Toric and tropical Bertini theorems in positive characteristic

TORIC AND TROPICAL BERTINI THEOREMS IN POSITIVE CHARACTERISTIC

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Abstract. We generalize the toric Bertini theorem of Fuchs, Mantova, and Zannier [FMZ18] to positive characteristic. A key part of the proof is a new algebraically closed field containing the field $\Bbbk(t_1, \ldots, t_d)$ of rational functions over an algebraically closed field $\Bbbk$ of prime characteristic. As a corollary, we extend the tropical Bertini theorem of Maclagan and Yu [MY21] to arbitrary characteristic, which removes the characteristic dependence from the d-connectivity result for tropical varieties from that paper.

1. Introduction

Bertini's theorem, which states that a general hyperplane section of an irreducible variety is again irreducible, is a basic result in algebraic geometry. This has been generalized in many different ways, most notably for this paper by Fuchs, Mantova, and Zannier [FMZ18], who replace hyperplane sections by certain subtori when the variety is a subvariety of an algebraic torus in characteristic zero. Our main result removes this characteristic assumption, at the expense of some precision.

Theorem 1.1 (Toric Bertini)**.** *Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Let X be a d-dimensional irreducible subvariety of $(\Bbbk^*)^n$ with $d \geq 2$, and let $\pi\colon (\Bbbk^*)^n \to (\Bbbk^*)^d$ be a morphism with $\pi|_X$ dominant and finite. Suppose that the pullback of $\pi|_X$ along any isogeny $\mu\colon (\Bbbk^*)^d \to (\Bbbk^*)^d$ is irreducible. Then for every $1 \leq r \leq d-1$ the set of r-dimensional subtori $T \subseteq (\Bbbk^*)^d$ with $\pi^{-1}(\boldsymbol{\theta} \cdot T) \cap X$ irreducible for all $\boldsymbol{\theta} \in (\Bbbk^*)^d$ is dense in the Grassmannian $\mathrm{Gr}(r, d)$.*

In characteristic zero [FMZ18] show that the conclusion holds for subtori T in a generic (Zariski open) set, rather than just in a dense set. More precisely, there are a finite number of exceptional subtori that T must avoid, and any other T will have irreducible preimage. A similar "generic" conclusion may hold in characteristic p; we are not aware of a counterexample, though our current techniques are insufficient for a proof. The techniques of [FMZ18] cannot be extended to characteristic p as they rely on results that are false in characteristic p.

The condition on the irreducibility of pullbacks along isogenies appears already in [FMZ18] and is necessary for a generic result; for example, for the projection $\pi : \mathrm{V}(x - y^2z^2) \to (\Bbbk^*)^2$ onto the first two coordinates, the preimage of any subtorus of $(\Bbbk^*)^2$ of the form (t^{2a}, t^b) is reducible. Thus the set of desired subtori cannot be open. However the denseness conclusion may still hold without this pullback hypothesis. We do not have a counterexample; see Remark 4.8.

One consequence of Theorem 1.1 is that we can remove the characteristic assumption in the Tropical Bertini theorem of Maclagan and Yu [MY21].

Theorem 1.2 (Tropical Bertini)**.** *Let $X \subset (\Bbbk^*)^n$ be an irreducible d-dimensional variety, with $d \geq 2$, over an algebraically closed valued field $\Bbbk$ with $\mathbb{Q}$ contained in the value group. The set of rational affine hyperplanes H in $\mathbb{R}^n$ for which the intersection $\mathrm{trop}(X) \cap H$ is the tropicalization of an irreducible variety is dense in the Euclidean topology on $\mathbb{P}^n_{\mathbb{Q}}$.*

The characteristic zero case of the Tropical Bertini theorem was originally introduced in [MY21] to prove a higher connectivity result for tropicalizations of irreducible varieties [MY21, Theorem 1]. Theorem 1.2 removes the characteristic assumption from that connectivity theorem.

The key ingredient in the proof of Theorem 1.1 is a new field $K_{\mathbf{w}}$ that contains the algebraic closure of $\Bbbk(t_1, \ldots, t_d)$. This field is smaller, in some crucial aspects, than previously constructed algebraically closed fields containing $\Bbbk(t_1, \ldots, t_d)$ when $\Bbbk$ has characteristic p.

The use for this field is best illustrated by the case when the variety X of Theorem 1.1 is a hypersurface, the map $\pi\colon X \to (\Bbbk^*)^d$ is projection onto the first d coordinates, and the subtorus $T \subseteq (\Bbbk^*)^d$ is one-dimensional: $T = \{(t^{n_1}, \ldots, t^{n_d}) : t \in \Bbbk^*\}$ for some $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$. This is, in fact, the core case of the proof. Given an irreducible $f \in \Bbbk[t_1^{\pm 1}, \ldots, t_d^{\pm 1}, y]$, we wish to show that the set of $\mathbf{n} \in \mathbb{Z}^d$, for which the substitution $g \in \Bbbk[x^{\pm 1}, y]$ given by

$$g(x, y) = f(x^{n_1}, \ldots, x^{n_d}, y)$$

remains irreducible, is dense in $\mathbb{P}^{d-1}_{\mathbb{Q}}$. The key idea is to regard f as a polynomial in y with coefficients in $\Bbbk(t_1, \ldots, t_d)$. The polynomial f then factors as

$$f = \prod_{i=1}^{s} (y - \alpha_i),$$

where α_i are in the algebraic closure of $\Bbbk(t_1, \ldots, t_d)$. A precise description of the algebraic closure is still unknown but several fields containing it are known; see [Chr79, Ked01, Ked17, AB12, McD95, GP00, AI09, AR19, Saa17]. When $d = 1$ and $\mathrm{char}(\Bbbk) = 0$, the algebraic closure of $\Bbbk(t_1)$ is contained in the field of Puiseux series $\Bbbk\{\{t_1\}\} = \bigcup_{n \geq 1} \Bbbk(\!(t_1^{1/n})\!)$. The exponents appearing in a particular Puiseux series all have a fixed common denominator. In characteristic p this is relaxed to allow arbitrary powers of p in the denominator, subject to the requirement, which is automatic for Puiseux series, that the set of all exponents is well ordered. As we recall in Section 2, there are also multivariate generalizations of this, so we may regard the roots α_i of f as multivariate Laurent series with fractional exponents.

Given this description of the roots of f, the natural expectation is that the roots of g as a polynomial in y are obtained from the series α_i by the same specialization $t_i = x^{n_i}$. To show that g remains irreducible for most $\mathbf{n}$ it then suffices to show that none of these specializations of α_is, or the elementary symmetric polynomials in them, which are the coefficients of any factors of g, are polynomials in x, as opposed to generalized Puiseux series.

However it is far from clear that this specialization map is well defined. For example, given the multivariate series

$$\alpha = \sum_{j \geq 1} t_1^{1-1/2^j} t_2^{1/2^j}$$

over a field of characteristic 2, we cannot make the substitution $t_1 = t_2 = x$. A contribution of this paper is to define a field $K_{\mathbf{w}}$ that contains the algebraic closure of $\Bbbk(t_1, \ldots, t_d)$, and has natural subrings on which this specialization map is a well-defined ring homomorphism, so this plan of attack goes through.

The structure of the paper is as follows. The new field $K_{\mathbf{w}}$ is introduced in Section 2, and it is shown to be algebraically closed in Section 3. Theorem 1.1 is proved in Section 4, while Theorem 1.2 is proved in Section 5.

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Notation. Throughout this paper, by a cone in $\mathbb{R}^d$ we mean a convex set closed under multiplication by positive scalars. It is *pointed* if its closure does not contain a line. For a cone C in $\mathbb{R}^d$, the *dual cone* $C^\vee$ is $\{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{y} \geq 0 \text{ for all } \mathbf{y} \in C\}$. We denote by $\text{int}(C)$ the interior of a cone C, and by $\mathbb{R}^d_{li}$ elements of $\mathbb{R}^d$ whose coordinates are linearly independent over $\mathbb{Q}$.

2. Field Families and p-discreteness

In this section we construct the field $K_\mathbf{w}$ containing $\Bbbk(t_1, \ldots, t_d)$ that plays a key role in the proof of Theorem 1.1. We will prove in Section 3 that $K_\mathbf{w}$ is algebraically closed.

2.1. Field families.

We first recall previous constructions of algebraically closed fields containing $\Bbbk(t_1, \ldots, t_d)$.

When $\text{char}(\Bbbk) = 0$ and $d = 1$, the field of Puiseux series $\Bbbk\{\{t\}\} := \bigcup_{n \geq 0} \Bbbk((t^{1/n}))$ is algebraically closed and contains $\Bbbk(t)$. Puiseux series are not algebraically closed when $\text{char}(\Bbbk) = p > 0$. This was first observed by Chevalley [Che63], who observed that the Artin–Schreier polynomial $x^p - x - t^{-1}$ has no Puiseux series root. Abhyankar [Abh55] showed that the series $\sum_{j \geq 0} t^{1/p^j}$ is a root. This was generalized by Rayner [Ray68], who showed that the collection

$$\left\{ \sum c_\alpha t^\alpha : \{\alpha : c_\alpha \neq 0\} \text{ is well ordered, and there is } N > 0 \text{ with } \alpha \in \bigcup_{j \geq 0} \frac{1}{Np^j}\mathbb{Z} \text{ for all } \alpha \right\} \tag{1}$$

forms an algebraically closed field. Rayner introduced the idea of a *field family*, which gives conditions on the possibilities of supports for power series of this form. We now recall this in our setting.

Definition 2.1. Let $\mathcal{A}$ be a family of subsets of an ordered abelian group $(\Gamma, \preceq)$. For $A \in \mathcal{A}$, let $S(A)$ denote the semigroup generated by A under addition in Γ. Then $\mathcal{A}$ is called a *field family* with respect to Γ if the following axioms hold:

(i) Every $A \in \mathcal{A}$ is well ordered, i.e., every subset has a least element with respect to $\preceq$.
(ii) The elements of members of $\mathcal{A}$ generate Γ.
(iii) If $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$.
(iv) If $A \in \mathcal{A}$ and $B \subseteq A$ then $B \in \mathcal{A}$.
(v) Given $A \in \mathcal{A}$ and $\gamma \in \Gamma$, then $\gamma + A := \{\gamma + a : a \in A\} \in \mathcal{A}$.
(vi) If $A \in \mathcal{A}$, with $a \succeq 0$ for all $a \in A$, then $S(A) \in \mathcal{A}$.

Given any field $\Bbbk$, let $\Bbbk^\Gamma$ denote the set of all mappings $\varphi : \Gamma \to \Bbbk$. The *support* of $\varphi \in \Bbbk^\Gamma$ is the set $\text{supp}\, \varphi = \{\gamma \in \Gamma : \varphi(\gamma) \neq 0\}$. For a family $\mathcal{A}$ of subsets of Γ define

$$\Bbbk^\Gamma(\mathcal{A}) = \{\varphi \in \Bbbk^\Gamma : \text{supp}\, \varphi \in \mathcal{A}\}.$$

We think of elements in $\Bbbk^\Gamma(\mathcal{A})$ as formal power series,

$$\varphi = \sum_{\gamma \in \Gamma} \varphi(\gamma) \mathbf{t}^\gamma \in \Bbbk^\Gamma(\mathcal{A}).$$

For $\varphi, \psi \in \Bbbk^\Gamma(\mathcal{A})$ and $\gamma \in \Gamma$, addition is defined as

$$(\varphi + \psi)(\gamma) = \varphi(\gamma) + \psi(\gamma)$$

and multiplication is defined as

$$(\varphi \cdot \psi)(\gamma) = \sum_{\gamma_1+\gamma_2=\gamma} \varphi(\gamma_1)\psi(\gamma_2).$$

Theorem ([Ray68, Theorem 1])**.** *If $\mathcal{A}$ is a field family then $\Bbbk^\Gamma(\mathcal{A})$ is a field.*

The field in (1) is the case when $\Gamma = \mathbb{Q}$ and $\mathcal{A}$ consists of all well-ordered subsets of $\mathbb{Q}$ that lie in $\bigcup_{j\geq 0} \frac{1}{Np^j}\mathbb{Z}$ for some $N \in \mathbb{N}$.

When $\Gamma = \mathbb{Q}^d$ the field elements can be regarded as generalized Puiseux series in d variables $t_1, \ldots, t_d$. For $\mathrm{char}(\Bbbk) = 0$, in [McD95] McDonald constructed multivariate Puiseux series solutions g for equations of the form $f(t_1, \ldots, t_d, g) = 0$ where $f \in \Bbbk[t_1, \ldots, t_d, y]$. The supports of the solution series are contained in sets of the form $\frac{1}{N}\mathbb{Z}^d \cap C$ where $N \in \mathbb{N}$ and C is a pointed cone.

Using Rayner's field family formalism Aroca and Ilardi constructed algebraically closed fields containing $\Bbbk(t_1, \ldots, t_d)$ and McDonald's series [AI09]. For $\mathrm{char}(\Bbbk) = p > 0$ Saavedra gave an analogous construction of algebraically closed fields containing $\Bbbk(t_1, \ldots, t_d)$ [Saa17]. The supports of Saavedra's series are contained in sets of the form $\bigcup_{j\geq 0} \frac{1}{Np^j}\mathbb{Z}^d \cap C$ where $N \in \mathbb{N}$ and C is a pointed cone.

All of these fields contain elements that are not algebraic over $\Bbbk(t_1, \ldots, t_d)$. While there is some work on characterizing the algebraic closure of $\Bbbk(t_1, \ldots, t_d)$ [AB12, AR19, Chr79, Ked01, Ked17] there is currently no complete answer in multiple variables and arbitrary characteristic.

