ..., S_n)$ is the possibilistic grounded equilibrium of M iff $\mathrm{S} = (\bigcup S_i)$ is a fuzzy equilibrium of $\psi(M)$, by Theorem 6, the definition of $\mathcal{ACC}_P$ (see Example 2), and Definition 16. ■

The following theorem indicates that the possibilistic grounded equilibria of any normal poss-MCS M correspond one-to-one to the fuzzy equilibria of the corresponding FMCS $\psi(M)$.

Theorem 8. *A possibilistic belief state $\tilde{S} = (S_1, ..., S_n)$ is a possibilistic grounded equilibrium of a normal poss-MCS M iff $\mathrm{S} = (\bigcup S_i)$ is a fuzzy equilibrium of $\psi(M)$.*

Proof Sketch: Let $M = \{C_1, ..., C_n\}$ with $C_i = (\mathcal{A}_i, K_i, B_i)$ be a normal poss-MCS and $\tilde{S} = (S_1, ..., S_n)$ a possibilistic belief state. Then, the corresponding FMCS is $\psi(M) = \{(L_P, kb, \emptyset)\}$ where $kb = \bigcup(K_i \cup \psi(B_i))$ and $\psi(B_i) = \{\psi(r) \mid r \in B_i\}$.

As M is a normal poss-MCS, kb is a normal PASP program over the set $\bigcup \mathcal{A}_i$ of atoms and $\psi(M)$ is a reducible FMCS.

By Definitions 28, the reduct of M *w.r.t.* $\tilde{S}^* = (S_1^*, ..., S_n^*)$ with $S_i^* = \{a \mid (a, \alpha) \in S_i\}$ is the definite poss-MCS $M^{\tilde{S}^*} = \{C_1^{\tilde{S}^*}, ..., C_n^{\tilde{S}^*}\}$. Furthermore, according to Definition 30, the corresponding FMCS of $M^{\tilde{S}^*}$ is $\psi(M^{\tilde{S}^*}) = \{(L_P, kb', \emptyset)\}$.

By Definition 20, the $\mathrm{S}-$reduct of $\psi(M)$ is a definite FMCS $\psi(M)^{\mathrm{S}} = \{(L_P, kb^{\bigcup S_i^*}, \emptyset)\}$.

Firstly, due to the definition of possibilistic reduct (see Example 2), Definition 28, and Remark 3, we can show that $\psi(M^{\tilde{S}^*}) = \psi(M)^{\mathrm{S}}$. That is, $kb' = kb^{\bigcup S_i^*}$.

Then, according to Definition 29, Theorem 7, and Definition 16, we have that $\tilde{S}$ is a possibilistic grounded equilibrium of M iff $\mathrm{S} = (\bigcup S_i)$ is a fuzzy equilibrium of $\psi(M)^{\mathrm{S}}$. ■

As shown above, both the definite poss-MCS and the normal poss-MCS can be taken as a particular FMCS. According to Theorems 7 and 8, the following corollary holds.

Corollary 3. *A possibilistic belief state $\tilde{S} = (S_1, ..., S_n)$ is a possibilistic grounded equilibrium of a poss-MCS M iff $\mathrm{S} = (\bigcup S_i)$ is a fuzzy equilibrium of $\psi(M)$.*

The head and body of a possibilistic bridge rule of the form (6) consist of proposition atoms, such that fuzzy bridge rules of the form (4) can not be translated into possibilistic bridge rules. Additionally, poss-MCSs are built on PASP logic, whereas FMCSs are built on various logics captured by abstract logic framework.

Hence that FMCSs can not be translated into poss-MCSs.

C. Relationship to p-MCSs

In this subsection, we show that p-MCSs can be embedded into FMCSs, but not vice versa.

1) p-MCSs:
A p-MCS $M = \{C_1, ..., C_n\}$ [7] consists of probabilistic contexts C_i based on propositional probabilistic logics [19]. The reasoning methods in all probabilistic contexts are the same.

