Available online at www.ijournalse.org

Emerging Science Journal

(ISSN: 2610-9182)

Vol. 6, No. 4, August, 2022

Estimating Simultaneous Confidence Intervals for Multiple Contrasts of Means of Normal Distribution with Known Coefficients of Variation

Kanyanatthanin Sodanin [1], Sa-Aat Niwitpong [1*], Suparat Niwitpong [1*]

[1] *Department of Statistics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand.*

Abstract

This study investigated the performance of simultaneous confidence intervals (SCIs) to differentiate the means of multiple normal population distributions with known coefficients of variation (CVs). The researchers aim to find the means of several normal distributions with known coefficients of variation, SCI_{MOVER}, SCI_s, and SCI_k, which are extended to k populations. The authors constructed SCIs for the difference between multiple normal means with known coefficients of variation. There are three approaches: the method of variance estimates recovery approach (MOVER), and two central limit theorem approaches (CLT). A Monte Carlo simulation was used to evaluate the performance of the coverage probabilities and expected lengths of the methods. The simulation results indicate that the MOVER approach is more desirable than the CLT approaches in terms of the coverage probability. The performance of the proposed approaches is also compared using an example with real data. Moreover, the coverage probability results for SCIMOVER were over the nominal level of 0.95, indicating that it is more stable than SCI_s and SCI_k and was thus more appropriate for use in this scenario. Finally, the researchers suggest using the MOVER approach for constructing the SCIs to determine the variation to achieve the best solution in related fields in the near future.

Keywords:

Simultaneous Confidence Interval;
Normal Distribution;
Method of Variance Estimates Recovery;
Approach (MOVER);
Central Limit Theorem Approach (CLT);
Coefficient of Variation.

Article History:

Received:	21	December	2021
Revised:	10	April	2022
Accepted:	07	May	2022
Available online:	28	May	2022

1- Introduction

This paper is based on the inspiration of the contents about the impact of confidence intervals for normal population means, which have been studied in many fields, especially in the health care industry. Many previous scholars have estimated the mean with a known value of the coefficient of variation (CV). In addition, previous studies have demonstrated methods for calculating the efficient normal mean with a known CV [1, 2]. According to previous researchers, the mean estimator with a known CV for two new confidence intervals and for the difference of normal means on a maximum likelihood estimator and the t-test statistic is proposed [3, 4]. Moreover, other researchers have studied the estimation of the mean of a normal distribution with a known coefficient of variation in many practical areas [5-8].

In past studies, the confidence intervals have been obtained from practical experiments, and the problems have been solved with this technique and discussed multiple times. Therefore, simultaneous confidence intervals (SCIs) have been proposed to compare two or more groups in drug trials in the pharmaceutical industry. Furthermore, SCIs procedures have been suggested in a number of various areas. The SCIs were presented for exponential distributions [9, 10] while SCIs for the ratio of normal means were suggested to use [11, 12]. Many researchers have proposed SCIs for lognormal distributions [13-16].

* **CONTACT**: sa-aat.n@sci.kmutnb.ac.th; suparat.n@sci.kmutnb.ac.th

DOI: http://dx.doi.org/10.28991/ESJ-2022-06-04-04

Due to the papers in the related fields, SCIs use for estimating the difference of means of normal distributions with unknown CV [17, 18]. In several realistic situations, it has been found that the performance of SCIs procedures can be seriously degraded in comparison with using the nominal rate. Nevertheless, many SCIs procedures have been proposed [19-22] and it is a good method to use with data from several different settings and in the comparison of multiple parameters. The purpose of this study is to estimate SCIs for the means of several normally distributed populations with known coefficients of variation using the method of variance estimates recovery (MOVER) and two central limit theorem (CLT) approaches. Confidence intervals for estimating the mean using the MOVER approach were first introduced to improve the effective approach [23]. And, since then several researchers have constructed and used them [24-32]. As mentioned above, the developing approaches will be prepared for the circumstances with the appropriate tools and processes. In this study, the MOVER approach is applied to construct new SCIs for the means of populations that are normally distributed with known coefficients of variation.

2- Theoretical Approach

Figure 1 shows the flowchart of the research methodology through which the objectives of this study were achieved.

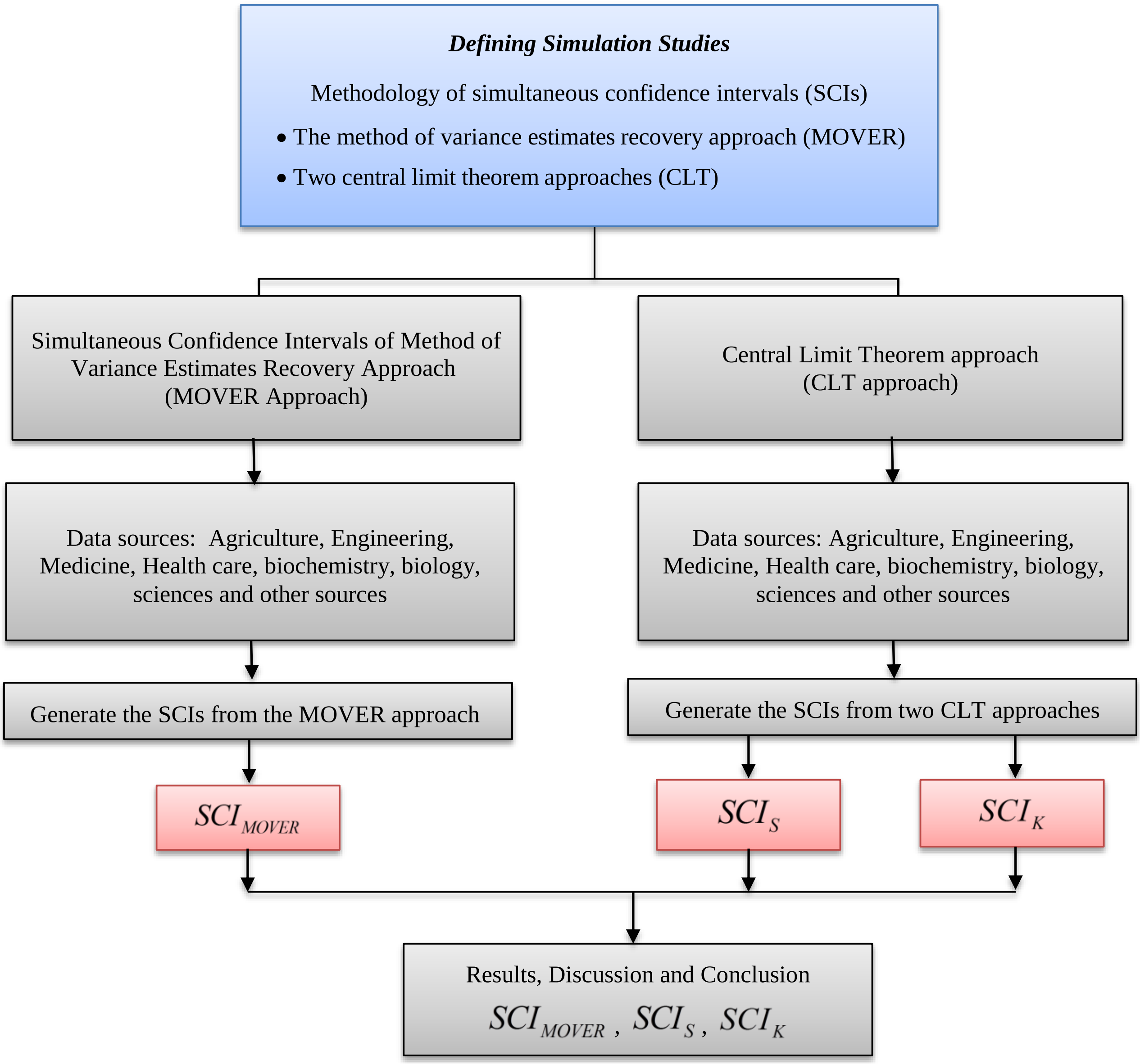

Figure 1. Flowchart of the research methodology

2-1- Notations and Motivations

The independent random sample, $X = (X_1, X_2, \dots, X_n)$ is a normally distributed population (μ, σ^2: mean and variance). Let $\tau = \sigma/\mu$ be the coefficient of variation (CV). The SCIs $100(1-\alpha)\%$ for μ with a known CV are to be constructed. The estimator $\bar{X}^* = (n+\tau^2)^{-1} \sum_{i=1}^{n} X_i$ was proposed and the following MMSE estimator (minimum mean-squared error) for normal mean [1]:

$$\hat{\mu} = \frac{\sum_{i=1}^{n} X_i}{n+\tau^2} = \frac{\bar{X}}{1+\left(\sigma^2/n\mu^2\right)} = \frac{n\bar{X}}{n+\left(\sigma^2/\mu^2\right)} \tag{1}$$

Let $X_j = (X_{j1}, X_{j2}, \dots, X_{jn_j})$ be independent random samples from the $j-th$ normal population $N(\mu_j, \sigma_j^2), j = 1, 2, \dots, k$:

$$\hat{\mu}_j = \frac{n_j \bar{X}_j}{n_j + \left(\sigma_j^2/\mu_j^2\right)}. \tag{2}$$

In this study, we consider the SCIs for $\hat{\mu}_{jl} = \hat{\mu}_j - \hat{\mu}_l,\ j, l = 1, 2, \dots, k,\ j \neq l$. The estimation of the SCIs are approximated the normal mean with known CVs ; $\hat{\mu}_{jl},\ j, l = 1, 2, \dots, k,\ j \neq l$ can be written as;

$$\hat{\mu}_{jl} = \hat{\mu}_j - \hat{\mu}_l = \frac{n_j \bar{X}_j}{n_j + \left(\sigma_j^2/\mu_j^2\right)} - \frac{n_l \bar{X}_l}{n_l + \left(\sigma_l^2/\mu_l^2\right)} \tag{3}$$

where μ_j and μ_l are the mean with a known coefficient of variation based on the *j-th* and *l-th* samples, respectively.

2-2- Simultaneous Confidence Intervals of Method of Variance Estimates Recovery Approach (MOVER Approach)

The estimators of the normal mean with a known coefficient of variation are:

$$\hat{\mu}_j = \frac{n_j \bar{X}_j}{n_j + \left(\sigma_j^2/\mu_j^2\right)} \quad \text{and} \quad \hat{\mu}_l = \frac{n_l \bar{X}_l}{n_l + \left(\sigma_l^2/\mu_l^2\right)} \tag{4}$$

The estimators of the normal mean with a known CVs : mean and variance defined as:

$$\bar{X}_j \sim N(\mu_j, \frac{\sigma_j^2}{n_j}) \quad \text{and} \quad \bar{X}_l \sim N(\mu_l, \frac{\sigma_l^2}{n_l})$$

According to Student's t-statistic with $n-1$ degrees of freedom can be used to defined the distribution of the random variable T [35], such that;

$$T = \frac{\left(\bar{X}-\mu\right)/\left(\sigma/\sqrt{n}\right)}{\sqrt{S^2/\sigma^2}} = \frac{Z}{\sqrt{V/(n-1)}}, \tag{5}$$

$$T = \frac{Z}{\sqrt{\frac{\chi_{n-1}^2}{n-1}}} = \frac{\frac{\bar{X}-\mu}{\sqrt{Var(\mu)}}}{\sqrt{\frac{(n-1)S^2}{(n-1)\sigma^2}}} = \frac{\bar{X}-\mu}{\sqrt{Var(\mu)}}\left(\frac{\sigma}{S}\right), \text{ where } \frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2.$$

Therefore, the $100(1-\alpha)\%$ two-sided confidence interval for the mean of normal distribution with known coefficient of variation $\hat{\mu}_j$, for $j = 1, 2, \dots, k$ is obtained as:

$$l_j = \bar{X}_j - t_{1-\alpha/2}\frac{S_j}{\sigma_j}\sqrt{Var\left(\hat{\mu}_j\right)}, \text{ and;} \tag{6}$$

$$u_j = \bar{X}_j + t_{1-\alpha/2}\frac{S_j}{\sigma_j}\sqrt{Var\left(\hat{\mu}_j\right)}. \tag{7}$$

Similarly, with $\hat{\mu}_l$, for $l = 1, 2, \dots, k$,

$$l_l = \bar{X}_l - t_{1-\alpha/2}\frac{S_l}{\sigma_l}\sqrt{Var\left(\hat{\mu}_l\right)}, \text{ and;} \tag{8}$$

$$u_l = \bar{X}_l + t_{1-\alpha/2}\frac{S_l}{\sigma_l}\sqrt{Var\left(\hat{\mu}_l\right)}. \tag{9}$$

Now, $Var(\mu)$ from Equation 5, becomes:

$$Var\left(\hat{\mu}_j\right) = \left(\frac{\mu_j}{1+\left(\frac{\sigma_j^2}{n_j\mu_j^2+\sigma_j^2}\right)\left(1+\frac{2\sigma_j^2+4n_j\mu_j^2\sigma_j^2}{\left(n_j\mu_j^2+\sigma_j^2\right)^2}\right)}\right)^2 \left(\frac{\sigma_j^2}{n_j\mu_j^2} + \frac{\left(\frac{n_j\sigma_j^2}{n_j\mu_j^2+\sigma_j^2}\right)^2\left(\frac{2}{n_j}+\frac{2\sigma_j^4+4n_j\mu_j^2\sigma_j^2}{\left(n_j\mu_j^2+\sigma_j^2\right)^2}\right)}{\left(n_j+\left(\frac{n_j\sigma_j^2}{n_j\mu_j^2+\sigma_j^2}\right)\left(1+\frac{2\sigma_i^4+4n_j\mu_j^2\sigma_j^2}{\left(n_j\mu_i^2+\sigma_j^2\right)^2}\right)\right)^2}\right)$$

The MOVER is proposed to construct a $100(1-\alpha)\%$ two-sided confidence interval (L_{12}, U_{12}) for $\mu_1 - \mu_2$, where μ_1 and μ_2 denote the two parameters of interest and L and U denote the lower limit and upper limit of the confidence interval, respectively [24]. (l_i, u_i) contains the parameter values for μ_i, for $i = 1, 2$. The lower limit L_{12} and upper limit U_{12} are respectively defined as;

$$L_{12} = \mu_1 - \mu_2 - \sqrt{(\mu_1 - l_1)^2 + (u_2 - \mu_2)^2} \text{ and;} \tag{10}$$

$$U_{12} = \mu_1 - \mu_2 + \sqrt{(u_1 - \mu_1)^2 + (\mu_2 - l_2)^2} \tag{11}$$

It is reasonable to extend the concept of Donner and Zou (2010) to construct the $100(1-\alpha)\%$ two-sided confidence interval (L_{jl}, U_{jl}) for $\hat{\mu}_{jl} = \hat{\mu}_j - \hat{\mu}_l, j, l = 1, 2, \dots, k,\ j \neq l$. The lower limit L_{jl} and upper limit U_{jl} are respectively obtained as;