2.2. **The field of p-discrete series.** We now use the formalism of Definition 2.1 to construct a field containing $\Bbbk(t_1, \ldots, t_d)$ but contained in Saavedra's field. Let $\mathbb{R}^d_{li}$ denote the set of vectors in $\mathbb{R}^d$ whose coordinates are linearly independent over $\mathbb{Q}$. For $\mathbf{w} \in \mathbb{R}^d_{li}$ we define an ordering $\preceq_{\mathbf{w}}$ on the abelian group $\mathbb{Q}^d$ as follows:

$$\mathbf{u} \preceq_{\mathbf{w}} \mathbf{v} \quad \text{when} \quad \mathbf{w}\cdot\mathbf{u} \leq \mathbf{w}\cdot\mathbf{v}.$$

The condition that the entries of $\mathbf{w}$ are linearly independent over the rationals ensures that $\preceq_{\mathbf{w}}$ is a total order.

Definition 2.2 (p-discreteness)**.** For a subset $A \subseteq \mathbb{Q}^d$, $\boldsymbol{\gamma} \in \mathbb{Q}^d$, and $\mathbf{w} \in \mathbb{R}^d_{li}$, set

$$A^+_{\boldsymbol{\gamma},\mathbf{w}} = \{\mathbf{a} \in A : \mathbf{w}\cdot\mathbf{a} > \mathbf{w}\cdot\boldsymbol{\gamma}\} \text{ and } A^-_{\boldsymbol{\gamma},\mathbf{w}} = \{\mathbf{a} \in A : \mathbf{w}\cdot\mathbf{a} < \mathbf{w}\cdot\boldsymbol{\gamma}\}.$$

We say that a set $A \subseteq \mathbb{Q}^d$ is *p-discrete with respect to* $\mathbf{w}$ if the following conditions are satisfied.

(a) There exists an open cone σ containing $\mathbf{w}$ for which the set $\{\mathbf{w}'\cdot\mathbf{a} : \mathbf{a} \in A\}$ is well ordered when $\mathbf{w}' \in \sigma \cap \mathbb{R}^d_{li}$.

(b) There is $N > 0$, $\boldsymbol{\gamma} \in \mathbb{Q}^d$, a pointed cone C with $\mathbf{w} \in \mathrm{int}(C^\vee)$ such that
$$A \subseteq (\boldsymbol{\gamma} + C) \cap \left(\bigcup_{j\geq 0} \frac{1}{Np^j}\mathbb{Z}^d\right).$$

(c) For any sequence $\{\mathbf{a}_i\}$ in A, if the sequence $\{\mathbf{w}\cdot\mathbf{a}_i\}$ converges in $\mathbb{R}$ then $\{\mathbf{a}_i\}$ converges to a point of $\mathbb{Q}^d$.

(d) For all $\boldsymbol{\gamma}' \in \mathbb{Q}^d$ there is an open cone $\sigma_{\boldsymbol{\gamma}'}$ containing $\mathbf{w}$ for which for all $\mathbf{w}' \in \sigma_{\boldsymbol{\gamma}'} \cap \mathbb{R}^d_{li}$, we have $A^+_{\boldsymbol{\gamma}',\mathbf{w}'} = A^+_{\boldsymbol{\gamma}',\mathbf{w}}$ and $A^-_{\boldsymbol{\gamma}',\mathbf{w}'} = A^-_{\boldsymbol{\gamma}',\mathbf{w}}$.

For a given p-discrete set A we may choose the closure of σ in condition (a) and the dual cone $C^\vee$ of C in condition (b) to coincide, at the expense of at least one of them not being the largest possible, since σ can be replaced with a smaller open cone and C can be replaced with a larger pointed cone.

In condition (d) it is possible that a different $\sigma_{\boldsymbol{\gamma}'}$ is needed for each $\boldsymbol{\gamma}'$. Note that condition (d) is equivalent to the existence of open cones $\sigma_{\boldsymbol{\gamma}'}$ containing $\mathbf{w}$ such that that $A^-_{\boldsymbol{\gamma}',\mathbf{w}} \subseteq \boldsymbol{\gamma}' - \sigma^\vee_{\boldsymbol{\gamma}'}$, and $A^+_{\boldsymbol{\gamma}',\mathbf{w}} \subseteq \boldsymbol{\gamma}' + \sigma^\vee_{\boldsymbol{\gamma}'}$.

Definition 2.3. For fixed $\mathbf{w} \in \mathbb{R}^d_{li}$, we define $\mathcal{A}_\mathbf{w}$ to be the set

$$(*) \qquad \mathcal{A}_\mathbf{w} = \{A \subset \mathbb{Q}^d : A \text{ is } p\text{-discrete with respect to } \mathbf{w}\}.$$

Remark 2.4. Saavedra's field family [Saa17, Proposition 5.1] satisfies condition (b), and also a weaker version of condition (a), where the well ordering is required only for $\mathbf{w}$. The stronger condition (a) is needed in Section 4 to show that some specialization maps $t_i = x^{n_i}$ are well defined. Condition (d) is needed in the proof of algebraic closure in Section 3 and is also used in Section 4. Condition (c) is needed in the proof of Lemma 2.6 to show that these series form a field family.

In the rest of this section we show that $\mathcal{A}_\mathbf{w}$ is a field family with respect to $(\mathbb{Q}^d, \preceq_\mathbf{w})$.

Lemma 2.5. *Fix $\mathbf{w} \in \mathbb{R}^d_{li}$, and $\boldsymbol{\gamma} \in \mathbb{Q}^d$. Suppose $A \in \mathcal{A}_\mathbf{w}$ with $\mathbf{w}\cdot\mathbf{a} \geq 0$ for all $\mathbf{a} \in A$. Let $S(A)$ be the semigroup generated by A under addition in $\mathbb{Q}^d$ and let $S(A)^-_{\boldsymbol{\gamma},\mathbf{w}}$ be as in Definition 2.2. Then there is $M > 0$ such that for any nonzero $\mathbf{s} = \sum_{i=1}^m \mathbf{a}_i \in S(A)^-_{\boldsymbol{\gamma},\mathbf{w}}$, where $\mathbf{a}_i \in A \setminus \{0\}$, the number m of summands is at most M.*

Proof. We have $\mathbf{w}\cdot\mathbf{s} = \sum_{i=1}^m \mathbf{w}\cdot\mathbf{a}_i$, so

$$0 < \min_i\{\mathbf{w}\cdot\mathbf{a}_i\} \leq \frac{\mathbf{w}\cdot\mathbf{s}}{m} < \frac{\mathbf{w}\cdot\boldsymbol{\gamma}}{m},$$

where the last inequality holds because $\mathbf{s} \in S(A)^-_{\boldsymbol{\gamma},\mathbf{w}}$. If for all $m > 0$ there is $\mathbf{s} \in S(A)^-_{\boldsymbol{\gamma},\mathbf{w}}$ that is the sum of at least m nonzero terms, we would get a contradiction to well-ordering of the set $\{\mathbf{w}\cdot\mathbf{a} \colon \mathbf{a} \in A\}$ given in condition (a). ☐

Lemma 2.6. *Let $A \in \mathcal{A}_\mathbf{w}$ and let $S(A)$ be the semigroup generated by elements of A under addition. Suppose that $\mathbf{w}\cdot\mathbf{a} \geq 0$ for all $\mathbf{a} \in A$, and conditions (a), (b), (c) of p-discreteness hold for $S(A)$. Then $S(A)$ also satisfies condition (d), so $S(A)$ is in $\mathcal{A}_\mathbf{w}$.*

Proof. Fix $\boldsymbol{\gamma}' \in \mathbb{Q}^d$. It suffices to show there are open cones σ_1 and σ_2 containing $\mathbf{w}$ for which $S(A)^+_{\boldsymbol{\gamma}',\mathbf{w}} \subseteq S(A)^+_{\boldsymbol{\gamma}',\mathbf{w}'}$ for all $\mathbf{w}' \in \sigma_1 \cap \mathbb{R}^d_{li}$ and $S(A)^-_{\boldsymbol{\gamma}',\mathbf{w}} \subseteq S(A)^-_{\boldsymbol{\gamma}',\mathbf{w}'}$ for all $\mathbf{w}' \in \sigma_2 \cap \mathbb{R}^d_{li}$.

We first show the existence of σ_1. Since $\{\mathbf{w}\cdot\mathbf{a} : \mathbf{a} \in S(A)^+_{\boldsymbol{\gamma}',\mathbf{w}}\}$ is well ordered by (a) there is $\mathbf{a} \in S(A)^+_{\boldsymbol{\gamma}',\mathbf{w}}$ achieving the minimum value of $\mathbf{w}\cdot\mathbf{a}$, so

$$\delta := \min\{\mathbf{w}\cdot\mathbf{a} : \mathbf{a} \in S(A)^+_{\boldsymbol{\gamma}',\mathbf{w}}\} - \mathbf{w}\cdot\boldsymbol{\gamma}'$$

is positive. By condition (b), we have $S(A) \subseteq \boldsymbol{\gamma} + C$ for a pointed cone C with $\mathbf{w} \in \operatorname{int}(C^\vee)$ and $\boldsymbol{\gamma} \in \mathbb{Q}^d$. If $\mathbf{w}\cdot\boldsymbol{\gamma}' \leq \mathbf{w}\cdot\boldsymbol{\gamma}$ then we may take σ_1 to be the interior of $\{\mathbf{x} : \mathbf{x}\cdot\boldsymbol{\gamma} \geq \mathbf{w}'\cdot\boldsymbol{\gamma}'\} \cap C^\vee$. We now suppose that $\mathbf{w}\cdot\boldsymbol{\gamma}' > \mathbf{w}\cdot\boldsymbol{\gamma}$. We may assume that C is full-dimensional here, so there is a rational point $\tilde{\boldsymbol{\gamma}}$ in $\boldsymbol{\gamma} + C \cap \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{w}\cdot\boldsymbol{\gamma}' < \mathbf{w}\cdot\mathbf{x} < \mathbf{w}\cdot\boldsymbol{\gamma}' + \delta\}$. Let C' be the closure of the cone generated by $\{\mathbf{a} - \tilde{\boldsymbol{\gamma}} : \mathbf{a} \in S(A)^+_{\boldsymbol{\gamma}',\mathbf{w}}\}$. The cone C' is pointed, as it is contained in the cone with vertex $\tilde{\boldsymbol{\gamma}}$ over the intersection $\boldsymbol{\gamma} + C \cap \{\mathbf{x} : \mathbf{w}\cdot\mathbf{x} = \mathbf{w}\cdot\boldsymbol{\gamma}' + \delta\}$, which is bounded. It also contains $\mathbf{w}$ in its dual. Thus any open cone σ_1 containing $\mathbf{w}$ and contained in $\operatorname{int}(C'^\vee) \cap \{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{x}\cdot(\tilde{\boldsymbol{\gamma}} - \boldsymbol{\gamma}') \geq 0\}$ will have the property that $S(A)^+_{\boldsymbol{\gamma}',\mathbf{w}} \subseteq S(A)^+_{\boldsymbol{\gamma}',\mathbf{w}'}$ for all $\mathbf{w}' \in \sigma_1$.

We next show the existence of σ_2.

By Lemma 2.5 there is a bound M on the number of summands m in any element $\mathbf{a} \in S(A)^-_{\boldsymbol{\gamma}',\mathbf{w}}$. Let

$$B_m = \left\{(\mathbf{a}_1, \ldots, \mathbf{a}_m) \in A^m : \sum_{i=1}^m \mathbf{a}_i \in S(A)^-_{\boldsymbol{\gamma}',\mathbf{w}}\right\}.$$

We show by induction on m that there is a cone C_m with $\mathbf{w} \in \operatorname{int}(C_m^\vee)$ and $\{\sum_{i=1}^m \mathbf{a}_i \in S(A)^-_{\boldsymbol{\gamma}',\mathbf{w}}\} \subseteq \boldsymbol{\gamma}' - C_m$. Any $\sigma_2 \subseteq \operatorname{int}(\bigcap_{m=1}^M C_m^\vee)$ containing $\mathbf{w}$ will then have the property that $S(A)^-_{\boldsymbol{\gamma}',\mathbf{w}} \subseteq S(A)^-_{\boldsymbol{\gamma}',\mathbf{w}'}$ for all $\mathbf{w}' \in \sigma_2$.

The base case is $m = 1$, where the claim follows since $B_1 \subseteq A$, so $B_1 \in \mathcal{A}_\mathbf{w}$, for which axiom (d) holds. Now suppose that $m > 1$, and the claim is true for smaller m. If

$$s := \sup\left\{\mathbf{w} \cdot \sum_{i=1}^m \mathbf{a}_i : (\mathbf{a}_1, \dots, \mathbf{a}_m) \in B_m\right\}$$

is less than $\mathbf{w} \cdot \boldsymbol{\gamma}'$, then the closure C_m of the cone generated by $\{\boldsymbol{\gamma}' - \sum_{i=1}^m \mathbf{a}_i : (\mathbf{a}_1, \dots, \mathbf{a}_m) \in B_m\}$ is pointed and has the required form. We may thus assume that $s = \mathbf{w} \cdot \boldsymbol{\gamma}'$.

