

Using Model Theory to Find Decidable and Tractable Description Logics with Concrete Domains

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Abstract

Concrete domains have been introduced in the area of Description Logic (DL) to enable reference to concrete objects (such as numbers) and predefined predicates on these objects (such as numerical comparisons) when defining concepts. Unfortunately, in the presence of general concept inclusions (GCIs), which are supported by all modern DL systems, adding concrete domains may easily lead to undecidability.

To regain decidability of the DL \mathcal{ALC} in the presence of GCIs, quite strong restrictions, called ω -admissibility, were imposed on the concrete domain. On the one hand, we generalize the notion of ω -admissibility from concrete domains with only binary predicates to concrete domains with predicates of arbitrary arity. On the other hand, we relate ω -admissibility to well-known notions from model theory. In particular, we show that finitely bounded homogeneous structures yield ω -admissible concrete domains. This allows us to show ω -admissibility of concrete domains using existing results from model theory. When integrating concrete domains into lightweight DLs of the \mathcal{EL} family, achieving decidability of reasoning is not enough. One wants the resulting DL to be tractable. This can be achieved by using so-called p-admissible concrete domains and restricting the interaction between the DL and the concrete domain. We investigate p-admissibility from an algebraic point of view. Again, this yields strong algebraic tools for demonstrating p-admissibility. In particular, we obtain an expressive numerical p-admissible concrete domain based on the rational numbers. Although ω -admissibility and p-admissibility are orthogonal conditions that are almost exclusive, our algebraic characterizations of these two properties allow us to locate an infinite class of p-admissible concrete domains whose integration into \mathcal{ALC} yields decidable DLs.

DL systems that can handle concrete domains allow their users to employ a fixed set of predicates of one or more fixed concrete domains when modelling concepts. They do not provide their users with means for defining new predicates, let alone new concrete domains. The good news is that finitely bounded homogeneous structures offer precisely that. We show that integrating concrete domains based on finitely bounded homogeneous structures into \mathcal{ALC} yields decidable DLs even if we allow predicates specified by first-order formulas. This class of structures also provides effective means for defining new ω -admissible concrete domains with at most binary predicates. The bad news is that defining ω -admissible concrete domains with predicates of higher arities is computationally hard. We obtain two new lower bounds for this meta-problem, but leave its decidability open. In contrast, we prove that there is no algorithm that would facilitate defining p-admissible concrete domains already for binary signatures.

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Contents

1. Introduction	1
2. Preliminaries	5
3. Description Logics with Concrete Domains	9
3.1. Basic definitions and undecidability results	9
3.2. Decidable and tractable DLs with concrete domains	16
4. A Model-Theoretic Analysis of ω-Admissibility	23
4.1. Homomorphism ω -compactness via ω -categoricity	23
4.2. Patchworks via homogeneity	24
4.3. JDJEPD via decomposition into orbits	27
4.4. Upper bounds via finite boundedness	28
4.5. ω -admissible finitely bounded homogeneous structures	32
4.6. ω -admissible homogeneous cores with a decidable CSP	34
4.7. Coverage of the developed sufficient conditions	36
4.8. Closure properties: homogeneity & finite boundedness	39
5. A Model-Theoretic Analysis of p-Admissibility	47
5.1. Convexity via square embeddings	47
5.2. Convex ω -categorical structures	50
5.3. Convex numerical structures	52
5.4. Ages defined by forbidden substructures	54
5.5. Ages defined by forbidden homomorphic images	56
5.6. (Non-)closure properties of convexity	59
6. Towards user-definable concrete domains	61
6.1. A proof-theoretic perspective	65
6.2. Universal Horn sentences and the JEP	66
6.3. Universal sentences and the AP: the Horn case	77
6.4. Universal sentences and the AP: the general case	90
7. Conclusion	99
7.1. Contributions and future outlook	99
A. Concrete Domains without Equality	103

Contents

Bibliography	107
List of Figures	115
Alphabetical Index	117

Introduction

Description Logics (DLs) [5, 9] are a well-investigated family of logic-based knowledge representation languages, which are frequently used to formalize ontologies for application domains such as the Semantic Web [57] or biology and medicine [56]. To define the important notions of such an application domain as formal concepts, DLs state necessary and sufficient conditions for an individual to belong to a concept. These conditions can be Boolean combinations of atomic properties required for the individual (expressed by concept names) or properties that refer to relationships with other individuals and their properties (expressed as role restrictions). For example, the concept of a father that has only daughters can be formalized by the concept description $C_{ex} := \neg Female \sqcap \exists child.Human \sqcap \forall child.Female$, which uses the concept names *Female* and *Human* and the role name *child* as well as the concept constructors negation (\neg), conjunction (\sqcap), existential restriction ($\exists r.D$), and value restriction ($\forall r.D$). The GCIs $Human \sqsubseteq \forall child.Human$ and $\exists child.Human \sqsubseteq Human$ say that humans have only human children, and they are the only ones that can have human children.

