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QUASI-INVARIANT GAUSSIAN MEASURES FOR THE NONLINEAR WAVE EQUATION IN THREE DIMENSIONS

TRISHEN S. GUNARATNAM, TADAHIRO OH, NIKOLAY TZVETKOV, AND HENDRIK WEBER

Abstract. We prove quasi-invariance of Gaussian measures supported on Sobolev spaces under the dynamics of the three-dimensional defocusing cubic nonlinear wave equation. As in the previous work on the two-dimensional case, we employ a simultaneous renormalization on the energy functional and its time derivative. Two new ingredients in the three-dimensional case are (i) the construction of the weighted Gaussian measures, based on a variational formula for the partition function inspired by Barashkov and Gubinelli (2018), and (ii) an improved argument in controlling the growth of the truncated weighted Gaussian measures, where we combine a deterministic growth bound of solutions with stochastic estimates on random distributions.

Contents

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1. Introduction

1.1. **Main result.** We consider the following defocusing cubic nonlinear wave equation (NLW) on the three-dimensional torus $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$:

$$\partial_t^2 u - \Delta u + u^3 = 0, \tag{1.1}$$

where $u : \mathbb{T}^3 \times \mathbb{R} \to \mathbb{R}$ is the unknown function. With $v = \partial_t u$, we rewrite (1.1) in the following vectorial form:

$$\begin{cases} \partial_t u = v \\ \partial_t v = \Delta u - u^3. \end{cases} \tag{1.2}$$

Given $\sigma \in \mathbb{R}$, let $H^\sigma(\mathbb{T}^3)$ denote the classical L^2-based Sobolev space of order σ defined by the norm:

$$\|u\|_{H^\sigma} = \|\langle n \rangle^\sigma \widehat{u}(n)\|_{\ell^2(\mathbb{Z}^3)},$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$ and $\widehat{u}$ denotes the Fourier transform of u. A classical argument yields global well-posedness of the Cauchy problem (1.2) in the Sobolev spaces:

$$\vec{H}^\sigma(\mathbb{T}^3) \stackrel{\text{def}}{=} H^\sigma(\mathbb{T}^3) \times H^{\sigma-1}(\mathbb{T}^3)$$

for $\sigma \geq 1$ and, consequently, admits a global flow Φ_{NLW} (see Lemma 2.4 below) on these spaces.

Given $s \in \mathbb{R}$, let $\vec{\mu}_s$ denote the Gaussian measure with Cameron-Martin space $\vec{H}^{s+1}(\mathbb{T}^3)$. Denoting $\vec{u} = (u, v)$, the Gaussian measure $\vec{\mu}_s$ has a formal density:

$$\begin{aligned} d\vec{\mu}_s &= Z_s^{-1} e^{-\frac{1}{2}\|\vec{u}\|^2_{\vec{H}^{s+1}}} d\vec{u} \\ &= \prod_{n \in \mathbb{Z}^3} Z_{s,n}^{-1} e^{-\frac{1}{2}\langle n \rangle^{2(s+1)}|\widehat{u}(n)|^2} e^{-\frac{1}{2}\langle n \rangle^{2s}|\widehat{v}(n)|^2} d\widehat{u}(n) d\widehat{v}(n). \end{aligned}$$

Samples $\vec{u}^\omega = (u^\omega, v^\omega)$ from $\vec{\mu}_s$ can be constructed via the following Karhunen-Loève expansions:[1]

$$u^\omega(x) = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle^{s+1}} e^{in\cdot x} \qquad \text{and} \qquad v^\omega(x) = \sum_{n \in \mathbb{Z}^3} \frac{h_n(\omega)}{\langle n \rangle^{s}} e^{in\cdot x}, \tag{1.3}$$

where $\{g_n\}_{n \in \mathbb{Z}^3}$ and $\{h_n\}_{n \in \mathbb{Z}^3}$ are collections of standard complex-valued Gaussian variables which are independent modulo the condition[2] $g_n = \overline{g_{-n}}$ and $h_n = \overline{h_{-n}}$. It is easy to see that the series (1.3) converge in $L^2(\Omega; \vec{H}^\sigma(\mathbb{T}^3))$ for

$$\sigma < s - \frac{1}{2} \tag{1.4}$$

and therefore the map

$$\omega \in \Omega \longmapsto (u^\omega, v^\omega)$$

induces the Gaussian measure $\vec{\mu}_s$ as a probability measure on $\vec{H}^\sigma(\mathbb{T}^3)$ for the same range of σ. Our main goal in this paper is to study the transport property of the Gaussian measure $\vec{\mu}_s$ under the dynamics of (1.2). We state our main result.

[1] Henceforth, we drop the harmless factor 2π.

[2] In particular, we impose that g_0 and h_0 are real-valued.

Theorem 1.1. *Let $s \geq 4$ be an even integer. Then, $\vec{\mu}_s$ is quasi-invariant under the dynamics of the defocusing cubic NLW (1.2) on $\mathbb{T}^3$. More precisely, for any $t \in \mathbb{R}$, the Gaussian measure $\vec{\mu}_s$ and its pushforward under $\Phi_{\text{NLW}}(t)$ are mutually absolutely continuous.*

Theorem 1.1 ensures the propagation of almost sure properties of $\vec{\mu}_s$ along the flow. This is important because, in infinite dimensions, many interesting properties concerning small-scale behavior under a Gaussian measure hold true with probability 0 or 1. This is an implication of Fernique's theorem (Theorem 2.7 in [12]); under a Gaussian measure, any given norm is finite with probability 0 or 1. For example, samples $\vec{u}$ of the Gaussian measure $\vec{\mu}_s$ almost surely belong to the L^p-based Sobolev spaces $\vec{W}^{\sigma,p}(\mathbb{T}^3)$ for any $p \geq 1$ and more generally to the Besov spaces, $\vec{B}^{\sigma}_{p,q}(\mathbb{T}^3)$ for any $p, q \geq 1$, including the case $p = q = \infty$ (Hölder-Besov space), provided that σ satisfies (1.4). Theorem 1.1 then implies that these L^p-based regularities are transported along the nonlinear flow. An analogous statement for deterministic initial data is expected to fail in general. See [18, 30, 34].

Theorem 1.1 is an addition to a series of recent results [36, 27, 25, 29, 26] that has made significant progress in the study of transport properties of Gaussian measures under nonlinear Hamiltonian PDEs. The general strategy, as introduced by the third author in [36], is to study quasi-invariance of the Gaussian measures $\vec{\mu}_s$ indirectly by studying weighted Gaussian measures, where the weight corresponds to a correction term that arises due to the presence of the nonlinearity. See Subsection 3.2. The two key steps in this strategy are (i) the construction of the weighted Gaussian measure and (ii) an energy estimate on the time derivative of the modified energy (that is, the energy of the Gaussian measure plus the correction term). In [29], the second and third authors employed this strategy and proved the analogue of Theorem 1.1 in the two-dimensional case. This was done by introducing a simultaneous renormalization on the modified energy functional and its time derivative and then performing a delicate analysis centered on a quadrilinear Littlewood-Paley expansion.

As pointed out in [29], the argument in the two-dimensional case does not extend to the current three-dimensional setting. The proof of Theorem 1.1 uses two new key ingredients. The first is the use of a variational formula in constructing weighted Gaussian measures, inspired by Barashkov and Gubinelli [2]. The second new ingredient appears in studying the growth of the truncated weighted Gaussian measures, where we combine a deterministic growth bound on solutions (as in a recent paper by Planchon, Visciglia, and the third author [31]) with stochastic estimates on random distributions (as in the two-dimensional case [29]). This hybrid argument allows us to use a softer energy estimate to prove quasi-invariance. Our simplification also comes from the use of Besov spaces in the spirit of [19]. This results in a significantly simpler proof of quasi-invariance in the harder, physically relevant three-dimensional case as compared with the two-dimensional case.

1.2. **Remarks and comments.** (i) A slight modification of the proof of Theorem 1.1 shows that the Gaussian measures $\vec{\mu}_s$ are also quasi-invariant under the nonlinear Klein-Gordon equation:

$$\begin{cases} \partial_t u = v \\ \partial_t v = (\Delta - 1)u - u^3. \end{cases} \tag{1.5}$$

It is easy to see that $\vec{\mu}_s$ is invariant under the linear Klein-Gordon equation, i.e. removing u^3 in (1.5), which trivially implies that almost sure properties of $\vec{\mu}_s$ are transported along the flow of the linear dynamics. The addition of a defocusing cubic nonlinearity into the equation destroys invariance but the quasi-invariance of $\vec{\mu}_s$ for (1.5) can be interpreted as saying that the nonlinear flow retains the small-scale properties of the linear flow.

In order to obtain invariance of $\vec{\mu}_s$ under the linear wave equation, one would need to replace $\langle \cdot \rangle$ with $|\cdot|$ in (1.3), which would raise an issue at the zeroth Fourier mode (see Remark 3.6). Nevertheless, in the study of small-scale properties of solutions, this issue is irrelevant and one can easily show that $\vec{\mu}_s$ is quasi-invariant under the linear wave equation. Theorem 1.1 then implies that the NLW dynamics also retains the small-scale properties of the linear wave dynamics.

(ii) The restriction that s is an even integer in Theorem 1.1 comes from an application of the classical Leibniz rule in order to derive the right correction term for the modified energy and the weighted Gaussian measure. In terms of regularity restrictions, the construction of the weighted Gaussian measure works for any real $s > \frac{3}{2}$ (Proposition 3.7). Our argument for the energy estimate (Proposition 3.8) only requires $s > \frac{5}{2}$ but, in our derivation of a modified energy, we also use the classical Leibniz rule for $(-\Delta)^{\frac{s}{2}}$ which only works if s is an even integer. It may be possible to relax this second condition using a fractional Leibniz rule to go below $s = 4$. At present, however, we do not know how to do this.

(iii) Our new hybrid argument in proving Theorem 1.1 requires a softer energy estimate than that in [29] and is also applicable to the two-dimensional case. We point out, however, that the argument in [29], involving heavier multilinear analysis, provides better quantitative information on the growth of the truncated weighted Gaussian measures. See Remark 3.12. For example, the argument in [29] allows us to prove higher L^p-integrability of the Radon-Nikodym derivative of the weighted Gaussian measures (with an energy cutoff), while our proof of Theorem 1.1 does not provide such extra information.

(iv) It would be of interest to investigate the quasi-invariance property of $\vec{\mu}_s$ for NLW with a higher order nonlinearity or in higher dimensions. Our techniques appear to carry over to higher order nonlinearities. This might even permit to analyze energy-supercritical equations (such as the three-dimensional septic NLW), where global well-posedness is not known. Consequently, one might aim to prove "local-in-time" quasi-invariance (as stated in [8]). See also [31] for an example of a local-in-time quasi-invariance result. See also Remark 3.4 below.

(v) Quasi-invariance results such as Theorem 1.1 are complimentary to the study of low regularity well-posedness with random initial data. Starting with the seminal work of Bourgain [6, 7], there has been intensive study on the random data Cauchy theory for nonlinear dispersive PDEs. There are two related directions in this study. The first one is the study of invariant measures associated with conservation laws such as Gibbs measures, in particular, the construction of almost sure global-in-time dynamics via the so-called Bourgain's invariant measure argument; see [27, 3] for the references therein. The other is the study of almost sure well-posedness with respect to random initial data. Here, one can often exploit the higher L^p_x-based regularity made accessible by randomization of initial data to establish well-posedness below critical thresholds, where equations are ill-posed in L^2-based Sobolev spaces. In the context of NLW, see the work [9, 10] by Burq and the third author for almost sure local well-posedness. There are also globalization arguments in this probabilistic setting; see [10, 32, 23, 24]. See also a general review [3] on the subject.

As for the defocusing cubic NLW (1.2) on $\mathbb{T}^3$, the scaling symmetry induces the critical regularity $\sigma_{\text{crit}} = \frac{1}{2}$. It is known that (1.2) is locally well-posed in $\vec{H}^\sigma(\mathbb{T}^3)$ for $\sigma \geq \frac{1}{2}$, while it is ill-posed for $\sigma < \frac{1}{2}$; see [17, 11, 9, 22]. In [9, 10], Burq and the third author proved almost sure global well-posedness of (1.2) with respect to the random initial data in (1.3) for $s > \frac{1}{2}$, namely for $\sigma > 0$. In this regime, the flow Φ_{NLW} exists almost surely globally in time. Then, it is natural to ask the following question.

Problem. *Study the transport property of the Gaussian measures $\vec{\mu}_s$ for low values of $s > \frac{1}{2}$, in particular in the regime where the global-in-time dynamics is constructed only probabilistically.*

1.3. **Organization.** In Section 2, we introduce basic tools in our proof: Besov spaces, the Wiener chaos estimate, the classical well-posedness theory of (1.2), and also deterministic growth bounds. In Section 3, we present the proof of Theorem 1.1 assuming (i) the construction of the weighted Gaussian measures (Proposition 3.7) and (ii) the energy estimate (Proposition 3.8). Section 4 is devoted to the construction of the weighted Gaussian measures and, finally, Section 5 deals with the energy estimate.

2. Analytic and stochastic toolbox

2.1. **On the phase space.** Given $N \in \mathbb{N}$, we denote by π_N the frequency projector on the (spatial) frequencies $\{|n| \leq N\}$:

$$(\pi_N u)(x) = \sum_{|n| \leq N} \widehat{u}_n \, e^{in\cdot x},$$

We then set

$$\mathcal{E}_N = \pi_N L^2(\mathbb{T}^3).$$

Namely, $\mathcal{E}_N$ is the finite-dimensional vector space of real-valued trigonometric polynomials of degree $\leq N$ endowed with the restriction of the $L^2(\mathbb{T}^3)$ scalar product. The product space $\mathcal{E}_N \times \mathcal{E}_N$ is a finite dimensional real inner-product space and thus there is a canonical Lebesgue measure on this space, which we denote by L_N. We also use $(\mathcal{E}_N \times \mathcal{E}_N)^\perp$ to denote the orthogonal complement of $\mathcal{E}_N \times \mathcal{E}_N$ in $\vec{H}^\sigma(\mathbb{T}^3)$, $\sigma < s - \frac{1}{2}$.

2.2. **Besov spaces.** Let $B(\xi, r)$ denote the ball in $\mathbb{R}^3$ of radius $r > 0$ centered at $\xi \in \mathbb{R}^3$ and let $\mathcal{A}$ denote the annulus $B(0, \frac{4}{3}) \setminus B(0, \frac{3}{8})$. Letting $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we define a sequence $\{\chi_j\}_{j \in \mathbb{N}_0}$ by setting

$$\chi_0 = \widetilde{\chi}, \qquad \chi_j(\,\cdot\,) = \chi(2^{-j}\,\cdot\,), \qquad \text{and} \qquad \sum_{j=0}^\infty \chi_j \equiv 1$$

for some suitable $\widetilde{\chi}, \chi \in C_c^\infty(\mathbb{R}^3; [0,1])$ such that $\text{supp}(\widetilde{\chi}) \subset B(0, \frac{4}{3})$ and $\text{supp}(\chi) \subset \mathcal{A}$. We then define the Littlewood-Paley projector $\mathbf{P}_j$, $j \in \mathbb{N}_0$, by setting

$$\mathbf{P}_j u(x) = \sum_{n \in \mathbb{Z}^3} \chi_j(n) \widehat{u}(n) e^{in\cdot x}$$

for $u \in \mathcal{D}'(\mathbb{T}^3)$.

Given $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the Besov space $B^s_{p,q}(\mathbb{T}^3)$ is the set of distributions $u \in \mathcal{D}'(\mathbb{T}^3)$ such that

$$\|u\|_{B^s_{p,q}} = \Big\| \big\{ 2^{sj} \|\mathbf{P}_j u\|_{L^p_x} \big\}_{j \in \mathbb{N}_0} \Big\|_{\ell^q_j} < \infty. \tag{2.1}$$

We use the conventions $\vec{B}^s_{p,q}(\mathbb{T}^3) = B^s_{p,q}(\mathbb{T}^3) \times B^{s-1}_{p,q}(\mathbb{T}^3)$ and $\vec{\mathcal{C}}^s(\mathbb{T}^3) = \mathcal{C}^s(\mathbb{T}^3) \times \mathcal{C}^{s-1}(\mathbb{T}^3)$, where $\mathcal{C}^s(\mathbb{T}^3) = B^s_{\infty,\infty}(\mathbb{T}^3)$ denotes the Hölder-Besov space. Note that (i) the parameter s measures differentiability and p measures integrability, (ii) $H^s(\mathbb{T}^3) = B^s_{2,2}(\mathbb{T}^3)$, and (iii) for $s > 0$ and not an integer, $\mathcal{C}^s(\mathbb{T}^3)$ coincides with the classical Hölder spaces; see [15].