For each $n > 0$, the set

$$\mathcal{S}_n^{(0)} = \left\{\mathbf{a}_1 : (\mathbf{a}_1, \dots, \mathbf{a}_m) \in B_m, \mathbf{w} \cdot \sum_{i=1}^m \mathbf{a}_i > \mathbf{w} \cdot \boldsymbol{\gamma}' - 1/n\right\}$$

is a subset of A, so by axiom (a) there is some $\mathbf{s}_n^{(0)} \in \mathcal{S}_n^{(0)}$ with $\mathbf{w} \cdot \mathbf{s}_n^{(0)}$ minimal. We claim that the set $\{\mathbf{w} \cdot \mathbf{s}_n^{(0)} : n \geq 1\}$ is weakly increasing, and bounded above by $\mathbf{w} \cdot \boldsymbol{\gamma}'$. The bound comes from the fact that $\mathbf{w} \cdot \mathbf{a}_i \geq 0$ for all $\mathbf{a}_i$ by assumption, which implies that $\mathbf{w} \cdot \mathbf{s}_n^{(0)} = \mathbf{w} \cdot \mathbf{a}_1 \leq \mathbf{w} \cdot \sum_{i=1}^m \mathbf{a}_i < \mathbf{w} \cdot \boldsymbol{\gamma}'$. Thus the sequence $\{\mathbf{w} \cdot \mathbf{s}_n^{(0)}\}$ converges to $\ell_0 \in \mathbb{R}$. We now iterate, at each stage constructing sets

$$\mathcal{T}_j = \{(\mathbf{a}_1, \dots, \mathbf{a}_m) \in B_m : \mathbf{w} \cdot \mathbf{a}_1 > \ell_{j-1}\},$$

and

$$\mathcal{S}_n^{(j)} = \left\{\mathbf{a}_1 : (\mathbf{a}_1, \dots, \mathbf{a}_m) \in \mathcal{T}_j, \mathbf{w} \cdot \sum_{i=1}^m \mathbf{a}_i > \mathbf{w} \cdot \boldsymbol{\gamma}' - 1/n\right\}.$$

with sequences $\{\mathbf{s}_n^{(j)}\}$, for which $\mathbf{w} \cdot \mathbf{s}_n^{(j)}$ converges to ℓ_j. Note that $\ell_j > \ell_{j-1}$ by construction.

We claim that this process must terminate for some j, with $\sup\{\mathbf{w} \cdot \sum_{i=1}^m \mathbf{a}_i : (\mathbf{a}_1, \dots, \mathbf{a}_m) \in \mathcal{T}_j\} < \mathbf{w} \cdot \boldsymbol{\gamma}'$. To see this, suppose that we can construct an infinite sequence

$$\ell_0 < \ell_1 < \ell_2 < \ell_3 < \dots$$

For each $j > 1$, fix $0 < \epsilon_j < \min\{(\ell_{j+1} - \ell_j)/2, (\ell_j - \ell_{j-1})/2\}$. Since for each j, the sequence $\{\mathbf{w} \cdot \mathbf{s}_n^{(j)}\}$ converges to ℓ_j there is $N_j > 0$ for which $|\mathbf{w} \cdot \mathbf{s}_n^{(j)} - \ell_j| < \epsilon_j$ for $n > N_j$. Fix $n_j > N_j$ with $1/n_j < \epsilon_j$ and pick $\mathbf{c}_j = (\mathbf{a}_1, \dots, \mathbf{a}_m) \in \mathcal{T}_j$ with $\mathbf{a}_1 = \mathbf{s}_{n_j}^{(j)}$. Then $\mathbf{w} \cdot \sum_{i=1}^m \mathbf{a}_i > \mathbf{w} \cdot \boldsymbol{\gamma}' - \epsilon_j$. Because $\sum_{i=1}^m \mathbf{a}_i \in S(A)^-_{\boldsymbol{\gamma}',\mathbf{w}}$, we also have the condition that $\mathbf{w} \cdot \sum_{i=1}^m \mathbf{a}_i < \mathbf{w} \cdot \boldsymbol{\gamma}'$. Subtracting $\mathbf{w} \cdot \mathbf{s}_{n_j}^{(j)} = \mathbf{w} \cdot \mathbf{a}_1$ we get

$$\mathbf{w} \cdot \boldsymbol{\gamma}' - \epsilon_j - \mathbf{w} \cdot \mathbf{s}_{n_j}^{(j)} < \sum_{i=2}^m \mathbf{w} \cdot \mathbf{a}_i < \mathbf{w} \cdot \boldsymbol{\gamma}' - \mathbf{w} \cdot \mathbf{s}_{n_j}^{(j)}.$$

Hence, since $-\ell_j + \epsilon_j > -\mathbf{w} \cdot \mathbf{s}_{n_j}^{(j)} \geq -\ell_j$ and by the choice of ϵ_j, we have

$$\mathbf{w} \cdot \boldsymbol{\gamma}' - \ell_{j+1} < \mathbf{w} \cdot \boldsymbol{\gamma}' - \ell_j - \epsilon_j < \sum_{i=2}^m \mathbf{w} \cdot \mathbf{a}_i < \mathbf{w} \cdot \boldsymbol{\gamma}' - \ell_j + \epsilon_j < \mathbf{w} \cdot \boldsymbol{\gamma}' - \ell_{j-1}.$$

But this implies that the subset $\{\sum_{i=2}^m \mathbf{a}_i : \mathbf{c}_j = (\mathbf{a}_1, \dots, \mathbf{a}_m)\}$ of $S(A)$ is not well ordered, contradicting our assumption that condition (a) holds. From this contradiction we conclude that the process terminates, so there is j for which $\sup\{\mathbf{w}\cdot\sum_{i=1}^m \mathbf{a}_i : (\mathbf{a}_1,\ldots,\mathbf{a}_m)\in\mathcal{T}_j\} < \mathbf{w}\cdot\boldsymbol{\gamma}'$. Write L for this j and set

$$\epsilon := \mathbf{w}\cdot\boldsymbol{\gamma}' - \sup\left\{\mathbf{w}\cdot\sum_{i=1}^m \mathbf{a}_i : (\mathbf{a}_1,\ldots,\mathbf{a}_m)\in\mathcal{T}_L\right\},$$

which is positive.

For each $0\le j\le L-1$, set

$$\mathcal{T}'_j = \left\{(\mathbf{a}_1,\ldots,\mathbf{a}_m)\in B_m : \mathbf{w}\cdot\sum_{i=2}^m \mathbf{a}_i > \mathbf{w}\cdot\boldsymbol{\gamma}' - \ell_j\right\}.$$

The set $\{\mathbf{w}\cdot\sum_{i=2}^m \mathbf{a}_i : (\mathbf{a}_1,\ldots,\mathbf{a}_m)\in\mathcal{T}'_j\}$ is well ordered by (a), so

$$\mu_j := \min\left\{\mathbf{w}\ \cdot\ \sum_{i=2}^m \mathbf{a}_i - \mathbf{w}\cdot\boldsymbol{\gamma}' + \ell_j : (\mathbf{a}_1,\ldots,\mathbf{a}_m)\in\mathcal{T}'_j\right\}$$

exists and is positive. Choose $n\in\mathbb{N}$ with $1/n<\min_j(\mu_j,\epsilon)$, and $n'>n$ with $\ell_j - \mathbf{w}\cdot\mathbf{s}_{n'}^{(j)} < 1/n$ for all $0\le j<L$.

Set $\mathcal{T}_0 = B_m$, and fix $0\le j<L$. Since ℓ_j is the limit of the sequence $\{\mathbf{w}\cdot\mathbf{s}_n^{(j)}\}$, for each j, by condition (c) we have $\ell_j = \mathbf{w}\cdot\tilde{\boldsymbol{\gamma}}_j$ for $\tilde{\boldsymbol{\gamma}}_j\in\mathbb{Q}^d$. We first consider the case that $(\mathbf{a}_1,\ldots,\mathbf{a}_m)\in\mathcal{T}_j\setminus\mathcal{T}_{j+1}$ satisfies

$$\mathbf{w}\cdot\sum_{i=1}^m \mathbf{a}_i > \mathbf{w}\cdot\boldsymbol{\gamma}' - 1/n'. \tag{2}$$

In that case $\mathbf{w}\cdot\mathbf{a}_1 \ge \mathbf{w}\cdot\mathbf{s}_{n'}^{(j)} > \ell_j - 1/n$. Thus $\mathbf{w}\cdot\sum_{i=2}^m \mathbf{a}_i < \mathbf{w}\cdot\boldsymbol{\gamma}' - \ell_j + 1/n < \mathbf{w}\cdot\boldsymbol{\gamma}' - \ell_j + \mu_j$, so by the definition of μ_j we have $\mathbf{w}\cdot\sum_{i=2}^m \mathbf{a}_i \le \mathbf{w}\cdot\boldsymbol{\gamma}' - \ell_j$, and thus $\mathbf{w}\cdot\sum_{i=2}^m \mathbf{a}_i \le \mathbf{w}\cdot\boldsymbol{\gamma}' - \mathbf{w}\cdot\tilde{\boldsymbol{\gamma}}_j$.

By induction there is a cone $C_{j,m-1}$ with $\mathbf{w}\in\operatorname{int}(C_{j,m-1}^\vee)$ and

$$\left\{\sum_{i=2}^m \mathbf{a}_i : (\mathbf{a}_1,\ldots,\mathbf{a}_m)\in B_m, \mathbf{w}\cdot\sum_{i=2}^m \mathbf{a}_i \le \mathbf{w}\cdot\boldsymbol{\gamma}' - \mathbf{w}\cdot\tilde{\boldsymbol{\gamma}}_j\right\} \subseteq \boldsymbol{\gamma}' - \tilde{\boldsymbol{\gamma}}_j - C_{j,m-1}.$$

By (d) for A there is an open cone C'_j with $\mathbf{w}\in\operatorname{int}({C'_j}^\vee)$ and $\{\mathbf{a}_1 : \mathbf{w}\cdot\mathbf{a}_1 \le \ell_j = \mathbf{w}\cdot\tilde{\boldsymbol{\gamma}}_j\}\subseteq\tilde{\boldsymbol{\gamma}}_j - C'_j$. Thus if $(\mathbf{a}_1,\ldots,\mathbf{a}_m)\in\mathcal{T}_j\setminus\mathcal{T}_{j+1}$ with $\mathbf{w}\cdot\sum_{i=1}^m \mathbf{a}_i > \mathbf{w}\cdot\boldsymbol{\gamma}' - 1/n'$ then

$$\sum_{i=1}^m \mathbf{a}_i \in \boldsymbol{\gamma}' - (C'_j + C_{j,m-1}).$$

Let C'' be the Minkowski sum $\sum_{j=0}^{L-1}(C'_j + C_{j,m-1})$. Note that for $(\mathbf{a}_1,\ldots,\mathbf{a}_m)\in\mathcal{T}_L$ we have $\mathbf{w}\cdot\sum_{i=1}^m \mathbf{a}_i < \mathbf{w}\cdot\boldsymbol{\gamma}' - 1/n'$. Thus for $(\mathbf{a}_1,\ldots,\mathbf{a}_m)\in B_m$ with $\mathbf{w}\cdot\sum_{i=1}^m \mathbf{a}_i > \mathbf{w}\cdot\boldsymbol{\gamma}' - 1/n'$ we have

$$\sum_{i=1}^m \mathbf{a}_i \in \boldsymbol{\gamma}' - C''.$$

The closure of the cone C''' generated by

$$\left\{\boldsymbol{\gamma}' - \sum_{i=1}^m \mathbf{a}_i : (\mathbf{a}_1,\ldots,\mathbf{a}_m)\in B_m, \mathbf{w}\cdot\sum_{i=1}^m \mathbf{a}_i \le \mathbf{w}\cdot\boldsymbol{\gamma}' - 1/n'\right\}$$

is pointed, $\mathbf{w}\in\operatorname{int}({C'''}^\vee)$, and $\{\sum_{i=1}^m \mathbf{a}_i : (\mathbf{a}_1,\ldots,\mathbf{a}_m)\in B_m, \mathbf{w}\cdot\sum_{i=1}^m \mathbf{a}_i \le \mathbf{w}\cdot\boldsymbol{\gamma}' - 1/n'\}\subseteq\boldsymbol{\gamma}' - C'''$.

Finally, let $C_m = C'' + C'''$. Then $\mathbf{w}\in\operatorname{int}(C_m^\vee)$, and for all $(\mathbf{a}_1,\ldots,\mathbf{a}_m)\in B_m$ with $\mathbf{w}\cdot\sum_{i=1}^m \mathbf{a}_i < \mathbf{w}\cdot\boldsymbol{\gamma}'$, we have $\sum_{i=1}^m \mathbf{a}_i \in \boldsymbol{\gamma}' - C_m$ as required. □

Theorem 2.7. *The set $\mathcal{A}_{\mathbf{w}}$ is a field family with respect to $(\mathbb{Q}^d, \preceq_{\mathbf{w}})$.*

Proof. We check each of the six axioms of a field family given in Definition 2.1.

Axiom (i): Each $A \in \mathcal{A}_{\mathbf{w}}$ is well ordered by condition (a) of Definition 2.2.

Axiom (ii): Every $\boldsymbol{\gamma} \in \mathbb{Q}^d$ is in $\mathcal{A}_{\mathbf{w}}$.

Axiom (iii): Fix $A, B \in \mathcal{A}_{\mathbf{w}}$. Condition (b) of p-discreteness holds for $A \cup B$ by [Saa17, Proposition 5.1]. Let σ_A, σ_B be the respective open cones containing $\mathbf{w}$ guaranteed by condition (a) of p-discreteness for A and B. The open cone $\sigma = \sigma_A \cap \sigma_B$ makes condition (a) hold for $A \cup B$. Similarly, for condition (d) we may use the intersection of the respective guaranteed open cones for A and B.

For (c), suppose $\{\mathbf{c}_i\}$ is a sequence in $A \cup B$ with $\{\mathbf{w} \cdot \mathbf{c}_i\}$ converging to $L \in \mathbb{R}$. Let $\{\mathbf{a}_j\}$ and $\{\mathbf{b}_j\}$ be the subsequences consisting of points in A and in B, respectively. If one of these subsequences is finite, then $\{\mathbf{c}_i\}$ converges to the limit of the other sequence. Otherwise, $\{\mathbf{w} \cdot \mathbf{a}_j\}$ and $\{\mathbf{w} \cdot \mathbf{b}_j\}$ also converge to L, so $\{\mathbf{a}_j\}$ converges to $\mathbf{a}$ and $\{\mathbf{b}_j\}$ converges to $\mathbf{b}$ with $\mathbf{a}, \mathbf{b} \in \mathbb{Q}^d$. We have $\mathbf{w} \cdot \mathbf{a} = \mathbf{w} \cdot \mathbf{b} = L$, so $\mathbf{w} \cdot (\mathbf{a} - \mathbf{b}) = 0$. Since $\mathbf{w} \in \mathbb{R}^d_{li}$, this implies that $\mathbf{a} - \mathbf{b} = 0$, so $\mathbf{a} = \mathbf{b}$, and $\{\mathbf{c}_i\}$ converges to the point $\mathbf{a} = \mathbf{b}$ in $\mathbb{Q}^d$ as required.

