



THE UNIVERSITY *of* EDINBURGH

Edinburgh Research Explorer

Exponentially-improved asymptotics and numerics for the (un)perturbed first Painlevé equation

Citation for published version:

Olde Daalhuis, AB 2022, 'Exponentially-improved asymptotics and numerics for the (un)perturbed first Painlevé equation', *Journal of Physics A: Mathematical and Theoretical*, vol. 55, no. 30, 304004.
<https://doi.org/10.1088/1751-8121/ac7bbb>

Digital Object Identifier (DOI):

[10.1088/1751-8121/ac7bbb](https://doi.org/10.1088/1751-8121/ac7bbb)

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Peer reviewed version

Published In:

Journal of Physics A: Mathematical and Theoretical

General rights

Copyright for the publications made accessible via the Edinburgh Research Explorer is retained by the author(s) and / or other copyright owners and it is a condition of accessing these publications that users recognise and abide by the legal requirements associated with these rights.

Take down policy

The University of Edinburgh has made every reasonable effort to ensure that Edinburgh Research Explorer content complies with UK legislation. If you believe that the public display of this file breaches copyright please contact openaccess@ed.ac.uk providing details, and we will remove access to the work immediately and investigate your claim.



EXPONENTIALLY-IMPROVED ASYMPTOTICS AND NUMERICS FOR THE (UN)PERTURBED FIRST PAINLEVÉ EQUATION

ADRI B. OLDE DAALHUIS

Dedicated to Sir Michael V. Berry on the occasion of his 80th birthday.

ABSTRACT. The solutions of the perturbed first Painlevé equation $y'' = 6y^2 - x^\mu$, $\mu > -4$, are uniquely determined by the free constant C multiplying the exponentially small terms in the complete large x asymptotic expansions. Full details are given, including the nonlinear Stokes phenomenon, and the computation of the relevant Stokes multipliers. We derive asymptotic approximations, depending on C , for the locations of the singularities that appear on the boundary of the sectors of validity of these exponentially-improved asymptotic expansions. Several numerical examples illustrate the power of the approximations. For the tri-tronquée solution of the unperturbed first Painlevé equation we give highly accurate numerics for the values at the origin and the locations of the zeros and poles.

1. INTRODUCTION

The perturbed first Painlevé equation

$$(1.1) \quad y'' = 6y^2 - x^\mu, \quad \mu \in \mathbb{R}, \quad \mu > -4,$$

was discussed in [5]. In the case $\mu = 0$ the solutions are just Weierstrass \wp functions, the case $\mu = 1$ is the unperturbed first Painlevé equation, and in the case $\mu = 2$ two solutions are $y(x) = \pm x/\sqrt{6}$, and no asymptotics is needed, although the transseries solutions below are still valid.

Before we start discussing the asymptotics of the solutions of (1.1) we first briefly discuss the possible singularities in the complex x plane. Obviously there could be a complicated branch-point at $x = 0$. The other singularities seem to be double poles, but a local analysis shows that the local behaviour near such a point is of the form

$$(1.2) \quad \begin{aligned} y(x) = & \frac{1}{(x - x_j)^2} + \frac{1}{10}x_j^\mu (x - x_j)^2 + \frac{\mu}{6}x_j^{\mu-1} (x - x_j)^3 \\ & + \left(h_j - \frac{\mu(\mu-1)}{14}x_j^{\mu-2} \ln(x - x_j) \right) (x - x_j)^4 \\ & - \frac{\mu(\mu-1)(\mu-2)}{48}x_j^{\mu-3} (x - x_j)^5 + \dots, \end{aligned}$$

in which the only free constants are the location x_j and the coefficient h_j . Note that the coefficient of $(x - x_j)^4$ does contain a logarithm. This logarithm is absent in the known cases $\mu = 0$ (Weierstrass \wp), $\mu = 1$ (first Painlevé equation). In all other case these ‘poles’ are actually branch-points. Similar observations have been made before. For example in [13] and [7] it is shown that for all of the solutions of $y''(x) = 6y(x)^2 - f(x)$ to be single-valued about all movable

2020 *Mathematics Subject Classification.* 33E17, 34E05, 34M35.

Key words and phrases. Painlevé equation, asymptotic expansions, Stokes phenomenon, hyperasymptotics, singularities.

singularities we need $f''(x) = 0$. In expansion (1.2) the next logarithm will appear in front of $(x - x_j)^8$, and even higher powers of $\ln(x - x_j)$ will appear in the tail of this expansion.

The dominant behaviours of the solutions that we will consider is $y_{\pm}(x) \sim \pm \sqrt{\frac{x^{\mu}}{6}}$. In the next section we consider both behaviours, but in the remainder we will focus on $y_{-}(x)$. Note that we have

$$(1.3) \quad y_{+}(x) = e^{\frac{4\pi i}{\mu+4}} y_{-}\left(xe^{\frac{2\pi i}{\mu+4}}\right).$$

Hence, our results for $y_{-}(x)$ can be translated to $y_{+}(x)$ via a rotation and a multiplication.

Rigorous results for the case $\mu = 1$ are given in [6]. Some of their techniques can also be applied in the case $\mu \neq 1$. Proposition 2 of that paper gives us that for any sector of angle less than $\frac{4\pi}{\mu+4}$, there exist a solution $y(x)$ of (1.1) such that $y(x) \sim -\sqrt{\frac{x^{\mu}}{6}}$ as $|x| \rightarrow \infty$ in this sector. More importantly, their Theorem 3 shows us that there exist a unique solution $y_{-}(x)$ of (1.1) such that $y_{-}(x) \sim -\sqrt{\frac{x^{\mu}}{6}}$ as $|x| \rightarrow \infty$ in the sector $|\arg x| < \frac{4\pi}{\mu+4}$. For this solution the constant beyond all orders $C = 0$. It can also be labelled as being the Borel-Laplace transform of its asymptotic expansion.

Once we have chosen the first term in the asymptotic expansion, the remaining terms are fixed, and the free constant, say C , multiplies exponentially-small terms. In §2 we discuss the formal series solutions including all the exponentially small terms in so-called transseries. We will determine the sector of validity of these transseries. The free constant C will determine the location of the singularities that will appear on the boundary of the sector of validity. At the end of §2 we will give asymptotic approximations for the locations of these singularities. In the numerical sections of this paper it will be demonstrated that these approximations are very good even for the singularities that are closest to the origin.