$$L_{jl} = \hat{\mu}_j - \hat{\mu}_l - \sqrt{(\hat{\mu}_j - l_j)^2 + (u_l - \hat{\mu}_l)^2} \text{ and;} \tag{12}$$

$$U_{jl} = \hat{\mu}_j - \hat{\mu}_l + \sqrt{(u_j - \hat{\mu}_j)^2 + (\hat{\mu}_l - l_l)^2} \tag{13}$$

where μ_j, μ_l, and μ_{jl} are defined in Equation 3; l_j and l_l are defined in Equations 6 and 8, respectively; and u_j and u_l are defined in Equations 7 and 9, respectively. Therefore, the $100(1-\alpha)\%$ two-sided simultaneous confidence interval for the differences between normal means with known coefficients of variation μ_{jl} based on the MOVER approach is defined as $SCI_{jl(MOVER)} = [L_{jl}, U_{jl}]$;

$$SCI_{jl(MOVER)} = \left(\hat{\mu}_j - \hat{\mu}_l - \sqrt{(\hat{\mu}_j - l_j)^2 + (u_l - \hat{\mu}_l)^2}, \hat{\mu}_j - \hat{\mu}_l + \sqrt{(u_j - \hat{\mu}_j)^2 + (\hat{\mu}_l - l_l)^2}\right) \tag{14}$$

Theorem 1: Let $X_{ji} = N(\mu_j, \sigma_j^2)$ be a normal distribution, for $j = 1, 2, \dots, k$ and $i = 1, 2, \dots, n$ Let $\hat{\mu}_j = \frac{n_j \bar{X}_j}{n_j + (\sigma_j^2/\mu_j^2)}$ and $\hat{\mu}_l = \frac{n_l \bar{X}_l}{n_l + (\sigma_l^2/\mu_l^2)}$ be the respective estimators of the means of normal distributions based on the j-th and l-th samples, then the lower limit and the upper limit of the confidence interval for $\hat{\mu}_{jl} = \hat{\mu}_j - \hat{\mu}_l$ can be defined as:

$$L_{jl} = \hat{\mu}_j - \hat{\mu}_l - \sqrt{(\hat{\mu}_j - l_j)^2 + (u_l - \hat{\mu}_l)^2} \text{ and;}$$

$$U_{jl} = \hat{\mu}_j - \hat{\mu}_l + \sqrt{(u_j - \hat{\mu}_j)^2 + (\hat{\mu}_l - l_l)^2} \text{ for } j, l = 1, 2, \dots, k, \text{ and } j \neq l.$$

$$P\{L_{jl} \leq \hat{\mu}_{jl} \leq U_{jl}, \forall j \neq l\} \to 1 - \alpha, \tag{15}$$

according to the study [27].

Hence, the confidence interval for μ_{jl} is given by (L_{jl}, U_{jl}), where:

$$(L_{jl}, U_{jl}) = \left((\hat{\mu}_j - \hat{\mu}_l) - z_{\alpha/2}^2 \sqrt{\frac{(\hat{\mu}_j - l_j)^2}{z_{\alpha/2}^2} + \frac{(u_l - \hat{\mu}_l)^2}{z_{\alpha/2}^2}}, (\hat{\mu}_j - \hat{\mu}_l) + z_{\alpha/2}^2 \sqrt{\frac{(u_j - \hat{\mu}_j)^2}{z_{\alpha/2}^2} + \frac{(\hat{\mu}_l - l_l)^2}{z_{\alpha/2}^2}}\right) \tag{16}$$

$$(L_{jl}, U_{jl}) = \left(\hat{\mu}_{jl} - z_{\alpha/2}^2 \sqrt{Var(\hat{\mu}_j) + Var(\hat{\mu}_l)},\ \hat{\mu}_{jl} + z_{\alpha/2}^2 \sqrt{Var(\hat{\mu}_j) + Var(\hat{\mu}_l)}\right) \tag{17}$$

Therefore, $P(L_{jl} \leq \hat{\mu}_{jl} \leq U_{jl}) = P\left\{\hat{\mu}_{jl} \in \left(\hat{\mu}_{jl} \pm z_{\alpha/2}^2 \sqrt{Var(\hat{\mu}_j) + Var(\hat{\mu}_l)}\right), \forall j \neq l\right\}$,

$$\text{then } P\left(\hat{\mu}_{jl} \in \left(\hat{\mu}_{jl} \pm z_{\alpha/2} \sqrt{Var(\hat{\mu}_j) + Var(\hat{\mu}_l)}\right), \forall j \neq l\right) \to 1 - \alpha. \tag{18}$$

The variance estimates for $\hat{\mu}_j$ at $\hat{\mu}_j = l_j$ and $\hat{\mu}_l$ at $\hat{\mu}_l = l_l$ are:

$$Var(\hat{\mu}_j) = \frac{(\hat{\mu}_j - l_l)^2}{z_{\alpha/2}^2} \text{ and } Var(\hat{\mu}_l) = \frac{(\hat{\mu}_l - l_l)^2}{z_{\alpha/2}^2}.$$

The variance estimates for $\hat{\mu}_j$ at $\hat{\mu}_j = u_j$ and $\hat{\mu}_l$ at $\hat{\mu}_l = u_l$ are:

$$Var(\mu_j) = \frac{(u_j - \mu_j)^2}{z_{\alpha/2}^2} \text{ and } Var(\hat{\mu}_l) = \frac{(u_l - \hat{\mu}_l)^2}{z_{\alpha/2}^2}.$$

2-3- Central Limit Theorem Approach (CLT Approach)

The estimator for the difference between the confidence intervals for two normal population means with known coefficients of variation was constructed [4]. The estimator $\bar{X}^* = (n+\tau^2)\sum_{i=1}^{n} X_i$ based on the MMSE for a normal mean with a variance of $\frac{\sqrt{n}s_i}{n+\tau_i^2}$ [1].

Considering Equation 3, then $\hat{\mu}_{jl} = \hat{\mu}_j - \hat{\mu}_l = \frac{n_j \bar{X}_j}{n_j + \left(\sigma_j^2/\mu_j^2\right)} - \frac{n_l \bar{X}_l}{n_l + \left(\sigma_l^2/\mu_l^2\right)}$.

Using the CLT for $\mu_{jl} = \mu_j - \mu_l,\ j,l = 1,2,\dots,k,\ j \neq l,$ the $100(1-\alpha)\%$ confidence interval for μ_{jl} when τ_j, τ_l are known is given by;

$$CI_S = \left[\left(\hat{\mu}_j - \hat{\mu}_l\right) - d\sqrt{\frac{nS_j^2}{\left(n+\tau_j^2\right)^2} + \frac{mS_l^2}{\left(m+\tau_l^2\right)^2}},\ \left(\hat{\mu}_j - \hat{\mu}_l\right) + d\sqrt{\frac{nS_j^2}{\left(n+\tau_j^2\right)^2} + \frac{mS_l^2}{\left(m+\tau_l^2\right)^2}} \right], \tag{19}$$

where, as mentioned earlier, $\bar{X}^* = \left(n+\tau_i^2\right)^{-1}\sum_{i=1}^{n} X_i$ and $Var\left(X^*\right) = \frac{\sqrt{n}S_i}{n+\tau_i^2}$, and d is the upper $(1-\alpha/2)th$ percentile of the standard normal distribution.