We first introduce some notations in an indexed probabilistic context C_i. $F(\mathcal{A}_i)$ is the set of all propositional formulas over a finite atomic propositions set $\mathcal{A}_i$ and $(i, F)\mu_F$ is named as a p-labeled formula where $F \in F(\mathcal{A}_i)$ and $\mu_F \in [0, 1]$. $(i, F)\mu_F$ stands for the probability of F in C_i is μ_F. It is substantially a probability formula (F, μ_F) in C_i.

Definition 31. [7] *A p-MCS $M = \{C_1, ..., C_n\}$ is a tuple of probabilistic contexts $C_i = (R_i, B_i)$. For each $1 \leq i \leq n$, C_i consists of the following components:*

(1) *R_i is a set of p-labeled formulas, building on a finite atomic propositions set $\mathcal{A}_i$, and*
(2) *B_i is a collection of probabilistic bridge rules of the form*

$$(i, F)\mu_F \leftarrow (c_1 : F_1)\mu_{F_1}, ..., (c_m : F_m)\mu_{F_m} \quad (8)$$

where for each $1 \leq k \leq m$, $1 \leq c_k \leq n$, $(i, F)\mu_F$ and $(c_k, F_k)\mu_{F_k}$ are p-labeled formulas, and existing c_k such that $i \neq c_k$.

Definition 32. [7] *Given a p-MCS $M = \{C_1, ..., C_n\}$ where for each $1 \leq i \leq n$, $C_i = (R_i, B_i)$ and R_i is built on a finite set $\mathcal{A}_i$ of atomic propositions. $w \subseteq \mathcal{A}_i$ is a possible word for C_i. $W_i = 2^{\mathcal{A}_i}$ denotes the set of possible words for C_i.*

(1) *A contextual world probability density function for a probabilistic context C_i is a mapping $WP_i : W_i \longrightarrow [0, 1]$ such that $\Sigma_{w \in W_i} WP_i(w) = 1$.*
(2) *A contextual probabilistic interpretation for C_i is a function $\mathcal{I}_{WP_i} : F(\mathcal{A}_i) \longrightarrow [0, 1]$ defined as $\mathcal{I}_{WP_i}(F) = \Sigma_{w \models F} WP_i(w)$ for each $F \in F(\mathcal{A}_i)$.*

Definition 33. [7] *Let $M = \{C_1, ..., C_n\}$ be a p-MCS and for each $1 \leq i \leq n$, WP_i a contextual world probability density function on W_i for C_i. A p-labeled chain of M is a sequence $\hat{S} = (S_1, ..., S_n)$ where $S_i = \{(w, WP_i(w)) \mid w \in W_i\}$.*

Note that a p-labeled chain $\hat{S}$ is essentially a sequence consisting of a world probability density function WP_i for each probabilistic context C_i.

Definition 34. [7] *Let $M = \{C_1, ..., C_n\}$ with $C_i = (R_i, B_i)$ be a p-MCS and $\hat{S} = (S_1, ..., S_n)$ with $S_i = \{(w, WP_i(w)) \mid w \in W_i\}$ a p-labeled chain. $\hat{S}$ satisfies M if for each $C_i = (R_i, B_i)$,*

(1) *whenever $(i, \phi)\mu_\phi \in R_i$, $\Sigma_{w \models \phi} WP_i(w) = \mu_\phi$, and*
(2) *whenever $r \in B_i$, $\Sigma_{w \models F} WP_i(w) = \mu_F$ and $\Sigma_{w \models F_j} WP_{c_j}(w) = \mu_{F_j}$ for $1 \leq j \leq m$.*

Definition 35. [7] *A p-labeled chain $\hat{S} = (S_1, ..., S_n)$ of a p-MCS M is said to be a p-labeled solution chain if $\hat{S}$ satisfies M.*

2) FMCSs Generalize p-MCSs:
From Example 2, we know that propositional probabilistic logic can be translated into the abstract logic $L'_P = (\mathcal{KB}'_P, \mathcal{U}'_P, \mathcal{L}'_P, \mathcal{FBS}'_P, \mathcal{ACC}'_P)$. Any probabilistic bridge rule can be written as a fuzzy bridge rule.