In §3 we discuss the special solution $y_{-}(x)$, including the computation of its Stokes multipliers, the level 1 hyperasymptotic approximation, which will include the Stokes-smoothing of the Stokes phenomenon, and the location of its singularities. In the numerical illustrations it seems to be the case that this special solution has no singularities on the positive real x -axis. This is known to be the case when $\mu = 0$ and $\mu = 1$. As far as we can see the techniques of [6] can not be used for $\mu \neq 1$. The Borel transform of this formal series is discussed in §7, and it follows that this Borel-Laplace transform $y_{-}(x)$ is well defined for $x > \tilde{\sigma}(\mu)$. For $\tilde{\sigma}(\mu)$ see Figure 3.

The numerical tools are introduced in §4. They are analytical continuation via the Taylor-series method for analytic differential equations, and contour integral representations for the locations of the singularities. These integrals are evaluated via the trapezoidal rule. Note that once we have a reasonable guess for the location of a singularity, say x_j , and we can evaluate $y(x)$ near that point, then (1.2) can also be used to obtain a much better approximation for x_j . These methods are used in §5 for the cases $\mu = \frac{15}{7}$ and $\mu = 4$. Finally, in §6 we illustrate the power of these simple methods by obtaining approximations to a precision of 60 significant digits for the unperturbed first Painlevé equation, $\mu = 1$, and in this way check some of the results in the recent literature.

The change of variable $z = \lambda x^{\frac{\mu}{4}+1}$ and $y(x) = \sqrt{\frac{x^{\mu}}{6}} u(z)$, with $\lambda = \frac{8 \cdot 6^{-\frac{1}{4}}}{\mu+4}$, will give us the differential equation

$$(1.4) \quad u''(z) + 2\nu \frac{u'(z)}{z} + \frac{4}{5}\nu \left(\frac{6}{5}\nu - 1\right) \frac{u(z)}{z^2} = \frac{3}{2} (u^2(z) - 1),$$

in which we use the notation $\nu = \frac{5\mu}{2(\mu+4)}$. From an asymptotics point of view this differential equation is slightly simpler than (1.1). Note that because we take $\mu > -4$ we will have $\nu < \frac{5}{2}$.

2. FORMAL SERIES SOLUTIONS

Differential equation (1.4) has formal solutions of the form

$$(2.1) \quad u_0(z) \sim \sum_{n=0}^{\infty} \frac{a_{n,0}}{z^n},$$

with

$$(2.2) \quad a_{0,0} = \pm 1, \quad a_{2n+1,0} = 0, \quad a_{2,0} = \frac{4}{15}\nu \left(\frac{6}{5}\nu - 1\right), \quad a_{4,0} = 2 \left(\frac{2}{15}\nu - 1\right) \left(\frac{3}{5}\nu - 1\right) a_{0,0} a_{2,0},$$

and the recurrence relation

$$(2.3) \quad 3a_{0,0}a_{n,0} = (n-2)(n-1-2\nu)a_{n-2,0} - \frac{3}{2} \sum_{m=3}^{n-3} a_{m,0}a_{n-m,0}, \quad n = 5, 6, 7, \dots$$

This recurrence relation is consistent with $a_{2n+1,0} = 0$. We do give $a_{4,0}$ explicitly. It does not follow from the recurrence relation (2.3).

Our formal solution (2.1) has no free constants. For the free constants we have to start considering exponentially small perturbations for our solution, that is, solutions of the form $u(z) = u_0(z) + Cu_1(z)$, in which $u_1(z)$ is exponentially small compared to $u_0(z)$. However, our differential equations (1.1) and (1.4) are nonlinear and once we start considering exponentially small terms we will immediately obtain terms that are double-, triple-, ... exponentially small. Hence, we will end up with a transseries

$$(2.4) \quad u(z) = \sum_{k=0}^{\infty} C^k u_k(z),$$

in which the u_k are solutions of the linear differential equations

$$(2.5) \quad u_k''(z) + 2\nu \frac{u_k'(z)}{z} + 3 \left(\frac{a_{2,0}}{z^2} - u_0(z) \right) u_k(z) = \frac{3}{2} \sum_{\ell=1}^{k-1} u_\ell(z) u_{k-\ell}(z), \quad k \geq 1,$$

with formal solutions

$$(2.6) \quad u_k(z) \sim e^{-k\sqrt{3a_{0,0}}z} \sum_{n=0}^{\infty} \frac{a_{n,k}}{z^{n+k\nu}}.$$

In the case $k = 1$ differential equation (2.5) is linear and homogeneous, and hence, there are no restrictions on $a_{0,1}$. We put that freedom in C and fix $a_{0,1} = 1$. For the other coefficients we have

$$(2.7) \quad 2\sqrt{3a_{0,0}}na_{n,1} = (n-1+\nu)(\nu-n)a_{n-1,1} + 3 \sum_{m=4}^{n+1} a_{m,0}a_{n-m+1,1}, \quad n = 1, 2, 3, \dots$$

The remaining coefficients are determined by

$$(2.8) \quad (k^2 - 1)a_{0,0}a_{0,k} = \frac{1}{2} \sum_{\ell=1}^{k-1} a_{0,\ell}a_{0,k-\ell}, \quad \implies \quad a_{0,k} = \frac{k}{(12a_{0,0})^{k-1}}, \quad k = 1, 2, 3, \dots,$$

and

$$\begin{aligned}
& 3(k^2 - 1)a_{0,0}a_{n,k} + 2k\sqrt{3a_{0,0}}(n - 1 + (k - 1)\nu)a_{n-1,k} \\
& + (n - 2 + k\nu)(n - 1 + (k - 2)\nu)a_{n-2,k} - 3\sum_{m=4}^n a_{m,0}a_{n-m,k} \\
(2.9) \quad & = \frac{3}{2}\sum_{\ell=1}^{k-1}\sum_{m=0}^n a_{m,\ell}a_{n-m,k-\ell}.
\end{aligned}$$

Transseries expansion (2.4) lives in the half-plane $\Re(\sqrt{3a_{0,0}}z) > 0$. In this half-plane the terms decay exponentially, compare (2.6). Below we will resum the transseries, and this is especially interesting on the boundary of this sector. In the opposite half-plane $\Re(\sqrt{3a_{0,0}}z) < 0$ we have the transseries

$$(2.10) \quad u(z) = \sum_{k=0}^{\infty} C^k u_{-k}(z),$$

in which

$$(2.11) \quad u_{-k}(z) \sim e^{k\sqrt{3a_{0,0}}z} \sum_{n=0}^{\infty} \frac{(-1)^n a_{n,k}}{z^{n+k\nu}}.$$