Lemma 2.1. *The following estimates hold.*
(i) (interpolation) *For $0 < s_1 < s_2$, we have*[3]

$$\|u\|_{H^{s_1}} \lesssim \|u\|_{H^{s_2}}^{\frac{s_1}{s_2}} \|u\|_{L^2}^{\frac{s_2 - s_1}{s_2}}. \tag{2.2}$$

(ii) (immediate embeddings) *Let $s_1, s_2 \in \mathbb{R}$ and $p_1, p_2, q_1, q_2 \in [1, \infty]$. Then, we have*

$$\begin{aligned} \|u\|_{B^{s_1}_{p_1,q_1}} &\lesssim \|u\|_{B^{s_2}_{p_2,q_2}} && \text{for } s_1 \le s_2,\ p_1 \le p_2, \text{ and } q_1 \ge q_2, \\ \|u\|_{B^{s_1}_{p_1,q_1}} &\lesssim \|u\|_{B^{s_2}_{p_1,\infty}} && \text{for } s_1 < s_2, \\ \|u\|_{B^{0}_{p_1,\infty}} &\lesssim \|u\|_{L^{p_1}} \lesssim \|u\|_{B^{0}_{p_1,1}}. \end{aligned} \tag{2.3}$$

(iii) (algebra property) *Let $s > 0$. Then, we have*

$$\|uv\|_{\mathcal{C}^s} \lesssim \|u\|_{\mathcal{C}^s} \|v\|_{\mathcal{C}^s}. \tag{2.4}$$

(iv) (Besov embedding) *Let $1 \le p_2 \le p_1 \le \infty$, $q \in [1, \infty]$, and $s_2 = s_1 + 3\big(\frac{1}{p_2} - \frac{1}{p_1}\big)$. Then, we have*

$$\|u\|_{B^{s_1}_{p_1,q}} \lesssim \|u\|_{B^{s_2}_{p_2,q}}. \tag{2.5}$$

(v) (duality) *Let $s \in \mathbb{R}$ and $p, p', q, q' \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{p'} = \frac{1}{q} + \frac{1}{q'} = 1$. Then, we have*

$$\bigg| \int_{\mathbb{T}^3} uv\, dx \bigg| \le \|u\|_{B^s_{p,q}} \|v\|_{B^{-s}_{p',q'}}, \tag{2.6}$$

where $\int_{\mathbb{T}^3} uv\, dx$ denotes the duality pairing between $B^s_{p,q}(\mathbb{T}^3)$ and $B^{-s}_{p',q'}(\mathbb{T}^3)$.
(vi) (fractional Leibniz rule) *Let $p, p_1, p_2, p_3, p_4 \in [1, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4} = \frac{1}{p}$. Then, for every $s > 0$, we have*

$$\|uv\|_{B^s_{p,q}} \lesssim \|u\|_{B^s_{p_1,q}} \|v\|_{L^{p_2}} + \|u\|_{L^{p_3}} \|v\|_{B^s_{p_4,q}}. \tag{2.7}$$

(vi) (product estimate) *Let $s_1 < 0 < s_2$ such that $s_1 + s_2 > 0$. Then, we have*

$$\|uv\|_{\mathcal{C}^{s_1}} \lesssim \|u\|_{\mathcal{C}^{s_1}} \|v\|_{\mathcal{C}^{s_2}}. \tag{2.8}$$

Proof. While these estimates are standard, we briefly discuss their proofs for readers' convenience. See also [1] for details of the proofs in the non-periodic case. The log convexity inequality (2.2) and the duality (2.6) follow from Hölder's inequality. The first estimate in (2.3) is immediate from the definition (2.1), while the second one in (2.3) follows from the ℓ^{q_1}-summability of $\big\{2^{(s_1 - s_2)j}\big\}_{j \in \mathbb{N}_0}$ for $s_1 < s_2$. The last estimate in (2.3) follows from

[3] We use the convention that the symbol $\lesssim$ indicates that inessential constants are suppressed in the inequality.

the boundedness of the Littlewood-Paley projector $\mathbf{P}_j$ and Minkowski's inequality. The Besov embedding (2.5) is a direct consequence of Bernstein's inequality:

$$\|\mathbf{P}_j u\|_{L^{p_1}} \lesssim 2^{3j(\frac{1}{p_2}-\frac{1}{p_1})}\|\mathbf{P}_j u\|_{L^{p_2}}.$$

The algebra property (2.4) is immediate from the following paraproduct decomposition due to Bony [4]:

$$uv = \sum_{j\in\mathbb{N}_0} \mathbf{P}_j u \cdot S_j v + \sum_{j\in\mathbb{N}_0}\sum_{|j-k|\le 1} \mathbf{P}_j u \cdot \mathbf{P}_k v + \sum_{k\in\mathbb{N}_0} S_k u \cdot \mathbf{P}_k v \tag{2.9}$$

with Hölder's inequality. Here, S_j is given by

$$S_j u = \sum_{k\le j-2} \mathbf{P}_k u.$$

The fractional Leibniz rule (2.7) also follows from the paraproduct decomposition (2.9). In proving (2.7) for the resonant product, i.e. the second term on the right-hand side of (2.9), one needs to proceed slightly more carefully:

$$\begin{aligned}\bigg\| 2^{sm}\Big\|\mathbf{P}_m\Big(\sum_{j\in\mathbb{N}_0}\sum_{|j-k|\le 1}\mathbf{P}_j u\cdot\mathbf{P}_k v\Big)\Big\|_{L^p}\bigg\|_{\ell^q_m} &\lesssim \bigg\|\sum_{j\ge m-10} 2^{s(m-j)}2^{sj}\|\mathbf{P}_j u\|_{L^{p_1}}\|\mathbf{P}_j v\|_{L^{p_2}}\bigg\|_{\ell^q_m}\\ &\lesssim \|u\|_{B^s_{p_1,q}}\|v\|_{L^{p_2}},\end{aligned}$$

where we used Young's and Hölder's inequalities together with the embedding: $L^{p_2}(\mathbb{T}^3) \hookrightarrow B^0_{p_2,\infty}(\mathbb{T}^3)$ in the last step. See also Lemma 2.84 in [1]. Lastly, the product estimate (2.8) follows from a similar consideration. $\square$

2.3. **Wiener chaos estimate.** Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of independent standard Gaussian random variables defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, where $\mathcal{F}$ is the σ-algebra generated by this sequence. Given $k\in\mathbb{N}_0$, we define the homogeneous Wiener chaoses $\mathcal{H}_k$ to be the closure (under $L^2(\Omega)$) of the span of Fourier-Hermite polynomials $\prod_{n=1}^\infty H_{k_n}(g_n)$, where H_j is the Hermite polynomial of degree j and $k=\sum_{n=1}^\infty k_n$.[4] Then, we have the following Ito-Wiener decomposition:

$$L^2(\Omega,\mathcal{F},\mathbb{P}) = \bigoplus_{k=0}^\infty \mathcal{H}_k.$$

See Theorem 1.1.1 in [21]. We have the following classical Wiener chaos estimate.

Lemma 2.2. *Let $k\in\mathbb{N}_0$. Then, we have*

$$\Big(\mathbb{E}\big[|X|^p\big]\Big)^{\frac{1}{p}} \le (p-1)^{\frac{k}{2}}\Big(\mathbb{E}\big[|X|^2\big]\Big)^{\frac{1}{2}} \tag{2.10}$$

for any random variable $X\in\mathcal{H}_k$ and any $2\le p<\infty$.

The estimate (2.10) is a direct corollary to the hypercontractivity of the Ornstein-Uhlenbeck semigroup due to Nelson [20] and the fact that any element $X\in\mathcal{H}_k$ is an eigenfunction for the Ornstein-Uhlenbeck operator with eigenvalue $-k$.

[4] This implies that $k_n=0$ except for finitely many n's.

For our purpose, we need the following three facts: (i) If Z is a linear combination of $\{g_n\}$, then $Z \in \mathcal{H}_1$. (ii) For $Z \in \mathcal{H}_1$, the random variable $Z^2 - \mathbb{E}[Z^2] \in \mathcal{H}_2$. (iii) If $Y, Z \in \mathcal{H}_1$ are independent, then $YZ \in \mathcal{H}_2$.

The next lemma gives a regularity criterion for stationary random distributions. Recall that a random distribution u on $\mathbb{T}^d$ is said to be stationary if $u(\,\cdot\,)$ and $u(x_0 + \cdot\,)$ have the same law for any $x_0 \in \mathbb{T}^d$. Moreover, we say that $u \in \mathcal{H}_k$ if $u(\varphi) \in \mathcal{H}^k$ for any test function $\varphi \in C^\infty(\mathbb{T}^d)$.

Lemma 2.3. (i) *Let u be a stationary random distribution on $\mathbb{T}^d$, belonging to $\mathcal{H}_k$ for some $k \in \mathbb{N}_0$. Suppose that there exists $s_0 \in \mathbb{R}$ such that*

$$\mathbb{E}\big[|\widehat{u}(n)|^2\big] \lesssim \langle n \rangle^{-d-2s_0} \tag{2.11}$$

for any $n \in \mathbb{Z}^d$. Then, for any $s < s_0$ and finite $p \geq 2$, we have $u \in L^p(\Omega; \mathcal{C}^s(\mathbb{T}^d))$.
(ii) *Let $\{u_N\}_{N\in\mathbb{N}}$ be a sequence of stationary random distributions on $\mathbb{T}^d$, belonging to $\mathcal{H}_k$ for some $k \in \mathbb{N}_0$. Suppose that there exists $s_0 \in \mathbb{R}$ such that u_N satisfies* (2.11) *for each $N \in \mathbb{N}$. Moreover, suppose that there exists $\theta > 0$ such that*

$$\mathbb{E}\big[|\widehat{u}_N(n) - \widehat{u}_M(n)|^2\big] \lesssim N^{-2\theta} \langle n \rangle^{-d-2s_0}$$

for any $n \in \mathbb{Z}^d$ and any $M \geq N \geq 1$. Then, for any $s < s_0$ and finite $p \geq 2$, u_N converges to some u in $L^p(\Omega; \mathcal{C}^s(\mathbb{T}^d))$.

The proof is a straightforward computation with the Wiener chaos estimate (Lemma 2.2). See [19, Proposition 3.6] for details of the proof of Part (i). Part (ii) follows from similar considerations.

2.4. **Truncated NLW dynamics: well-posedness and approximation.** In the following, we often work at the level of the truncated dynamics in order to rigorously justify calculations. As such, in this subsection, we briefly go over the well-posedness theory and approximation results of the following Cauchy problem for the truncated NLW on $\mathbb{T}^3$:

$$\begin{cases} \partial_t u = v \\ \partial_t v = \Delta u - \pi_N\big((\pi_N u)^3\big) \\ (u, v)|_{t=0} = (u_0, v_0), \end{cases} \tag{2.12}$$

where $N \geq 1$ and π_N denotes the projector onto spatial frequencies $\{|n| \leq N\}$. We also use the following shorthand notations:

$$u_N = \pi_N u \qquad \text{and} \qquad v_N = \pi_N v.$$

We allow $N = \infty$ with the convention $\pi_\infty = \text{Id}$, which reduces (2.12) to (1.2).

For the (untruncated) NLW (1.2), the conserved energy is given by

$$E(\vec{u}) = \frac{1}{2}\int_{\mathbb{T}^3} \big(|\nabla u|^2 + v^2\big) + \frac{1}{4}\int_{\mathbb{T}^3} u^4.$$

The truncated system (2.12) also has the following conserved energy:

$$E_N(\vec{u}) = \frac{1}{2}\int_{\mathbb{T}^3} \big(|\nabla u|^2 + v^2\big) + \frac{1}{4}\int_{\mathbb{T}^3} (\pi_N u)^4. \tag{2.13}$$

In the following two lemmas, we state the classical well-posedness theory for (2.12) and the relevant dynamical properties.

Lemma 2.4. *Let $\sigma \geq 1$ and $N \in \mathbb{N} \cup \{\infty\}$. Then, the truncated NLW* (2.12) *is globally well-posed in $\vec{H}^\sigma(\mathbb{T}^3)$. Namely, given any $(u_0, v_0) \in \vec{H}^\sigma(\mathbb{T}^3)$, there exists a unique global solution to* (2.12) *in $C(\mathbb{R}; \vec{H}^\sigma(\mathbb{T}^3))$, where the dependence on initial data is continuous. Moreover, if we denote by $\Phi_N(t)$ the data-to-solution map at time t, then $\Phi_N(t)$ is a continuous bijection on $\vec{H}^\sigma(\mathbb{T}^3)$ for every $t \in \mathbb{R}$, satisfying the semigroup property:*

$$\Phi_N(t + \tau) = \Phi_N(t) \circ \Phi_N(\tau)$$

for any $t, \tau \in \mathbb{R}$.

The global well-posedness result stated in Lemma 2.4 follows from a standard local well-posedness theory along with the conservation of the truncated energy $E_N(\vec{u})$. See [29, Lemma 2.1] for the proof in the two-dimensional case.[5] The same proof applies to the three-dimensional case in view of the Sobolev embedding $H^1(\mathbb{T}^3) \subset L^6(\mathbb{T}^3)$ (with a small modification at the zeroth frequency).

Lemma 2.5. (i) (Growth bound) *Given $\sigma \geq 1$, we denote by B_R the ball of radius $R > 0$ in $\vec{H}^\sigma(\mathbb{T}^3)$ centered at the origin. Then, for any given $T > 0$, there exists $C(R, T) > 0$ such that*

$$\Phi_N(t)(B_R) \subset B_{C(R,T)} \tag{2.14}$$

for any $t \in [0, T]$ and $N \in \mathbb{N} \cup \{\infty\}$.

(ii) (Approximation) *Let $\sigma \geq 1$, $T > 0$, and K be a compact set in $\vec{H}^\sigma(\mathbb{T}^3)$. Then, for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that*

$$\|\Phi(t)(\vec{u}) - \Phi_N(t)(\vec{u})\|_{\vec{H}^\sigma(\mathbb{T}^3)} < \varepsilon$$

for any $t \in [0, T]$, $\vec{u} \in K$, and $N \geq N_0$. Hence, we have

$$\Phi(t)(K) \subset \Phi_N(t)(K + B_\varepsilon).$$

for any $t \in [0, T]$ and $N \geq N_0$. Here, $\Phi(t)$ denotes the solution map $\Phi_\infty(t) = \Phi_{\mathrm{NLW}}(t)$ for the (*untruncated*) *NLW* (1.2).

Proof. The solution $\vec{u} = (u, v)$ to (2.12) satisfies the following Duhamel formulation:

$$\begin{aligned} u(t) &= S(t)(u_0, v_0) - \int_0^t \frac{\sin((t - t')|\nabla|)}{|\nabla|} \pi_N\big((\pi_N u)^3\big)(t') dt', \\ v(t) &= \partial_t S(t)(u_0, v_0) - \int_0^t \cos((t - t')|\nabla|) \pi_N\big((\pi_N u)^3\big)(t') dt', \end{aligned} \tag{2.15}$$

where $S(t)$ denotes the linear wave propagator given by

$$S(t)(u_0, v_0) = \cos(t|\nabla|)u_0 + \frac{\sin(t|\nabla|)}{|\nabla|} v_0.$$

From the fractional Leibniz rule (2.7) and (2.5), we have

$$\|u^3\|_{H^{\sigma-1}} \lesssim \|u\|_{B^{\sigma-1}_{6,2}} \|u\|^2_{L^6} \lesssim \|u\|_{H^\sigma} \|u\|^2_{H^1} \tag{2.16}$$

[5] This is in the context of the nonlinear Klein-Gordon equation but the proof can be easily adapted.

for $\sigma \geq 1$. Then, from (2.15) and (2.16) with the conservation of the truncated energy E_N in (2.13), we have[6]

$$\begin{aligned} \|\vec{u}(t)\|_{\vec{H}^\sigma} &\leq \|(u_0, v_0)\|_{\vec{H}^\sigma} + C(1+|t|)\int_0^t \|u(t')\|_{H^\sigma}\|u(t')\|_{H^1}^2 dt' \\ &\leq \|(u_0, v_0)\|_{\vec{H}^\sigma} + C(1+|t|)\cdot E_N(u_0, v_0)\int_0^t \|(u,v)(t')\|_{\vec{H}^\sigma} dt'. \end{aligned}$$

Hence, the growth bound (2.14) follows from Gronwall's inequality.

The approximation property (ii) follows from a modification of the local well-posedness argument. Since the argument is standard, we omit details. See, for example, our previous works: Proposition 2.7 in [36] and Lemma 6.20/B.2 in [27]. $\square$

3. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. We first present a general framework of the strategy. We then introduce a renormalized energy and discuss further refinements required for our problem. In Subsection 3.4, we prove Theorem 1.1 by assuming the construction of the weighted Gaussian measure (Proposition 3.7) and the renormalized energy estimate (Proposition 3.8). We present the proofs of Propositions 3.7 and 3.8 in Sections 4 and 5.