The maximum likelihood estimator of μ and $\tau = \sigma/\mu$ for a known coefficient of variation was derived [7]. The estimator $\bar{X} = \left[\sqrt{4\tau^2 S^2 + \left(1+4\tau^2\right)\bar{X}^2} - \bar{X}^2\right]/2\tau^2$ is normal $N\left(\mu, \tau^2\mu^2/m\left(1+2\tau^2\right)\right)$, where $\tau^2\mu^2/m\left(1+2\tau^2\right)$ is the Cramer-Rao bound.

In addition, there is another similar estimation of the difference of the confidence for the normal mean. The research's proof is similar to this one [3]:

$$CI_K = \left[\left(\hat{\mu}_j - \hat{\mu}_l\right) - d\sqrt{\frac{S_j^2}{n\left(1+2\tau_j^2\right)^2} + \frac{S_l^2}{m\left(1+\tau_l^2\right)^2}},\left(\hat{\mu}_j - \hat{\mu}_l\right) + d\sqrt{\frac{S_j^2}{n\left(1+2\tau_j^2\right)^2} + \frac{S_l^2}{m\left(1+\tau_l^2\right)^2}} \right] \tag{20}$$

where $S_j^2 = n^{-1}\sum_{j=1}^{k}\left(X_j - \bar{X}_j\right)^2$ and $S_l^2 = n^{-1}\sum_{l=1}^{k}\left(X_l - \bar{X}_l\right)^2$.

Theorem 2: Let $X_{ji} = N\left(\mu_j, \sigma_j^2\right)$ be a normal distribution, for $j = 1,2,\dots,k$ and $i = 1,2,\dots,n$ and:

$$L_{jl} = \left(\bar{X}_j - \bar{X}_l\right) - d\sqrt{\frac{nS_j^2}{\left(n+\tau_j^2\right)^2} + \frac{mS_l^2}{\left(m+\tau_l^2\right)^2}} \ \ and\ \ U_{jl} = \left(\bar{X}_j - \bar{X}_l\right) + d\sqrt{\frac{nS_j^2}{\left(n+\tau_j^2\right)^2} + \frac{mS_l^2}{\left(m+\tau_l^2\right)^2}}$$

be the respective lower and upper limits of the confidence interval for the difference between normal means with known coefficients of variation, where $j,l = 1,2,\dots,k,\ j \neq l,$ by applying the CLT, then

$$P\left(L_{jl} \le \hat{\mu}_{jl} \le U_{jl}, \forall j \neq l\right) \to 1-\alpha. \tag{21}$$

The proof of this theorem is similar to the one in the study [17]:

$P\left(L_{jl} \le \hat{\mu}_{jl} \le U_{jl}\right) = P\left\{\hat{\mu}_{jl} \in \left(\hat{\mu}_{jl} \pm z_{\alpha/2}^2\sqrt{Var\left(\hat{\mu}_j\right) + Var\left(\hat{\mu}_l\right)}\right), \forall j \neq l\right\}$, and so

$P\left(\hat{\mu}_{jl} \in \left(\hat{\mu}_{jl} \pm d\sqrt{Var\left(\hat{\mu}_j\right) + Var\left(\hat{\mu}_l\right)}\right),\ \forall j \neq l\right) \to 1-\alpha,$ where

$Var\left(\hat{\mu}_j\right) = \frac{nS_j^2}{\left(n+\tau_j^2\right)^2}$ and $Var\left(\hat{\mu}_l\right) = \frac{mS_l^2}{\left(m+\tau_l^2\right)^2}$.

Theorem 3: Let $X_{ji}^* = N(\mu_j, \sigma_j^2)$ be a normal distribution, for $j = 1,2,\dots,k$ and $i = 1,2,\dots,n$ and $L_{jl} = \left(\bar{X}_j^* - \bar{X}_l^*\right) - d\sqrt{\frac{S_j^2}{n\left(1+2\tau_j^2\right)} + \frac{S_l^2}{m\left(1+2\tau_l^2\right)}}$ and $U_{jl} = \left(\bar{X}_j^* - \bar{X}_l^*\right) + d\sqrt{\frac{S_j^2}{n\left(1+2\tau_j^2\right)} + \frac{S_l^2}{m\left(1+2\tau_l^2\right)}}$ be the lower and upper limits of the CI for difference between normal means with known CVs, for $j,l = 1,2,\dots,k,\ j \neq l,$ [7].

The proof is similar to this one [27]:

$$P\left(L_{jl} \le \hat{\mu}_{jl} \le U_{jl}, \forall j \ne l\right) \to 1-\alpha. \text{ Therefore,} \tag{22}$$

$$P\left(L_{jl} \le \hat{\mu}_{jl} \le U_{jl}\right) = P\left\{\hat{\mu}_{jl} \in \left(\hat{\mu}_{jl} \pm z^2_{\alpha/2}\sqrt{Var\left(\hat{\mu}_j\right)+Var\left(\hat{\mu}_l\right)}\right), \forall j \ne l\right\} \text{ and so:}$$

$$P\left(\hat{\mu}_{jl} \in \left(\hat{\mu}_{jl} \pm d\sqrt{Var\left(\hat{\mu}_j\right)+Var\left(\hat{\mu}_l\right)}\right), \forall j \ne l\right) \to 1-\alpha, \text{ where;}$$

$$Var\left(\hat{\mu}_j\right) = \frac{S_j^2}{n\left(1+2\tau^2\right)} \text{ and } Var\left(\hat{\mu}_l\right) = \frac{S_l^2}{m\left(1+2\tau^2\right)}.$$

3- Simulation Studies and Results

We conducted simulation studies to compare the SCI_{MOVER} and SCI_S, SCI_k from the MOVER approach and the two CLT approaches .Their performances were evaluated through their attained coverage probabilities (CP) and expected lengths (EL).

3-1- Computing Algorithm:

1. Generate X_j and random sample n_j from a normal population with parameters μ_j and σ_j^2, for $j = 1, 2, \dots, k$ and then calculate $\bar{x}_j$ and s_j the observed values of $\bar{X}_j$ and S_j.
2. Generate the SCIs from the MOVER approach $\left(SCI_{jl(MOVER)}\right)$, $\hat{\mu}_{jl} = \hat{\mu}_j - \hat{\mu}_l$, for $j, l = 1, 2, \dots, k,\ j \ne l$;
3. Generate the SCIs from the two CLT approaches $\left(SCI_{jl(S)}, SCI_{jl(K)}\right)$, $\hat{\mu}_{jl} = \hat{\mu}_j - \hat{\mu}_l$, for $j, l = 1, 2, \dots, k,\ j \ne l$;
4. Replicate steps 1–4 for M=5000, and $\hat{\mu}_{jl} = \hat{\mu}_j - \hat{\mu}_l$ for $j, l = 1, 2, \dots, k,\ j \ne l$, to provide an estimate of the coverage probabilities.