When we combine (2.4) with (2.6) we obtain a double sum which can be resummed as

$$(2.12) \quad u(z) \sim \sum_{n=0}^{\infty} \frac{G_n(X(z))}{z^n},$$

in which $X(z) = Ce^{-\sqrt{3a_{0,0}}z}z^{-\nu}$ and

$$(2.13) \quad G_n(X) = \sum_{k=0}^{\infty} a_{n,k}X^k.$$

In the case of $n = 0$ we can use (2.8) to determine $G_0(X)$. However, we can also substitute (2.12) into (1.4) and use $X'(z) = -(\sqrt{3a_{0,0}} + \frac{\nu}{z})X(z)$. This will give us a power series expansion. The coefficient of z^0 can be evaluated as

$$(2.14) \quad 3a_{0,0}(X^2G_0''(X) + XG_0'(X)) = \frac{3}{2}(G_0^2(X) - 1),$$

and the coefficient of z^{-1} can be evaluated as

$$(2.15) \quad 3a_{0,0}(X^2G_1''(X) + XG_1'(X)) + 2\nu\sqrt{3a_{0,0}}X^2G_0''(X) = 3G_0(X)G_1(X).$$

Recall that $a_{0,0}^2 = 1$. We combine (2.14) with the initial data $G_0(0) = a_{0,0}$, $G_0'(0) = a_{0,1} = 1$ and obtain

$$(2.16) \quad G_0(X) = a_{0,0} + \frac{144X}{(X - 12a_{0,0})^2}.$$

Similarly, we combine (2.15) with the initial data $G_1(0) = 0$, $G_1'(0) = a_{1,1} = \nu(\nu - 1)/(2\sqrt{3a_{0,0}})$ and obtain

$$(2.17) \quad G_1(X) = \frac{\nu\sqrt{3a_{0,0}}X(288(1 - \nu) - 8(3\nu - 19)a_{0,0}X + 2X^2 - \frac{a_{0,0}}{90}X^3)}{(X - 12a_{0,0})^3}.$$

We already know from (1.2) that our solution can have double pole type singularities. For a fixed C we can use (2.16) to obtain a first approximation. The double poles should satisfy the approximation $X(z) \approx 12a_{0,0}$.

To obtain an extra term $X(z) \approx 12a_{0,0} + \frac{\alpha}{z}$ in this approximation we will look for double poles of $G_0(X) + z^{-1}G_1(X)$, that is, we determine constant α such that $G_0(X) + z^{-1}G_1(X)$ does not have a triple pole at level z^{-1} . We expand

$$(2.18) \quad \begin{aligned} G_0(X) &= a_{0,0} + \frac{144X}{\left(X - 12a_{0,0} - \frac{\alpha}{z} + \frac{\alpha}{z}\right)^2} \\ &= a_{0,0} + \frac{144X}{\left(X - 12a_{0,0} - \frac{\alpha}{z}\right)^2} - \frac{288X\alpha z^{-1}}{\left(X - 12a_{0,0} - \frac{\alpha}{z}\right)^3} + \mathcal{O}(z^{-2}), \end{aligned}$$

and determine α such that the triple pole in (2.18) cancels the triple pole in $z^{-1}G_1(X)$. The solution will be a function of z^{-1} , but we are only interested in the constant part: $\alpha = -\sqrt{3a_{0,0}}\nu \left(2\nu - \frac{124}{15}\right)$. Hence, we expect double pole type singularities near solutions of

$$(2.19) \quad Ce^{-\sqrt{3a_{0,0}}z}z^{-\nu} = 12a_{0,0} - \sqrt{3a_{0,0}}\nu \left(2\nu - \frac{124}{15}\right) z^{-1}.$$

The analysis above is similar to the one in [1, §6.6a].

3. THE CASE $y \sim -x^{\mu/2}/\sqrt{6}$

With the notation of the previous section we have $a_{0,0} = -1$, and we take $\sqrt{3a_{0,0}} = i\sqrt{3}$. Our starting point will be the positive real axis and we consider the Borel-Laplace transform of the formal series

$$(3.1) \quad y_-(x) \sim \sqrt{\frac{x^\mu}{6}} \sum_{n=0}^{\infty} \frac{a_{n,0}\lambda^n}{x^{(\mu+4)n/4}},$$

that is, the free constant $C = 0$ when $x \rightarrow \infty$ along the positive real axis. According to (2.6) the ‘exponentially-small’ terms are oscillatory on the positive real axis, that is, the positive real axis is an anti-Stokes line. In the z -plane the imaginary axes will be active Stokes lines and the negative real axis will be the boundary for the sector of validity of asymptotic expansion (2.1). Hence, the sector of validity is $|\arg z| < \pi$, that is, $|\arg x| < \frac{4\pi}{\mu+4}$.

The nonlinear Stokes phenomenon is the switching on of the exponentially small terms when the imaginary axes are crossed, that is, in the transseries (2.4) the constant C switches from $C = 0$ to $C = K_\pm$ when we cross the positive/negative imaginary axis, respectively. The constants K_\pm are the Stokes multipliers. The details are very similar to the special case $\mu = 1$ which is discussed in [9]. Hence, the transseries expansions for this function are

$$(3.2) \quad y_-(x) \sim \begin{cases} \sqrt{\frac{x^\mu}{6}} \sum_{k=0}^{\infty} K_-^k u_k(z), & \frac{-4\pi}{\mu+4} < \arg x < \frac{-2\pi}{\mu+4}, \\ \sqrt{\frac{x^\mu}{6}} u_0(z), & \frac{-2\pi}{\mu+4} < \arg x < \frac{2\pi}{\mu+4}, \\ \sqrt{\frac{x^\mu}{6}} \sum_{k=0}^{\infty} K_+^k u_{-k}(z), & \frac{2\pi}{\mu+4} < \arg x < \frac{4\pi}{\mu+4}. \end{cases}$$

Near the boundaries of the sector of validity $|\arg x| = \frac{4\pi}{\mu+4}$ the $u_k(z)$ are not exponential small anymore and it makes sense to resum the transseries. Taking only the first term (2.6) and using (2.8) we obtain for x near the boundary $\arg x = \frac{4\pi}{\mu+4}$ that

$$(3.3) \quad y_-(x) \sim \sqrt{\frac{x^\mu}{6}} \sum_{k=0}^{\infty} \frac{a_{0,k} K_+^k e^{ik\sqrt{3}z}}{z^{k\nu}} = \sqrt{\frac{x^\mu}{6}} \left(-1 + \frac{K_+ e^{i\sqrt{3}z} z^{-\nu}}{\left(1 + \frac{1}{12} K_+ e^{i\sqrt{3}z} z^{-\nu}\right)^2} \right)$$

Hence, the transseries contain information about singularities near the boundary of the sector of validity. In this case we can see that we expect a double poles near the solutions of $12 + K_+ e^{i\sqrt{3}z} z^{-\nu} = 0$. We know already from (1.2) that in the case $\mu \neq 0, 1$ these singularities are actually log singularities, but the dominant term is a double pole. We will verify all of this in the numerical sections below.