3.1. **General framework.**

In [36], the third author introduced a general strategy, combining PDE techniques and stochastic analysis to prove quasi-invariance of Gaussian measures under nonlinear Hamiltonian PDE dynamics. In the following, we briefly describe a rough idea behind this method [36, 29], using NLW on $\mathbb{T}^d$ as an example. See also [28] for a survey on this subject. Note that we keep our discussion at a formal level and that some steps need to be justified by working at the level of the truncated dynamics (2.12).

Let $\Phi = \Phi_{\text{NLW}}$ as in the previous section. In order to prove quasi-invariance of $\vec{\mu}_s$ under Φ, we would like to show $\vec{\mu}_s(\Phi(t)(A)) = 0$ for any $t \in \mathbb{R}$ and any measurable set $A \subset \vec{H}^\sigma(\mathbb{T}^d)$ with $\vec{\mu}_s(A) = 0$. Here, $\sigma < s + 1 - \frac{d}{2}$ denotes the regularity of samples on $\mathbb{T}^d$ under $\vec{\mu}_s$. The main idea is to study the evolution of

$$\vec{\mu}_s(\Phi(t)(A)) = Z_s^{-1}\int_{\Phi(t)(A)} e^{-\frac{1}{2}\|\vec{u}\|^2_{\vec{H}^{s+1}}} d\vec{u}$$

for a general measurable set $A \subset \vec{H}^\sigma(\mathbb{T}^d)$ and to control the growth of $\vec{\mu}_s(\Phi(t)(A))$ in time. Here, the main goal is show a differential inequality of the form:

$$\frac{d}{dt}\vec{\mu}_s(\Phi(t)(A)) \leq Cp^\beta\big\{\vec{\mu}_s(\Phi(t)(A))\big\}^{1-\frac{1}{p}} \tag{3.1}$$

for some $0 \leq \beta \leq 1$. Once (3.1) could be established, Yudovich's argument [39] or its refinement [29] when $\beta = 1$ would then yield quasi-invariance for short times. Iterating the argument and using time-reversibility of the equation yields quasi-invariance for all $t \in \mathbb{R}$. In this argument, the linear power of p in the prefactor of the right-hand side of (3.1) is crucial.

[6]The factor $1 + |t|$ appears in controlling the zeroth frequency: $\frac{\sin((t-t')|\nabla|)}{|\nabla|} = t - t'$.

By applying a change-of-variable formula, we have

$$\vec{\mu}_s(\Phi(t)(A)) \text{``}=\text{''} Z_s^{-1} \int_A e^{-\frac{1}{2}\|\Phi(t)(\vec{u})\|^2_{\vec{H}^{s+1}}} d\vec{u}. \tag{3.2}$$

For the truncated dynamics (2.12), the formula (3.2) can be justified via invariance of the Lebesgue measure and bijectivity of the flow Φ_N. See Lemma 3.9 below. Fix $t_0 \in \mathbb{R}$. Then, by taking a time derivative, we arrive at

$$\begin{aligned} \frac{d}{dt}\vec{\mu}_s(\Phi(t)(A))\bigg|_{t=t_0} &= -\frac{1}{2} Z_s^{-1} \int_{\Phi(t_0)(A)} \frac{d}{dt}\Big(\|\Phi(t)(\vec{u})\|^2_{\vec{H}^{s+1}}\Big) e^{-\frac{1}{2}\|\Phi(t)(\vec{u})\|^2_{\vec{H}^{s+1}}} d\vec{u}\bigg|_{t=0} \\ &= -\frac{1}{2} \int_{\Phi(t_0)(A)} \frac{d}{dt}\Big(\|\Phi(t)(\vec{u})\|^2_{\vec{H}^{s+1}}\Big)\bigg|_{t=0} d\vec{\mu}_s. \end{aligned} \tag{3.3}$$

This reduction of the analysis to that at $t = 0$, exploiting the group property $\Phi(t_0 + t) = \Phi(t)\Phi(t_0)$ was inspired from the work [37]. Suppose that we had an effective energy estimate (with smoothing) of the form:

$$\frac{d}{dt}\|\Phi(t)(\vec{u})\|^2_{\vec{H}^{s+1}}\bigg|_{t=0} \text{``}\le\text{''} C(\|\vec{u}\|_{\vec{H}^1})\|\vec{u}\|^\theta_{\vec{\mathcal{C}}^\sigma} \tag{3.4}$$

for some $\theta \le 2$. Then, the desired estimate (3.1) would follow from (3.2), (3.3), and (3.4) along with the Wiener chaos estimate (Lemma 2.2). Note that, in the energy estimate (3.4), we can afford to place two factors of $\vec{u}$ in the stronger Hölder-Besov $\vec{\mathcal{C}}^\sigma$-norm, while we need to place all the other factors in the (weaker) $\vec{H}^1$-norm, which is controlled by the conserved energy $E(\vec{u})$ in (2.13).

In [36], the third author established an energy estimate of the form (3.4) for the BBM equation by consideration in the spirit of quasilinear hyperbolic PDEs (namely, integration by parts in x). Unfortunately, an energy estimate of the form (3.4) does not hold in general for nonlinear Hamiltonian PDEs. In [27, 29], the second and third authors circumvented this problem by introducing a modified energy:

$$E_s(\vec{u}) = \frac{1}{2}\|\vec{u}\|^2_{\vec{H}^{s+1}} + R_s(\vec{u})$$

with a suitable correction term $R_s(\vec{u})$ such that the desired energy estimate of the form (3.4) holds for this modified energy. By following the strategy described above, they first established quasi-invariance of the weighted Gaussian measure associated with this modified energy:

$$d\vec{\rho}_s = Z_s^{-1} e^{-E_s(\vec{u})} d\vec{u} = Z_s^{-1} e^{-R_s(\vec{u})} d\vec{\mu}_s$$

(with a cutoff on a conserved quantity). Then, quasi-invariance of $\vec{\mu}_s$ followed from the mutual absolute continuity of $\vec{\mu}_s$ and $\vec{\rho}_s$.

For Schrödinger-type equations, modified energies were introduced by the normal form method (namely, integration by parts in time); see [27, 25, 14]. In [29], the second and third authors derived a modified energy for NLW on $\mathbb{T}^2$ based on integration by parts in x but a certain renormalization was needed to control singularity. We will describe the details of this derivation in the next subsection.

Summary: The study of quasi-invariance has therefore been reduced to two steps: (i) the construction of the weighted Gaussian measure $\vec{\rho}_s$ and (ii) establishing an effective energy estimate on $\partial_t E_s(\vec{u})\big|_{t=0}$.

3.2. **Renormalized energy for NLW.** In this subsection, we present a discussion on a modified energy for our problem. See (3.18) below for the full modified energy. In the following, we fix $\sigma = s + 1 - \frac{d}{2} - \varepsilon \geq 1$ for some small $\varepsilon > 0$ and let B_R denotes the ball of radius $R > 0$ in $\vec{H}^\sigma(\mathbb{T}^d)$ centered at the origin. Fix a frequency cutoff size N and, instead of using (a suitable truncated version of) the energy of $\vec{\mu}_s$, let us consider the following natural energy to work with for the wave equation (see Remark 3.6):

$$\frac{1}{2}\int_{\mathbb{T}^d}(D^s v_N)^2 + \frac{1}{2}\int_{\mathbb{T}^d}(D^{s+1}u_N)^2,$$

where $D^s = (-\Delta)^{\frac{s}{2}}$ denotes the Riesz potential of order s. Fix an even integer $s \geq 4$ and let $\vec{u} = (u, v)$ be a solution to the truncated NLW (2.12). Then, the Leibniz rule yields

$$\begin{aligned}\partial_t\bigg[\frac{1}{2}\int_{\mathbb{T}^d}(D^s v_N)^2 + \frac{1}{2}\int_{\mathbb{T}^d}(D^{s+1}u_N)^2\bigg] &= \int_{\mathbb{T}^d}(D^{2s}v_N)(-u_N^3)\\ &= -3\int_{\mathbb{T}^d} D^s v_N D^s u_N\, u_N^2\\ &\quad + \sum_{\substack{|\alpha|+|\beta|+|\gamma|=s\\ |\alpha|,|\beta|,|\gamma|<s}} c_{\alpha,\beta,\gamma}\int_{\mathbb{T}^d} D^s v_N\cdot\partial^\alpha u_N\cdot\partial^\beta u_N\cdot\partial^\gamma u_N\end{aligned}\tag{3.5}$$

for some combinatorial constants $c_{\alpha,\beta,\gamma}$ that depend only on s, where ∂^α denotes $\partial_{x_1}^{\alpha_1}\cdots\partial_{x_d}^{\alpha_d}$ for a multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$. Samples $\vec{u}$ under the Gaussian measure $\vec{\mu}_s$ belong almost surely to $\vec{\mathcal{C}}^\sigma(\mathbb{T}^d)\setminus\vec{\mathcal{C}}^{s+1-\frac{d}{2}}(\mathbb{T}^d)$ for $\sigma < s + 1 - \frac{d}{2}$. The main issue is how to treat $D^s v_N$ on the right-hand side of (3.5) due to its low regularity $\sigma - 1$. It turns out that all but the first term on the right-hand side of (3.5) can be treated by integration by parts. See Remark 3.3. As for the first term, recalling from (2.12) that $v_N = \partial_t u_N$, we have

$$-3\int_{\mathbb{T}^d} D^s v_N D^s u_N\, u_N^2 = -\frac{3}{2}\partial_t\bigg[\int_{\mathbb{T}^d}(D^s u_N)^2 u_N^2\bigg] + 3\int_{\mathbb{T}^d}(D^s u_N)^2\, v_N u_N. \tag{3.6}$$

The terms on the right-hand side of (3.6) are better behaved than that on the left-hand side since D^s no longer falls on the less regular term v. This motivates us to define a modified energy with a correction term of the form:

$$R_s(\vec{u}) = \frac{3}{2}\int_{\mathbb{T}^d}(D^s u_N)^2 u_N^2.$$

When $d = 1$, this choice of the correction term allows us to define a suitable modified energy and to construct the weighted Gaussian measure associated with this modified energy (modulo an issue at the zeroth frequency). When $d = 2$ or 3, however, we have $u \notin \mathcal{C}^s(\mathbb{T}^d)$ almost surely and thus the limiting expression $(D^s u)^2$ is ill defined since it is the square of a distribution of negative regularity. Moreover, the singular term $(D^s u)^2$ appears in both terms on the right-hand side of (3.6). As such, we have issues at the level of both the energy and its time derivative, which propagate to both the construction of the weighted Gaussian measure and the energy estimate.

Motivated by Euclidean quantum field theory, we introduce a renormalization. This amounts to replacing $(D^s u)^2$ by $(D^s u)^2 - \infty$, suitably interpreted; given $N \in \mathbb{N}$, we replace

$(D^s u_N)^2$ in (3.6) by $Q_{s,N}(u_N)$, where

$$Q_{s,N}(f) \stackrel{\text{def}}{=} (D^s f)^2 - \sigma_N \tag{3.7}$$

and σ_N is given by

$$\sigma_N \stackrel{\text{def}}{=} \mathbb{E}_{\vec{\mu}_s}\Big[(D^s \pi_N u)^2\Big] \sim \sum_{\substack{n\in\mathbb{Z}^d\\ 1\le|n|\le N}} \frac{1}{|n|^2} \sim \begin{cases} \log N & \text{for } d=2,\\ N & \text{for } d=3,\end{cases} \tag{3.8}$$

as $N\to\infty$. The crucial observation in [29] is that the effect of the renormalization for the two terms on the right-hand side in (3.6) precisely cancels each other, since

$$-\frac{3}{2}\sigma_N \partial_t\bigg[\int_{\mathbb{T}^d} u_N^2\bigg] + 3\sigma_N\int_{\mathbb{T}^d} v_N u_N = 0,$$

where we used the equation (2.12). As a result, we obtain

$$-3\int_{\mathbb{T}^d} D^s v_N D^s u_N\, u_N^2 = -\frac{3}{2}\partial_t\bigg[\int_{\mathbb{T}^d} Q_{s,N}(u_N)u_N^2\bigg] + 3\int_{\mathbb{T}^d} Q_{s,N}(u_N)v_N u_N. \tag{3.9}$$

In view of (3.5) and (3.9), we define the renormalized energy $\mathscr{E}_{s,N}(\vec{u})$ by

$$\mathscr{E}_{s,N}(\vec{u}) = \frac{1}{2}\int_{\mathbb{T}^d}(D^{s+1}u)^2 + \frac{1}{2}\int_{\mathbb{T}^d}(D^s v)^2 + \frac{3}{2}\int_{\mathbb{T}^d} Q_{s,N}(u_N)u_N^2. \tag{3.10}$$

Then, we have

$$\begin{aligned}\partial_t\mathscr{E}_{s,N}(\vec{u}) &= 3\int_{\mathbb{T}^d} Q_{s,N}(u_N)v_N u_N\\ &\quad + \sum_{\substack{|\alpha|+|\beta|+|\gamma|=s\\ |\alpha|,|\beta|,|\gamma|<s}} c_{\alpha,\beta,\gamma}\int_{\mathbb{T}^d} D^s v_N\cdot\partial^\alpha u_N\cdot\partial^\beta u_N\cdot\partial^\gamma u_N.\end{aligned} \tag{3.11}$$

Note that we have renormalized both the energy and its time derivative at the same time. The considerations above motivate the definition of the renormalized weighted Gaussian measure:

$$d\vec{\rho}_{s,r,N} = Z_{s,N,r}^{-1}\mathbf{1}_{\{E_N(\vec{u})\le r\}} e^{-\mathscr{E}_{s,N}(\vec{u})}d\vec{u}, \tag{3.12}$$

where $E_N(\vec{u})$ is as in (2.13). The energy cutoff in (3.12) is necessary to construct this measure due to an issue with the zeroth frequency (see Remark 3.6).

Remark 3.1. If $\vec{u}$ is distributed according to the Gaussian measure $\vec{\mu}_s$, then we can apply Wick renormalization to $(D^s u_N)^2$ and obtain the Wick power $:(D^s u_N)^2:$. Here, Wick renormalization corresponds the orthogonal projection onto a (second) homogeneous Wiener chaos under $L^2(\vec{\mu}_s)$. In this case, we have

$$:(D^s u_N)^2: = Q_{s,N}(u_N).$$

This renormalization allows us to take a limit $:(D^s u)^2: = \lim_{N\to\infty} :(D^s u_N)^2:$ in a suitable space (see Lemmas 4.1 and 4.6 below). In the discussion above for deriving the renormalized energy $\mathscr{E}_{s,N}$, however, $\vec{u}$ denotes a solution to (2.12) and a notation such as $:(D^s u_N)^2:$ is not well defined. This is the reason we needed to introduce $Q_{s,N}$ in (3.7).

Remark 3.2. This simultaneous renormalization of the energy and its time derivative does not introduce any modification to the original truncated equation (2.12) since its Hamiltonian $E_N(\vec{u})$ remains unchanged. We also point out two (related) interesting observations: (i) renormalization is usually applied in the handling of rough functions, whereas we use renormalization in the context of high regularity solutions, and (ii) the simultaneous renormalization is introduced only as a tool to prove Theorem 1.1.

Remark 3.3. In view of the regularity of $\vec{u}$ under $\vec{\mu}_s$, it may seem that some of the lower order terms under the sum on the right-hand side of (3.11) are divergent as $N \to \infty$: for example, when $|\alpha| = s-1$, $|\beta| = 1$, and $\gamma = 0$. However, by integration by parts (in x) and the independence of u and v, they turn out to be convergent without any renormalization. See the proof of Proposition 3.8.

• **Problem (i): Construction of the weighted Gaussian measure.** The problem of constructing the limiting weighted Gaussian measure measure $\vec{\tilde{\rho}}_{s,r} = \lim_{N\to\infty} \vec{\tilde{\rho}}_{s,r,N}$ bears some similarity with the problem of constructing the Φ^4-measures. First of all, the need for renormalization in (3.10) means that the positivity of the random variable $\int (D^s u)^2 u^2$ is destroyed. Moreover, there is a similarity between the measures themselves; despite not having the simple algebraic structure of the Φ^4-measure, the term $\int (D^s u)^2 u^2$ is quartic in u. In [29], the second and third authors exploited these similarities and modified Nelson's construction of the Φ^4_2-measure to construct the desired weighted Gaussian measure $\vec{\tilde{\rho}}_{s,r}$ in the two-dimensional case. The construction in [29] heavily uses the logarithmic divergence rate (3.8) of the renormalization constants and uses the energy cutoff $\mathbf{1}_{\{E_N(u,v)\le r\}}$, while they did not make use of the positive quartic potential energy term $\frac{1}{4}\int u^4$.

The analogy between $\vec{\tilde{\rho}}_{s,r}$ and the Φ^4-measures starts to break down in the three-dimensional case. On the one hand, Nelson's construction fails for both. For the measure $\vec{\tilde{\rho}}_{s,r}$, this is due to the algebraic divergence rate (3.8) of the renormalization constants σ_N; see Remark 3.6 in [29]. For the Φ^4_3-measure, the issue is more subtle and further renormalization beyond Wick renormalization is required. As a consequence, the resulting Φ^4_3-measure is expected to be singular with respect to its underlying Gaussian measure. We point out that one expects a priori that the renormalizations necessary for $\vec{\tilde{\rho}}_{s,r}$ are different from the Φ^4_3-measure since the singular term in $\int (D^s u)^2 u^2$ is quadratic, not quartic, in u.