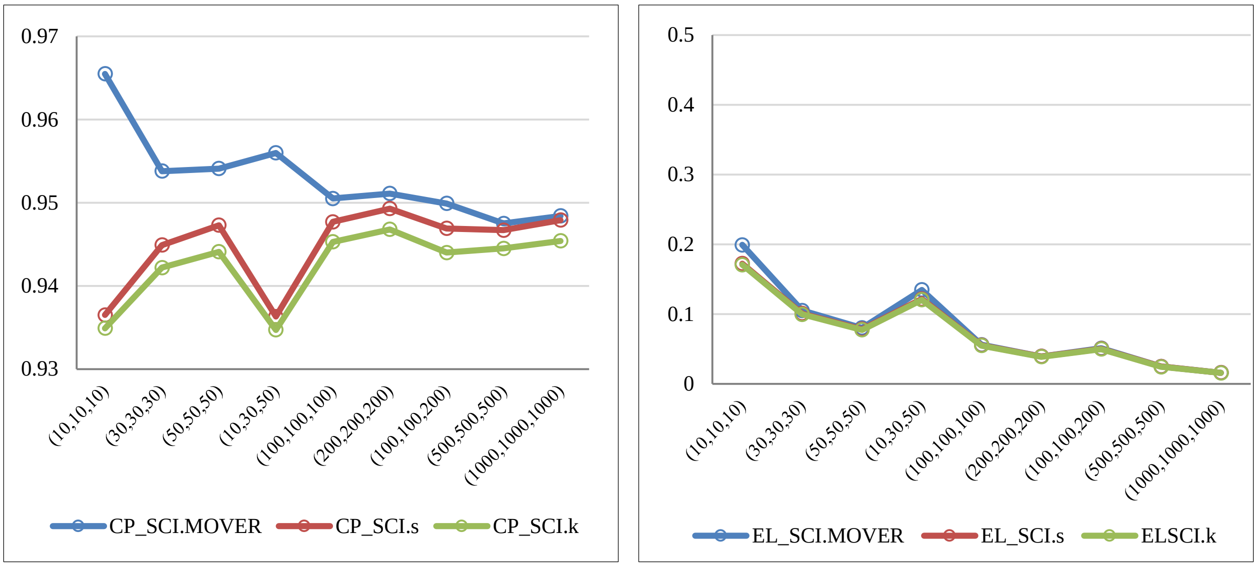

Figure 1. The CP and the EL of the 95%two-sided SCIs for the means of normal distributions with known CVs: (τ_1, τ_2, τ_3)= (0.1,0.1,0.1), 3 sample cases (n_1, n_2, n_3)

A simulation study was conducted to calculate the coverage probabilities (CP) and expected lengths (EL) of the SCIs for MOVER and the two CLT approaches (SCI_S, SCI_k) with sample cases $k = 3$ and $k = 5$, population means $\mu_1 = \mu_2 = \dots = \mu_k = 1$, population standard deviations $\sigma_1, \sigma_2, \dots, \sigma_k$, and sample sizes $n_1, n_2, \dots, n_k$. The results of the CP and EL for each set of parameters using M = 5,000 (5000 random samples) simulation runs and the coefficient of the CI set at $1 - \alpha = 0.95$ are reported in Tables 1-2 and illustrated in Figures 2-5.

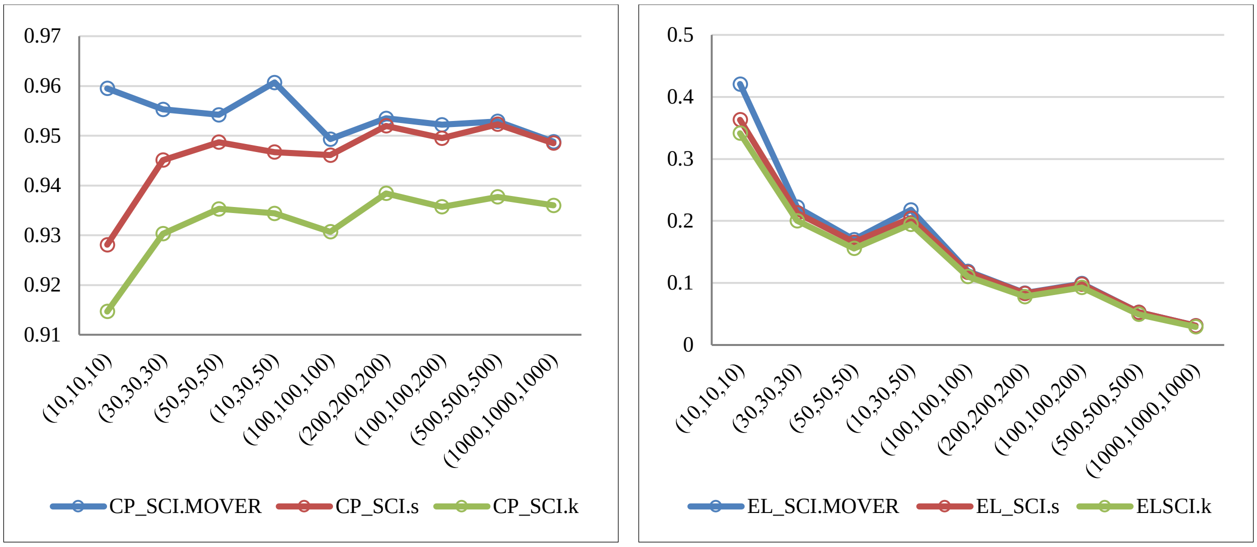

Figure 2. The CP and the EL of the 95% two-sided SCIs for the means of normal distributions with known CVs: (τ_1, τ_2, τ_3)= (0.1,0.2,0.3), 3 sample cases (n_1, n_2, n_3)

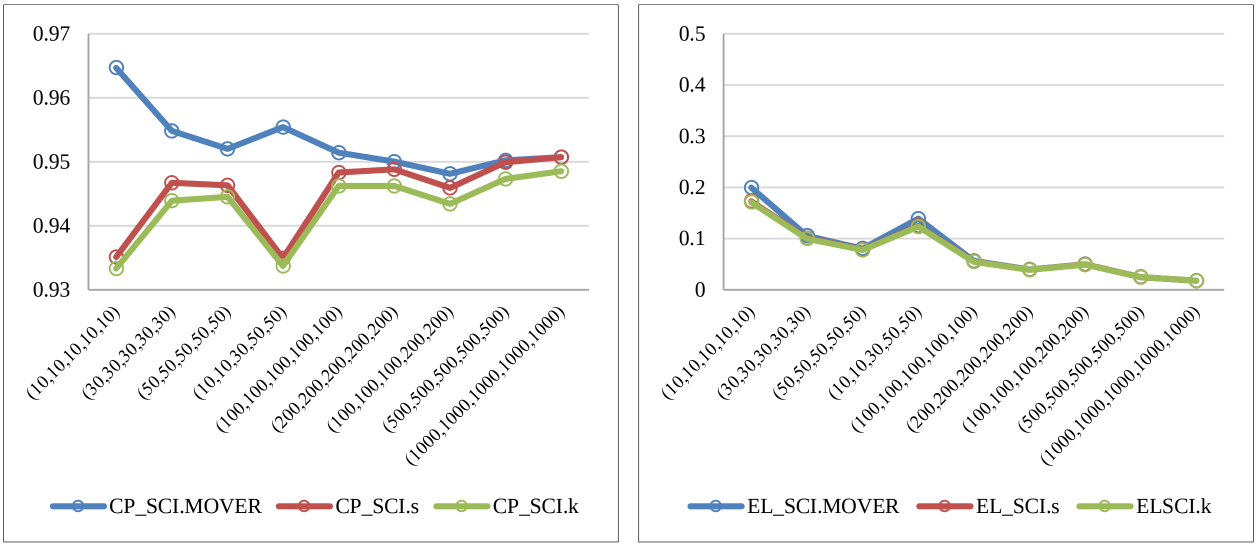

Figure 3. The CP and the EL of the 95% two-sided SCIs for the means of normal distributions with known CVs: $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$= (0.1,0.1,0.1,0.1,0.1), 5 sample cases $(n_1, n_2, n_3, n_4, n_5)$