Remark 4. A probabilistic bridge rule r of the form (8) can be translated into the following fuzzy bridge rule $\pi(r)$

$$(F, \mu_F) \leftarrow \langle c_1 : (F_1, \mu_{F_1})\rangle, ..., \langle c_m : (F_m, \mu_{F_m})\rangle. \quad (9)$$

Definition 36. *Given a p-MCS $M = \{C_1, ..., C_n\}$ where for $1 \leq i \leq n$, $C_i = (R_i, B_i)$ and R_i is built on a finite set $\mathcal{A}_i$ of atomic propositions. The corresponding FMCS $\pi(M) = \{\pi(C_1), ..., \pi(C_n)\}$ comprises the following components, for each $1 \leq i \leq n$,*

(1) *$\pi(C_i) = (L'_{P_i}, kb_i, br_i)$,*
(2) *L'_{P_i} is the abstract logic corresponding propositional probabilistic logic over $\mathcal{A}_i$, and*
(3) *$kb_i = \{(F, \mu_F) \mid (i, F)\mu_F \in R_i\}$ and $br_i = \{\pi(r) \mid r \in B_i\}$.*

The following theorem indicates that the p-labeled solution chains of any p-MCS M correspond to the fuzzy equilibria of the corresponding FMCS $\pi(M)$.

Theorem 9. *Given a p-MCS $M = \{C_1, ..., C_n\}$ where for $1 \leq i \leq n$, $C_i = (R_i, B_i)$ and R_i is built on a finite set $\mathcal{A}_i$ of atomic propositions. If a p-labeled chain $\hat{S} = (S_1, ..., S_n)$ with $S_i = \{(w, WP_i(w)) \mid w \in W_i\}$ is a p-labeled solution chain of M then $\mathrm{S} = (\mathcal{S}_1, ..., \mathcal{S}_n)$ with $\mathcal{S}_i = \{(\varphi, \mu_\varphi) \mid \varphi \in F(\mathcal{A}_i)$ and $\mu_\varphi = \Sigma_{w \models \varphi} WP_i(w)\}$ is a fuzzy equilibrium of $\pi(M)$.*

Proof Sketch: Let $\hat{S} = (S_1, ..., S_n)$ with $S_i = \{(w, WP_i(w)) | w \in W_i\}$ be a p-labeled solution chain of M.

By Definition 36, we have that
$\pi(M) = \{\pi(C_1), ..., \pi(C_n)\}$ with $\pi(C_i) = (L'_{P_i}, kb_i, br_i)$.

By Definitions 15, 34 and 35, we can obtain that for each probabilistic bridge rule r, the corresponding fuzzy bridge rule $\pi(r)$ in $\pi(M)$ is applicable in $\mathrm{S} = (\mathcal{S}_1, ..., \mathcal{S}_n)$.

Therefore, for each kb_i in $\pi(M)$,

$$kb_i \cup H(br_i, \mathrm{S}) = kb_i \cup \{h(\pi(r)) \mid \pi(r) \in br_i\}.$$

By Definition 32 and the definition of probability distribution (see Example 2), we can obtain that for each $\pi(C_i)$ in $\pi(M)$, WP_i satisfies the following equations with constraints

(1) $\Sigma_{w \models \phi} WP_i(w) = \mu_\phi$ for each $(\phi, \mu_\phi) \in kb_i$,
(2) $\Sigma_{w \models F} WP_i(w) = \mu_F$ for each $(F, \mu_F) \in \{h(\pi(r)) \mid \pi(r) \in br_i\}$,
(3) $\Sigma_{w \in W_i} WP_i(w) = 1$ and $WP_i(w) \in [0, 1]$.