In order to construct $\vec{\tilde{\rho}}_{s,r}$, we use the techniques introduced in a recent paper [2] by Barashkov and Gubinelli, where the partition functions of the Φ^4_2- and Φ^4_3-measures were analyzed by way of variational formulas. In particular, we show that the measures $\vec{\tilde{\rho}}_{s,r}$ are still absolutely continuous with respect to the underlying Gaussian measure.[7] One technical issue with the construction of $\vec{\tilde{\rho}}_{s,r}$ is that it is not clear whether the term $\int (D^s u)^2 u^2$ is good enough to control the large-scale behavior (= low frequency part) of u. In the following, we circumvent this problem by introducing a new renormalized energy $E_{s,N}(\vec{u})$ in (3.18) by adding the energy $E_N(\vec{u})$ in (2.13) (plus an extra term controlling the zeroth Fourier coefficient of u) to the renormalized energy $\mathscr{E}_{s,N}(\vec{u})$ in (3.10). This allows us to use the potential energy term $\frac{1}{4}\int u_N^4$ in (2.13) to get rid of the need of the energy cutoff $\mathbf{1}_{\{E_N(\vec{u})\le r\}}$.

[7] In order to avoid an issue at the zeroth frequency, we need to make a modification to the renormalized energy $\mathscr{E}_{s,N}(\vec{u})$. This leads to a slightly different weighted Gaussian measure. See (3.18), (3.20), and (3.21) below.

The effect is to change the underlying Gaussian measure $\vec{\mu}_s$ to a different Gaussian measure $\vec{\nu}_s$, which will be shown to be equivalent to $\vec{\mu}_s$ by Kakutani's theorem. See Lemma 3.5 below. The measures that we construct are simple yet interesting examples of measures that require only Wick renormalization but for which Nelson's construction fails.

• **Problem (ii): Energy estimate.** In the two-dimensional case [29], it was not possible to establish an energy estimate of the form (3.4). Instead, it was shown that

$$\Big|\partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))|_{t=0}\Big| \lesssim C(\|\vec{u}\|_{\vec{H}^1})F(\vec{u}). \tag{3.13}$$

for a suitable renormalized energy. Here, $F(\vec{u})$ denotes complicated expressions that contain high regularity information on $\vec{u}$ such as the $\vec{W}^{\sigma,\infty}$-norm as well as the renormalized second power $\int_{\mathbb{T}^2} Q_{s,N}(u_N)$. As mentioned above, all but two factors need to be placed in the weaker H^1-norm so that $F(\vec{u})$ is at most quadratic in $\vec{u}$, which implies that $F(\vec{u}) \in \mathcal{H}_2$. This allows us to obtain the right growth bound of the form (3.1) after applying the Wiener chaos estimate (Lemma 2.2). Here, it is crucial to study the energy estimate (3.13) at time $t = 0$ to exploit the Gaussian initial data in in (1.3). In [29], the energy estimate (3.13) involved a delicate quadrilinear Littlewood-Paley expansion balancing the interplay between the energy conservation and the higher order regularity. As pointed out in [29], the estimate of the form (3.13) fails for the three-dimensional case.

In a recent paper [31], Planchon, Visciglia, and the third author proved quasi-invariance of the Gaussian measures under the dynamics of the (super-)quintic nonlinear Schrödinger equations (NLS) on $\mathbb{T}$ by establishing a novel energy estimate. The idea is to exploit a deterministic growth bound (2.14) on solutions. Then, the required energy estimate takes the following form:[8]

$$\Big|\partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))\Big| \le C\big(1 + \|\Phi_N(t)(\vec{u})\|_{\vec{H}^\sigma}^k\big). \tag{3.14}$$

Here, $k > 0$ can be *any* positive number. The main point is that if we start dynamics with a measurable set $A \subset B_R$, then (3.14) with the growth bound (2.14) yields

$$\Big|\mathbf{1}_A(\vec{u}) \cdot \partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))\Big| \le C\Big|\mathbf{1}_{B_{C(R,T)}}(\vec{u}) \cdot \big(1 + \|\vec{u}\|_{\vec{H}^\sigma}^k\big)\Big| \le C(R)^k$$

for any $t \in [0, T]$ and $N \in \mathbb{N} \cup \{\infty\}$. This control allows us to prove quasi-invariance for each measurable set $A \subset B_R$ (in the sense of (3.24) below). Then, by a soft argument, we can conclude quasi-invariance of the Gaussian measure $\vec{\mu}_s$. The main advantage of this argument is that we are allowed to place any power k in the stronger $\vec{H}^\sigma$-norm. Note that the energy estimate (3.14) is entirely deterministic and hence there is no need to reduce the analysis to time $t = 0$.

In this paper, we combine these two approaches described above and establish an energy estimate of the form:

$$\Big|\mathbf{1}_{B_R}(\vec{u}) \cdot \partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))|_{t=0}\Big| \le C(\|\vec{u}\|_{\vec{H}^\sigma})F(\vec{u}),$$

where we use the deterministic growth bound (2.14) to control $C(\|\vec{u}\|_{\vec{H}^\sigma})$, while we use the Wiener chaos estimate (Lemma 2.2) to control $F(\vec{u})$. The fact that we have access to the stronger $\vec{H}^\sigma$-norm (rather than $\vec{H}^1$-norm as in (3.13)) allows us to get by with a

[8]In the case of NLS, we have u instead of $\vec{u} = (u, v)$. For the sake of presentation, we keep the notation adapted to the NLW context.

softer energy estimate. Moreover, in our case, $F(\vec{u})$ is given in an explicit manner (see Proposition 5.1). It contains products of derivatives of u_N and v_N as well as the $\mathcal{C}^{-1-\varepsilon}$-norm of the Wick power $Q_{s,N}(u_N) = (D^s u_N)^2 - \sigma_N$. By proceeding as in [19], we establish regularity properties of these random distributions in Proposition 4.3. These two points lead to a significantly simpler proof of quasi-invariance than the two-dimensional case [29].

Remark 3.4. Following the discussion of Remark (iv) in Subsection 1.2, one might attempt to implement an analogous construction of weighted Gaussian measure in the case of NLW with a higher order nonlinearity or in higher dimensions. Higher order nonlinearities would result in a higher power of the regular part of the renormalized energy, while the singular part would remain quadratic, i.e. $(D^s u)^2$. Thus, the construction of these measures seems tractable. This is in sharp contrast with the construction of the Φ_3^{2n} measures, where higher order nonlinearities result in higher powers of distributions which makes the construction of such measures impossible (for $n \geq 3$). Higher dimensions would result in a more singular quadratic part.

3.3. **Statements of key results.** In the remaining part of this paper, we fix $d = 3$. In this subsection, we introduce a new renormalized energy and then state the key propositions in proving Theorem 1.1.

We first introduce a new Gaussian measure, whose energy is more suitable for analysis on NLW (but still controls the zeroth frequency). Define a Gaussian measure $\vec{\nu}_s$ via the following Karhunen-Loève expansions:

$$\begin{aligned} u^\omega(x) &= g_0(\omega) + \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} \frac{g_n(\omega)}{(|n|^2 + |n|^{2s+2})^{\frac{1}{2}}} e^{in\cdot x}, \\ v^\omega(x) &= \sum_{n \in \mathbb{Z}^3} \frac{h_n(\omega)}{(1 + |n|^{2s})^{\frac{1}{2}}} e^{in \cdot x}, \end{aligned} \tag{3.15}$$

where $\{g_n\}_{n\in\mathbb{Z}^3}$ and $\{h_n\}_{n\in\mathbb{Z}^3}$ are as in (1.3). Then, the formal density of $\vec{\nu}_s$ is given by

$$d\vec{\nu}_s = Z_s^{-1} e^{-H_s(\vec{u})} d\vec{u},$$

where

$$H_s(\vec{u}) = \frac{1}{2}\bigg(\int_{\mathbb{T}^3} u\bigg)^2 + \frac{1}{2}\int_{\mathbb{T}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{T}^3} (D^{s+1}u)^2 + \frac{1}{2}\int_{\mathbb{T}^3} v^2 + \frac{1}{2}\int_{\mathbb{T}^3} (D^s v)^2. \tag{3.16}$$

Lemma 3.5. *Let $s > \frac{3}{4}$. Then, the Gaussian measures $\vec{\mu}_s$ and $\vec{\nu}_s$ are equivalent.*

The proof of this lemma is based on a simple application of Kakutani's theorem [16]; see the proof of Lemma 6.1 in [29] for details in the two-dimensional case.

Remark 3.6. The linear wave equation conserves the homogeneous Sobolev norm:

$$\|\vec{u}\|^2_{\dot{\vec{H}}^{s+1}} = \int_{\mathbb{T}^3} (D^{s+1}u)^2 + \int_{\mathbb{T}^3} (D^s v)^2.$$

Hence, we would like to work with Gaussian measures with formal density $e^{-\frac{1}{2}\|u\|^2_{\dot{\vec{H}}^{s+1}}}$. These measures do not exist as probability measures since the zeroth frequency is not controlled. This is the reason we chose to include $g_0(\omega)$ in (3.15), giving rise to the first term in $H_s(\vec{u})$ defined in (3.16).

As we see below, we add the truncated energy $E_N(\vec{u})$ in (2.13) to construct the full renormalized energy, which explains the appearance of the terms with $|\nabla u|^2$ and v^2 in (3.16). This addition of the truncated energy $E_N(\vec{u})$ allows us to include the quartic potential energy $\frac{1}{4}\int u_N^4$ without changing the time derivative of the renormalized energy; see (3.19). We point out that this quartic homogeneity plays an important role in the construction of the weighted Gaussian measure.

Given $N \in \mathbb{N}$, we *redefine* the parameter σ_N, adapted to the new Gaussian measure $\vec{\nu}_s$, by

$$\sigma_N \stackrel{\text{def}}{=} \mathbb{E}_{\vec{\nu}_s}\Big[(D^s u_N)^2\Big] = \sum_{\substack{n\in\mathbb{Z}^3\\ 1\le |n|\le N}} \frac{|n|^{2s}}{|n|^2+|n|^{2s+2}} \sim N \longrightarrow \infty \tag{3.17}$$

as $N \to \infty$. We also redefine the operator $Q_{s,N}$ in (3.7) with this new definition of σ_N. In the remaining part of this paper, we will use these new definitions for σ_N and $Q_{s,N}$.

We now define the full renormalized energy $E_{s,N}(\vec{u})$ by

$$E_{s,N}(\vec{u}) = \mathscr{E}_{s,N}(\vec{u}) + E_N(\vec{u}) + \frac{1}{2}\bigg(\int_{\mathbb{T}^3} u_N\bigg)^2, \tag{3.18}$$

where $\mathscr{E}_{s,N}$ is as in (3.10) and E_N is the truncated energy in (2.13). Then, it follows from (3.11) and the conservation of the truncated energy that

$$\begin{aligned} \partial_t E_{s,N}(\vec{u}) = 3\int_{\mathbb{T}^3} & Q_{s,N}(u_N)v_N u_N \\ &+ \sum_{\substack{|\alpha|+|\beta|+|\gamma|=s\\ |\alpha|,|\beta|,|\gamma|<s}} c_{\alpha,\beta,\gamma}\int_{\mathbb{T}^3} D^s v_N\cdot\partial^\alpha u_N\cdot\partial^\beta u_N\cdot\partial^\gamma u_N \\ &+ \bigg(\int_{\mathbb{T}^3} u_N\bigg)\bigg(\int_{\mathbb{T}^3} v_N\bigg) \end{aligned} \tag{3.19}$$

for any solution $\vec{u}$ to the truncated NLW (2.12). Moreover, from (3.16), we have

$$E_{s,N}(\vec{u}) = H_s(\vec{u}) + R_{s,N}(u),$$

where

$$\begin{aligned} R_{s,N}(u) &= \frac{3}{2}\int_{\mathbb{T}^3} Q_{s,N}(u_N)u_N^2 + \frac{1}{4}\int_{\mathbb{T}^3} u_N^4 \\ &= \frac{3}{2}\int_{\mathbb{T}^3}\Big((D^s u_N)^2 - \sigma_N\Big)u_N^2 + \frac{1}{4}\int_{\mathbb{T}^3} u_N^4. \end{aligned} \tag{3.20}$$

We are now ready to state the two key ingredients for proving Theorem 1.1: (i) the construction of the weighted Gaussian measures and (ii) the renormalized energy estimate.

Define the weighted Gaussian measure $\vec{\rho}_{s,N}$ by

$$d\vec{\rho}_{s,N}(\vec{u}) = \mathscr{Z}_{s,N}^{-1} e^{-R_{s,N}(u)} d\vec{\nu}_s(\vec{u}), \tag{3.21}$$

where $\mathscr{Z}_{s,N}$ is the normalization constant. The following proposition establishes uniform integrability of the density $e^{-R_{s,N}(u)}$ in (3.21), which allows us to construct the limiting weighted Gaussian measure $\vec{\rho}_s$ by

$$d\vec{\rho}_s(\vec{u}) = \mathscr{Z}_s^{-1} e^{-R_s(u)} d\vec{\nu}_s(\vec{u}),$$

where $R_s(u)$ is a limit of $R_{s,N}(u)$; see Lemma 4.1.

Proposition 3.7 (Construction of the weighted Gaussian measure)**.** *Let $s > \frac{3}{2}$. Then, the weighted Gaussian measures $\vec{\rho}_{s,N}$ converges strongly to $\vec{\rho}_s$. Namely, we have*

$$\lim_{N\to\infty} \vec{\rho}_{s,N}(A) = \vec{\rho}_s(A)$$

for any measurable set $A \subset \vec{H}^\sigma(\mathbb{T}^3)$, $\sigma < s - \frac{1}{2}$. Moreover, given any finite $p \geq 1$, the sequence $\{e^{-R_{s,N}(u)}\}_{N\in\mathbb{N}}$ and $e^{-R_s(u)}$ are uniformly bounded in $L^p(\vec{\nu}_s)$. As a consequence, $\vec{\rho}_s$ is equivalent to $\vec{\nu}_s$.

Next, we state the key renormalized energy estimate. Recall that B_R denotes the ball of radius $R > 0$ in $\vec{H}^\sigma(\mathbb{T}^3)$ centered at the origin. We denote by $\Phi_N(t)$ the flow of the truncated NLW dynamics (2.12).

Proposition 3.8 (Renormalized energy estimate)**.** *Let $s \geq 4$ be an even integer. Then, given $R > 0$, there is a constant $C = C(R) > 0$ such that*

$$\bigg\{ \int \mathbf{1}_{B_R}(\vec{u}) \cdot \Big| \partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))|_{t=0} \Big|^p d\vec{\nu}_s(\vec{u}) \bigg\}^{\frac{1}{p}} \leq Cp$$

for any finite $p \geq 1$ and any $N \in \mathbb{N}$.

Before we state the main proposition on the evolution of the truncated measures $\vec{\rho}_{s,N}$, let us state the following change-of-variable formula. Given $N \in \mathbb{N}$, let $\mathcal{E}_N = \pi_N L^2(\mathbb{T}^3)$ and we endow $\mathcal{E}_N \times \mathcal{E}_N$ with the Lebesgue measure L_N as in Section 2. Then, by viewing the Gaussian measure $\vec{\nu}_s$ as a product measure on $(\mathcal{E}_N \times \mathcal{E}_N) \times (\mathcal{E}_N \times \mathcal{E}_N)^\perp$, we can write the truncated weighted Gaussian measure $\vec{\rho}_{s,N}$ defined in (3.21) as

$$\begin{aligned} d\vec{\rho}_{s,N}(\vec{u}) &= \mathcal{Z}_{s,N}^{-1}\, e^{-R_{s,N}(\pi_N u)} d\vec{\nu}_s(\vec{u}), \\ &= \hat{Z}_{s,N}^{-1}\, e^{-E_{s,N}(\pi_N \vec{u})}\, dL_N \otimes d\vec{\nu}_{s;N}^{\perp}(\vec{u}), \end{aligned} \tag{3.22}$$

where $\hat{Z}_{s,N}$ denotes the normalization constant and $\vec{\nu}_{s;N}^{\perp}$ denotes the marginal Gaussian measure of $\vec{\nu}_s$ on $(\mathcal{E}_N \times \mathcal{E}_N)^\perp$. Then, we have the following change-of-variable formula.

Lemma 3.9. *Let $s > \frac{3}{2}$ and $N \in \mathbb{N}$. Then, we have*

$$\vec{\rho}_{s,N}(\Phi_N(t)(A)) = \hat{Z}_{s,N}^{-1} \int_A e^{-E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))}\, dL_N \otimes d\vec{\nu}_{s;N}^{\perp}(\vec{u})$$

for any $t \in \mathbb{R}$ and any measurable set $A \subset \vec{H}^\sigma(\mathbb{T}^3)$ with $\sigma < s - \frac{1}{2}$.