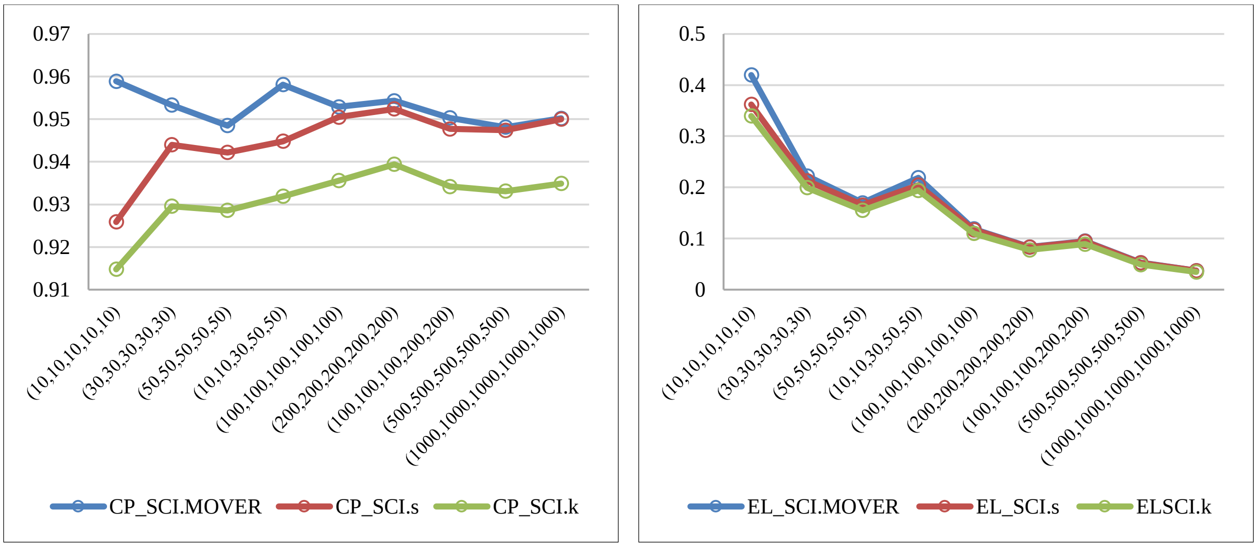

Figure 4. The CP and the EL of the 95% two-sided SCIs for the means of normal distributions with known CVs: $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$= (0.1,0.1,0.2,0.3,0.3), 5 sample cases $(n_1, n_2, n_3, n_4, n_5)$

Table 1. The CP and the EL of the 95% two-sided SCIs for the means of normal distributions with known CVs: 3 sample cases

(n_1, n_2, n_3)	(τ_1, τ_2, τ_3)	SCI_{MOVER}	SCI_S	SCI_k
(10,10,10)	(0.1,0.1,0.1)	0.9655 (0.1991)	0.9365 (0.1724)	0.9349 (0.1706)
	(0.1,0.2,0.3)	0.9595 (0.4208)	0.9281 (0.3631)	0.9147 (0.3414)
(30,30,30)	(0.1,0.1,0.1)	0.9538 (0.1051)	0.9449 (0.1007)	0.9422 (0.0997)
	(0.1,0.2,0.3)	0.9553 (0.2223)	0.9451 (0.2129)	0.9303 (0.2002)
(50,50,50)	(0.1,0.1,0.1)	0.9541 (0.0802)	0.9473 (0.0782)	0.9441 (0.0774)
	(0.1,0.2,0.3)	0.9542 (0.1699)	0.9487 (0.1657)	0.9353 (0.1558)
(10,30,50)	(0.1,0.1,0.1)	0.9560 (0.1347)	0.9363 (0.1217)	0.9347 (0.1205)
	(0.1,0.2,0.3)	0.9607 (0.2178)	0.9467 (0.2045)	0.9344 (0.1946)
(100,100,100)	(0.1,0.1,0.1)	0.9505 (0.0560)	0.9477 (0.0554)	0.9453 (0.0548)
	(0.1,0.2,0.3)	0.9493 (0.1188)	0.9461 (0.1173)	0.9307 (0.1103)
(200,200,200)	(0.1,0.1,0.1)	0.9511 (0.0394)	0.9493 (0.0392)	0.9468 (0.0388)
	(0.1,0.2,0.3)	0.9535 (0.0836)	0.9520 (0.0831)	0.9384 (0.0781)
(100,100,200)	(0.1,0.1,0.1)	0.9499 (0.0510)	0.9469 (0.0504)	0.9440 (0.0499)
	(0.1,0.2,0.3)	0.9522 (0.0988)	0.9495 (0.0978)	0.9357 (0.0926)
(500,500,500)	(0.1,0.1,0.1)	0.9475 (0.0248)	0.9467 (0.0248)	0.9445 (0.0245)
	(0.1,0.2,0.3)	0.9529 (0.0527)	0.9523 (0.0526)	0.9377 (0.0494)
(1000,1000,1000)	(0.1,0.1,0.1)	0.9484 (0.0160)	0.9479 (0.0160)	0.9454 (0.0158)
	(0.1,0.2,0.3)	0.9488 (0.0310)	0.9485 (0.0310)	0.9360 (0.0293)

The results from Table 1 for $k = 3$ indicate that the SCIs for MOVER (SCI_{MOVER}) performed satisfactorily in terms of the CP and the expected lengths. The highest CP for SCI_{MOVER} was over the nominal level of 0.95, while that of SCI_s was closer to the nominal level of 0.95 than SCI_k. The highest CP for SCI_{MOVER} was slightly closer to the nominal level of 0.95 than the others for moderate and large sample sizes ($n \geq 100$). In almost all cases, the CP for SCI_{MOVER} were better than SCI_s and SCI_k. However, the expected lengths of SCI_s and SCI_k. were slightly shorter than those of SCI_{MOVER} for small and moderate sample sizes ($n \geq 200$). The SCIs for SCI_s and SCI_k are different from each other in that SCI_k was less stable than SCI_s and SCI_{MOVER} The performance of SCI_s was better than SCI_k even though the expected lengths of the latter were shorter.

For $k=5$, the results in Table 2 for SCI_{MOVER} were similar to those in Table 1 ($k=3$).The highest coverage probability for SCI_{MOVER} was slightly over the nominal level of 0.95. The CP of SCI_{MOVER} indicate better performance than SCI_s and SCI_k. The CP for SCI_k show that it was less stable than SCI_s and SCI_{MOVER}, and the CP of SCI_s were higher than those of SCI_k. The expected lengths of SCI_k and SCI_s were slightly shorter than those of SCI_{MOVER}, with those of SCI_k being the shortest. Therefore, the MOVER approach can be considered as a better alternative to SCI_s and SCI_k for constructing the SCI_s for the difference between the means of multiple normal distributions with known CVs.