To determine the Stokes multipliers K_{\pm} we can use the asymptotic formula

$$(3.4) \quad a_{n,0} \sim \frac{K_+}{2\pi i} \sum_{m=0}^{\infty} (-1)^m a_{m,1} \frac{\Gamma(n-m-\nu)}{(-i\sqrt{3})^{n-m-\nu}} - \frac{K_-}{2\pi i} \sum_{m=0}^{\infty} a_{m,1} \frac{\Gamma(n-m-\nu)}{(i\sqrt{3})^{n-m-\nu}},$$

as $n \rightarrow \infty$. Compare [9, (4.3)]. Since $a_{2n+1,0} = 0$ it follows that

$$(3.5) \quad K_+ = \overline{K_-},$$

and hence,

$$(3.6) \quad a_{2n,0} \sim \frac{-K_-}{\pi i} \sum_{m=0}^{\infty} a_{m,1} \frac{\Gamma(2n-m-\nu)}{(i\sqrt{3})^{2n-m-\nu}},$$

as $n \rightarrow \infty$. In this final result the optimal number of terms is n , and this formula can be used to compute the Stokes multipliers numerically to any precision.

The details for the first hyperasymptotic re-expansion are very similar to the case $\mu = 1$ discussed in [9]. The optimal number of terms of expansions (2.1) and (3.1) is N such that $N - \sqrt{3}|z| = \mathcal{O}(1)$ as $z \rightarrow \infty$. With this N we have

$$(3.7) \quad \sqrt{\frac{6}{x^\mu}} y_-(x) = \sum_{n=0}^{N-1} \frac{a_{n,0}}{z^n} + \mathcal{O}\left(e^{-\sqrt{3}|z|} |z|^{1/2}\right),$$

as $z \rightarrow \infty$ in the sector $|\arg z| < \frac{1}{2}\pi$. The level 1 re-expansion will be

$$(3.8) \quad \begin{aligned} \sqrt{\frac{6}{x^\mu}} y_-(x) &= \sum_{n=0}^{2N-1} \frac{a_{n,0}}{z^n} + z^{1-2N} \frac{K_+}{2\pi i} \sum_{n=0}^{N-1} (-1)^n a_{n,1} F^{(1)}\left(z; \begin{matrix} 2N-n-\nu \\ i\sqrt{3} \end{matrix}\right) \\ &- z^{1-2N} \frac{K_-}{2\pi i} \sum_{n=0}^{N-1} a_{n,1} F^{(1)}\left(z; \begin{matrix} 2N-n-\nu \\ -i\sqrt{3} \end{matrix}\right) + \mathcal{O}\left(e^{-2\sqrt{3}|z|} |z|^1\right), \end{aligned}$$

as $z \rightarrow \infty$ again in the sector $|\arg z| < \frac{1}{2}\pi$. Compare [9, (5.3)]. The first hyperterminant function can be expressed in terms of the incomplete gamma function $F^{(1)}\left(z; \begin{matrix} N+1 \\ \sigma \end{matrix}\right) = -e^{\sigma z} (-z)^N \Gamma(N+1)\Gamma(-N, \sigma z)$. It is the simplest function with a Stokes phenomenon. For more details see [8].

The level 1 expansion (3.8) can be used to determine solution $y_-(x)$ uniquely. The order estimate in (3.8) is double exponentially small. Hence, the term $Cu_1(z)$ is clearly not present in the transseries expansion for $y_-(x)$, that is, $C = 0$.

The first array of poles in the lower half-plane are located near solutions of

$$(3.9) \quad K_- \frac{e^{-i\sqrt{3}\lambda x^{(\mu+4)/4}}}{\lambda^\nu x^{\nu(\mu+4)/4}} = -12 - \frac{i\sqrt{3}\nu \left(2\nu - \frac{124}{15}\right)}{\lambda x^{(\mu+4)/4}},$$

and the first array of poles in the upper half-plane are located near solutions of

$$(3.10) \quad K_+ \frac{e^{i\sqrt{3}\lambda x^{(\mu+4)/4}}}{\lambda^\nu x^{\nu(\mu+4)/4}} = -12 + \frac{i\sqrt{3}\nu \left(2\nu - \frac{124}{15}\right)}{\lambda x^{(\mu+4)/4}}.$$

Note that the sign in front of the second term on the right-hand sides of (3.9) and (3.10) are different. This is a consequence of the $(-1)^n$, with $n = 1$, in (2.11).

4. THE NUMERICS.

To obtain very accurate numerical approximations we start with a large x_0 on an anti-Stokes line and use an optimally truncated asymptotic expansion. In the case of $y \sim -x^{\mu/2}/\sqrt{6}$ we will start on the positive real x -axis and use (3.1) to determine $y_-(x)$ and its derivative, and in the case $y \sim +x^{\mu/2}/\sqrt{6}$ we will start with a x such that $\arg x = \frac{2\pi}{\mu+4}$. Once we have determined the function and its first derivative we can combine the original differential equation (1.1) with the Taylor-series method (see [4, §3.7(ii)]) and ‘walk’ in the direction of the origin along the anti-Stokes line. Thus initially we compute the first 2 Taylor coefficients in

$$(4.1) \quad y(x) = \sum_{m=0}^{\infty} b_m (x - x_0)^m,$$

via an optimally truncated asymptotic expansion, and the higher coefficients via

$$(4.2) \quad (m+2)(m+1)b_{m+2} = 6 \sum_{\ell=0}^m b_\ell b_{m-\ell} - (-1)^m x_0^{\mu-m} \frac{(-\mu)_m}{m!}, \quad m = 0, 1, 2, \dots$$

We do control the step-size *step* and the number of Taylor coefficients that we use in (4.1). At each step we take $x_1 = x_0 + \text{step}$ and compute $y(x_1)$ and $y'(x_1)$ via (4.1) and take $x_0 = x_1$ in (4.2) to compute the higher coefficients.

To study the numerical stability of this process we can linearise (1.1) near x_0 and in that way we observe that the worst that can happen is that the numerics is polluted with a little bit of a solution of (2.5) ($k = 1$). However, the solutions of that equation will be oscillatory along the anti-Stokes line. Hence, the numerical integration should be stable.