The proof of Lemma 3.9 is based on (i) the invariance of the Lebesgue measure L_N under (the low frequency part of) the truncated NLW dynamics $\pi_N\Phi_N(t)$, (ii) the conservation of the truncated energy $E_N(\vec{u})$ under $\Phi_N(t)$ and (iii) the bijectivity of the solution map $\Phi_N(t)$. As it follows from similar considerations presented in [36, 27], we omit details of the proof.

We now state and prove the main proposition, essentially establishing the differential inequality (3.1). This proposition allows us to control the growth of the pushforward measure $\vec{\rho}_{s,N}(\Phi_N(t)(A))$ of a given measurable set $A \subset \vec{H}^\sigma(\mathbb{T}^3)$ uniformly in $N \in \mathbb{N}$, provided that the set A lies in the ball $B_R \subset \vec{H}^\sigma(\mathbb{T}^3)$ of radius $R > 0$. Namely, it only provides a *set-dependent* control. This dependence on $R>0$, however, does not cause any trouble in establishing quasi-invariance of the Gaussian measure $\vec{\nu}_s$ (and hence of $\vec{\mu}_s$).

Proposition 3.10. *Let $s\geq 4$ be an even integer and $\sigma\in\big(1,s-\frac{1}{2}\big)$. Then, given $R>0$ and $T>0$, there exists $C_{R,T}>0$ such that*

$$\frac{d}{dt}\vec{\rho}_{s,N}(\Phi_N(t)(A))\leq C_{R,T}\cdot p\left\{\vec{\rho}_{s,N}(\Phi_N(t)(A))\right\}^{1-\frac{1}{p}}$$

for any $p\geq 2$, any $N\in\mathbb{N}$, any $t\in[0,T]$, and any measurable set $A\subset B_R\subset\vec{H}^\sigma(\mathbb{T}^3)$.

In [29], there is an analogous statement, controlling the evolution of the truncated measures (without the restriction on B_R); see [29, Lemma 5.2]. The main idea of the proof of Lemma 5.2 in [29] is to reduce the analysis to that at $t=0$, which provides access to the random distributions in (3.15). On the other hand, the main idea in [31] at this step is to use the *deterministic* control (2.14) on the growth of solutions. In the following, we combine both of these ideas, thus introducing a hybrid argument which works more effectively than each of the two methods.

Proof. Fix $R,T>0$ and $t_0\in[0,T]$. Let $A\subset B_R$ be a measurable set in $\vec{H}^\sigma(\mathbb{T}^3)$. Using the flow property of $\Phi_N(t)$, we have

$$\begin{aligned}\frac{d}{dt}\vec{\rho}_{s,N}(\Phi_N(t)(A))\bigg|_{t=t_0}&=\mathcal{Z}_{s,N}^{-1}\frac{d}{dt}\int_{\Phi_N(t)(A)}e^{-R_{s,N}(\pi_N u)}d\vec{\nu}_s(\vec{u})\bigg|_{t=t_0}\\&=\mathcal{Z}_{s,N}^{-1}\frac{d}{dt}\int_{\Phi_N(t)(\Phi_N(t_0)(A))}e^{-R_{s,N}(\pi_N u)}d\vec{\nu}_s(\vec{u})\bigg|_{t=0}.\end{aligned}$$

The change-of-variable argument (Lemma 3.9), (3.22), and the growth bound (2.14) in Lemma 2.5 yield

$$\begin{aligned}&\frac{d}{dt}\vec{\rho}_{s,N}(\Phi_N(t)(A))\bigg|_{t=t_0}\\&=\hat{Z}_{s,N}^{-1}\frac{d}{dt}\int_{\Phi_N(t_0)(A)}e^{-E_{s,N}(\pi_N\Phi_N(t)(u,v))}dL_N\otimes d\vec{\nu}_{s;N}^{\perp}\bigg|_{t=0}\\&=-\mathcal{Z}_{s,N}^{-1}\int_{\Phi_N(t_0)(A)}\partial_t E_{s,N}(\pi_N\Phi_N(t)(\vec{u}))\big|_{t=0}\,e^{-R_{s,N}(\pi_N u)}d\vec{\nu}_s(\vec{u})\\&\leq\mathcal{Z}_{s,N}^{-1}\int_{B_{C(R,T)}}\Big|\partial_t E_{s,N}(\pi_N\Phi_N(t)(\vec{u}))\big|_{t=0}\Big|\,e^{-R_{s,N}(\pi_N u)}d\vec{\nu}_s(\vec{u}).\end{aligned}$$

Then, from Hölder's inequality, we obtain

$$\begin{aligned}\frac{d}{dt}\vec{\rho}_{s,N}(\Phi_N(t)(A))\bigg|_{t=t_0}&\leq\Big\|\mathbf{1}_{B_{C(R,T)}}(\vec{u})\cdot\partial_t E_{s,N}(\pi_N\Phi_N(t)(\vec{u}))\big|_{t=0}\Big\|_{L^p(\vec{\rho}_{s,N})}\\&\qquad\times\left\{\vec{\rho}_{s,N}(\Phi_N(t_0)(A))\right\}^{1-\frac{1}{p}}.\end{aligned}$$

Finally, by Cauchy-Schwarz inequality together with the uniform exponential moment bound on $R_{s,N}(u)$ in Proposition 3.7 and Proposition 3.8, we obtain

$$\begin{aligned}
&\Big\| \mathbf{1}_{B_{C(R,T)}}(u,v)\cdot \partial_t E_{s,N}(\pi_N\Phi_N(t)(\vec{u}))\big|_{t=0}\Big\|_{L^p(\vec{\rho}_{s,N})}\\
&\le \mathcal{Z}_{s,N}^{-\frac{1}{p}} \Big\| \mathbf{1}_{B_{C(R,T)}}(\vec{u})\cdot \partial_t E_{s,N}(\pi_N\Phi_N(t)(\vec{u}))\big|_{t=0}\Big\|_{L^{2p}(\vec{\nu}_{s,N})} \Big\| e^{-R_{s,N}(u)}\Big\|_{L^2(\vec{\nu}_s)}^{\frac{1}{p}} \\
&\le C_{R,T}\cdot p.
\end{aligned} \tag{3.23}$$

Here, we used the boundedness of $\mathcal{Z}_{s,N}^{-1}$, uniformly in $N\in\mathbb{N}$ (recall that $\mathcal{Z}_{s,N}\to\mathcal{Z}_s>0$ as $N\to\infty$). This completes the proof of Proposition 3.10. $\square$

3.4. **Proof of Theorem 1.1.** We conclude this section by presenting the proof of Theorem 1.1. Our aim is to show that for each *fixed* $R>0$, we have

$$\vec{\nu}_s(A)=0 \qquad \text{implies} \qquad \vec{\nu}_s\big(\Phi(t)(A)\big)=0 \tag{3.24}$$

for any measurable set $A\subset B_R\subset \vec{H}^\sigma(\mathbb{T}^3)$, $\sigma\in(1,s-\frac{1}{2})$ and any $t>0$.[9] Since the choice of $R>0$ is arbitrary, this yields quasi-invariance of $\vec{\nu}_s$ under the NLW dynamics. Then, we invoke Lemma 3.5 to conclude quasi-invariance of $\vec{\mu}_s$ (Theorem 1.1).

Arguing as in [29], Proposition 3.10 allows us to establish quasi-invariance of the truncated weighted Gaussian measures $\vec{\rho}_{s,N}$ with the uniform control in $N\in\mathbb{N}$ (but with dependence on $R>0$). See Proposition 5.3 in [29]. By the approximation property of the truncated NLW dynamics (Lemma 2.5 (ii)) and the strong convergence of $\vec{\rho}_{s,N}$ to $\vec{\rho}_s$ (Proposition 3.7), we can upgrade this to the $N=\infty$ case, thus establishing quasi-invariance of the untruncated weighted Gaussian measure $\vec{\rho}_s$ under the NLW dynamics. See Lemma 5.5 in [29] for the proof.

Lemma 3.11. *Given any $R>0$, there exists $t_*=t_*(R)\in[0,1]$ such that for any $\varepsilon>0$, there exists $\delta>0$ with the following property; if a measurable set $A\subset B_R\subset\vec{H}^\sigma(\mathbb{T}^3)$, $\sigma\in\big(1,s-\frac{1}{2}\big)$ satisfies*

$$\vec{\rho}_s(A)<\delta,$$

then we have

$$\vec{\rho}_s(\Phi(t)(A))<\varepsilon$$

for any $t\in[0,t_*]$.

Finally, we establish (3.24) by exploiting the mutual absolute continuity between $\vec{\rho}_s$ and $\vec{\nu}_s$ for each fixed $R>0$. Let $A\subset B_R$ be such that $\vec{\nu}_s(A)=0$. By the mutual absolute continuity of $\vec{\nu}_s$ and $\vec{\rho}_s$, we have

$$\vec{\rho}_s(A)=0.$$

Now, fix a target time $T>0$ and let $C(R,T)$ be as in Lemma 2.5 (i). Namely, we have

$$\Phi(t)(A)\subset B_{C(R,T)} \tag{3.25}$$

for all $t\in[0,T]$. Then, by applying Lemma 3.11 with R replaced by $C(R,T)$, we obtain

$$\vec{\rho}_s(\Phi(t)(A))=0 \tag{3.26}$$

[9] In view of the time reversibility of the equation (1.2), it suffices to consider positive times.

for $t \in [0, t_*]$, where $t_* = t_*(C(R,T))$. In view of (3.25), we can iterate this argument and conclude that (3.26) holds for any $t \in [0, T]$. Since the choice of $T > 0$ was arbitrary, we obtain (3.26) for any $t > 0$. Finally, by invoking the mutual absolute continuity of $\vec{\nu}_s$ and $\vec{\rho}_s$ once again, we have

$$\vec{\nu}_s(\Phi(t)(A)) = 0$$

for any $t > 0$. This proves (3.24) and hence Theorem 1.1.

Remark 3.12. While this new hybrid argument allows us to establish quasi-invariance of the Gaussian measure $\vec{\nu}_s$ (and hence $\vec{\mu}_s$) under the NLW dynamics even in the three-dimensional case, it does not provide as good of a quantitative bound as the two-dimensional argument. For example, in the two-dimensional case, the argument in [29] yielded

$$\vec{\rho}_s(\Phi(t)(A)) \lesssim \big(\vec{\rho}_s(A)\big)^{\frac{1}{c^{1+|t|}}} \tag{3.27}$$

for a weighted Gaussian measure $\vec{\rho}_{s,r}$ with an energy cutoff $\mathbf{1}_{\{E(u,v)\leq r\}}$, where $c = c(r) > 0$; see Remark 5.6 in [29]. Our present understanding does not provide an analogous bound to (3.27) in three dimensions.

4. Construction of the weighted Gaussian measure

In this section, we prove Proposition 3.7 by establishing uniform integrability of the densities $R_{s,N}(u)$ of the weighted Gaussian measures $\vec{\rho}_{s,N}$ in (3.21). In Subsection 4.1, we first prove some regularity properties of random distributions (Proposition 4.3) and then the L^p-convergence of $R_{s,N}(u)$ in (3.20). We split the proof of the main result (Proposition 4.2) into two parts. In Subsection 4.2, we follow the argument by Barashkov and Gubinelli [2] and express the partition function $\mathcal{Z}_{s,N}$ in terms of a minimization problem involving a stochastic control problem (Proposition 4.4). In Subsection 4.3, we then study the minimization problem and establish boundedness of the partition function $\mathcal{Z}_{s,N}$, uniformly in $N \in \mathbb{N}$.

Let $N \geq 1$. Recall that $\vec{\rho}_{s,N}$ has density $e^{-R_{s,N}(u)}$ with respect to $\vec{\nu}_s$. In particular, note that the non-Gaussian part of $\vec{\rho}_{s,N}$ depends only on u. This motivates the following reduction; define $H_s^{(1)}(u)$ and $H_s^{(2)}(v)$ by

$$\begin{aligned} H_s^{(1)}(u) &= \frac{1}{2}\bigg(\int_{\mathbb{T}^3} u\bigg)^2 + \frac{1}{2}\int_{\mathbb{T}^3} |\nabla u|^2 + \frac{1}{2}\int_{\mathbb{T}^3} (D^{s+1}u)^2, \\ H_s^{(2)}(v) &= \frac{1}{2}\int_{\mathbb{T}^3} v^2 + \frac{1}{2}\int_{\mathbb{T}^3} (D^s v)^2. \end{aligned}$$

Then, define Gaussian measures $\nu_s^{(j)}$, $j = 1, 2$, with formal densities:

$$d\nu_s^{(1)} = Z_{1,s}^{-1} e^{-H_s^{(1)}(u)} du \qquad \text{and} \qquad d\nu_s^{(2)} = Z_{2,s}^{-1} e^{-H_s^{(2)}(v)} dv.$$

Since $H_s(\vec{u}) = H_s(u, v)$ in (3.16) is now written as

$$H_s(\vec{u}) = H_s^{(1)}(u) + H_s^{(2)}(v),$$

the Gaussian measure $\vec{\nu}_s$ can be rewritten as

$$d\vec{\nu}_s(\vec{u}) = d\nu_s^{(1)}(u) \otimes d\nu_s^{(2)}(v). \tag{4.1}$$

From decomposition (4.1), we have

$$d\vec{\rho}_{s,N}(\vec{u}) = d\rho_{s,N}(u) \otimes d\nu_s^{(2)}(v),$$

where $\rho_{s,N}$ is given by

$$d\rho_{s,N}(u) = \mathcal{Z}_{s,N}^{-1} e^{-R_{s,N}(u)} d\nu_s^{(1)}(u).$$

The partition function $\mathcal{Z}_{s,N}$ is now expressed as

$$\mathcal{Z}_{s,N} = \int e^{-R_{s,N}(u)} d\nu_s^{(1)}(u). \tag{4.2}$$

In the following, we denote $\nu_s^{(1)}$ by ν_s and prove various statements in terms of ν_s but they can be trivially upgraded to the corresponding statement for $\vec{\nu}_s$.

Lemma 4.1. *Let $s > \frac{3}{2}$. Then, given any finite $p < \infty$, $R_{s,N}$ defined in* (3.20) *converges to some R_s in $L^p(\nu_s)$ as $N \to \infty$.*

The goal of this section is to prove the following proposition on uniform (in $N \in \mathbb{N}$) integrability of the density $e^{-R_{s,N}(u)}$ for $\vec{\rho}_{s,N}$, which allows us to construct the limiting measure $\vec{\rho}_s$. As a consequence of our construction, the weighted Gaussian measure $\vec{\rho}_s$ is equivalent to $\vec{\nu}_s$ (and hence to $\vec{\mu}_s$ in view of Lemma 3.5).

Proposition 4.2. *Let $s > \frac{3}{2}$. Then, given any finite $p < \infty$, there exists $C_p > 0$ such that*

$$\sup_{N\in\mathbb{N}} \Big\| e^{-R_{s,N}(u)} \Big\|_{L^p(\nu_s)} \le C_p < \infty. \tag{4.3}$$

Moreover, we have

$$\lim_{N\to\infty} e^{-R_{s,N}(u)} = e^{-R_s(u)} \qquad \text{in } L^p(\nu_s). \tag{4.4}$$

While the first part of Proposition 3.7 follows from Proposition 4.2 with $p = 1$, we need to have the uniform bound (4.3) for some $p > 1$ for the proof of Proposition 3.10. See (3.23). Note that this requirement on a higher integrability for some $p > 1$ is analogous to the situation in Bourgain's construction on invariant Gibbs measures for Hamiltonian PDEs [6], where, as in (3.23), the analysis of the weighted Gaussian measure needs to be reduced to that of the underlying Gaussian measure by Cauchy-Schwarz inequality. Since the argument is identical for any $p \ge 1$, we only present details for the case $p = 1$. We point out that the L^p-convergence (4.4) is a consequence of the uniform exponential moment bound (4.3) and the softer convergence in measure (as a consequence of Lemma 4.1). See Remark 3.8 in [35]. Therefore, we focus on proving the uniform bound (4.3).

In the next subsection, we prove Lemma 4.1. The subsequent subsections are devoted to the proof of Proposition 4.2.

4.1. **Regularity of random distributions.** Let u be distributed according to ν_s and $Q_{s,N}$ be as in (3.7) with σ_N in (3.17). In this case, we have

$$:(D^s u_N)^2: = Q_{s,N}(u_N),$$

where the left-hand side is the standard notation for the Wick renormalization.

We first state and prove the regularity properties of (products of) certain random distributions. The proof of Lemma 4.1 is presented at the end of this subsection.