Axiom (iv): All conditions of p-discreteness are inherited from A by $B \subseteq A$.

Axiom (v): All conditions for a set $A \subseteq \mathbb{Q}^d$ to be p-discrete are invariant under translation by a point in $\mathbb{Q}^d$.

Axiom (vi): Fix $A \in \mathcal{A}_{\mathbf{w}}$ with $\mathbf{w} \cdot \mathbf{a} \geq 0$ for all $\mathbf{a} \in A$. Let $S(A)$ be the semigroup generated by A under addition. Let σ be as in condition (a) of p-discreteness for A, and let $\sigma_{\mathbf{0}}$ be as in condition (d) for $\mathbf{0}$. For any $\mathbf{w}' \in \sigma \cap \sigma_{\mathbf{0}}$ we have $A^+_{0,\mathbf{w}'} = A^+_{0,\mathbf{w}} = A$, since $\mathbf{w} \cdot \mathbf{a} \geq 0$ for all $\mathbf{a} \in A$, and A is p-discrete. Thus the fact that $S(A)$ is well ordered with respect to $\mathbf{w}'$ follows from [Neu49, Theorem 3.4]. Condition (b) for $S(A)$ follows as in [Saa17, Proposition 5.1].

We now prove condition (c) holds. Let $\{\mathbf{s}_i\}$ be a sequence in $S(A)$ such that $\{\mathbf{w} \cdot \mathbf{s}_i\}$ converges to some $L \in \mathbb{R}$. We want to show that $\{\mathbf{s}_i\}$ converges to some $\mathbf{s} \in \mathbb{Q}^d$. Pick $\boldsymbol{\gamma} \in \mathbb{Q}^d$ such that $\mathbf{w} \cdot \boldsymbol{\gamma} > L$. There exists an integer N such that for all $i > N$, we have $|\mathbf{w} \cdot \mathbf{s}_i - L| < \mathbf{w} \cdot \boldsymbol{\gamma} - L$. Consequently, for all $i > N$ we have $\mathbf{s}_i \in S(A)^-_{\boldsymbol{\gamma},\mathbf{w}}$. When $\mathbf{s}_i \neq \mathbf{0}$, Lemma 2.5 says that the number m of summands of $\mathbf{s}_i = \sum_{j=1}^m \mathbf{a}_i^{(j)}$ is bounded by some $M > 0$. When $\mathbf{s}_i = \mathbf{0}$, the assumption that $\mathbf{w} \cdot \mathbf{a} \geq 0$ for all $\mathbf{a} \in A$ implies that $\mathbf{s}_i = \sum_{j=1}^m \mathbf{a}_i^{(j)}$ has just one summand, namely $\mathbf{0} \in A$. Thus by passing to a subsequence $\{\mathbf{s}_{n(i)}\}$ of $\{\mathbf{s}_i\}$ we may assume that each $\mathbf{s}_{n(i)}$ is a sum of exactly m terms $\mathbf{s}_{n(i)} = \sum_{j=1}^m \mathbf{a}_i^{(j)}$ for some $m \leq M$, and the sequence $\{\mathbf{w} \cdot \mathbf{a}_i^{(j)}\}_i$ is weakly increasing for each $1 \leq j \leq m$. By hypothesis we have $0 \leq \mathbf{w} \cdot \mathbf{a}_i^{(j)} < \mathbf{w} \cdot \boldsymbol{\gamma}$. Hence for each fixed $1 \leq j \leq m$ the weakly increasing and bounded sequence $\{\mathbf{w} \cdot \mathbf{a}_i^{(j)}\}_i$ converges. Since A is p-discrete, each $\{\mathbf{a}_i^{(j)}\}_i$ converges to some $\mathbf{a}^{(j)} \in \mathbb{Q}^d$, so $\{\mathbf{s}_{n(i)}\}$ converges to $\mathbf{s} = \sum_{j=1}^m \mathbf{a}^{(j)} \in \mathbb{Q}^d$.

So far we have shown that $\{\mathbf{s}_i\}$ has a subsequence converging to $\mathbf{s} \in \mathbb{Q}^d$. We claim that the entire sequence $\{\mathbf{s}_i\}$ also converges to $\mathbf{s}$. Suppose there exists an $\epsilon > 0$ for which for all $n \in \mathbb{N}$, there is an index $i_n > n$ such that $|\mathbf{s}_{i_n} - \mathbf{s}| > \epsilon$. Note that $\{\mathbf{w} \cdot \mathbf{s}_{i_n}\}_n$ also converges to L. Hence, by applying the same argument as above, we deduce that the sequence $\{\mathbf{s}_{i_n}\}_n$ itself has a convergent subsequence to $\mathbf{s}$, contradicting the ϵ distance of the $\mathbf{s}_{i_n}$ away from $\mathbf{s}$.

Condition (d) of p-discreteness now holds by Lemma 2.6, which completes the proof. ☐

Definition 2.8. Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$ and let $\mathbf{w} \in \mathbb{R}^d_{li}$. We denote by $K_{\mathbf{w}}$ the field $\Bbbk^{\mathbb{Q}^d}(\mathcal{A}_{\mathbf{w}})$ of p-discrete series in direction $\mathbf{w}$ with coefficients in $\Bbbk$, with variables $t_1, \ldots, t_d$. Specifically, the elements of $K_{\mathbf{w}}$ are formal power series in variables $t_1, \ldots t_d$, with exponents in the field family $\mathcal{A}_{\mathbf{w}}$ and coefficients in $\Bbbk$.

3. The field of p-discrete series is algebraically closed

In this section, we prove that $K_{\mathbf{w}}$ is algebraically closed. We begin with some preliminary results.

Lemma 3.1. *Let $A \in \mathcal{A}_{\mathbf{w}}$, and assume that $\mathbf{w} \cdot \mathbf{a} < 0$ for all $\mathbf{a} \in A$. Then there is an $M > 0$ such that $|\mathbf{a}| \leq M$ for all $\mathbf{a} \in A$. That is, A is bounded.*

Proof. Since $A \in \mathcal{A}_{\mathbf{w}}$, the set A is contained in the polyhedron $\boldsymbol{\gamma} + C$ for some $\boldsymbol{\gamma} \in \mathbb{Q}^d$ and pointed cone C with $\mathbf{w} \in \operatorname{int}(C^\vee)$. The polyhedron $(\boldsymbol{\gamma} + C) \cap \{\mathbf{x} : \mathbf{w} \cdot \mathbf{x} \leq 0\}$ is a polytope, as its facet normals span $\mathbb{R}^d$, so it is bounded, and thus A is also bounded. □

Lemma 3.2. *Let $A \in \mathcal{A}_{\mathbf{w}}$, and assume that $\mathbf{w} \cdot \mathbf{a} < 0$ for all $\mathbf{a} \in A$. Then $S := \bigcup_{i=1}^{\infty} p^{-i} A$ is also in the field family $\mathcal{A}_{\mathbf{w}}$.*

Proof. We check that S satisfies the conditions of Definition 2.2.

Condition (a): We use similar ideas to [Saa17, Lemma 5.2].

Since condition (a) holds for A, there exists an open cone σ containing $\mathbf{w}$ for which $Q_{i,\mathbf{w}'} := \{\mathbf{w}' \cdot (p^{-i}\mathbf{a}) : p^{-i}\mathbf{a} \in p^{-i}A\}$ is well ordered for any $i \in \mathbb{Z}_{\geq 0}$ and $\mathbf{w}' \in \sigma \cap \mathbb{R}^d_{li}$. Since condition (d) holds for A, there is an open cone $\sigma_{\mathbf{0}}$ such that $A^-_{\mathbf{0},\mathbf{w}} = \{\mathbf{a} \in A : \mathbf{w} \cdot \mathbf{a} < 0\}$ is equal to $A^-_{\mathbf{0},\mathbf{z}}$ for any $\mathbf{z} \in \sigma_{\mathbf{0}} \cap \mathbb{R}^d_{li}$. Since $A = A^-_{\mathbf{0},\mathbf{w}}$ by assumption, we have that $\mathbf{a} \cdot \mathbf{z} < 0$ for any $\mathbf{a} \in A$ and any $\mathbf{z} \in \sigma_{\mathbf{0}}$. Set $V = \sigma \cap \sigma_{\mathbf{0}}$.

Now, let $\mathbf{w}' \in V \cap \mathbb{R}^d_{li}$ and assume that there exists an infinite strictly decreasing sequence $T := \{\mathbf{w}' \cdot (p^{-k_j}\mathbf{a}_j)\}_j$, where each $\mathbf{a}_j \in A$, and each $k_j > 0$. Then there must be infinitely many distinct integers k_j in this sequence, or else some $Q_{i,\mathbf{w}'}$ would not be well ordered. So, there is a strictly increasing subsequence $\{k_{n(j)}\}_j$ of the sequence $\{k_j\}_j$. Consider the associated subsequence of T:

$$\mathbf{w}' \cdot (p^{-k_{n(0)}}\mathbf{a}_{n(0)}) > \mathbf{w}' \cdot (p^{-k_{n(1)}}\mathbf{a}_{n(1)}) > \cdots > \mathbf{w}' \cdot (p^{-k_{n(j)}}\mathbf{a}_{n(j)}) > \cdots . \tag{3}$$

Since the set $\{\mathbf{w}' \cdot \mathbf{a}_{n(j)}\}_j$ is well ordered, as $\mathbf{w}' \in V \cap \mathbb{R}^d_{li}$, it has a smallest element. In particular, there must exist indices $s < t$ with $\mathbf{w}' \cdot \mathbf{a}_{n(s)} \leq \mathbf{w}' \cdot \mathbf{a}_{n(t)}$. Finally $\mathbf{w}' \cdot \mathbf{a}_{n(t)}$ is negative, as $\mathbf{w}' \in \sigma_{\mathbf{0}} \cap \mathbb{R}^d_{li}$, and $k_{n(s)} < k_{n(t)}$, so we have

$$\mathbf{w}' \cdot (p^{-k_{n(s)}}\mathbf{a}_{n(s)}) \leq \mathbf{w}' \cdot (p^{-k_{n(s)}}\mathbf{a}_{n(t)}) < \mathbf{w}' \cdot (p^{-k_{n(t)}}\mathbf{a}_{n(t)}),$$

which contradicts (3). Hence there is no infinite strictly decreasing sequence T, and the set $\{\mathbf{w}' \cdot \mathbf{s} : \mathbf{s} \in S\}$ is well ordered.

Condition (b): Since $A \in \mathcal{A}_{\mathbf{w}}$, there is $N > 0$, $\boldsymbol{\gamma} \in \mathbb{Q}^d$, and a pointed cone C such that $\mathbf{w} \in \operatorname{int}(C^\vee)$ and A is contained in the intersection

$$(\boldsymbol{\gamma} + C) \cap \left(\bigcup_{j \geq 0} \frac{1}{Np^j} \mathbb{Z}^d \right).$$

Since $\mathbf{w} \cdot \mathbf{a} < 0$ for $\mathbf{w} \in A$, we must have that $\mathbf{w} \cdot \boldsymbol{\gamma} < 0$. Let C_1 be the convex hull of C and the ray spanned by $-\boldsymbol{\gamma}$. This is a pointed cone, and $\mathbf{w} \in C_1^\vee$ because $\mathbf{w} \in C^\vee$ and $-\mathbf{w} \cdot \boldsymbol{\gamma} > 0$. For

$\mathbf{a} \in A$, since $\mathbf{a} - \boldsymbol{\gamma} \in C$, we have that $p^{-i}\mathbf{a} - p^{-i}\boldsymbol{\gamma} \in C$, and that $(p^{-i}\mathbf{a} - p^{-i}\boldsymbol{\gamma}) + (1 - p^{-i})(-\boldsymbol{\gamma}) = p^{-i}\mathbf{a} - \boldsymbol{\gamma} \in C_1$. Thus, $p^{-i}\mathbf{a} \in \boldsymbol{\gamma} + C_1$, and we conclude that $p^{-i}A$ is contained in

$$(\boldsymbol{\gamma} + C_1) \cap \left(\bigcup_{j \geq 0} \frac{1}{Np^j}\mathbb{Z}^d \right). \tag{4}$$

Since C_1 does not depend on the particular $i > 0$, we conclude that S is contained in the set (4).

Condition (c): Suppose that $\{p^{-k_j}\mathbf{a}_j\}_j$ is a sequence in S with each $\mathbf{a}_j \in A$ such that $\{\mathbf{w} \cdot (p^{-k_j}\mathbf{a}_j)\}_j$ converges to an element of $\mathbb{R}$.

If $\{p^{-k_j}\mathbf{a}_j\}_j$ is contained in $p^{-i}A$ for some fixed i then we may apply condition (c) for A to conclude that the sequence converges to an element of $\mathbb{Q}^d$. If the sequence $\{p^{-k_j}\mathbf{a}_j\}_j$ is contained in a finite union of the sets $p^{-i}A$, then we may apply the condition (c) part of the argument in item (iii) of the proof of Theorem 2.7 to conclude that our original sequence $\{p^{-k_j}\mathbf{a}_j\}_j$ converges to an element of $\mathbb{Q}^d$.