However, we are dealing with a nonlinear differential equation and do not control the locations of the singularities. Note that in the previous sections we did note that the origin can be a complicated branch-point and we did make predictions of the locations of possible double ‘poles’. Remarkably these predictions seem to be good even for small values of x .

In the case that $\mu = 1$ we obtain from (1.2) that the residue at $x = x_j$ of $-\frac{1}{2}xy'(x)/y(x)$ is x_j , and the reader can verify that the residue at $x = x_j$ of $\frac{1}{56}y'(x)^3/y(x)$ is h_j . Hence, we can use loop integrals to evaluate the position of the pole and the constant h_j . We do not know the exact location of the poles, but we will need only reasonably good predictions, say \tilde{x}_j , which we do obtain from Padé approximants, or from the solutions of (3.9) and (3.10). Our loops will be circles $|x - \tilde{x}_j| = r$ because we are going to use the trapezoidal rule, and according to [12] the right-hand side of

$$(4.3) \quad \frac{1}{2\pi i} \oint_{|\tau|=r} F(\tau) d\tau \approx \frac{1}{2M} \sum_{m=0}^{2M-1} w_m F(w_m), \quad \text{where } w_m = re^{\pi i m/M},$$

converges exponentially fast to the left-hand side as $M \rightarrow \infty$, as long as $(\tau - a)F(\tau)$ is analytic in a disc $|\tau| \leq \tilde{r}$, with $r < \tilde{r}$ and $|a| < r$. Once we know $y(x)$ and $y'(x)$ at, say, $x = \tilde{x}_j + w_0$ then we use (4.2) again to compute many Taylor coefficients, and use them in (4.1) to evaluate $y(x)$ and $y'(x)$ at $x = \tilde{x}_j + w_1$. We can continue this process to evaluate $y(x)$ and $y'(x)$ at all $x = \tilde{x}_j + w_m$.

In the case that $\mu \neq 0, 1$ this method to determine x_j does not work, because $y(x)$ will have a logarithmic singularity at x_j . However, from (1.2) we obtain the local expansion

$$(4.4) \quad \frac{-xy'(x)}{2y(x)} = \frac{x_j}{x - x_j} + \frac{3}{14}\mu(\mu - 1)x_j^{\mu-1} (x - x_j)^5 \ln(x - x_j) + \dots + \text{reg}(x - x_j),$$

in which $\text{reg}(x - x_j)$ denotes a function that is analytic at $x = x_j$. Let \tilde{x}_j be a reasonable approximation for x_j and let $r > 0$ be small enough such that $x = x_j$ is the only singularity

contained in the disk $|x - \tilde{x}_j| \leq r$ then we can approximate the integral

$$(4.5) \quad \frac{1}{2\pi i} \oint \frac{-xy'(x)}{2y(x)} dx \approx x_j + \frac{1}{28}\mu(\mu-1)x_j^{\mu-1}(r + \tilde{x}_j - x_j)^6,$$

where we integrate along the contour $x = \tilde{x}_j + re^{2\pi i\theta}$, $\theta \in [0, 1]$. Hence, the closer we are at x_j the smaller the impact of this logarithm. We will use (4.3) with $F(\tau) = -\frac{1}{2}xy'(x)/y(x)$, in which $x = \tau + \tilde{x}_j$, and decreasing values of r .

Note that we will ignore the second term on the right-hand side of (4.5). However, the only unknown on the right-hand side of (4.5) is x_j . Hence, we can also evaluate the left-hand side of (4.5) numerically, and use the full approximation (4.5) to compute x_j . This will result in slightly better approximations.

5. EXAMPLE 1: $\mu = \frac{15}{7}$ AND $\nu = 4$

In the first example we take $\mu = \frac{15}{7}$, a non-integer. We will have $\nu = \frac{75}{86}$. In this section we will aim to obtain approximations to a precision of 10 significant digits. As a check we will do the same calculations with larger starting points and considerably more steps and Taylor coefficients. In this way we can check that the digits that we give below are actually correct.

To determine the Stokes multipliers we use (3.6) with $n = 15$ and 15 terms on its right-hand side. We obtain

$$(5.1) \quad K_- = 0.07069725039 + 0.01439846034i.$$

For the remaining numerics in this section we use $x = 6$ as our starting point. The optimal number of terms in asymptotic approximation (3.1) is approximately 11. We obtain

$$(5.2) \quad b_0 = y_-(6) = -2.7837507946, \quad b_1 = y'_-(6) = -0.4971881751.$$

The Taylor-series method is described in the previous section. We will take 20 Taylor coefficients in (4.1) and ‘walk’ in 100 steps to $x = 2$. We obtain

$$(5.3) \quad y_-(2) = -0.8564979712, \quad y'_-(2) = -0.4608802105.$$

We compute the first 200 Taylor coefficients at $x = 2$ via (4.2) and use this Taylor series to compute a Padé approximant of order $[99, 100]$ about the point $x = 2$. In Figure 1 (left) we can see the distribution of the poles of this Padé approximant. The accumulation of poles near the origin clearly indicates that the origin is a branch-point (see [10]), but we can also see that there are poles at approximately $p_1 \approx -2.75 + 1.7i$ and at $p_2 \approx -3.2 + 3.05i$.

To obtain better numerical approximations for these poles we will ‘walk’ along a straight line from $x = 2$ to $x = p_j + r$, with $r = \frac{1}{2}$, and use the contour integral method described at the end of §4.

In the case $j = 1$, using 20 Taylor coefficients we ‘walk’ in 1000 steps to the first pole and obtain

$$(5.4) \quad y_-(p_1 + \frac{1}{2}) = 4.261757203 + 0.011948086i, \quad y'_-(p_1 + \frac{1}{2}) = -16.66671859 - 1.52622461i.$$