Proposition 4.3. *Let $s \geq 1$ and $\varepsilon > 0$. Then, there exists $C = C(s, \varepsilon) > 0$ such that for any $N \in \mathbb{N}$ and any $2 \leq p < \infty$, we have*

$$\| :(D^s u_N)^2: \|_{L^p(\nu_s, \mathcal{C}^{-1-\varepsilon})} \leq Cp, \tag{4.5}$$

$$\|\partial^\kappa v_N \, \partial^\alpha u_N\|_{L^p(\vec{\nu}_s(u,v), \mathcal{C}^{-1-\varepsilon})} \leq Cp \qquad \text{for } |\kappa| = s-1 \text{ and } |\alpha| = s, \tag{4.6}$$

$$\|\partial^\kappa v_N \, \partial^\alpha u_N\|_{L^p(\vec{\nu}_s(u,v), \mathcal{C}^{-\frac{1}{2}-\varepsilon})} \leq Cp \qquad \text{for } |\kappa| = s-1 \text{ and } |\alpha| \leq s-1, \tag{4.7}$$

where $u_N = \pi_N u$ and $v_N = \pi_N v$. Moreover, as $N \to \infty$, the sequences above converge to limits denoted by $:(D^s u)^2:$ and $\partial^\kappa v \, \partial^\alpha u$ with respect to the same topologies.

We will also use this proposition in proving the renormalized energy estimate in Section 5.

Proof. We only prove (4.5) in the following. The other estimates (4.6) and (4.7) follow in a similar manner, with the simplification that no renormalization is needed due to the independence of u and v under $\vec{\nu}_s$. The regularity $-1-\varepsilon$ in (4.6) is naturally expected in view of the regularities $< -\frac{1}{2}$ for each of $\partial^\kappa v_N$ and $\partial^\alpha u_N$. A similar comment applies to (4.7), where the regularity of $\partial^\kappa v$ is less than $-\frac{1}{2}$.

Noting that

$$\frac{|n|^s}{(|n|^2 + |n|^{2s+2})^{\frac{1}{2}}} \lesssim \frac{1}{\langle n \rangle}$$

for any $n \in \mathbb{Z}^3 \setminus \{0\}$, it follows from the Karhunen-Loève expansion (3.15) that

$$\begin{aligned} \mathbb{E}_{\nu_s}\Big[\big|\mathcal{F}\big\{ :(D^s u_N)^2: \big\}(n)\big|^2\Big] &\lesssim \sum_{\substack{n_1, n_2 \in \mathbb{Z}^3 \\ |n_j| \leq N}} \frac{\big|\mathbb{E}[g_{n_1} g_{n-n_1} g_{-n_2} g_{-n+n_2}]\big|}{\langle n_1 \rangle \langle n - n_1 \rangle \langle n_2 \rangle \langle n - n_2 \rangle} \mathbf{1}_{\{n \neq 0\}} \\ &\quad + \sum_{\substack{n_1, n_2 \in \mathbb{Z}^3 \\ |n_j| \leq N}} \frac{\big|\mathbb{E}\big[(|g_{n_1}|^2 - 1)(|g_{n_2}|^2 - 1)\big]\big|}{\langle n_1 \rangle^2 \langle n_2 \rangle^2} \mathbf{1}_{\{n = 0\}} \end{aligned} \tag{4.8}$$

for any $n \in \mathbb{Z}^3$, where $\mathcal{F}$ denotes Fourier transform. In the first sum on the right-hand side of (4.8), we note that due to the independence (modulo the conjugates) of the g_n's and by Wick's theorem, all non-vanishing terms must satisfy $n_1 = n_2$ or $n_1 = n - n_2$. Thus, we obtain

$$\sum_{\substack{n_1, n_2 \in \mathbb{Z}^3 \\ |n_j| \leq N}} \frac{\big|\mathbb{E}[g_{n_1} g_{n-n_1} g_{-n_2} g_{-n+n_2}]\big|}{\langle n_1 \rangle \langle n - n_1 \rangle \langle n_2 \rangle \langle n - n_2 \rangle} \mathbf{1}_{\{n \neq 0\}} \lesssim \sum_{n_1 \in \mathbb{Z}^3} \frac{1}{\langle n_1 \rangle^2 \langle n - n_1 \rangle^2} \lesssim \frac{1}{\langle n \rangle} \tag{4.9}$$

uniformly in $N \in \mathbb{N}$, where in the last inequality we used a standard result on discrete convolutions (see Lemma 4.2 in [19]). In the second sum on the right-hand side of (4.8), we note that, by Wick's theorem, the contribution from $|n_1| \neq |n_2|$ vanishes. Thus, we obtain

$$\sum_{\substack{n_1, n_2 \in \mathbb{Z}^3 \\ |n_j| \leq N}} \frac{\big|\mathbb{E}\big[(|g_{n_1}|^2 - 1)(|g_{n_2}|^2 - 1)\big]\big|}{\langle n_1 \rangle^2 \langle n_2 \rangle^2} \mathbf{1}_{\{n = 0\}} \lesssim 1, \tag{4.10}$$

uniformly in $N \in \mathbb{N}$. Putting (4.9) and (4.10) together, we obtain

$$\mathbb{E}\Big[\big|\mathcal{F}\big\{ :(D^s u_N)^2: \big\}(n)\big|^2\Big] \lesssim \frac{1}{\langle n \rangle}$$

for any $n \in \mathbb{Z}^3$ and $N \in \mathbb{N}$.

By a similar computation, we have

$$\mathbb{E}\Big[\big|\mathcal{F}\big\{ :(D^s u_N)^2: - :(D^s u_M)^2: \big\}(n)\big|^2\Big] \lesssim \frac{1}{N^\theta \langle n \rangle^{1-\theta}}$$

for any $n \in \mathbb{Z}^3$, any $M \geq N \geq 1$, and $\theta \in [0,1]$. Note that $:(D^s u_N)^2:$ lies in the second homogeneous Wiener chaos $\mathcal{H}_2$. Hence, by Lemma 2.3 with $\theta > 0$ sufficiently small, we conclude that $:(D^s u_N)^2:$ converges to some $:(D^s u)^2:$ in $L^p(\nu_s; \mathcal{C}^{-1-\varepsilon}(\mathbb{T}^3))$ for any finite $p \geq 2$. □

We now present the proof of Lemma 4.1.

Proof of Lemma 4.1. For $s > \frac{3}{2}$, Lemma 2.3 implies u_N converges to u in $L^p(\nu_s; \mathcal{C}^\sigma)$ for any finite $p \geq 2$ and any $\sigma < s - \frac{1}{2}$. In the following, we choose $\sigma > 0$ sufficiently close to $s - \frac{1}{2}$. Then, by the algebra property (2.4), we see that u_N^2 (and u_N^4, respectively) converges to u^2 (and u^4, respectively) in $L^p(\nu_s; \mathcal{C}^\sigma)$ for any finite $p \geq 2$.

Proposition 4.3 asserts that $:(D^s u_N)^2:$ converges to $:(D^s u)^2: \in L^p(\nu_s, \mathcal{C}^{-1-\varepsilon}(\mathbb{T}^3))$ for any $\varepsilon > 0$. Recall from (2.8) that the bilinear multiplication map from $\mathcal{C}^{s_1} \times \mathcal{C}^{s_2}$ to $\mathcal{C}^{s_1}$ is a continuous operation for $s_1 < 0 < s_2$ such that $s_1 + s_2 > 0$. Therefore, by choosing $\sigma > 1 + \varepsilon$ (which is possible since $s > \frac{3}{2}$), we conclude that

$$:(D^s u)^2: u^2 = \lim_{N \to \infty} :(D^s u_N)^2: u_N^2$$

exists as an element in $L^p(\nu_s; \mathcal{C}^{-1-\varepsilon}(\mathbb{T}^3))$ for all finite $p \geq 2$. This means that

$$\frac{3}{2} :(D^s u)^2: u^2 + \frac{1}{4} u^4 \in L^p(\nu_s, \mathcal{C}^{-1-\varepsilon}(\mathbb{T}^3)). \tag{4.11}$$

Lemma 4.1 then follows from (4.11). □

4.2. **Variational formulation.** In this subsection, we follow the argument in [2] and derive a variational formula for the normalization constant $\mathcal{Z}_{s,N}$ in (4.2). Given small $\varepsilon > 0$, let $\Omega_\varepsilon = C(\mathbb{R}_+, \mathcal{C}^{-\frac{3}{2}-\varepsilon}(\mathbb{T}^3))$ equipped with its Borel σ-algebra. Denote by[10] $\{X_t\}$ the coordinate process on Ω_ε and consider the probability measure $\mathbb{P}$ that makes $\{X_t\}$ a cylindrical Brownian motion in $L^2(\mathbb{T}^3)$. Namely, we have

$$X_t = \sum_{n \in \mathbb{Z}^3} B_t^n e^{in\cdot x},$$

where $\{B_t^n\}_{n \in \mathbb{Z}^3}$ is a sequence of independent complex-valued[11] Brownian motions such that $\overline{B_t^n} = B_t^{-n}$, $n \in \mathbb{Z}^3$. Then, define a centered Gaussian process $\{Y_t\}$ by

$$Y_t = \mathcal{J}^{-s-1} X_t \overset{\text{def}}{=} B_t^0 + \sum_{n \in \mathbb{Z}^3 \setminus \{0\}} \frac{B_t^n}{(|n|^2 + |n|^{2s+2})^{\frac{1}{2}}} e^{in\cdot x}. \tag{4.12}$$

Then, in view of (3.15), we have $\mathrm{Law}_{\mathbb{P}}(Y_1) = \nu_s$. By truncating the sum in (4.12), we also define the truncated process $Y_t^N = \pi_N Y_t$ with the property $\mathrm{Law}_{\mathbb{P}}(Y_1^N) = \mathrm{Law}_{\nu_s}(\pi_N u)$. Note that we have $\mathbb{E}[(D^s Y_1^N)^2] = \sigma_N$, where σ_N is as in (3.17). For simplicity of notations, we suppress dependence on $N \in \mathbb{N}$ when it is clear from the context.

[10] In the remaining part of this section, we use the standard notation in stochastic analysis where subscripts denote parameters for stochastic processes.

[11] We normalize B_t^n so that $\mathrm{Var}(B_t^n) = t$. Moreover, we impose that B_t^0 is real-valued.

Let $\mathbb{H}_a$ denote the space of progressively measurable processes that belong to $L^2([0,1]; L^2(\mathbb{T}^3))$, $\mathbb{P}$-almost surely. We say that an element θ of $\mathbb{H}_a$ is a *drift*. Given a drift $\theta \in \mathbb{H}_a$, we define the measure $\mathbb{Q}^\theta$ whose Radon-Nikodym derivative with respect to $\mathbb{P}$ is given by the following stochastic exponential:

$$\frac{d\mathbb{Q}^\theta}{d\mathbb{P}} = e^{\int_0^1 \langle \theta_t, dX_t\rangle - \frac{1}{2}\int_0^1 \|\theta_t\|_{L^2_x}^2 dt}. \tag{4.13}$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product on $L^2(\mathbb{T}^3)$. Then, by letting $\mathbb{H}_c$ denote the space of drifts such that $\mathbb{Q}^\theta(\Omega_\varepsilon) = 1$, it follows from Girsanov's theorem ([12, Theorem 10.14] and [33, Theorems 1.4 and 1.7 in Chapter VIII]) that the process X_t is a semimartingale under $\mathbb{Q}^\theta$ with a decomposition:

$$X_t = X_t^\theta + \int_0^t \theta_{t'} dt', \tag{4.14}$$

where X_t^θ is now a cylindrical Brownian motion in $L^2(\mathbb{T}^3)$ under the new measure $\mathbb{Q}^\theta$. From (4.14), we also obtain the decomposition:

$$Y_t = Y_t^\theta + I_t(\theta), \tag{4.15}$$

where $Y_t^\theta = \mathcal{J}^{-s-1} X_t^\theta$ and $I_t(\theta) = \int_0^t \mathcal{J}^{-s-1}\theta_{t'} dt'$. In the following, we use $\mathbb{E}$ to denote an expectation with respect to $\mathbb{P}$, while we use $\mathbb{E}_\mathbb{Q}$ for an expectation with respect to some other probability measure $\mathbb{Q}$.

Before proceeding further, let us recall the following estimate ([13, Lemma 2.6]):

$$\int_0^1 \|\theta_t\|_{L^2_x}^2 dt \le 2H(\mathbb{Q}^\theta|\mathbb{P}), \tag{4.16}$$

where $H(\mathbb{Q}^\theta|\mathbb{P})$ denotes the relative entropy of $\mathbb{Q}^\theta$ with respect to $\mathbb{P}$ defined by

$$H(\mathbb{Q}^\theta|\mathbb{P}) = \mathbb{E}_{\mathbb{Q}^\theta}\bigg[\log \frac{d\mathbb{Q}^\theta}{d\mathbb{P}}\bigg] = \mathbb{E}\bigg[\frac{d\mathbb{Q}^\theta}{d\mathbb{P}} \log \frac{d\mathbb{Q}^\theta}{d\mathbb{P}}\bigg].$$

With the notations introduced above, we have the following variational characterization of the partition function $\mathcal{Z}_{s,N}$ defined in (4.2).

Proposition 4.4. *For any $N \in \mathbb{N}$, we have*

$$-\log \mathcal{Z}_{s,N} = \inf_{\theta \in \mathbb{H}_c} \mathbb{E}_{\mathbb{Q}^\theta}\bigg[R_{s,N}(Y_1^\theta + I_1(\theta)) + \frac{1}{2}\int_0^1 \|\theta_t\|_{L^2_x}^2 dt\bigg]. \tag{4.17}$$

Proof. As a preliminary step, we first derive bounds on $\mathcal{Z}_{s,N}$ and

$$\mathbb{E}\bigg[\frac{e^{-R_{s,N}(Y_1)}}{\mathcal{Z}_{s,N}} \log\bigg(\frac{e^{-R_{s,N}(Y_1)}}{\mathcal{Z}_{s,N}}\bigg)\bigg].$$

Note that these bounds imply that the measure $\frac{e^{-R_{s,N}(Y_1)}}{\mathcal{Z}_{s,N}} d\mathbb{P}$ has a finite relative entropy with respect to $\mathbb{P}$.

From (4.2), Jensen's inequality, and (3.20), there exists finite $C(N) > 0$ such that

$$\mathcal{Z}_{s,N} \ge e^{-\mathbb{E}[R_{s,N}(Y_1)]} \ge e^{-\mathbb{E}\big[\frac{3}{2}\int (D^s Y_1^N)^2 (Y_1^N)^2 dx + \frac{1}{4}\int (Y_1^N)^4 dx\big]} \ge C(N). \tag{4.18}$$

In view of the following pointwise lower bound:

$$\begin{aligned}\frac{3}{2}(D^sY_1^N)^2(Y_1^N)^2 - \frac{3}{2}\sigma_N(Y_1^N)^2 + \frac{1}{4}(Y_1^N)^4 &\ge -\frac{3}{2}\sigma_N(Y_1^N)^2 + \frac{1}{4}(Y_1^N)^4\\ &\ge -\frac{9}{2}\sigma_N^2 + \frac{1}{8}(Y_1^N)^4 \ge -C(N) > -\infty,\end{aligned} \tag{4.19}$$

it follows from (4.18), Cauchy's inequality, and Lemma 4.1 that there exists finite $C(N) > 0$ such that

$$\begin{aligned}\mathbb{E}\bigg[\frac{e^{-R_{s,N}(Y_1)}}{\mathcal{Z}_{s,N}}\log\bigg(\frac{e^{-R_{s,N}(Y_1)}}{\mathcal{Z}_{s,N}}\bigg)\bigg] &\le C(N)\mathbb{E}\Big[e^{-R_{s,N}(Y_1)}\big(1+\log e^{-R_{s,N}(Y_1)}\big)\Big]\\ &\le C(N)\mathbb{E}\Big[e^{-2R_{s,N}(Y_1)} + |R_{s,N}(Y_1)|^2 + 1\Big]\\ &\le C(N) < \infty.\end{aligned} \tag{4.20}$$

Now, fix $\theta \in \mathbb{H}_c$. We show that

$$-\log \mathcal{Z}_{s,N} \le \mathbb{E}_{\mathbb{Q}^\theta}\bigg[R_{s,N}(Y_1^\theta + I_1(\theta)) + \frac{1}{2}\int_0^1 \|\theta_t\|_{L_x^2}^2 dt\bigg]. \tag{4.21}$$

Suppose that $\mathbb{E}_{\mathbb{Q}^\theta}\Big[\int_0^1 \|\theta_t\|_{L_x^2}^2 dt\Big] = \infty$. Then, (4.21) holds trivially since it follows from the decomposition (4.15) of Y_t under $\mathbb{Q}^\theta$ and Cauchy's inequality with Lemma 4.1, (4.18), and (4.19) that

$$\mathbb{E}_{\mathbb{Q}^\theta}\Big[|R_{s,N}(Y_1^\theta + I_1(\theta))|\Big] = \mathbb{E}\bigg[|R_{s,N}(Y_1)|\frac{e^{-R_{s,N}(Y_1)}}{\mathcal{Z}_{s,N}}\bigg] < \infty.$$

Next, suppose that

$$\mathbb{E}_{\mathbb{Q}^\theta}\bigg[\int_0^1 \|\theta_t\|_{L_x^2}^2 dt\bigg] < \infty. \tag{4.22}$$

Note that $\mathcal{Z}_{s,N} = \mathbb{E}[e^{-R_{s,N}(Y_1)}]$. Then, by changing the measure with (4.13), Jensen's inequality, and applying the decompositions (4.14) and (4.15) of X_t and Y_t under $\mathbb{Q}^\theta$, we obtain

$$\begin{aligned}-\log \mathcal{Z}_{s,N} &\le \mathbb{E}_{\mathbb{Q}^\theta}\bigg[R_{s,N}(Y_1) + \int_0^1 \langle \theta_t, dX_t\rangle - \frac{1}{2}\int_0^1 \|\theta_t\|_{L_x^2}^2 dt\bigg]\\ &= \mathbb{E}_{\mathbb{Q}^\theta}\bigg[R_{s,N}(Y_1^\theta + I_1(\theta)) + \int_0^1 \langle \theta_t, dX_t^\theta\rangle + \frac{1}{2}\int_0^1 \|\theta_t\|_{L_x^2}^2 dt\bigg].\end{aligned} \tag{4.23}$$

From (4.22), we see that the process $\int_0^t \langle \theta_{t'}, dX_{t'}^\theta\rangle$ is a $\mathbb{Q}^\theta$-martingale and hence we conclude that

$$\mathbb{E}_{\mathbb{Q}^\theta}\bigg[\int_0^1 \langle \theta_t, dX_t^\theta\rangle\bigg] = 0. \tag{4.24}$$

Therefore, from (4.23) and (4.24), we obtain (4.21).