Assume that the sequence is not contained in a finite union of sets $p^{-i}A$. Then, there is a strictly increasing subsequence $\{k_{n(j)}\}_j$ of the sequence $\{k_j\}_j$. Since the elements of $\{\mathbf{a}_j\}_j$ are bounded in length by Lemma 3.1, the associated subsequence $\{p^{-k_{n(j)}}\mathbf{a}_{n(j)}\}_j$ of $\{p^{-k_j}\mathbf{a}_j\}_j$ converges to $\mathbf{0}$. Therefore, since $\{\mathbf{w} \cdot (p^{-k_j}\mathbf{a}_j)\}_j$ converges by assumption, it must converge to 0. We will show that this forces our original sequence $\{p^{-k_j}\mathbf{a}_j\}_j$ to converge to $\mathbf{0}$.

Suppose otherwise. Then there is an $\epsilon > 0$ and a subsequence $\{p^{-k_{m(j)}}\mathbf{a}_{m(j)}\}_j$ of $\{p^{-k_j}\mathbf{a}_j\}_j$ such that $p^{-k_{m(j)}}|\mathbf{a}_{m(j)}| > \epsilon$ for all $j \geq 0$. By Lemma 3.1, there exists an $M > 0$ such that $|\mathbf{a}_{m(j)}| < M$ for each $j \geq 0$. Consequently, $p^{-k_{m(j)}}M > p^{-k_{m(j)}}|\mathbf{a}_{m(j)}| > \epsilon$, and we conclude that $p^{-k_{m(j)}} > \epsilon/M$ for all $j \geq 0$.

Because $\{\mathbf{w} \cdot (p^{-k_{m(j)}}\mathbf{a}_{m(j)})\}_j$ converges to 0, and the coefficients $p^{-k_{m(j)}}$ are bounded below by ϵ/M, we have that $\{\mathbf{w} \cdot \mathbf{a}_{m(j)}\}$ converges to 0. As $A \in \mathcal{A}_{\mathbf{w}}$, it follows that $\{\mathbf{a}_{m(j)}\}_j$ converges to an element $\mathbf{z} \in \mathbb{Q}^d$ such that $\mathbf{w} \cdot \mathbf{z} = 0$. But the only $\mathbf{z} \in \mathbb{Q}^d$ such that $\mathbf{w} \cdot \mathbf{z} = 0$ is $\mathbf{z} = \mathbf{0}$ since $\mathbf{w} \in \mathbb{R}^d_{li}$. This is a contradiction. Hence our original sequence $\{p^{-k_j}\mathbf{a}_j\}_j$ converges to $\mathbf{0}$.

Condition (d): Since A satisfies condition (d), for each $\boldsymbol{\delta} \in \mathbb{Q}^d$, there is an open cone $\sigma_{\boldsymbol{\delta}}$ containing $\mathbf{w}$ such that for all $\mathbf{w}' \in \sigma_{\boldsymbol{\delta}} \cap \mathbb{R}^d_{li}$, we have $A^+_{\boldsymbol{\delta},\mathbf{w}'} = A^+_{\boldsymbol{\delta},\mathbf{w}}$ and $A^-_{\boldsymbol{\delta},\mathbf{w}'} = A^-_{\boldsymbol{\delta},\mathbf{w}}$. Fix $\boldsymbol{\gamma}' \in \mathbb{Q}^d$. We consider three cases: (i) $\mathbf{w} \cdot \boldsymbol{\gamma}' > 0$, (ii) $\mathbf{w} \cdot \boldsymbol{\gamma}' = 0$, and (iii) $\mathbf{w} \cdot \boldsymbol{\gamma}' < 0$.

When $\mathbf{w} \cdot \boldsymbol{\gamma}' > 0$, we have $S = S^-_{\boldsymbol{\gamma}',\mathbf{w}}$, and $S^+_{\mathbf{w},\boldsymbol{\gamma}'} = \emptyset$, since $\mathbf{w} \cdot \mathbf{s} < 0$ for all $\mathbf{s} \in S$. Let $V := \{\mathbf{v} \in \mathbb{R}^d : \mathbf{v} \cdot \boldsymbol{\gamma}' > 0\}$. Then, for $\mathbf{a} \in A$ and $\mathbf{w}' \in (\sigma_{\mathbf{0}} \cap V) \cap \mathbb{R}^d_{li}$, we have $\mathbf{w}' \cdot \mathbf{a} < 0$ since $A = A^-_{\mathbf{0},\mathbf{w}} = A^-_{\mathbf{0},\mathbf{w}'}$. So, $\mathbf{w}' \cdot \mathbf{s} < 0$ for any $\mathbf{s} \in S$. Since $\mathbf{w}' \in V$, we have $\mathbf{w}' \cdot \mathbf{s} < \mathbf{w}' \cdot \boldsymbol{\gamma}'$ for every $\mathbf{s} \in S$. Hence, $S = S^-_{\boldsymbol{\gamma}',\mathbf{w}'}$ and $S^+_{\boldsymbol{\gamma}',\mathbf{w}'} = \emptyset$. We conclude that for each $\mathbf{w}' \in (\sigma_{\mathbf{0}} \cap V) \cap \mathbb{R}^d_{li}$, we have $S^+_{\boldsymbol{\gamma}',\mathbf{w}'} = S^+_{\boldsymbol{\gamma}',\mathbf{w}}$ and $S^-_{\boldsymbol{\gamma}',\mathbf{w}'} = S^-_{\boldsymbol{\gamma}',\mathbf{w}}$.

Next consider the case when $\mathbf{w} \cdot \boldsymbol{\gamma}' = 0$. Then $\boldsymbol{\gamma}' = \mathbf{0}$ since $\mathbf{w} \in \mathbb{R}^d_{li}$. Then for any $\mathbf{w}' \in \sigma_{\mathbf{0}} \cap \mathbb{R}^d_{li}$, we have $S = S^-_{\mathbf{0},\mathbf{w}} = S^-_{\mathbf{0},\mathbf{w}'}$ and $\emptyset = S^+_{\mathbf{0},\mathbf{w}} = S^+_{\mathbf{0},\mathbf{w}'}$.

Finally, consider the case $\mathbf{w} \cdot \boldsymbol{\gamma}' < 0$. Let $i_0 > 0$ be such that for all $i \geq i_0$, we have $\mathbf{w} \cdot (p^{-i}\mathbf{a}) > \mathbf{w} \cdot \boldsymbol{\gamma}'$, for all $\mathbf{a} \in A$. Such an i_0 exists because $\mathbf{w} \cdot \boldsymbol{\gamma}' < 0$ and the set $\{\mathbf{w} \cdot \mathbf{a} : \mathbf{a} \in A\}$ is well ordered.

Let $\mathbf{w}' \in \left(\bigcap_{i=1}^{i_0} \sigma_{p^i\boldsymbol{\gamma}'} \right) \cap \mathbb{R}^d_{li}$. Then,

(1) for each $1 \leq i \leq i_0$, we have $(p^{-i}A)^+_{\boldsymbol{\gamma}',\mathbf{w}} = (p^{-i}A)^+_{\boldsymbol{\gamma}',\mathbf{w}'}$ and $(p^{-i}A)^-_{\boldsymbol{\gamma}',\mathbf{w}} = (p^{-i}A)^-_{\boldsymbol{\gamma}',\mathbf{w}'}$, because $\mathbf{w}' \in \sigma_{p^i\boldsymbol{\gamma}'}$;

(2) by the choice of i_0 and since $\mathbf{w'} \in \sigma_{p^{i_0}\gamma'}$, we have that $p^{-i_0}A = (p^{-i_0}A)^+_{\gamma',\mathbf{w}} = (p^{-i_0}A)^+_{\gamma',\mathbf{w'}}$. Furthermore, for each $i > i_0$ and each $\mathbf{a} \in A$, we have $\mathbf{w'} \cdot (p^{-i}\mathbf{a}) > \mathbf{w'} \cdot (p^{-i_0}\mathbf{a})$. Thus, for each $i > i_0$, we have $(p^{-i}A)^+_{\gamma',\mathbf{w'}} = p^{-i}A$ and $(p^{-i}A)^-_{\gamma',\mathbf{w'}} = \emptyset$.

Therefore, for all $i \geq 1$, we have $(p^{-i}A)^+_{\gamma',\mathbf{w}} = (p^{-i}A)^+_{\gamma',\mathbf{w'}}$ and $(p^{-i}A)^-_{\gamma',\mathbf{w}} = (p^{-i}A)^-_{\gamma',\mathbf{w'}}$. It follows that $S^+_{\gamma',\mathbf{w}} = S^+_{\gamma',\mathbf{w'}}$ and $S^-_{\gamma',\mathbf{w}} = S^-_{\gamma',\mathbf{w'}}$. □

Let $\nu : K_\mathbf{w} \to \mathbb{R} \cup \{\infty\}$ be the valuation defined by

$$\nu(f) := \min\{\mathbf{w} \cdot \mathbf{a} : \mathbf{a} \in \operatorname{supp}(f)\}$$

for $f \neq 0$ and $\nu(0) := \infty$.

Remark 3.3 (Properties of $K_\mathbf{w}$ with the valuation ν)**.**

(i) The valued field $K_\mathbf{w}$ has valuation ring $R_\mathbf{w} := \{f \in K_\mathbf{w} : \nu(f) \geq 0\}$. The valuation ring has maximal ideal $\mathfrak{m} = \{f \in K_\mathbf{w} : \nu(f) > 0\}$. The residue field $R_\mathbf{w}/\mathfrak{m}$ is isomorphic to the field of coefficients $\Bbbk$; the map which sends $\overline{f} \in R_\mathbf{w}/\mathfrak{m}$ to the constant term of $f \in R_\mathbf{w}$ gives the isomorphism.

(ii) The value group of $K_\mathbf{w}$ is $\mathbb{Q}^d$, which is n-divisible for all n.

(iii) By [Ray68, Theorem 2], $K_\mathbf{w}$ is a *Henselian valued field*. This means that $R_\mathbf{w}$ satisfies Hensel's lemma: if $g(X) \in R_\mathbf{w}[X]$ and its reduction mod $\mathfrak{m}$, $\overline{g}(X) \in \Bbbk[X]$ has a simple root $a \in \Bbbk$, then there exists a unique $b \in R_\mathbf{w}$ such that $g(b) = 0$ and $\overline{b} = a \in \Bbbk$.

We can now prove the main theorem of this section. Our proof is essentially the same as [Saa17, Theorem 5.3] except that the third bullet point in Saavedra's proof is replaced by our Lemma 3.2.

Theorem 3.4. *Fix an algebraically closed field $\Bbbk$ of characteristic $p > 0$. The field $K_\mathbf{w} = \Bbbk^{\mathbb{Q}^d}(\mathcal{A}_\mathbf{w})$ is algebraically closed.*

Proof. We proceed by contradiction and assume that $K_\mathbf{w}$ is not algebraically closed. Then $K_\mathbf{w}$ admits a proper extension of finite degree. By [Ray68, Lemma 4] there exists $f \in K_\mathbf{w}$ with $\nu(f) < 0$ such that the polynomial $X^p - X - f \in K_\mathbf{w}[X]$ is irreducible. We express f as the sum of two elements $f^+, f^- \in K_\mathbf{w}$ as follows:

$$f^+(\mathbf{v}) = \begin{cases} f(\mathbf{v}) & \mathbf{v} \in \operatorname{supp}(f), \mathbf{w} \cdot \mathbf{v} \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad f^-(\mathbf{v}) = \begin{cases} f(\mathbf{v}) & \mathbf{v} \in \operatorname{supp}(f), \mathbf{w} \cdot \mathbf{v} < 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since $\operatorname{supp}(f^+)$ and $\operatorname{supp}(f^-)$ are subsets of $\operatorname{supp}(f) \in \mathcal{A}_\mathbf{w}$, it follows that $\operatorname{supp}(f^+), \operatorname{supp}(f^-) \in \mathcal{A}_\mathbf{w}$. It suffices to prove the following two claims:

Claim 1. There exists $g \in K_\mathbf{w}$ that is a root of $X^p - X - f^+$.

Claim 2. There exists $h \in K_\mathbf{w}$ that is a root of $X^p - X - f^-$.

Indeed, for such g and h we have that $g+h$ is a root of $X^p - X - f$ which contradicts the irreducibility of $X^p - X - f$. Hence $K_\mathbf{w}$ is algebraically closed.

Proof of Claim 1. We have that f^+ belongs to the valuation ring $R_\mathbf{w}$ of the valued field $K_\mathbf{w}$, since $\nu(f^+) \geq 0$ by definition. We will use the Henselian property of $R_\mathbf{w}$ to find the root. Indeed, the reduction of $X^p - X - f^+$ modulo the maximal ideal $\mathfrak{m} \subseteq R_\mathbf{w}$ is $X^p - X - f^+(\mathbf{0}) \in \Bbbk[X]$. Because $\Bbbk$ is algebraically closed, $X^p - X - f^+(\mathbf{0})$ factors. Furthermore, $X^p - X - f^+(\mathbf{0})$ has simple roots because its derivative is -1. By the Henselian property each of these roots lifts to a distinct root of $X^p - X - f^+$, so the desired root g exists. This completes the proof of Claim 1.

Proof of Claim 2. Define $h : \mathbb{Q}^d \to \Bbbk$ by

$$h(\mathbf{v}) = \begin{cases} \sum_{j=1}^{\infty} \left(f^-(p^j\mathbf{v})\right)^{p^{-j}} & \mathbf{w}\cdot\mathbf{v} < 0 \\ 0 & \text{otherwise.} \end{cases}$$

We will see in (5) that this is exactly the root we want.

Subclaim a. h is well defined.

Since $\Bbbk$ is algebraically closed of characteristic p, the p^jth roots exist and are unique. We now show that the infinite sum in the definition of h is in fact a finite sum. Assume for contradiction that there are infinitely many $p^j\mathbf{v}$s in $\operatorname{supp}(f^-)$. Since $\mathbf{w}\cdot\mathbf{v} < 0$, these infinitely many $p^j\mathbf{v}$s form an infinite strictly decreasing sequence. This is a contradiction: $\operatorname{supp}(f^-) \in \mathcal{A}_{\mathbf{w}}$ and so $\operatorname{supp}(f^-)$ is well ordered by item (i) of the definition of field family (Definition 2.1).