The size of these values indicates that we are close to the singularity. The location of the pole is determined via (4.3) in which we take $F(\tau) = -\frac{1}{2}xy'(x)/y'(x)$, with $x = \tau + p_1$, $r = \frac{1}{2}$ and $M = 1000$. We will need $y(p_1 + w_m)$. Again, we will use the Taylor-series method with 20 coefficients and step $w_m - w_{m-1}$. We obtain the approximation

$$(5.5) \quad r = \frac{1}{2}, \quad p_1 = -2.743854960 + 1.711273828i.$$

We repeat this process, starting at $x = 2$, with this better guess for p_1 and $r = \frac{1}{10}$

$$(5.6) \quad r = \frac{1}{10}, \quad p_1 = -2.740061378 + 1.709843142i,$$

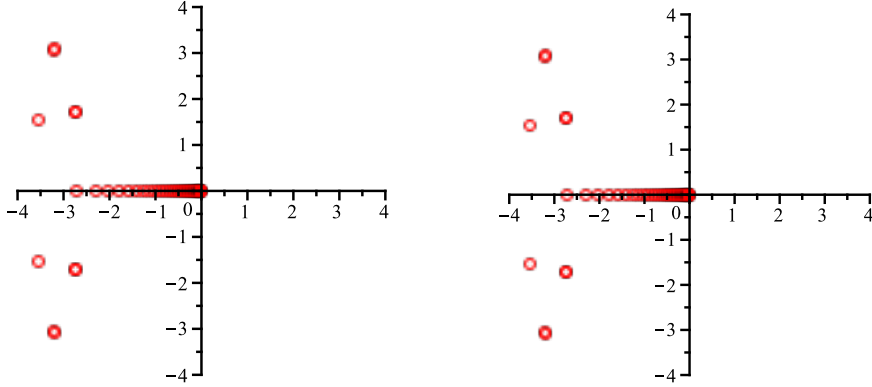


FIGURE 1. The poles of the Padé approximants of $y_-(x)$ in the cases $\mu = \frac{15}{7}$ (left) and $\mu = 4$ (right).

and again

$$(5.7) \quad r = \frac{1}{100}, \quad p_1 = -2.740061121 + 1.709843110i,$$

At the end of §3 we did mention that the poles should approximately satisfy (3.10). This approximation was constructed for the large poles. When we solve (3.10) for x near p_1 we obtain the solution $x = -2.736 + 1.705i$, with relative error 0.002. Hence, even for the small poles we obtain reasonable approximations via (3.10).

In a similar manner we can obtain a numerical approximation for the second pole

$$(5.8) \quad \begin{aligned} r &= \frac{1}{2}, & p_2 &= -3.206143009 + 3.079481200i, \\ r &= \frac{1}{10}, & p_2 &= -3.200868582 + 3.074868336i, \\ r &= \frac{1}{100}, & p_2 &= -3.200868242 + 3.074868282i, \end{aligned}$$

and when we solve (3.10) for x near p_2 we obtain the solution $x = -3.199 + 3.074i$, with relative error 0.0004.

In the introduction we do mention that the double pole expansion (1.2) can also be used to obtain a very good approximation for the location of the pole. Say that we start for the second pole with the approximation originating from (3.10), that is $p_2 \approx -3.199 + 3.074i$ and with the Taylor series method, mentioned above, we evaluate $y_-(p_2 + \frac{1}{100}) = 6986.503356 + 1027.767205i$. Then we can use the approximation

$$(5.9) \quad y_-(x) \approx \frac{1}{(x - x_j)^2} + \frac{1}{10}x_j^\mu (x - x_j)^2 + \frac{\mu}{6}x_j^{\mu-1} (x - x_j)^3,$$

with $x = p_2 + \frac{1}{100}$ and solve for x_j near p_2 . We obtain $x_j = -3.200868241 + 3.074868282i$. Note that compared with the final result in (5.8) only the final digit is different.

We did mention above that in Figure 1 (left) it is clearly visible that the origin is a branch-point, but it is not obvious that the other poles are actually also (weak) branch-points. For that reason we do include some details for the case $\mu = 4$, that is $\nu = \frac{5}{4}$. In that case the origin is a regular point. We compute the first 120 Taylor coefficients at $x = 0$ via (4.2) and use this Taylor series to compute a Padé approximant of order [59, 60] about the origin. In Figure 1 (right) we can see the distribution of the poles of this Padé approximant. The poles start to accumulate at p_0 and p_1 , indicating that these are branch-points. With the same numerical steps as described above we obtain that $K_- = 0.5297382962 - 0.2194247868i$, and that there are ‘poles’ at $p_0 = -1.182001651$, $p_1 = -0.895391503 + 2.352132859i$, and $p_2 = -0.745388754 + 3.344311527i$.

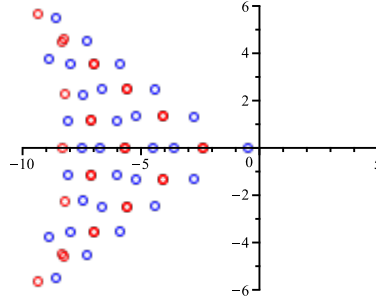


FIGURE 2. The zeros (blue) and poles (red) of the Padé approximants of $y_-(x)$ in the case $\mu = 1$.

6. EXAMPLE 2: THE UNPERTURBED FIRST PAINLEVÉ EQUATION

In this section we will aim to obtain approximations to a precision of 60 significant digits. The main reason for this is that we want to check some of the results in the recent literature. The unperturbed first Painlevé equation is the case $\mu = 1$, $\nu = \frac{1}{2}$. In this case the singularities in the complex plane are double poles. Hence, they are not branch-points and the contour integral method to determine the locations of the poles and zeros will be much more efficient. The Stokes multiplier is known to be $K_- = -\frac{3^{1/4}}{\sqrt{5\pi}}(1+i)$, see [11]. Taking $n = 100$ in (3.6) and 100 terms on its right-hand side we would obtain an approximation for K_- to a precision of 63 significant digits.

For the remaining numerics in this section we use $x = 33$ as our starting point. The optimal number of terms in asymptotic approximation (3.1) is approximately 70. We obtain

$$(6.1) \quad \begin{aligned} b_0 = y_-(33) &= -2.345227006792405252263591282624246998603914831899264653960958, \\ b_1 = y'_-(33) &= -0.035532293810222842527936052573825449588186033237794348317154. \end{aligned}$$

We will take 40 Taylor coefficients in (4.1) and ‘walk’ in 1000 steps to the origin. We obtain

$$(6.2) \quad \begin{aligned} y_-(0) &= -0.187554308340494893838681757595444367707042203291560247736544, \\ y'_-(0) &= -0.304905560261228856534104124988845544022671489625676976089364. \end{aligned}$$

Accurate values for this tri-tronquée solution at the origin are also given in [2]. They claim 64 digit precision, but comparing their results with (6.2) we see that they did obtain 30 digit precision.

We compute the first 100 Taylor coefficients at $x = 0$ via (4.2) and use this Taylor series to compute a Padé approximant of order [49, 50] about the point $x = 0$. In Figure 2 we can see the distribution of the zeros and poles of this Padé approximant.