Next, we show that the infimum in (4.17) is indeed achieved for a special choice of drift. Given $N \in \mathbb{N}$, define $\mathbb{Q}^N$ by the density

$$\frac{d\mathbb{Q}^N}{d\mathbb{P}} = \frac{e^{-R_{s,N}(Y_1)}}{\mathcal{Z}_{s,N}}. \tag{4.25}$$

By the Brownian martingale representation theorem ([33, Proposition 1.6 in Chapter VIII]), there exists a drift $\widetilde{\theta}^N \in \mathbb{H}_c$ such that

$$\frac{d\mathbb{Q}^N}{d\mathbb{P}} = e^{\int_0^1 \widetilde{\theta}_t^N dX_t - \frac{1}{2}\int_0^1 \|\widetilde{\theta}_t^N\|_{L_x^2}^2 dt}. \tag{4.26}$$

Then, from (4.25) and(4.26), we obtain

$$-\log \mathcal{Z}_{s,N} = R_{s,N}(Y_1) + \int_0^1 \langle \widetilde{\theta}_t^N, dX_t\rangle - \frac{1}{2}\int_0^1 \|\widetilde{\theta}_t^N\|_{L_x^2}^2 dt. \tag{4.27}$$

Taking expectations of (4.27) with respect to $\mathbb{Q}^N$ and using the decompositions (4.14) and (4.15) of X_t and Y_t under $\mathbb{Q}^N$, we obtain

$$-\log \mathcal{Z}_{s,N} = \mathbb{E}_{\mathbb{Q}^N}\bigg[R_{s,N}\big(Y_1^{\widetilde{\theta}^N} + I_1(\widetilde{\theta}^N)\big) + \int_0^1 \langle \widetilde{\theta}_t^N, dX_t^{\widetilde{\theta}^N}\rangle + \frac{1}{2}\int_0^1 \|\widetilde{\theta}_t^N\|_{L_x^2}^2 dt\bigg]. \tag{4.28}$$

On the other hand, from (4.25) and (4.20), we have

$$\mathbb{E}_{\mathbb{Q}^N}\bigg[\log\frac{d\mathbb{Q}^N}{d\mathbb{P}}\bigg] = \mathbb{E}\bigg[\frac{e^{-R_{s,N}(Y_1)}}{\mathcal{Z}_{s,N}}\log\bigg(\frac{e^{-R_{s,N}(Y_1)}}{\mathcal{Z}_{s,N}}\bigg)\bigg] < \infty. \tag{4.29}$$

In particular, it follows from (4.29) and (4.16) that

$$\mathbb{E}_{\mathbb{Q}^N}\bigg[\int_0^1 \|\widetilde{\theta}_t^N\|_{L_x^2}^2 dt\bigg] < \infty.$$

This implies that the stochastic integral $\int_0^t \langle \widetilde{\theta}_{t'}^N, dX_{t'}^{\widetilde{\theta}^N}\rangle$ is a $\mathbb{Q}^N$-martingale. Therefore, from (4.28), we obtain

$$-\log \mathcal{Z}_{s,N} = \mathbb{E}_{\mathbb{Q}^N}\bigg[R_{s,N}\big(Y_1^{\widetilde{\theta}^N} + I_1(\widetilde{\theta}^N)\big) + \frac{1}{2}\int_0^1 \|\widetilde{\theta}_t^N\|_{L_x^2}^2 dt\bigg].$$

This completes the proof of Proposition 4.4. □

Remark 4.5. The material presented above differs from [2] in the following ways: (i) we do not need to introduce a time-dependent cutoff in the definition of $\{Y_t\}$ and (ii) we do not need to use the stronger Boué-Dupuis formula [5]:

$$-\log \mathcal{Z}_{s,N} = \inf_{\theta\in\mathbb{H}_a} \mathbb{E}\bigg[R_{s,N}(Y_1 + I_1(\theta)) + \frac{1}{2}\int_0^1 \|\theta_t\|_{L^2}^2 dt\bigg].$$

See [38] or Theorem 2 in [2] for further discussion.

4.3. **Exponential integrability.** In this subsection, we present the proof of Proposition 4.2 by studying the minimization problem (4.17) in Proposition 4.4. In particular, we show that the infimum in (4.17) is bounded away from $-\infty$, uniformly in $N \in \mathbb{N}$. Our strategy is to use pathwise stochastic bounds on Y_1^θ, uniform in the drift θ and use pathwise deterministic bounds on $I_1(\theta)$ independently of the drift (see Lemmas 4.6 and 4.7).

We first state two lemmas on the pathwise regularity estimates on Y_1^θ and $I_1(\theta)$.

Lemma 4.6. *Let $2 \le p < \infty$. Then, we have*

$$\sup_{\theta\in\mathbb{H}_c} \mathbb{E}_{\mathbb{Q}^\theta}\Big[\|D^s Y_1^\theta\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}^p + \|:(D^s Y_1^\theta)^2:\|_{\mathcal{C}^{-1-\varepsilon}}^p\Big] < \infty \tag{4.30}$$

for any $\varepsilon > 0$. Here, colons denote Wick renormalization.

Proof. Recall that $\{X_t^\theta\}$ under $\mathbb{Q}^\theta$ is a cylindrical Brownian motion in $L^2(\mathbb{T}^3)$ for any $\theta \in \mathbb{H}_c$. Thus, the supremum in (4.30) is superfluous since the law of $Y_1^\theta = \mathcal{J}^{-s-1}X_1^\theta$ under $\mathbb{Q}^\theta$ is invariant under a change of drifts. In particular, we have $\mathrm{Law}_{\mathbb{Q}^\theta}(Y_1^\theta) = \nu_s$. Then, (4.30) follows from the Hölder-Besov regularity of samples under ν_s and (4.5) in Proposition 4.3. $\square$

Lemma 4.7 (Cameron-Martin drift regularity)**.** *The drift term $\theta \in \mathbb{H}_c$ has the regularity of the Cameron-Martin space $H^{s+1}(\mathbb{T}^3)$:*

$$\|I_1(\theta)\|^2_{H^{s+1}} \le \int_0^1 \|\theta_t\|^2_{L^2} dt. \tag{4.31}$$

Proof. This is immediate from Minkowski's integral inequality followed by Cauchy-Schwarz inequality:

$$\|I_1(\theta)\|_{H^{s+1}} = \bigg\| \int_0^1 \theta_t dt \bigg\|_{L^2} \le \int_0^1 \|\theta_t\|_{L^2} dt \le \bigg(\int_0^1 \|\theta_t\|^2_{L^2} dt \bigg)^{\frac{1}{2}},$$

yielding (4.31). $\square$

We now present the proof of Proposition 4.2, using Proposition 4.4. Fixing an arbitrary drift $\theta \in \mathbb{H}_c$, the quantity that we wish to bound from below is

$$\mathcal{W}_N(\theta) = \mathbb{E}_{\mathbb{Q}^\theta}\bigg[R_{s,N}(Y_1^\theta + I_1(\theta)) + \frac{1}{2}\int_0^1 \|\theta_t\|^2_{L^2_x} dt \bigg]. \tag{4.32}$$

Since the drift $\theta \in \mathbb{H}_c$ is fixed, we suppress the dependence on the drift θ henceforth and denote $Y = Y_1^\theta$ and $\Theta = I_1(\theta)$. From the definition (3.20) of $R_{s,N}$, we have

$$\begin{aligned} R_{s,N}(Y+\Theta) &= \frac{3}{2}\int_{\mathbb{T}^3} :(D^sY)^2:(Y+\Theta)^2 + 2D^sY D^s\Theta(Y+\Theta)^2 + (D^s\Theta)^2(Y+\Theta)^2 \\ &\quad + \frac{1}{4}\int_{\mathbb{T}^3}(Y+\Theta)^4. \end{aligned} \tag{4.33}$$

The main strategy is to bound $\mathcal{W}_N(\theta)$ from below pathwise and independently of the drift by utilizing the positive terms:

$$\mathcal{U}_N(\theta) = \frac{3}{2}\int (D^s\Theta)^2\Theta^2 + \frac{1}{4}\int \Theta^4 + \frac{1}{2}\int_0^1 \|\theta_t\|^2_{L^2_x} dt. \tag{4.34}$$

In the following, we state three lemmas, controlling the other terms appearing in (4.33). The proofs of these lemmas follow from lengthy but straightforward computations and are presented at the end of this section. The first lemma handles the terms quadratic in D^sY.

Lemma 4.8 (Terms quadratic in D^sY)**.** *Let $s > \frac{3}{2}$. Then, given $\delta > 0$ sufficiently small, there exist small $\varepsilon > 0$ and $c(\delta) > 0$ such that*

$$\int_{\mathbb{T}^3} :(D^sY)^2: Y^2 \lesssim \| :(D^sY)^2: \|^2_{\mathcal{C}^{-1-\varepsilon}} + \|D^sY\|^4_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}, \tag{4.35}$$

$$\int_{\mathbb{T}^3} :(D^sY)^2: Y\Theta \le c(\delta)\Big(\| :(D^sY)^2: \|^4_{\mathcal{C}^{-1-\varepsilon}} + \|D^sY\|^4_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \Big) + \delta\|\Theta\|^2_{H^{s+1}}, \tag{4.36}$$

$$\int_{\mathbb{T}^3} :(D^sY)^2: \Theta^2 \le c(\delta)\| :(D^sY)^2: \|^4_{\mathcal{C}^{-1-\varepsilon}} + \delta\Big(\|\Theta\|^2_{H^{s+1}} + \|\Theta\|^4_{L^4} \Big). \tag{4.37}$$

The next lemma handles the terms linear in D^sY.

Lemma 4.9 (Terms linear in D^sY)**.** *Let $s>1$. Then, given $\delta>0$ sufficiently small, there exist small $\varepsilon>0$, $c(\delta)>0$, and $p_j=p_j(\varepsilon,s)>1$, $j=1,2$, such that*

$$\int_{\mathbb{T}^3} D^sY D^s\Theta Y^2 \le c(\delta)\|D^sY\|^6_{\mathcal{C}^{-\frac12-\varepsilon}} + \delta\|\Theta\|^2_{H^{s+1}}, \tag{4.38}$$

$$\int_{\mathbb{T}^3} D^sY D^s\Theta Y\Theta \le c(\delta)\Big(1+\|D^sY\|_{\mathcal{C}^{-\frac12-\varepsilon}}\Big)^{p_1} + \delta\Big(\|\Theta\|^2_{H^{s+1}} + \|\Theta\|^4_{L^4}\Big), \tag{4.39}$$

$$\begin{aligned}\int_{\mathbb{T}^3} D^sY D^s\Theta\Theta^2 &\le c(\delta)\Big(1+\|D^sY\|_{\mathcal{C}^{-\frac12-\varepsilon}}\Big)^{p_2}\\ &\qquad + \delta\Big(\|\Theta\|^2_{H^{s+1}} + \|\Theta\|^4_{L^4} + \|D^s\Theta\Theta\|^2_{L^2}\Big).\end{aligned} \tag{4.40}$$

Lastly, the third lemma controls the term quadratic in $D^s\Theta$.

Lemma 4.10 (Term quadratic in $D^s\Theta$)**.** *Let $s>1$. Then, given $\delta>0$, there exist small $\varepsilon>0$, $c(\delta)>0$, and $p=p(s,\varepsilon)>1$ such that*

$$\int_{\mathbb{T}^3} (D^s\Theta)^2 Y\Theta \le c(\delta)\|D^sY\|^p_{\mathcal{C}^{-\frac12-\varepsilon}} + \delta\Big(\|\Theta\|^2_{H^{s+1}} + \|\Theta\|^4_{L^4} + \|D^s\Theta\Theta\|^2_{L^2}\Big). \tag{4.41}$$

The regularity restriction $s>\frac32$ appears in controlling the terms quadratic in D^sY. We now prove Proposition 4.2, assuming Lemmas 4.8, 4.9, and 4.10.

First, note that the remaining terms left to treat in (4.33) are harmless. The terms $\int_{\mathbb{T}^3}(D^s\Theta)^2Y^2$, $\int_{\mathbb{T}^3} Y^4$, and $\int_{\mathbb{T}^3} Y^2\Theta^2$ are positive and thus can be discarded. The remaining two terms can be controlled by Young's inequality:

$$\int_{\mathbb{T}^3} Y^3\Theta + \int_{\mathbb{T}^3} Y\Theta^3 \le c(\delta)\|Y\|^4_{L^4} + \delta\|\Theta\|^4_{L^4}$$

for any $\delta>0$. We now apply the regularity estimates of Lemmas 4.6 and 4.7 to the bounds obtained in Lemmas 4.8, 4.9, and 4.10, and the bounds on the harmless terms. Then, from (4.32), (4.33), and (4.34), we conclude that, by choosing $\delta>0$ sufficiently small, there exists finite $C=C(\delta)>0$ such that

$$\sup_{N\in\mathbb{N}}\sup_{\theta\in\mathbb{H}_c} \mathcal{W}_N(\theta) \ge \sup_{N\in\mathbb{N}}\sup_{\theta\in\mathbb{H}_c}\Big\{-C(\delta)+\frac14\mathcal{U}_N(\theta)\Big\} \ge -C(\delta) > -\infty.$$

Therefore, by Proposition 4.4, this proves Proposition 4.2 (when $p=1$).

In the remaining part of this section, we present the proofs of Lemmas 4.8, 4.9, and 4.10.

Proof of Lemma 4.8. By duality (2.6) and the algebra property (2.4), we have

$$\text{LHS of (4.35)} \le \|:\!(D^sY)^2\!:\|_{B^{-1-2\varepsilon}_{1,1}}\|Y\|^2_{\mathcal{C}^{1+2\varepsilon}}.$$

Then, by choosing $\varepsilon>0$ sufficiently small, (4.35) follows from the trivial embeddings (2.3) and Cauchy's inequality, provided that $s>\frac32$.

By duality (2.6) and the fractional Leibniz rule (2.7), we have

$$\begin{aligned}\text{LHS of (4.36)} &\lesssim \|:\!(D^sY)^2\!:\|_{B^{-1-2\varepsilon}_{\infty,2}}\|Y\Theta\|_{B^{1+2\varepsilon}_{1,2}}\\ &\lesssim \|:\!(D^sY)^2\!:\|_{\mathcal{C}^{-1-\varepsilon}}\Big(\|Y\|_{B^{1+2\varepsilon}_{2,2}}\|\Theta\|_{L^2} + \|Y\|_{L^2}\|\Theta\|_{B^{1+2\varepsilon}_{2,2}}\Big).\end{aligned}$$

Then, by choosing $\varepsilon>0$ sufficiently small, (4.36) follows from (2.3) and Young's inequality, provided that $s>\frac32$.

Lastly, proceeding as above with (2.6) and (2.7), we have

$$\text{LHS of (4.37)} \lesssim \| :(D^s Y)^2: \|_{B^{-1-2\varepsilon}_{\infty,2}} \|\Theta\|_{B^{1+2\varepsilon}_{2,2}} \|\Theta\|_{L^2}.$$

Then, (4.37) follows from (2.3), $L^4(\mathbb{T}^3) \hookrightarrow L^2(\mathbb{T}^3)$, and Young's inequality. $\square$

Next, we present the proof of Lemma 4.9. The main idea is to use (i) $\|\Theta\|_{H^{s+1}}$ for controlling derivatives on Θ and (ii) $\|\Theta\|_{L^4}$ and $\|D^s\Theta\Theta\|_{L^2}$ for controlling homogeneity of Θ.