Table 2. The CP and the EL of the 95% two-sided SCIs for the means of normal distributions with known CVs: 5 sample cases

$(n_1, n_2, n_3, n_4, n_5)$	$(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)$	SCI_{MOVER}	SCI_S	SCI_k
(10,10,10,10,10)	(0.1,0.1,0.1,0.1,0.1)	0.9647 (0.1996)	0.9351 (0.1728)	0.9333 (0.1710)
	(0.1,0.1,0.2,0.3,0.3)	0.9589 (0.4197)	0.9259 (0.3620)	0.9148 (0.3394)
(30,30,30,30,30)	(0.1,0.1,0.1,0.1,0.1)	0.9548 (0.1052)	0.9467 (0.1008)	0.9439 (0.0998)
	(0.1,0.1,0.2,0.3,0.3)	0.9533 (0.2223)	0.9440 (0.2129)	0.9296 (0.1995)
(50,50,50,50,50)	(0.1,0.1,0.1,0.1,0.1)	0.9520 (0.0802)	0.9463 (0.0782)	0.9445 (0.0774)
	(0.1,0.1,0.2,0.3,0.3)	0.9485 (0.1696)	0.9422 (0.1654)	0.9286 (0.1550)
(10,10,30,50,50)	(0.1,0.1,0.1,0.1,0.1)	0.9554 (0.1387)	0.9349 (0.1246)	0.9337 (0.1233)
	(0.1,0.1,0.2,0.3,0.3)	0.9581 (0.2187)	0.9448 (0.2044)	0.9319 (0.1942)
(100,100,100,100,100)	(0.1,0.1,0.1,0.1,0.1)	0.9514 (0.0561)	0.9483 (0.0554)	0.9462 (0.0548)
	(0.1,0.1,0.2,0.3,0.3)	0.9529 (0.1187)	0.9505 (0.1173)	0.9356 (0.1099)
(200,200,200,200,200)	(0.1,0.1,0.1,0.1,0.1)	0.9500 (0.0394)	0.9488 (0.0392)	0.9462 (0.0388)
	(0.1,0.1,0.2,0.3,0.3)	0.9543 (0.0835)	0.9524 (0.0830)	0.9394 (0.0778)
(100,100,100,200,200)	(0.1,0.1,0.1,0.1,0.1)	0.9481 (0.0498)	0.9459 (0.0493)	0.9434 (0.0488)
	(0.1,0.1,0.2,0.3,0.3)	0.9503 (0.0951)	0.9477 (0.0943)	0.9342 (0.0890)
(500,500,500,500,500)	(0.1,0.1,0.1,0.1,0.1)	0.9502 (0.0248)	0.9499 (0.0248)	0.9473 (0.0245)
	(0.1,0.1,0.2,0.3,0.3)	0.9481 (0.0527)	0.9474 (0.0526)	0.9331 (0.0492)
(1000,1000,1000,1000,1000)	(0.1,0.1,0.1,0.1,0.1)	0.9507 (0.0175)	0.9507 (0.0175)	0.9485 (0.0174)
	(0.1,0.1,0.2,0.3,0.3)	0.9502 (0.0372)	0.9500 (0.0372)	0.9349 (0.0348)

4- Case Study

In this section, we use an example with real data to illustrate the confidence intervals using MOVER and two CLT approaches .For the data in Table 3, three diets were randomly assigned to groups of pregnant rats to study their effects on dietary residual zinc in the bloodstream (the amount of zinc in parts per million), and we determined the 95 % SCIs for the means of normal distributions with known coefficients of variation [35] .The summary statistics are as follows:

$\bar{x}_1 = 0.4960,\ \bar{x}_2 = 0.5420,\ \bar{x}_3 = 0.8280,\ s_1^2 = 0.0083,\ s_2^2 = 0.0244,\ s_3^2 = 0.0193,\ n_1, n_2, n_3 = 5.$ The comparison of SCI_{MOVER} and SCI_S, SCI_K is reported in Table 4. For the 15 observations of the pregnant rats ($\mu_{jl} = \mu_j - \mu_l,\ j,l = 1,2,...,k,\ j \neq l$), wthe results show that the MOVER approach performed much better than the two CLT approaches, while the expected lengths of SCI_{MOVER} were not much different from SCI_S and SCI_k. Moreover, the results for $n_i = 5$ and the different values of $\tau_i, i = 1,2,...,k$ are similar to those from the simulation study.

Table 3. The 15 observations of pregnant rats used to study the effect of three diets on dietary residual zinc in the bloodstream

Diet		
1	**2**	**3**
0.50	0.42	1.06
0.42	0.40	0.82
0.65	0.73	0.72
0.47	0.47	0.72
0.44	0.69	0.82

The amount of zinc is in ppm (parts per million)

Table 4. The 95% SCIs for the difference between means of normal distributions with known CVs

Parameters	SCI_{MOVER}		SCI_s		SCI_k	
	Lower	**Upper**	**Lower**	**Upper**	**Lower**	**Upper**
$\mu_2 - \mu_1$	-0.1761	0.2690	-0.1158	0.1968	-0.1025	0.1945
$\mu_3 - \mu_1$	0.1268	0.5387	0.1858	0.4756	0.1904	0.4736
$\mu_3 - \mu_2$	0.0289	0.5446	0.1091	0.4713	0.1125	0.4595

5- Conclusion

This study was carried out for the multiple contrasts of means of several normal distributions with known CVs: SCI_{MOVER} and SCI_s, SCI_k were extended to k populations, and these approaches were used to construct SCIs to differentiate multiple normal means with known coefficients of variation. The performance of these approaches was investigated using Monte Carlo simulation. The coverage probability results for SCI_{MOVER} were over the nominal level of 0.95 and indicate that it is more stable than SCI_s, SCI_k and was thus more appropriate for use in this scenario. In the academic and application spheres, continuous effort is made to develop SCIs by applying the tools of multiple contrasts of means of normal distribution with CV. The objective of this article is to process comprehensive research of SCIs as a tool that may improve the effectiveness of confidence intervals by using the method of variance estimates recovery (MOVER) and two central limit theorem (CLT) approaches. The MOVER and CLT methodologies were used to conduct quantitative and qualitative research in this study. There is the potential for positive effects on a variety of societies and fields. Confidence intervals for estimating the mean using the MOVER approach were first introduced to improve the effective approach, and since then, several researchers have constructed and used them. As mentioned above, the developing approaches are set to prepare for the circumstances with appropriate tools and processes. In this study, the MOVER approach is applied to construct new SCIs for the means of k populations that are normally distributed with known coefficients of variation. The benefits of the study can be determined on both practical and theoretical levels. From the practical point of view, the MOVER approach can specifically be used to create strategic documents in humanities, health, agriculture, education, and sciences. To sum up, review and practices can be applied in various industries and should be taken into consideration in future research.

6- Declarations

6-1- Author Contributions

Conceptualization, K.S. and S.N.; methodology, K.S. and S.N.; software, K.S.; validation, K.S., S.N. and S.N.; formal analysis, K.S.; investigation, K.S; resources, K.S. and S.N.; data curation, K.S. and S.N.; writing—original draft preparation, K.S.; writing—review and editing, K.S. and S.N.; supervision, K.S., S.N. and S.N.; All authors have read and agreed to the published version of the manuscript.

6-2- Data Availability Statement

The data presented in this study are available in the article.

6-3- Funding

The authors received no financial support for the research, authorship, and/or publication of this article.

6-4- Acknowledgements

The authors would like to thank Faculty of Applied Sciences and National Statistical Office for every support extended to us.

6-5- Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the authors.

7- References

[1] Searls, D. T. (1964). The Utilization of a Known Coefficient of Variation in the Estimation Procedure. Journal of the American Statistical Association, 59(308), 1225–1226. doi:10.1080/01621459.1964.10480765.

[2] Skrepnek, G. H., & Sahai, A. (2013). Improved methods in the efficient estimation of a normal population mean with a known coefficient of variation. American Journal of Mathematics and Statistics, 3(6), 389-393.

[3] Niwitpong, S. (2015). Confidence intervals for the normal mean with known coefficient of variation. Far East Journal of Mathematical Sciences, 97(6), 711–727. doi:10.17654/FJMSJul2015_711_727.

[4] Niwitpong, S., & Niwitpong, S. A. (2017). Confidence intervals for the difference between normal means with known coefficients of variation. Annals of Operations Research, 256(2), 237-251. doi:10.1007/s10479-016-2388-9.

[5] Bhat, K., & Rao, K. A. (2007). On tests for a normal mean with known coefficient of variation. International Statistical Review, 75(2), 170–182. doi:10.1111/j.1751-5823.2007.00019.x.