The location of the first zero, which is approximately at $z_1 \approx -\frac{1}{2}$, is determined via (4.3) in which we take $F(\tau) = \frac{1}{2}xy'(x)/y'(x)$, with $x = \tau - \frac{1}{2}$, $r = \frac{1}{2}$ and $M = 60$. We will need $y(w_m - \frac{1}{2})$. Again, we will use the Taylor-series method with 40 coefficients and step $w_m - w_{m-1}$. We obtain the approximation

$$(6.3) \quad \begin{aligned} z_1 &= -0.499912553551334521451561845356016137446077785951448892634807, \\ y'(z_1) &= -0.468865514339593121531937054555736186201504711389139130341116, \end{aligned}$$

in which we did obtain $y'(z_1)$ by walking in 10 steps from the origin to z_1 , taking 40 Taylor coefficients. Note that with a relatively small $M = 60$ we do already obtain more than 60 digits precision.

To approximate the first real pole we first walk to $x = -2$ in 300 steps, taking 40 Taylor coefficients and obtain

$$(6.4) \quad \begin{aligned} y_-(-2) &= 6.74868071988330557708652890818896015343487191457022993535053, \\ y'_-(-2) &= -35.3975621098672136235306591226332932218101982620590565238107. \end{aligned}$$

The location of the first pole p_1 and the corresponding h_1 (see (1.2)), is determined via (4.3) in which we take $F(\tau) = -\frac{1}{2}xy'(x)/y'(x)$ and $\frac{1}{56}y'(x)^3/y(x)$, respectively, with $x = \tau - \frac{5}{2}$, $r = \frac{1}{2}$ and $M = 200$. We obtain

$$(6.5) \quad \begin{aligned} p_1 &= -2.38416876956881663929914585244876719041040881473785051267724, \\ h_1 &= 0.0621357392261776408964901416400624601977407713738296636635327, \end{aligned}$$

verifying the results in [3], except the final digits. When we solve (3.10) for x near p_1 we obtain the solution $x = -2.365 + 0.002i$, with relative error 0.008.

Finally we compute also the location of the first complex pole. The details are the same as above, the same number of steps, the same M , and the centre for the circle will be $-4.0 + 1.3i$. The result is

$$(6.6) \quad \begin{aligned} p_2 &= -4.07105552317228805392886956167452318934557741897847147742812 \\ &\quad + 1.33555121517567079951876062434077312552294901369825871527178i, \end{aligned}$$

and when we solve (3.10) for x near p_2 we obtain the solution $x = -4.068 + 1.337i$, with relative error 0.0007.

7. THE BOREL TRANSFORM ON THE POSITIVE REAL LINE

In this section we will study the Borel transform via

$$(7.1) \quad u(z) = -1 + \int_0^\infty e^{-zt}b(t) dt.$$

When we start with differential equation (1.4) and multiply each term by z^2 then we obtain for the Borel transform $b(t)$ the differential-integral equation

$$(7.2) \quad (t^2 + 3)b''(t) + (2 - \nu)2tb'(t) + 2\left(\frac{4}{5}\nu - 1\right)\left(\frac{3}{5}\nu - 1\right)b(t) = \frac{3}{2} \int_0^t b'(\tau)b'(t - \tau) d\tau.$$

It can be checked that

$$(7.3) \quad b(t) = \sum_{n=0}^{\infty} \frac{a_{n+1,0}}{n!} t^n, \quad |t| < \sqrt{3},$$

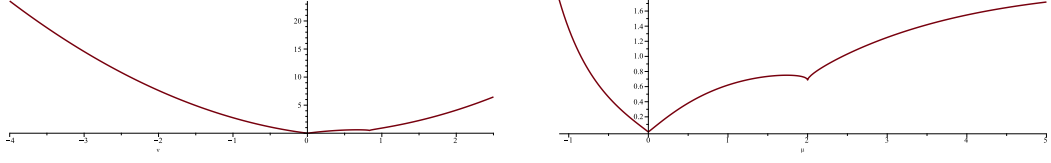
with $a_{0,0} = -1$ and $a_{n+1,0}$ defined in (2.2) and (2.3), is a solution of (7.2). In this section we will show that this solution is well defined and has a bound of the form $|b(t)| \leq ce^{\sigma(\nu)t}$, $t \geq 0$. Hence, our Borel-Laplace transform $u(z)$, defined in (7.1), is well-defined for $\Re(z) > \sigma(\nu)$. Our original perturbed first Painlevé equation is in terms of x and μ , and we obtain that the corresponding solution is well defined for $x > \tilde{\sigma}(\mu) = \left(\frac{6^{1/4}}{8}(\mu + 4)\sigma\right)^{1/(1+\mu/4)}$. For $\sigma(\nu)$ and $\tilde{\sigma}(\mu)$ see Figure 3.

To obtain a more convenient integral equation we use (1.4) directly in (7.1) and obtain

$$(7.4) \quad (t^2 + 3)b(t) = 2\nu \int_0^t \tau b(\tau) d\tau + 3a_{2,0}t - 3a_{2,0} \int_0^t (t - \tau)b(\tau) d\tau + \frac{3}{2} \int_0^t b(\tau)b(t - \tau) d\tau.$$

Dividing both sides by $t^2 + 3$ we have $b(t) = \mathcal{T}b(t)$ with

$$(7.5) \quad \mathcal{T}b(t) = \frac{2\nu}{t^2 + 3} \int_0^t \tau b(\tau) d\tau + \frac{3a_{2,0}t}{t^2 + 3} - \frac{3a_{2,0}}{t^2 + 3} \int_0^t (t - \tau)b(\tau) d\tau + \frac{\frac{3}{2}}{t^2 + 3} \int_0^t b(\tau)b(t - \tau) d\tau.$$

FIGURE 3. σ as a function of ν (left) and $\tilde{\sigma}$ as a function of μ (right).

We are going to show that this is a contraction mapping. Let c and σ be positive constants and define the norm

$$(7.6) \quad \|h\| = \inf \{M \mid |h(t)| \leq Mce^{\sigma t} \text{ for all } t \geq 0\}.$$

Denote by \mathcal{B}_σ the complex vector space of analytic function $h(t)$ on $[0, \infty)$ such that $\|h\|$ is bounded. Equipped with this norm, \mathcal{B}_σ becomes a Banach space.