Proof of Lemma 4.9. By duality (2.6) and the fractional Leibniz rule (2.7) with (2.3), we have

$$\begin{aligned}\text{LHS of (4.38)} &\lesssim \|D^s Y\|_{B^{-\frac{1}{2}-2\varepsilon}_{\infty,2}} \|D^s\Theta Y^2\|_{B^{\frac{1}{2}+2\varepsilon}_{1,2}}\\ &\lesssim \|D^s Y\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \Big(\|Y^2\|_{B^{\frac{1}{2}+2\varepsilon}_{2,2}} \|D^s\Theta\|_{L^2} + \|Y^2\|_{L^2} \|D^s\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,2}} \Big)\\ &\le \|D^s Y\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \|Y\|^2_{\mathcal{C}^{\frac{1}{2}+3\varepsilon}} \|\Theta\|_{H^{s+1}}.\end{aligned}$$

Then, by choosing $\varepsilon > 0$ sufficiently small, (4.38) follows from Cauchy's inequality, provided that $s > 1$.

By duality (2.6) and the fractional Leibniz rule (2.7) with (2.3) and (2.4), we have

$$\begin{aligned}\text{LHS of (4.39)} &\lesssim \|D^s Y\|_{B^{-\frac{1}{2}-2\varepsilon}_{\infty,2}} \|D^s\Theta Y\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{1,2}}\\ &\lesssim \|D^s Y\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \Big(\|Y\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,2}} \|D^s\Theta\|_{L^2} + \|Y\Theta\|_{L^2} \|D^s\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,2}} \Big)\\ &=: T_1 + T_2.\end{aligned}$$

By Hölder's inequality and (2.3), we have

$$\begin{aligned}T_2 &\lesssim \|D^s Y\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \|Y\|_{L^4} \|\Theta\|_{H^{s+1}} \|\Theta\|_{L^4}\\ &\lesssim \|D^s Y\|^2_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \|\Theta\|_{H^{s+1}} \|\Theta\|_{L^4}\end{aligned} \tag{4.42}$$

for $s > \frac{1}{2}$ and small $\varepsilon > 0$.

By (2.7), (2.3), and the interpolation (2.2), we have

$$\begin{aligned}\|Y\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,2}} &\lesssim \|Y\|_{B^{\frac{1}{2}+2\varepsilon}_{\infty,2}} \|\Theta\|_{L^2} + \|Y\|_{L^\infty} \|\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,2}}\\ &\lesssim \|Y\|_{\mathcal{C}^{\frac{1}{2}+3\varepsilon}} \|\Theta\|_{H^{\frac{1}{2}+2\varepsilon}}\\ &\lesssim \|Y\|_{\mathcal{C}^{\frac{1}{2}+3\varepsilon}} \|\Theta\|^{\gamma}_{H^{s+1}} \|\Theta\|^{1-\gamma}_{L^2}\end{aligned}$$

for some $\gamma = \gamma(s, \varepsilon) \in (0, 1)$. Thus, we have

$$T_1 \lesssim \|D^s Y\|^2_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \|\Theta\|^{1+\gamma}_{H^{s+1}} \|\Theta\|^{1-\gamma}_{L^4} \tag{4.43}$$

for $s > 1$ and small $\varepsilon > 0$. Hence, noting that $\frac{1}{2} + \frac{1}{4} < 1$ and $\frac{1+\gamma}{2} + \frac{1-\gamma}{4} < 1$ for $\gamma \in (0, 1)$, the desired estimate (4.39) follows from applying Young's inequality to (4.42) and (4.43).

Finally, we consider (4.40). By (2.6) and (2.7) with (2.3), we have

$$\begin{aligned}\text{LHS of (4.40)} &\lesssim \|D^s Y\|_{B^{-\frac{1}{2}-2\varepsilon}_{\infty,1}} \|D^s\Theta\Theta^2\|_{B^{\frac{1}{2}+2\varepsilon}_{1,\infty}} \\ &\lesssim \|D^s Y\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \Big(\|D^s\Theta\Theta\|_{L^2}\|\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,\infty}} + \|D^s\Theta\Theta\|_{B^{\frac{1}{2}+2\varepsilon}_{2,\infty}}\|\Theta\|_{L^2}\Big) \\ &=: T_3 + T_4.\end{aligned}$$

By the interpolation (2.2) with $L^4(\mathbb{T}^3) \hookrightarrow L^2(\mathbb{T}^3)$, there exists $\gamma_1 = \gamma_1(s,\varepsilon) \in (0,1)$ such that

$$T_3 \lesssim \|D^s Y\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \|D^s\Theta\Theta\|_{L^2} \|\Theta\|^{\gamma_1}_{H^{s+1}} \|\Theta\|^{1-\gamma_1}_{L^4}.$$

Noting that $\frac{1}{2} + \frac{\gamma_1}{2} + \frac{1-\gamma_1}{4} < 1$, we can apply Young's inequality to bound the contribution from T_3 by the right-hand side of (4.40).

It remains to estimate T_4. By the interpolation (2.2) and (2.7), we have

$$\begin{aligned}\|D^s\Theta\Theta\|_{H^{\frac{1}{2}+2\varepsilon}}\|\Theta\|_{L^2} &\lesssim \|D^s\Theta\Theta\|^{\gamma_2}_{H^1}\|D^s\Theta\Theta\|^{1-\gamma_2}_{L^2}\|\Theta\|_{L^2} \\ &\lesssim \Big(\|D^s\Theta\|_{B^1_{2,2}}\|\Theta\|_{L^\infty} + \|D^s\Theta\|_{L^6}\|\Theta\|_{B^1_{3,2}}\Big)^{\gamma_2} \\ &\quad \times \|D^s\Theta\Theta\|^{1-\gamma_2}_{L^2}\|\Theta\|_{L^4},\end{aligned} \tag{4.44}$$

where $\gamma_2 = \gamma_2(\varepsilon) \in (0,1)$ is given by

$$\gamma_2 = \frac{1}{2} + 2\varepsilon. \tag{4.45}$$

By Sobolev's inequality and the interpolation (2.2) (with $s > \frac{1}{2}$), we have

$$\|D^s\Theta\|_{B^1_{2,2}}\|\Theta\|_{L^\infty} + \|D^s\Theta\|_{L^6}\|\Theta\|_{B^1_{3,2}} \lesssim \|\Theta\|_{H^{s+1}}\|\Theta\|_{H^{\frac{3}{2}+\varepsilon}} \lesssim \|\Theta\|^{1+\gamma_3}_{H^{s+1}}\|\Theta\|^{1-\gamma_3}_{L^4}, \tag{4.46}$$

where $\gamma_3 = \gamma_3(s,\varepsilon) \in (0,1)$ is given by

$$\gamma_3 = \frac{3+2\varepsilon}{2(s+1)}. \tag{4.47}$$

Combining (4.44) and (4.46), we obtain

$$T_4 \lesssim \|D^s Y\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \|\Theta\|^{\gamma_2(1+\gamma_3)}_{H^{s+1}} \|D^s\Theta\Theta\|^{1-\gamma_2}_{L^2} \|\Theta\|^{1+\gamma_2(1-\gamma_3)}_{L^4}.$$

From (4.45) and (4.47), we observe that

$$\frac{\gamma_2(1+\gamma_3)}{2} + \frac{1-\gamma_2}{2} + \frac{1+\gamma_2(1-\gamma_3)}{4} < 1,$$

provided that $s > \frac{1}{2}$ and $\varepsilon > 0$ is sufficiently small. Therefore, we can apply Young's inequality to bound the contribution from T_4 by the right-hand side of (4.40). This completes the proof of Lemma 4.9. □

We conclude this section by presenting the proof of Lemma 4.10.

Proof of Lemma 4.10. By Cauchy's inequality, we have

$$\int_{\mathbb{T}^3} (D^s\Theta)^2 Y\Theta \le c(\delta)\int_{\mathbb{T}^3}(D^s\Theta)^2 Y^2 + \delta\|D^s\Theta\Theta\|^2_{L^2}. \tag{4.48}$$

By Hölder's and Sobolev's inequalities followed by the interpolation (2.2) with (2.3) and (2.4), we have

$$\begin{aligned}\int_{\mathbb{T}^3} (D^s\Theta)^2 Y^2 &\lesssim \|D^s\Theta\|_{L^3}^2 \|Y^2\|_{L^3} \lesssim \|\Theta\|^2_{H^{s+\frac{1}{2}}} \|Y^2\|_{H^{\frac{1}{2}}} \\ &\lesssim \|\Theta\|^{2\gamma}_{H^{s+1}} \|\Theta\|^{2(1-\gamma)}_{L^2} \|Y^2\|_{\mathcal{C}^{\frac{1}{2}+\varepsilon}} \\ &\lesssim \|\Theta\|^{2\gamma}_{H^{s+1}} \|\Theta\|^{2(1-\gamma)}_{L^4} \|D^s Y\|^2_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}\end{aligned} \tag{4.49}$$

for some $\gamma = \gamma(s) \in (0,1)$, provided that $s > 1$ and $\varepsilon > 0$ is sufficiently small. Noting that $\frac{2\gamma}{2} + \frac{2(1-\gamma)}{4} < 1$, (4.41) follows from (4.48), (4.49), and Young's inequality. $\square$

5. Renormalized energy estimate

Recall from (3.19) that

$$\partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))\Big|_{t=0} = F_1(\vec{u}_N) + F_2(\vec{u}_N) + F_3(\vec{u}_N),$$

where $\vec{u}_N = (u_N, v_N)$ and

$$\begin{aligned}F_1(\vec{u}_N) &= 3\int_{\mathbb{T}^3} Q_{s,N}(u_N) v_N u_N, \\ F_2(\vec{u}_N) &= \sum_{\substack{|\alpha|+|\beta|+|\gamma|=s \\ |\alpha|,|\beta|,|\gamma|<s}} c_{\alpha,\beta,\gamma} \int_{\mathbb{T}^3} D^s v_N \cdot \partial^\alpha u_N \cdot \partial^\beta u_N \cdot \partial^\gamma u_N, \\ F_3(\vec{u}_N) &= \bigg(\int_{\mathbb{T}^3} u_N\bigg)\bigg(\int_{\mathbb{T}^3} v_N\bigg).\end{aligned}$$

Proposition 5.1. *Let $s \geq 4$ be an even integer. Then, there exist $\sigma < s - \frac{1}{2}$ sufficiently close to $s - \frac{1}{2}$ and small $\varepsilon > 0$ such that*

$$\bigg| \partial_t E_{s,N}(\pi_N \Phi_N(t)(\vec{u}))\Big|_{t=0} \bigg| \leq \big(1 + \|\vec{u}_N\|^2_{\vec{H}^\sigma}\big) F(\vec{u}_N), \tag{5.1}$$

where

$$\begin{aligned}F(\vec{u}_N) = 1 &+ \|Q_{s,N}(u_N)\|_{\mathcal{C}^{-1-\varepsilon}} \\ &+ \sup_{\substack{|k|=s-1 \\ |\alpha|=s}} \|\partial^\kappa v_N\, \partial^\alpha u_N\|_{\mathcal{C}^{-1-\varepsilon}} + \sup_{\substack{|k|=s-1 \\ |\alpha|\leq s-1}} \|\partial^\kappa v_N\, \partial^\alpha u_N\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}}.\end{aligned}$$

Proposition 3.8 follows from Proposition 5.1, the cutoff in the $\vec{H}^\sigma$-norm, and the Wiener chaos estimate (Lemma 2.2).

Proof. In the following, we prove (5.1) uniformly in $N \in \mathbb{N}$. Thus, we drop the N-dependence and write $Q_s(u)$ for $Q_{s,N}(u_N)$.

First, note that the estimate for F_3 follows trivially from Cauchy-Schwarz inequality. Next, we treat F_1. By duality (2.6) and the fractional Leibniz rule (2.7), we have

$$\begin{aligned}\int_{\mathbb{T}^3} Q_s(u) uv &\lesssim \|Q_s(u)\|_{\mathcal{C}^{-1-\varepsilon}} \|uv\|_{\mathcal{B}^{1+\varepsilon}_{1,1}} \\ &\lesssim \|Q_s(u)\|_{\mathcal{C}^{-1-\varepsilon}} \|u\|_{H^\sigma} \|v\|_{H^{\sigma-1}},\end{aligned} \tag{5.2}$$

provided that $\sigma > 2+\varepsilon$. This is guaranteed by choosing σ sufficiently close to $s-\frac{1}{2}$, when $s > \frac{5}{2}$.

It remains to consider F_2. By integration by parts, it suffices to consider terms of the form:

$$\int_{\mathbb{T}^3} \partial^\kappa v\, \partial^\alpha u\, \partial^\beta u\, \partial^\gamma u,$$

where $|\kappa| = s-1$, $\max(\alpha,\beta,\gamma) \le s$, and $|\alpha|+|\beta|+|\gamma| = s+1$. Without loss of generality, we assume that $|\alpha| \ge |\beta| \ge |\gamma|$. The idea is to group the low regularity terms ($\partial^\kappa v$ and $\partial^\alpha u$) and treat them as one piece.

First, let us assume that $|\alpha| = s$. In this case, we have $|\beta| = 1$ and $|\gamma| = 0$. By duality (2.6) and the fractional Leibniz rule (2.7), we have

$$\bigg| \int_{\mathbb{T}^3} \partial^\kappa v \partial^\alpha u\, \partial u\, u \bigg| \lesssim \|\partial^\kappa v\, \partial^\alpha u\|_{\mathcal{C}^{-1-\varepsilon}} \|\partial u\, u\|_{\mathcal{B}^{1+\varepsilon}_{1,1}} \lesssim \|\partial^\kappa v\, \partial^\alpha u\|_{\mathcal{C}^{-1-\varepsilon}} \|u\|^2_{H^\sigma}, \tag{5.3}$$

provided that $\sigma > 2+\varepsilon$. By choosing $\varepsilon > 0$ sufficiently small, we can guarantee this condition if $s > \frac{5}{2}$.

This leaves the case $|\alpha| \le s-1$. Noting that $|\beta| \le \frac{s+1}{2}$ and $|\gamma| \le \frac{s+1}{3}$ (under $|\alpha| \ge |\beta| \ge |\gamma|$), we see that $\partial^\beta u, \partial^\gamma u \in H^{\frac{1}{2}+\varepsilon}(\mathbb{T}^3)$ for $s > 3$. Thus, by duality (2.6) and the fractional Leibniz rule (2.7), we have:

$$\bigg| \int_{\mathbb{T}^3} \partial^\kappa v\, \partial^\alpha u\, \partial^\beta u \partial^\gamma u \bigg| \lesssim \|\partial^\kappa v\, \partial^\alpha u\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \|\partial^\beta u\, u\|_{\mathcal{B}^{\frac{1}{2}+\varepsilon}_{1,1}} \lesssim \|\partial^\kappa v\, \partial^\alpha u\|_{\mathcal{C}^{-\frac{1}{2}-\varepsilon}} \|u\|^2_{H^\sigma}. \tag{5.4}$$

This completes the proof of Proposition 5.1. □

Remark 5.2. The restriction $s > 3$ in the last case appears only when $|\beta| = \frac{s+1}{2}$. In fact, when $|\beta| \le \frac{s}{2}$, the estimate (5.4) holds true for $s > 2$. On the other hand, when $|\beta| = \frac{s+1}{2}$, we must have $|\alpha| = |\beta| = \frac{s+1}{2}$. In this case, by applying dyadic decompositions and working with the Littlewood-Paley pieces $\mathbf{P}_{j_2}\partial^\alpha u\, \mathbf{P}_{j_3}\partial^\beta u$, we can move half a derivative from the third factor to the second factor, thus showing that a slight variant of (5.4) holds for $s > 2$. Therefore, the estimates (5.2) and (5.3) on F_1 and F_2 impose the regularity restriction $s > \frac{5}{2}$.

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Trishen S. Gunaratnam, Department of Mathematical Sciences, University of Bath, 4 West, Claverton Down, Bath, BA2 7AY, United Kingdom
E-mail address: T.Gunaratnam@bath.ac.uk

Tadahiro Oh, School of Mathematics, The University of Edinburgh, and The Maxwell Institute for the Mathematical Sciences, James Clerk Maxwell Building, The King's Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, United Kingdom
E-mail address: hiro.oh@ed.ac.uk

Nikolay Tzvetkov, Université de Cergy-Pontoise, 2, av. Adolphe Chauvin, 95302 Cergy-Pontoise Cedex, France
E-mail address: nikolay.tzvetkov@u-cergy.fr

Hendrik Weber, Department of Mathematical Sciences, University of Bath, 4 West, Claverton Down, Bath, BA2 7AY, United Kingdom
E-mail address: H.Weber@bath.ac.uk