Subclaim b. $\operatorname{supp}(h) \in \mathcal{A}_{\mathbf{w}}$.

Since $\operatorname{supp}(h)$ is a subset of $S = \bigcup_{i=1}^{\infty} p^{-i}\operatorname{supp}(f)$, and $S \in \mathcal{A}_{\mathbf{w}}$, by Lemma 3.2, we conclude that $\operatorname{supp}(h) \in \mathcal{A}_{\mathbf{w}}$ as $\mathcal{A}_{\mathbf{w}}$ is a field family.

Subclaim c. $h^p - h - f^- = 0$.

First suppose $\mathbf{w}\cdot\mathbf{v} < 0$. Then $\mathbf{w}\cdot p^{-1}\mathbf{v} < 0$. We claim that

$$h^p(\mathbf{v}) = h\left(p^{-1}\mathbf{v}\right)^p = \sum_{j=1}^{\infty}\left(f^-(p^{j-1}\mathbf{v})\right)^{p^{-(j-1)}}.$$

The first equality is the Frobenius homomorphism, and the second equality is by the definition of h. Hence, we have

$$\begin{aligned} h^p(\mathbf{v}) - h(\mathbf{v}) - f^-(\mathbf{v}) &= \sum_{j=1}^{\infty}\left(f^-(p^{j-1}\mathbf{v})\right)^{p^{-(j-1)}} - \sum_{j=1}^{\infty}\left(f^-(p^{j}\mathbf{v})\right)^{p^{-j}} - f^-(\mathbf{v}) \\ &= \sum_{j'=0}^{\infty}\left(f^-(p^{j'}\mathbf{v})\right)^{p^{-j'}} - \sum_{j=1}^{\infty}\left(f^-(p^{j}\mathbf{v})\right)^{p^{-j}} - f^-(\mathbf{v}) \\ &= f^-(\mathbf{v}) - f^-(\mathbf{v}) = 0. \end{aligned} \tag{5}$$

On the other hand, when $\mathbf{w}\cdot\mathbf{v} \geq 0$ we have $h^p(\mathbf{v}) = h(\mathbf{v}) = 0 = f^-(\mathbf{v})$ by definition. So h satisfies the polynomial $X^p - X - f^-$. ☐

4. Toric Bertini Theorems

In this section we prove the main theorem of this paper: Theorem 1.1.

If $\operatorname{char}(\Bbbk) > 0$ set $p = \operatorname{char}(\Bbbk)$; otherwise set $p = 1$. For $\mathbf{w} \in \mathbb{R}^d_{li}$, let $K_{\mathbf{w}}$ be the field of p-discrete series in direction $\mathbf{w}$ with coefficients in $\Bbbk$, with variables $t_1, \dots, t_d$, as defined in Definition 2.8. If $\operatorname{char}(\Bbbk) = 0$, we set $p = 1$ in this definition; this is still an algebraically closed field by [McD95, AI09]. For an open cone C containing $\mathbf{w}$, let K_C be the subring of $K_{\mathbf{w}}$ consisting of those elements whose supports are well ordered with respect to $\preceq_{\mathbf{w}'}$ for every $\mathbf{w}' \in C \cap \mathbb{R}^d_{li}$, and for which the only allowable denominators in exponents are powers of p. The fact that K_C is a subring follows from the fact that unions and sums of well-ordered subsets of $\mathbb{Q}^d$ are again well ordered. We will use the following properties of K_C:

(1) For $\alpha \in K_C$, there is $\boldsymbol{\gamma} \in \mathbb{Q}^d$ with $\operatorname{supp}(\alpha) \subset \boldsymbol{\gamma} + C^\vee$.
(2) Polynomials $\alpha \in \Bbbk[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ are in K_C for every cone C.
(3) If $C' \subseteq C$, then K_C is a subring of $K_{C'}$.

Let $\Bbbk\{\{x\}\}$ be the field of generalized Puiseux series in one variable x. This consists of all formal power series with coefficients in $\Bbbk$ whose support is a well-ordered subset of $\bigcup_{i=0}^{\infty} \frac{1}{Np^i}\mathbb{Z}$ for some integer N.

Definition 4.1. Fix $\mathbf{w} \in \mathbb{R}^d_{li}$, and an open cone C containing $\mathbf{w}$. For $\mathbf{n} = (n_1, \ldots, n_d) \in C \cap \mathbb{Z}^d$, and $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d) \in (\Bbbk^*)^d$ we define $\phi^{\boldsymbol{\theta}}_{\mathbf{n}} : K_C \to \Bbbk\{\{x\}\}$ by

$$\phi^{\boldsymbol{\theta}}_{\mathbf{n}} \left(\sum c_{\mathbf{u}} \mathbf{t}^{\mathbf{u}} \right) = \sum c_{\mathbf{u}} \boldsymbol{\theta}^{\mathbf{u}} x^{\mathbf{n} \cdot \mathbf{u}}.$$

Lemma 4.2. *For every $\mathbf{n} = (n_1, \ldots, n_d) \in C \cap \mathbb{Z}^d$, and every $\boldsymbol{\theta} \in (\Bbbk^*)^d$, the function $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}$ is well defined and is a ring homomorphism from K_C to $\Bbbk\{\{x\}\}$.*

Proof. We first observe that the condition on K_C that the denominators of exponents be only powers of p ensures that the expression $\boldsymbol{\theta}^{\mathbf{u}}$ is well defined, as pth roots are unique in characteristic p. Fix $\alpha \in K_C$. To finish showing that $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}$ is well defined, we need to check that the substitution does not map infinitely many terms of α to the same term in $\Bbbk\{\{x\}\}$. In other words, we need to check that the fibers of the map

$$\operatorname{supp}(\alpha) \to \mathbb{Q}, \quad \mathbf{v} \mapsto \mathbf{n} \cdot \mathbf{v} \tag{6}$$

are finite. Suppose on the contrary that for some $r \in \mathbb{Q}$ the preimage $\{\mathbf{v} \in \operatorname{supp}(\alpha) : \mathbf{n} \cdot \mathbf{v} = r\}$ is infinite. Consider an infinite non-repeating sequence in this set.

Since C is an open cone and $\mathbf{n} \in C$, there are vectors $\mathbf{w}_1, \ldots, \mathbf{w}_d \in C \cap \mathbb{R}^d_{li}$ such that $\mathbf{n} = a_1\mathbf{w}_1 + \cdots + a_d\mathbf{w}_d$, for some positive real numbers $a_1, \ldots, a_d$. Consider the infinite sequence above. Since it is well ordered with respect to $\preceq_{\mathbf{w}_1}$, we can pass to an increasing subsequence. We can now repeat this operation using the orders given by $\mathbf{w}_2, \ldots, \mathbf{w}_d$, to get a sequence that is increasing with respect to all the orders given by $\mathbf{w}_1, \ldots, \mathbf{w}_d$. Thus the dot product of the sequence with $\mathbf{n}$ must be strictly increasing as well, contradicting the assumption that it is constant.

This shows that the substitution is a well-defined map. The fact that $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}$ is a ring homomorphism follows from the fact that taking the dot product with $\mathbf{n}$ commutes with union and sum of subsets of $\mathbb{Q}^d$. □

Since $\Bbbk[t_1, \ldots, t_d] \subseteq K_C$ for all cones C, we can also define an analogous homomorphism from $\Bbbk[t_1, \ldots, t_d, y]$:

$$\phi^{\boldsymbol{\theta}}_{\mathbf{n}} \colon \Bbbk[t_1, \ldots, t_d, y] \to \Bbbk[x^{\pm 1}, y]$$

by

$$\phi^{\boldsymbol{\theta}}_{\mathbf{n}} \left(\sum c_{\mathbf{u},j} \mathbf{t}^{\mathbf{u}} y^j \right) = \sum \phi^{\boldsymbol{\theta}}_{\mathbf{n}}(c_{\mathbf{u},j} \mathbf{t}^{\mathbf{u}}) y^j.$$

We say an element $\alpha \in K_C$ has unbounded support if the support of α is not contained in a bounded region in $\mathbb{R}^d$.

Lemma 4.3. *Let $\alpha \in K_C$ have unbounded support and be algebraic over $\Bbbk(t_1, \ldots, t_d)$. Then there exists an open cone $C' \subseteq C$ containing $\mathbf{w}$ such that for any integer vector $\mathbf{n} \in C'$ and any $\boldsymbol{\theta} \in (\Bbbk^*)^d$ the image $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha)$ is not a polynomial in $\Bbbk[x^{\pm 1}]$.*

Proof. Since α is algebraic over $\Bbbk(t_1, \ldots, t_d)$, there is a polynomial $h \in \Bbbk[t_1, \ldots, t_d, y]$ such that $h(t_1, \ldots, t_d, \alpha) = 0$. Since $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}$ is a ring homomorphism for any $\mathbf{n} \in C \cap \mathbb{Z}^d$ and any $\boldsymbol{\theta} \in (\Bbbk^*)^d$, for such choices we then have

$$\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(h)(x, \phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha)) = h(\theta_1 x^{n_1}, \ldots, \theta_d x^{n_d}, \phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha)) = 0.$$

Suppose $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha)$ is a polynomial in $\Bbbk[x^{\pm 1}]$. In order to have $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(h)(x, \phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha)) = 0$, there must be two distinct monomials $x^{a_i}y^i$ and $x^{a_j}y^j$ in $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(h)$ that have the same maximal x-degree after plugging in $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha)$ for y. Then

$$a_i + i \cdot \deg_x \left(\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha)\right) = a_j + j \cdot \deg_x \left(\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha)\right)$$

so $i \neq j$ and we have $\deg_x \phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha) = \frac{a_i - a_j}{j-i}$. Since each monomial in $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(h)$ is the specialization of a monomial in h,

$$\deg_x \phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha) \leq \max \left\{ \mathbf{n} \cdot \left(\frac{\mathbf{u}_r - \mathbf{u}_s}{s-r} \right) : \mathbf{t}^{\mathbf{u}_r} y^r \text{ and } \mathbf{t}^{\mathbf{u}_s} y^s \text{ are monomials in } h \right\}.$$

For all directions $\mathbf{n}$ sufficiently close to $\mathbf{w}$, the maximum is attained at the same pair of monomials $\mathbf{t}^{\mathbf{u}_i}y^i$ and $\mathbf{t}^{\mathbf{u}_j}y^j$ in h. Explicitly, we may take $\mathbf{n}$ in the open cone C_1 containing $\mathbf{w}$ in the normal fan of the convex hull of the $\frac{\mathbf{u}_r - \mathbf{u}_s}{s-r}$, which is also the normal cone of $\frac{\mathbf{u}_i - \mathbf{u}_j}{j-i}$.

Since $\alpha \in K_C$, there is $\boldsymbol{\gamma} \in \mathbb{Q}^d$ with $\operatorname{supp}(\alpha) \subset \boldsymbol{\gamma} + C^\vee$. Choose an open cone $C_2 \subseteq C_1 \cap C$ whose closure is contained in the interior of C.

Then $(\boldsymbol{\gamma} + C^\vee) \setminus (\frac{\mathbf{u}_i - \mathbf{u}_j}{j-i} + {C_2}^\vee)$ is bounded, but $\operatorname{supp}(\alpha)$ is unbounded, so we can choose $\mathbf{v} \in \operatorname{supp}(\alpha) \cap \frac{\mathbf{u}_i - \mathbf{u}_j}{j-i} + {C_2}^\vee$. Then since $\mathbf{v} \in \frac{\mathbf{u}_i - \mathbf{u}_j}{j-i} + {C_2}^\vee$, for all $\mathbf{n} \in C_2$ we have $\mathbf{n} \cdot \mathbf{v} \geq \mathbf{n} \cdot \frac{\mathbf{u}_i - \mathbf{u}_j}{j-i}$. By axiom (d) of p-discreteness for $K_{\mathbf{w}}$, there is an open cone C_3 containing $\mathbf{w}$ for which for any $\mathbf{n} \in C_3 \cap \mathbb{Z}^d$ the exponent $\mathbf{v}$ is the only preimage of the map (6), so does not get cancelled after the substitution $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}$. Thus for $\mathbf{n} \in C_2 \cap C_3$ the monomial $x^{\mathbf{n} \cdot \mathbf{v}}$ appears in a term of $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha)$, contradicting the degree bound above. Thus $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha)$ cannot be a polynomial for $\mathbf{n} \in C' := C_2 \cap C_3$. □

Lemma 4.4. *Let $\alpha \in K_C$ be an element with infinite but bounded support. Then there exists an open cone $C' \subseteq C$ containing $\mathbf{w}$ and a sublattice $H \subset \mathbb{Z}^d$ such that for any integer vector $\mathbf{n} \in (C' \setminus H) \cap \mathbb{Z}^d$, the substitution $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha)$ is not a polynomial. Moreover, the sublattice H can be chosen to have an arbitrarily large index in $\mathbb{Z}^d$.*