For the terms on the right-hand side of (7.5) we have in the case $t \geq 0$ that

$$(7.7) \quad \left| \frac{2\nu}{t^2+3} \int_0^t \tau h(\tau) d\tau \right| \leq \frac{2|\nu|c\|h\|}{t^2+3} \int_0^t \tau e^{\sigma\tau} d\tau \leq \frac{2|\nu|c\|h\|t}{t^2+3} \int_0^t e^{\sigma\tau} d\tau \leq \frac{|\nu|\|h\|}{\sigma\sqrt{3}} ce^{\sigma t},$$

$$(7.8) \quad \left| \frac{3a_{2,0}t}{t^2+3} \right| \leq |a_{2,0}| (te^{-\sigma t}) e^{\sigma t} \leq \frac{|a_{2,0}|}{c\sigma} ce^{\sigma t},$$

$$(7.9) \quad \left| \frac{3a_{2,0}}{t^2+3} \int_0^t (t-\tau)h(\tau) d\tau \right| \leq \frac{3|a_{2,0}|t}{t^2+3} \int_0^t |h(\tau)| d\tau \leq \frac{\sqrt{3}|a_{2,0}|\|h\|}{2\sigma} ce^{\sigma t},$$

$$(7.10) \quad \left| \frac{\frac{3}{2}}{t^2+3} \int_0^t h_1(\tau)h_2(t-\tau) d\tau \right| \leq \frac{\frac{3}{2}t\|h_1\|\|h_2\|}{t^2+3} c^2 e^{\sigma t} \leq \frac{\sqrt{3}\|h_1\|\|h_2\|c}{4} ce^{\sigma t},$$

in which we have used several times $\frac{t}{t^2+3} \leq \frac{\sqrt{3}}{6}$.

Combining the inequalities above we obtain

$$(7.11) \quad \|\mathcal{T}h\| \leq \frac{|a_{2,0}|}{c\sigma} + \frac{\sqrt{3}}{4}c\|h\|^2 + \frac{|\nu| + \frac{3}{2}|a_{2,0}|}{\sigma\sqrt{3}}\|h\|,$$

and using the convolution identity $h_1 * h_1 - h_2 * h_2 = (h_1 + h_2) * (h_1 - h_2)$ we have

$$(7.12) \quad \|\mathcal{T}h_1 - \mathcal{T}h_2\| \leq \left(\frac{\sqrt{3}}{4}c\|h_1 + h_2\| + \frac{|\nu| + \frac{3}{2}|a_{2,0}|}{\sigma\sqrt{3}} \right) \|h_1 - h_2\|.$$

Recall that $a_{2,0} = \frac{4}{15}\nu(\frac{6}{5}\nu - 1)$. It is now possible to choose c and σ such that when $\|h\| \leq 1$ we will have from (7.11) that $\|\mathcal{T}h\| \leq 1$, and taking $\|h_j\| \leq 1$ we will have in (7.12) that the multiplier of $\|h_1 - h_2\|$ will be less than 1. Hence, we want to find a pair c, σ such that both

$$(7.13) \quad \frac{|a_{2,0}|}{c\sigma} + \frac{\sqrt{3}}{4}c + \frac{|\nu| + \frac{3}{2}|a_{2,0}|}{\sigma\sqrt{3}} \leq 1, \quad \frac{\sqrt{3}}{2}c + \frac{|\nu| + \frac{3}{2}|a_{2,0}|}{\sigma\sqrt{3}} \leq 1.$$

Once we have such a pair we have shown that $b(t) = \mathcal{T}b(t)$ has a unique solution with the bound $|b(t)| \leq ce^{\sigma t}$ for all $t \geq 0$. We want σ as small as possible, but $\sigma \sim \frac{4}{25}(\sqrt{3} + \frac{2}{c})\nu^2$ as $\nu \rightarrow -\infty$.

A reasonable choice seems to be $c = \frac{7}{10}$. It is also possible to use the optimal $c = \sqrt{\frac{3}{\alpha^2} + \frac{4}{\alpha} - \frac{\sqrt{3}}{\alpha}}$, with $\alpha = \left| \frac{\nu}{a_{2,0}} \right| + \frac{3}{2}$. The corresponding σ as a function of ν is displayed in Figure 3.

ACKNOWLEDGEMENT

The author wants to thank Nalini Joshi for stimulating discussions regarding the main topics of this paper, and thanks the Isaac Newton Institute for Mathematical Sciences for support during the program ‘Applicable resurgent asymptotics: towards a universal theory’ supported by EPSRC grant no. EP/R014604/1. The authors’ research was supported by a research grant 60NANB20D126 from the National Institute of Standards and Technology.

REFERENCES

- [1] O. COSTIN, *Asymptotics and Borel summability*, vol. 141 of Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, CRC Press, Boca Raton, FL, 2009.
- [2] O. COSTIN AND G. V. DUNNE, *Resurgent extrapolation: rebuilding a function from asymptotic data. Painlevé I*, J. Phys. A, 52 (2019), pp. 445205, 29.
- [3] ———, *Uniformization and Constructive Analytic Continuation of Taylor Series*, Comm. Math. Phys., 392 (2022), pp. 863–906.
- [4] *NIST Digital Library of Mathematical Functions*. <http://dlmf.nist.gov/>, Release 1.1.5 of 2022-03-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
- [5] N. JOSHI, *Tri-tronquée solution of perturbed first Painlevé equations*, Teoret. Mat. Fiz., 137 (2003), pp. 188–192.
- [6] N. JOSHI AND A. V. KITAEV, *On Boutroux’s tritronquée solutions of the first Painlevé equation*, Stud. Appl. Math., 107 (2001), pp. 253–291.
- [7] M. D. KRUSKAL AND P. A. CLARKSON, *The Painlevé-Kowalevski and poly-Painlevé tests for integrability*, Stud. Appl. Math., 86 (1992), pp. 87–165.
- [8] A. B. OLDE DAALHUIS, *Hyperterminants. II*, J. Comput. Appl. Math., 89 (1998), pp. 87–95.
- [9] ———, *Hyperasymptotics for nonlinear ODEs. II. The first Painlevé equation and a second-order Riccati equation*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 461 (2005), pp. 3005–3021.
- [10] H. STAHL, *The convergence of Padé approximants to functions with branch points*, J. Approx. Theory, 91 (1997), pp. 139–204.
- [11] Y. TAKEI, *On the connection formula for the first Painlevé equation—from the viewpoint of the exact WKB analysis*, no. 931, 1995, pp. 70–99. Painlevé functions and asymptotic analysis (Japanese) (Kyoto, 1995).
- [12] L. N. TREFETHEN AND J. A. C. WEIDEMAN, *The exponentially convergent trapezoidal rule*, SIAM Rev., 56 (2014), pp. 385–458.
- [13] H. WITTICH, *Eindeutige Lösungen der Differentialgleichungen $w'' = P(z, w)$* , Math. Ann., 125 (1953), pp. 355–365.

SCHOOL OF MATHEMATICS AND MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES, THE UNIVERSITY OF EDINBURGH, EDINBURGH EH9 3FD, UNITED KINGDOM

Email address: A.OldeDaalhuis@ed.ac.uk

URL: www.maths.ed.ac.uk/~adri