[6] Brazauskas, V., & Ghorai, J. (2007). Estimating the common parameter of normal models with known coefficients of variation: A sensitivity study of asymptotically efficient estimators. Journal of Statistical Computation and Simulation, 77(8), 663–681. doi:10.1080/10629360600578221.

[7] Khan, R. A. (2015). A remark on estimating the mean of a normal distribution with known coefficient of variation. Statistics, 49(3), 705–710. doi:10.1080/02331888.2013.809722.

[8] Singh, A., & Vishwakarma, G. K. (2019). Improved predictive estimation for mean using the Searls technique under ranked set sampling. Communications in Statistics - Theory and Methods, 50(9), 2015–2038. doi:10.1080/03610926.2019.1657456.

[9] Kharrati-Kopaei, M., Malekzadeh, A., & Sadooghi-Alvandi, M. (2013). Simultaneous fiducial generalized confidence intervals for the successive differences of exponential location parameters under heteroscedasticity. Statistics and Probability Letters, 83(6), 1547–1552. doi:10.1016/j.spl.2013.03.001.

[10] Krishnamoorthy, K., Waguespack, D., & Hoang-Nguyen-Thuy, N. (2019). Confidence interval, prediction interval and tolerance limits for a two-parameter Rayleigh distribution. Journal of Applied Statistics, 47(1), 160–175. doi:10.1080/02664763.2019.1634681.

[11] Dilba, G., Bretz, F., & Guiard, V. (2006). Simultaneous confidence sets and confidence intervals for multiple ratios. Journal of Statistical Planning and Inference, 136(8), 2640–2658. doi:10.1016/j.jspi.2004.11.009.

[12] Schmidt, S., & Brannath, W. (2019). Simultaneous confidence intervals for ratios with application to the gold standard design with more than one experimental treatment. Statistics in Medicine, 38(28), 5350–5360. doi:10.1002/sim.8365.

[13] Zhang, Q., Xu, J., Zhao, J., Liang, H., & Li, X. (2022). Simultaneous confidence intervals for ratios of means of zero-inflated log-normal populations. Journal of Statistical Computation and Simulation, 92(6), 1113–1132. doi:10.1080/00949655.2021.1986508.

[14] Abdel-Karim, A. H. (2015). Construction of simultaneous confidence intervals for ratios of means of lognormal distributions. Communications in Statistics: Simulation and Computation, 44(2), 271–283. doi:10.1080/03610918.2013.767911.

[15] Sadooghi-Alvandi, S. M., & Malekzadeh, A. (2014). Simultaneous confidence intervals for ratios of means of several lognormal distributions: A parametric bootstrap approach. Computational Statistics and Data Analysis, 69, 133–140. doi:10.1016/j.csda.2013.07.039.

[16] Maneerat, P., Niwitpong, S. A., & Niwitpong, S. (2021). Simultaneous confidence intervals for all pairwise comparisons of the means of deltalognormal distributions with application to rainfall data. PLoS ONE, 16(7), e0253935. doi:10.1371/journal.pone.0253935.

[17] Thangjai, W., Niwitpong, S., & Niwitpong, S. A. (2017). Confidence intervals for mean and difference of means of normal distributions with unknown coefficients of variation. Mathematics, 5(3), 39. doi:10.3390/math5030039.

[18] Panichkitkosolkul, W., & Tulyanitikul, B. (2020). Performance of Statistical Methods for Testing the Signal-to-Noise Ratio of a Log-Normal Distribution. 2020 IEEE 7th International Conference on Industrial Engineering and Applications (ICIEA). doi:10.1109/iciea49774.2020.9101993.

[19] Zhang, G. (2014). Simultaneous confidence intervals for several inverse Gaussian populations. Statistics & Probability Letters, 92, 125-131. doi:10.1016/j.spl.2014.05.013.

[20] Benjamini, Y., Madar, V., & Stark, P. B. (2011). Simultaneous Confidence Intervals with more Power to Determine Signs. University of California, Berkeley, United States.

[21] Segbehoe, L. S., Schaarschmidt, F., & Djira, G. D. (2022). Simultaneous confidence intervals for contrasts of quantiles. Biometrical Journal, 64(1), 7–19. doi:10.1002/bimj.202000077.

[22] Chen, T., & Hoppe, F. M. (2017). Simultaneous Confidence Intervals and Regions. Wiley StatsRef: Statistics Reference Online, 1–8. doi:10.1002/9781118445112.stat05961.pub2.

[23] Wang, H. (2008). Exact confidence coefficients of simultaneous confidence intervals for multinomial proportions. Journal of Multivariate Analysis, 99(5), 896–911. doi:10.1016/j.jmva.2007.05.003.

[24] Donner, A., & Zou, G. Y. (2012). Closed-form confidence intervals for functions of the normal mean and standard deviation. Statistical Methods in Medical Research, 21(4), 347–359. doi:10.1177/0962280210383082.

[25] Zhang, G. (2014). Simultaneous confidence intervals for several inverse Gaussian populations. Statistics & Probability Letters, 92, 125–131. doi:10.1016/j.spl.2014.05.013.

[26] Klingenberg, B., & Satopää, V. (2013). Simultaneous confidence intervals for comparing margins of multivariate binary data. Computational Statistics and Data Analysis, 64, 87–98. doi:10.1016/j.csda.2013.02.016.

[27] Thangjai, W., Niwitpong, S., & Niwitpong, S. (2019). Simultaneous confidence intervals for all differences of coefficients of variation of log-normal distributions. Hacettepe Journal of Mathematics and Statistics, 48(5), 1505–1521. doi:10.15672/hujms.454804.

[28] Xiong, S., & Mu, W. (2009). Simultaneous confidence intervals for one-way layout based on generalized pivotal quantities. Journal of Statistical Computation and Simulation, 79(10), 1235–1244. doi:10.1080/00949650802232641.

[29] Schaarschmidt, F., Gerhard, D., & Vogel, C. (2017). Simultaneous confidence intervals for comparisons of several multinomial samples. Computational Statistics & Data Analysis, 106, 65–76. doi:10.1016/j.csda.2016.09.004.

[30] Zheng, J., Yin, D., Yuan, M., & Chow, S. C. (2019). Simultaneous confidence interval methods for analytical similarity assessment. Journal of Biopharmaceutical Statistics, 29(5), 920–940. doi:10.1080/10543406.2019.1657142.

[31] Li, B. (2021). Simultaneous confidence intervals of estimable functions based on quasi-likelihood in generalized linear models for over-dispersed data. Journal of Statistical Computation and Simulation, 91(1), 108–127. doi:10.1080/00949655.2020.1807548.

[32] Yosboonruang, N., Niwitpong, S. A., & Niwitpong, S. (2021). Simultaneous confidence intervals for all pairwise differences between the coefficients of variation of rainfall series in Thailand. PeerJ, 9, e11651. doi:10.7717/peerj.11651.

[33] Yang, Z., Tian, G. L., Liu, X., & Ma, C. X. (2021). Simultaneous confidence interval construction for many-to-one comparisons of proportion differences based on correlated paired data. Journal of Applied Statistics, 48(8), 1442-1456. doi:10.1080/02664763.2020.1795815.

[34] Segbehoe, L. S., Schaarschmidt, F., & Djira, G. D. (2022). Simultaneous confidence intervals for contrasts of quantiles. Biometrical Journal, 64(1), 7–19. doi:10.1002/bimj.202000077.

[35] Walpole, R. E., Myers, R. H., Myers, S. L. & Ye, K. (2012). Probability and Statistics for Engineers and Scientists. Prentice Hall, Hoboken, United States.