Proof. The support of α must contain points in $\mathbb{Q}^d$ with coordinates whose denominators (in reduced form) are arbitrarily large because there are only finitely many elements in $\mathbb{Q}^d$ in a bounded region with denominators smaller than a given bound. Let $\mathbf{v}$ be an element of $\operatorname{supp}(\alpha)$ with a non-integer coordinate. By axiom (d) of p-discreteness, there is an open cone $C' \subseteq C$ containing $\mathbf{w}$ for which for any $\mathbf{n} \in C' \cap \mathbb{Z}^d$ the term $\mathbf{v}$ is the only preimage of the map (6), so does not get cancelled by the substitution $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}$. Let $H = \{\mathbf{n} \in \mathbb{Z}^d \mid \mathbf{n} \cdot \mathbf{v} \in \mathbb{Z}\}$. For all integer vectors $\mathbf{n} \in C'$ not in H, the map $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}$ sends the term $\mathbf{t}^{\mathbf{v}}$ in α to the term $\boldsymbol{\theta}^{\mathbf{v}} x^{\mathbf{n} \cdot \mathbf{v}}$ in $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha)$ with non-integer exponent, so $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha)$ is not a polynomial. The lattice H is a sublattice of $\mathbb{Z}^d$ but is not all of $\mathbb{Z}^d$ because if the denominator (in reduced form) of jth coordinate of $\mathbf{v}$ is $M > 1$, then $\mathbf{e}_j, 2\mathbf{e}_j, \ldots, (M-1)\mathbf{e}_j$ are not in H, so H has index at least M in $\mathbb{Z}^d$. The element $\mathbf{v} \in \operatorname{supp}(\alpha)$ can be chosen to have a coordinate with an arbitrarily large denominator, so the sublattice H to avoid can be chosen to have an arbitrarily large index. □

Following Amoroso-Sombra [AS19], we say that for an irreducible variety X, a map $\pi : X \to (\Bbbk^*)^d$ has the *PB property* if for every isogeny (surjective group homomorphism with finite kernel) λ of $(\Bbbk^*)^d$ the pullback $\lambda^* X := X \times_\lambda (\Bbbk^*)^d$ in (7) is irreducible.

$$\begin{array}{ccc} X \times_\lambda (\Bbbk^*)^d & \longrightarrow & X \\ \big\downarrow {\scriptstyle \lambda^*\pi} & & \big\downarrow {\scriptstyle \pi} \\ (\Bbbk^*)^d & \xrightarrow{\ \lambda\ } & (\Bbbk^*)^d \end{array} \tag{7}$$

Example 4.5. Consider the variety $X = \mathrm{V}(x^2 - yz^2)$. The map $\pi : X \to (\Bbbk^*)^2$ projecting onto the first two coordinates does not satisfy the PB property, since for the isogeny $\lambda : (\Bbbk^*)^2 \to (\Bbbk^*)^2$ given by $(x, y) \mapsto (x, y^2)$ we have that $X \times_\lambda (\Bbbk^*)^2 = \mathrm{V}(x^2 - y^2z^2) = \mathrm{V}(x - yz) \cup \mathrm{V}(x + yz)$ is reducible.

Theorem 4.6. *Let $f \in \Bbbk[t_1^{\pm 1}, \dots, t_d^{\pm 1}, y]$ be irreducible and monic in y. Suppose that the projection of $\mathrm{V}(f) \subseteq (\Bbbk^*)^d \times \mathbb{A}^1$ onto the first d coordinates has the PB property. Fix $\mathbf{w} \in \mathbb{R}^d_{li}$. Then there exists an open cone C containing $\mathbf{w}$ and finitely many sublattices $H_1, \dots, H_r \subset \mathbb{Z}^d$ such that for all vectors $\mathbf{n} \in (C \cap \mathbb{Z}^d) \setminus (H_1 \cup \cdots \cup H_r)$, and all $\boldsymbol{\theta} \in (\Bbbk^*)^d$ the Laurent polynomial $f(\theta_1 x^{n_1}, \dots, \theta_d x^{n_d}, y) \in \Bbbk[x^{\pm 1}, y]$ is irreducible. Moreover, the sublattices can be chosen to be of arbitrarily high index.*

Proof. Consider f as a polynomial in $K_{\mathbf{w}}[y]$. Since $K_{\mathbf{w}}$ is algebraically closed, we can write

$$f = \prod_{i=1}^{s} (y - \alpha_i)$$

where $\alpha_i \in K_{\mathbf{w}}$. By the construction of $K_{\mathbf{w}}$, there is $N > 0$ for which every denominator of an exponent appearing in some α_i can be put in the form Np^j for some $j \geq 0$, where $p = 1$ if $\mathrm{char}(\Bbbk) = 0$. The isogeny $\mu \colon (\Bbbk^*)^d \to (\Bbbk^*)^d$ given by $t_i \mapsto t_i^N$ extends to an inclusion of fields $\mu \colon K_{\mathbf{w}} \to K_{\mathbf{w}}$, and a map $K_{\mathbf{w}}[y] \to K_{\mathbf{w}}[y]$, sending y to y. We then have $g := \mu(f) = \prod_{i=1}^{s} (y - \mu(\alpha_i))$. It suffices to prove the theorem for g, as $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(g) = \phi^{\boldsymbol{\theta}}_{N\mathbf{n}}(f)$, and if $\phi^{\boldsymbol{\theta}}_{N\mathbf{n}}(f)$ is irreducible, so is $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(f)$. We thus henceforth assume that $N = 1$.

Since $N = 1$, all denominators are powers of p, so by axioms (a) and (b) of p-discreteness there is an open cone C with $\alpha_i \in K_C$ for all $1 \leq i \leq s$. Thus for all $\mathbf{n} \in C \cap \mathbb{Z}^d$ and all $\boldsymbol{\theta} \in (\Bbbk^*)^d$ we have

$$\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(f) = \prod_{i=1}^{s} (y - \phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha_i)),$$

so all roots of $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(f)$ have the form $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(\alpha_i)$ for a root α_i of f. Since $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(f)$ is monic, if it is not irreducible, then it can be factored into monic polynomials, and monic factors of $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}(f)$ are images under $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}$ of monic factors of $f \in K_{\mathbf{w}}[y]$. Thus we can consider all of the finitely many ways to factor f into two monic polynomials in $K_{\mathbf{w}}[y]$ and show that "most" substitutions of $\mathbf{n}$ do not make the factors into polynomials in $\Bbbk[x^{\pm 1}, y]$. Since f has $\deg_y(f)$ roots in $K_{\mathbf{w}}$, there are $m := 2^{\deg_y(f)-1} - 1$ ways to factor f into two monic polynomials.

Since the projection $\pi : \mathrm{V}(f) \to (\Bbbk^*)^d$ onto the first d coordinates has the PB property, f cannot be factored into polynomials in $K_{\mathbf{w}}[y]$ whose coefficients in $K_{\mathbf{w}}$ have finite support. If it did have such a factorization, then the isogeny $\lambda : t_i \mapsto t_i^{N'}$ would clear the common denominator N' of the exponents of the coefficients in the factorization, violating the PB property. If a factorization of f in $K_{\mathbf{w}}[y]$ involves a coefficient with unbounded support, then by Lemma 4.3 there is an open cone containing $\mathbf{w}$ such that any integer vector $\mathbf{n}$ in this cone gives a substitution that is not a polynomial factorization. If a factorization involves a coefficient whose support is infinite but bounded, by Lemma 4.4 we can choose a sublattice of index greater than m for $\mathbf{n}$ to avoid. Since the union of m lattices with index greater than m cannot cover all of $\mathbb{Z}^d$, the conclusion follows. □

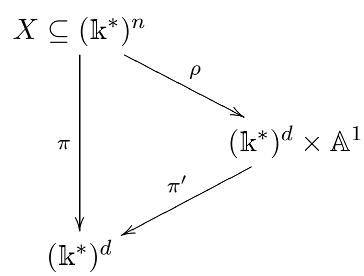
Remark 4.7. The use of the field $K_{\mathbf{w}}$, instead of one of the larger fields of generalized Puiseux series such as the one constructed in [Saa17], was crucial for Theorem 4.6. The fact that the polynomial f factors completely is a consequence of the fact that $K_{\mathbf{w}}$ is algebraically closed (Theorem 3.4), which used axiom (d) of Definition 2.2. The fact that the roots of the transformed polynomial g all live in K_C for some open cone C is a consequence of axiom (a) of Definition 2.2. Without this reduction, the specialization homomorphism $\phi^{\boldsymbol{\theta}}_{\mathbf{n}}$ might not be well defined.

We are now ready to prove the main theorem of this paper. Recall that a morphism $\psi\colon (\Bbbk^*)^r \to (\Bbbk^*)^d$ is given in coordinates by $\psi(t_1,\dots,t_r) = (c_1\mathbf{t}^{\mathbf{a}_1},\dots,c_d\mathbf{t}^{\mathbf{a}_d})$, where $c_i \in \Bbbk^*$ and $\mathbf{a}_i \in \mathbb{Z}^r$ for $1 \leq i \leq d$, and $c_i = 1$ for all i if ψ is an embedding of tori. Let A be the $r \times d$ matrix with columns the $\mathbf{a}_i$. The morphism ψ is an embedding if the matrix A has rank r, and the greatest common divisor of the $r \times r$ minors of A is one. Changes of coordinates on $(\Bbbk^*)^r$ correspond to row operations on A, so we conclude that r-dimensional subtori of $(\Bbbk^*)^d$ correspond to rational points in the Grassmannian $\mathrm{Gr}(r,d)$.

Theorem 1.1 (Toric Bertini)**.** *Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic. Let X be a d-dimensional irreducible subvariety of $(\Bbbk^*)^n$ where $d \geq 2$, and let $\pi\colon (\Bbbk^*)^n \to (\Bbbk^*)^d$ be a morphism with $\pi|_X$ dominant and finite. Suppose that $\pi|_X$ satisfies PB. Then for every $1 \leq r \leq d-1$ the set of r-dimensional subtori $T \subseteq (\Bbbk^*)^d$ with $\pi^{-1}(\boldsymbol{\theta}\cdot T)\cap X$ irreducible for all $\boldsymbol{\theta} \in (\Bbbk^*)^d$ is dense in* $\mathrm{Gr}(r,d)$.

Proof. The morphism π is given by $\pi(\mathbf{t})_j = c_j\mathbf{t}^{\mathbf{p}_j}$ for $1 \leq j \leq d$, $c_j \in \Bbbk^*$, and $\mathbf{p}_j \in \mathbb{Z}^n$. Let P be the $d \times n$ matrix with rows $\mathbf{p}_1,\dots,\mathbf{p}_d$. Integer row and column operations correspond to changes of coordinates on $(\Bbbk^*)^d$ and $(\Bbbk^*)^n$, so we may assume that P is in Smith normal form, and thus the morphism π is projection onto the first d coordinates followed by an isogeny $\lambda\colon t_i \mapsto t_i^{d_i}$ and multiplication by $\mathbf{c} := (c_1,\dots,c_d) \in (\Bbbk^*)^d$. Since $\pi|_X$ satisfies PB, $X \times_\lambda (\Bbbk^*)^d \subset (\Bbbk^*)^n$ is irreducible, and the map $\lambda^*\pi\colon X \times_\lambda (\Bbbk^*)^d \to (\Bbbk^*)^d$ in (7) is projection onto the first d coordinates followed by multiplication by $\mathbf{c}$. Furthermore, observe that $\lambda^*\pi$ satisfies PB. If $T \subset (\Bbbk^*)^d$ is an r-dimensional subtorus with $(\lambda^*\pi)^{-1}(\boldsymbol{\theta}\cdot T)$ irreducible for all $\boldsymbol{\theta} \in (\Bbbk^*)^d$, then $\pi|_X^{-1}(\lambda(\boldsymbol{\theta})\cdot T')$ is irreducible for $T' = \lambda(T)$. It thus suffices to prove the theorem for $X \times_\lambda (\Bbbk^*)^d$. As the factor $\mathbf{c}$ can be absorbed into $\boldsymbol{\theta}$, we may thus assume that π is projection onto the first d coordinates.

We now reduce to the case that X is a hypersurface. Fix $a_1,\dots,a_n \in \Bbbk$, and consider the morphism $\rho\colon (\Bbbk^*)^n \to (\Bbbk^*)^d \times \mathbb{A}^1$ given by $(t_1,\dots,t_n) \mapsto (t_1,\dots,t_d,\sum_{i=1}^n a_it_i)$. Writing $\pi'\colon (\Bbbk^*)^d \times \mathbb{A}^1 \to (\Bbbk^*)^d$ for the projection onto the first d coordinates, we have the following commuting diagram.



For generic $(a_1,\dots,a_n)$ the morphism ρ is birational (for example, by following the proof of [Har77, Proposition I.4.9]; note that this is independent of the characteristic of the field). Let $U \subset X$ be an open set on which ρ is an isomorphism, and let $Z = X \setminus U$. Since π is a proper morphism, as it is finite, we conclude that $\pi(Z)$ is a subvariety of $(\Bbbk^*)^d$ of dimension at most $d-1$. Since ρ is birational, the variety $\overline{\rho(X)}$ is d-dimensional, so is a hypersurface in $(\Bbbk^*)^d \times \mathbb{A}^1$, defined by a polynomial $f \in \Bbbk[t_1^{\pm1},\dots,t_d^{\pm1},y]$. As $\pi|_X$ is finite, the polynomial $\sum_{i=1}^n a_it_i \in \Bbbk[X]$ satisfies a monic polynomial with coefficients in $\Bbbk[t_1^{\pm1},\dots,t_d^{\pm1}]$. This monic polynomial must be a multiple of f, so f is monic in y.

We next show that $\pi' : \mathrm{V}(f) \to (\Bbbk^*)^d$ satisfies PB. Consider the Cartesian square

$$\begin{array}{ccc} \mathrm{V}(f) \times_\mu (\Bbbk^*)^d & \xrightarrow{\alpha} & \mathrm{V}(f) \subseteq (\Bbbk^*)^d \times \mathbb{A}^1 \\ \big\downarrow{\scriptstyle \mu^*\pi'} & & \big\downarrow{\scriptstyle \pi'} \\ (\Bbbk^*)^d & \xrightarrow{\quad\mu\quad} & (\Bbbk^*)^d \end{array} \tag{8}$$

where μ is an isogeny. Since $\mathrm{V}(f) \times_\mu (\Bbbk^*)^d$ is the hypersurface defined by the specialization, determined by μ, of the irreducible polynomial f, all irreducible components of $\mathrm{V}(f) \times_\mu (\Bbbk^*)^d$ are d-dimensional. This, together with the finiteness of α (which follows since μ is finite and finite morphisms are stable under base change), implies that all irreducible components of $\mathrm{V}(f) \times_\mu (\Bbbk^*)^d$ map by α onto $\mathrm{V}(f)$.