Hence, by the definition of $\mathcal{ACC}'_{P_i}$ (see Example 2), for each i, $\mathcal{S}_i \in \mathcal{ACC}'_{P_i}(kb_i \cup H(br_i, \mathrm{S}))$, which implies that $\mathrm{S} = (\mathcal{S}_1, ..., \mathcal{S}_n)$ is a fuzzy equilibrium of $\pi(M)$. ■

Probabilistic bridge rules of the form (8) contain no default negation, such that fuzzy bridge rules of the form (4) can not be translated into probabilistic bridge rules. Besides, p-MCSs are based on propositional probabilistic logic, whereas FMCSs are based on various logics covered by abstract logic framework.

As a result, FMCSs can not be embedded into p-MCSs.

VI. Conclusion Remarks and Future Works

In the present work, a generic framework of multi-context systems for dealing with multiple knowledge under imprecise environments was proposed. Firstly, we investigated the formal notions of abstract logics, monotonic abstract logics, and reducible abstract logics in our setting. A monotonic abstract logic is a special reducible abstract logic. Secondly, we formalized the fuzzy multi-context system based on abstract logics and established its syntactic and semantic framework. In more specific details, the (grounded) fuzzy equilibrium and the well-founded fuzzy equilibrium were introduced to capture the semantics of fuzzy multi-context systems. Furthermore, the fuzzy multi-context system along with its relevant theories are illustrated with different examples, especially using a simple medical diagnosis case scenario. Finally, we showed the following facts: (i) FMCSs are a generalization of MCSs;

(ii) p-MCSs can be embedded into FMCSs; and (iii) poss-MCSs can be translated into FMCSs.

FMCSs hardly suffer from the limitation of bi-valued logic, many-valued logic, monotonic logic, and non-monotonic logic. It is worth pointing out that we can choose distinct logics to handle the representation and reasoning of different types of knowledge. In other word, FMCSs contribute to the integration of different types of knowledge.

Although FMCSs have been presented here, it is just the beginning. Several interesting issues are considered as future works. On the one hand, to handle the inconsistency in FMCSs arising from information exchange, we intend to adapt the idea of inconsistency management of MCSs given in [15] to FMCSs. On the other hand, we will study algorithms and complexity related to fuzzy multi-context systems. The complexity related to FMCSs is challenging in general. As discussed in [2], however, we can similarly show that deciding if an FMCS whose abstract logics have poly-size kernels has a fuzzy equilibrium is one level above its underlying inference of logics. Formally, an abstract logic L has poly-size kernels if there exists a function g which assigns a set $g(kb, \mathrm{S}) \subseteq \mathrm{S}$ of size polynomial in the size of kb—the kernel of S—to each $kb \in \mathcal{KB}$ and $\mathrm{S} \in \mathcal{ACC}(kb)$, such that there exists a bijection f between the fuzzy belief sets in $\mathcal{ACC}(kb)$ and their kernels, that is, $\mathrm{S} \rightleftharpoons f(g(kb, \mathrm{S}))$ [2]. Furthermore, an abstract logic L has kernel reasoning in Δ_i^p if given an arbitrary knowledge base kb, an element s, and a set G of elements, deciding whether (1) $G = g(kb, \mathrm{S})$ for some $\mathrm{S} \in \mathcal{ACC}(kb)$ and (2) $s \in \mathrm{S}$ is in Δ_i^p [2]. Intuitively, given an FMCS M where each abstract logic L_i has poly-size kernels and kernel reasoning in Δ_i^p and each knowledge base kb_i and set br_i of fuzzy bridge rules are finite, deciding whether M has a fuzzy equilibrium is in Σ_{i+1}^p, which is the same as the conclusion in MCSs [2]. Finally, we will consider real-world applications where the fuzzy multi-context systems framework could be applied.

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