Since $\rho|_X$ is birational, there exist non-empty open sets $U \subseteq X$ and $W \subseteq \mathrm{V}(f)$ such that $\rho|_U : U \to W$ is an isomorphism. Because all irreducible components of $V(f) \times_\mu (\Bbbk^*)^d$ map by α onto the irreducible $V(f)$, we have $\overline{\alpha^{-1}(W)} = \mathrm{V}(f) \times_\mu (\Bbbk^*)^d$. Thus, to see that $\mathrm{V}(f) \times_\mu (\Bbbk^*)^d$ is irreducible, it suffices to show that $\alpha^{-1}(W) = W \times_\mu (\Bbbk^*)^d$ is irreducible. Since the isomorphism $\rho|_U : U \to W$ respects the maps to $(\Bbbk^*)^d$, we have $W \times_\mu (\Bbbk^*)^d \cong U \times_\mu (\Bbbk^*)^d$, so the irreducibility follows from the fact that $\pi|_X$ satisfies PB. Thus π' satisfies PB.

We next observe that it suffices to prove the theorem for $\pi'\colon \mathrm{V}(f) \to (\Bbbk^*)^d$. Let T_0 be any given r-dimensional subtorus of $(\Bbbk^*)^d$. Note that for every $\boldsymbol{\theta} \in (\Bbbk^*)^d$ every irreducible component Y of $\pi|_X^{-1}(\boldsymbol{\theta} \cdot T_0)$ satisfies $\pi(Y) = \boldsymbol{\theta} \cdot T_0$. Indeed, since $\boldsymbol{\theta} \cdot T_0$ is a complete intersection, by the Principal Ideal Theorem (see for example, [Eis95, Theorem 10.2]) we have $\dim(Y) \geq \dim(\boldsymbol{\theta} \cdot T_0)$. Since π is finite, $\pi(Y)$ is a closed set, and $\dim(\pi(Y)) = \dim(Y)$ (see for example, [Eis95, Proposition 9.2]), so we conclude that $\pi(Y) = \boldsymbol{\theta} \cdot T_0$. It thus follows that if T_0 is a subtorus of $(\Bbbk^*)^d$ with $\boldsymbol{\theta} \cdot T_0$ not contained in $\pi(Z)$ and $\pi|_X^{-1}(\boldsymbol{\theta} \cdot T_0)$ reducible for some $\boldsymbol{\theta} \in (\Bbbk^*)^d$, then every irreducible component of $\pi|_X^{-1}(\boldsymbol{\theta} \cdot T_0)$ intersects U, so $\pi'^{-1}(\boldsymbol{\theta} \cdot T_0) \cap \mathrm{V}(f) = \overline{\rho(\pi|_X^{-1}(\boldsymbol{\theta} \cdot T_0))}$ is reducible as well.

To prove the theorem, it thus suffices to prove that we can choose an r-dimensional subtorus T arbitrarily close to T_0 in $\mathrm{Gr}(r, d)$ with $\boldsymbol{\theta} \cdot T \not\subseteq \pi(Z)$, and $\pi'^{-1}(\boldsymbol{\theta} \cdot T) \cap \mathrm{V}(f)$ irreducible for all $\boldsymbol{\theta} \in (\Bbbk^*)^d$.

Note that if $\pi'^{-1}(\boldsymbol{\theta} \cdot T_0) \cap \mathrm{V}(f)$ is reducible, then for any one-dimensional subtorus $\tilde{T} \subset T_0$ the preimage $\pi'^{-1}(\boldsymbol{\theta} \cdot \tilde{T}) \cap \mathrm{V}(f)$ is also reducible if it is nonempty. To see this, consider a parameterization of the torus T_0 by coordinates $z_1, \dots, z_r$. Specializing f to this parameterization yields a reducible polynomial $f' \in \Bbbk[z_1^{\pm 1}, \dots, z_r^{\pm 1}, y]$. The pullback $\pi'^{-1}(\boldsymbol{\theta} \cdot \tilde{T})$ is a further nonzero specialization of f', so remains reducible.

Let A_0 be an $r \times d$ matrix corresponding to T_0, and let $\mathbf{w} \in \mathbb{R}^d_{li}$ be close to the first row of A_0 in the Euclidean topology on $\mathbb{P}^{d-1}_{\mathbb{Q}}$ By Theorem 4.6, there is an open cone C containing $\mathbf{w}$ and a finite collection $\mathcal{H}$ of lattices of arbitrarily high index for which for any $\mathbf{n} \in C$ not in any lattice in $\mathcal{H}$ the subtorus $\tilde{T} = (t^{n_1}, \dots, t^{n_d})$ has $\pi'^{-1}(\boldsymbol{\theta} \cdot \tilde{T}) \cap \mathrm{V}(f)$ irreducible for all $\boldsymbol{\theta} \in (\Bbbk^*)^d$. We assume that the lower bound on the index of the lattices has been chosen to guarantee that $\mathbb{Z}^d \setminus \bigcup_{H \in \mathcal{H}} H$ is nonempty. We have $\boldsymbol{\theta} \cdot \tilde{T} \subseteq \pi(Z)$ only if the tropicalization $\mathrm{trop}(\boldsymbol{\theta} \cdot \tilde{T})$, which is an affine line in $\mathbb{R}^d$ with direction vector $\mathbf{n}$, is contained in $\mathrm{trop}(\pi(Z))$. For generic $\mathbf{n}$, the polyhedral complex $\mathrm{trop}(\pi(Z))$ does not contain any lines in the direction $\mathbf{n}$.

Choose $\mathbf{n} \in C \setminus \bigcup_{H \in \mathcal{H}} H$ close to $\mathbf{w}$ in the Euclidean topology on $\mathbb{P}^{d-1}_{\mathbb{Q}}$, such that $\mathrm{trop}(\pi(Z))$ does not contain any lines in direction $\mathbf{n}$, and let A be the matrix obtained by replacing the first row of A_0 by $\mathbf{n}$. Let T be the r-dimensional subtorus of $(\Bbbk^*)^d$ corresponding to A. This contains the one-dimensional subtorus $\tilde{T} = \{(t^{n_1}, \dots, t^{n_d}) : t \in \Bbbk^*\}$, for which $\pi'^{-1}(\boldsymbol{\theta} \cdot \tilde{T}) \cap \mathrm{V}(f)$ is irreducible and $\boldsymbol{\theta} \cdot \tilde{T} \not\subseteq \pi(Z)$ for all $\boldsymbol{\theta} \in (\Bbbk^*)^d$, so $\pi'^{-1}(\boldsymbol{\theta} \cdot T) \cap \mathrm{V}(f)$ is irreducible, and thus $\pi|_X^{-1}(\boldsymbol{\theta} \cdot T)$ is irreducible for all such $\boldsymbol{\theta}$. As T is close to T_0 in $\mathrm{Gr}(r, d)$, this proves the theorem. □

Remark 4.8. Theorem 1.1 differs from the version in [FMZ18], in that we only show that the set of exceptional tori are the complement of a dense set, rather than essentially finite. As already discussed, the finiteness is not achievable without the PB condition we, and [FMZ18], impose. We do not know whether the PB condition is necessary for the conclusion of Theorem 1.1. In our proof the main use is to guarantee that the sublattices $H_1, \ldots, H_r$ to avoid do not cover $\mathbb{Z}^d$, so the "genericity" condition for $\mathbf{n}$ is nonempty. However we do not know an example of a projection π failing PB where the union of the lattices is all of $\mathbb{Z}^d$. In addition, it is possible that the stronger finiteness conclusion of [FMZ18] holds when we assume PB. To prove this using our techniques, we would need a deeper understanding of the structure of the cones C used in Theorem 4.6. In characteristic zero, under mild hypotheses, McDonald [McD95] relates these cones to the normal fan of the fiber polytope of a certain projection of the Newton polytope of the polynomial f. It would be interesting to understand to what extent that can be generalized to arbitrary characteristic.

5. Tropical Bertini Theorem

One application of Theorem 1.1 is that the tropical Bertini theorem of [MY21] holds in arbitrary characteristic, and thus the d-connectivity of the tropicalization of d-dimensional irreducible varieties holds in arbitrary characteristic.

The *tropicalization* of a variety $X \subseteq (\Bbbk^*)^n$, where $\Bbbk$ is a valued field with valuation $\mathrm{val}\colon \Bbbk^* \to \mathbb{R}$, is

$$\mathrm{trop}(X) = \mathrm{cl}((\mathrm{val}(x_1), \ldots, \mathrm{val}(x_n)) : (x_1, \ldots, x_n) \in X(L)),$$

where $L/\Bbbk$ is a nontrivially valued algebraically closed field extension, and cl() is the closure in the usual Euclidean topology on $\mathbb{R}^n$. By the Structure Theorem (see, for example, [MS15, Chapter 3], or [BG84]) $\mathrm{trop}(X)$ is the support of a connected polyhedral complex.

In [MY21] Maclagan and Yu proved a tropical Bertini theorem in characteristic zero. We now remove the characteristic assumption.

We identify rational affine hyperplanes in $\mathbb{R}^n$ with points of $\mathbb{P}^n_{\mathbb{Q}}$.

Theorem 1.2 (Tropical Bertini)**.** *Let $X \subset (\Bbbk^*)^n$ be an irreducible d-dimensional variety, with $d \geq 2$, over an algebraically closed field $\Bbbk$ with $\mathbb{Q}$ contained in the value group. The set of rational affine hyperplanes H in $\mathbb{R}^n$ for which the intersection* $\mathrm{trop}(X) \cap H$ *is the tropicalization of an irreducible variety is dense in the Euclidean topology on $\mathbb{P}^n_{\mathbb{Q}}$.*

Proof. We only need to replace a few sentences from the proof of [MY21, Theorem 5]. Explicitly, the proof holds essentially verbatim (with the forward reference to Theorem 8 replaced by a forward reference to our Theorem 1.1) until the sentence of [MY21, Theorem 5] "We can now apply Theorem 8 to $\rho|_Y \colon Y \to (\Bbbk^*)^d$", where "Theorem 8" should again be replaced by Theorem 1.1 of this paper. The proof then continues as follows.

By Theorem 1.1, since $\rho|_Y$ satisfies PB, and is dominant and finite, the set of $(d-1)$-dimensional subtori $T' \subseteq (\Bbbk^*)^d$ such that $\pi_X^{-1}(\boldsymbol{\theta} \cdot T')$ is irreducible for all $\boldsymbol{\theta} \in (\Bbbk^*)^d$ is dense in $\mathrm{Gr}(d-1, d) \cong \mathbb{P}^{d-1}_{\mathbb{Q}}$. By definition, this means that the set of affine hyperplanes $\overline{H} = \mathrm{trop}(\boldsymbol{\theta} \cdot T')$ for such $\boldsymbol{\theta}, T'$ is dense in $\mathbb{P}^d_{\mathbb{Q}}$.

Fix one such $\overline{H} = \mathrm{trop}(\boldsymbol{\theta} \cdot T')$, and let H be the hyperplane in $\mathbb{R}^n$ defined by $H = (\mathrm{trop}(\rho)^{-1}(\overline{H}))$. Since ρ is a monomial map, we have $H = \mathrm{trop}(\rho^{-1}(\boldsymbol{\theta} \cdot T'))$. As $\mathrm{trop}(\rho)$ is injective on every maximal face of $\mathrm{trop}(Y)$, the intersection $H \cap \mathrm{trop}(Y)$ is transverse, so by the Transverse Intersection Lemma [OP13, Theorem 1.1], [BJSST07, Lemma 15], [MS15, Theorem 3.4.12],

$$H \cap \mathrm{trop}(Y) = \mathrm{trop}(\rho^{-1}|_Y(\boldsymbol{\theta} \cdot T')).$$

Thus $H \cap \mathrm{trop}(Y)$ is the tropicalization of the irreducible variety $\rho|_Y^{-1}(\boldsymbol{\theta} \cdot T')$. Note that

$$\mathrm{trop}(\mu'(\rho|_Y^{-1}(\boldsymbol{\theta} \cdot T'))) = \mathrm{trop}(\mu')(H \cap \mathrm{trop}(Y)) = \mathrm{trop}(\mu')(H) \cap \mathrm{trop}(X) = \mathrm{trop}(\pi)^{-1}(\mathrm{trop}(\mu)(\overline{H})).$$

Since $\mathrm{trop}(\mu) \in \mathrm{GL}(n, \mathbb{Q})$, and the set of $\overline{H}$ is dense in $\mathbb{P}^d_{\mathbb{Q}}$, the set of $\mathrm{trop}(\mu)(\overline{H})$ is dense in $\mathbb{P}^d_{\mathbb{Q}}$ as required. □

In [MY21] Theorem 1.2 is used to prove a higher connectivity theorem for tropical varieties. While the statement of [MY21, Theorem 1] assumes that the field $\Bbbk$ has characteristic zero, as observed in [MY21, Remark 11] this is only needed to apply the tropical Bertini theorem, so in light of Theorem 1.2 we have the following generalization of [MY21, Theorem 1]. A pure polyhedral complex Σ of dimension d is d-connected through codimension one if it is still connected after removing any $d-1$ closed facets.

Theorem 5.1. *Let $\Bbbk$ be a field that is either algebraically closed, complete, or real closed with convex valuation ring. Let X be a d-dimensional irreducible subvariety of $(\Bbbk^*)^n$. Let Σ be a pure d-dimensional rational polyhedral complex with support $|\Sigma| = \mathrm{trop}(X)$. Write ℓ for the dimension of the lineality space of Σ. Then Σ is $(d-\ell)$-connected through codimension one.*

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