DL systems provide their users with reasoning services that allow them to derive implicit knowledge from the explicitly represented one. In our example, the above GCIs imply that elements of our concept C_{ex} also belong to the concept $D_{ex} := Human \sqcap \forall child.Human$, i.e., C_{ex} is subsumed by D_{ex} w.r.t. these GCIs. A specific DL is determined by which kind of concept constructors are available. A major goal of DL research was and still is to find a good compromise between expressiveness and the complexity of reasoning, i.e., to locate DLs that are expressive enough for interesting applications, but still have inference problems (like subsumption) that are decidable and preferably of a low complexity. For the DL \mathcal{ALC} , in which all the concept descriptions used in the above example can be expressed, the subsumption problem w.r.t. GCIs is EXPTIME-complete [9].

Classical DLs like \mathcal{ALC} cannot refer to concrete objects and predefined relations over these objects when defining concepts. For example, a constraint stating that parents are strictly older than their children cannot be expressed in \mathcal{ALC} . To overcome this deficit, a scheme for integrating certain well-behaved concrete domains, called admissible, into \mathcal{ALC} was introduced in [6], and it was shown that this integration leaves the relevant inference problems (such as subsumption) decidable. Basically, admissibility requires that the set of predicates of the concrete domain is closed under negation and that the constraint satisfaction problem (CSP) for the concrete domain is decidable. However, in this setting, GCIs were not considered since

1. Introduction

they were not a standard feature of DLs then,¹ though a combination of concrete domains and GCIs would be useful in many applications. For example, using the syntax employed in [70] and also in this thesis, the above constraint regarding the age of parents and their children could be expressed by the GCI $Human \sqcap \exists age, child age. (x_1 < x_2) \sqsubseteq \perp$, which says that there cannot be a human whose age is smaller than the age of one of their children. Here \perp is the bottom concept, which is always interpreted as the empty set, *age* is a feature that maps from the abstract domain populating concepts into the concrete domain of rational numbers, and $<$ is the usual “smaller than” predicate.

A first indication that concrete domains might be harmful for decidability was given in [8], where it was shown that adding transitive closure of roles to the extension of \mathcal{ALC} by an admissible concrete domain based on real arithmetics renders the subsumption problem undecidable. The proof of this result uses a reduction from the Post Correspondence Problem (PCP). It was shown in [68] that this proof can be adapted to the case where transitive closure of roles is replaced by GCIs, and it actually works for considerably weaker concrete domains, such as the rational numbers \mathbb{Q} or the natural number \mathbb{N} with a unary predicate $=_0$ for equality with zero, a binary equality predicate $=$, and a unary predicate $+_1$ for incrementation. In [10] it is shown, by a reduction from the halting problem of two-register machines, that undecidability even holds without $=$ and $=_0$.

To regain decidability, one option is to impose syntactic restrictions on how the DL can interact with the concrete domain [49, 76]. The main idea is to disallow paths (such as *child age* in our example), which has the effect that concrete domain predicates cannot compare properties (such as the age) of different individuals. Another option is to impose stronger restrictions than admissibility on the concrete domain. After first positive results for specific concrete domains (e.g., the rational numbers with order and equality [67, 69]), the notion of ω -admissible concrete domains was introduced in [70], and it was shown (by designing a tableau-based decision procedure) that integrating such a concrete domain into \mathcal{ALC} leaves reasoning decidable also in the presence of GCIs. In [10], we generalize the notion of ω -admissibility and the decidability result from concrete domains with only binary predicates as in [70] to concrete domains with predicates of arbitrary arity.

When integrating a concrete domain into a lightweight DL like \mathcal{EL} , one wants to preserve tractability rather than just decidability. To achieve this, the notion of p-admissible concrete domains was introduced in [3] and paths of length > 1 were disallowed in concrete domain restrictions. Regarding the latter condition, note that, in the above example, we have used the path *child age*, which has length 2. The restriction to paths of length 1 means (in our example) that we can no longer compare the ages of different humans, but we can still define concepts like teenager, using the GCI $Teenager \sqsubseteq Human \sqcap \exists age. \geq_{10}(x_1) \wedge \leq_{19}(x_1)$, where \geq_{10} and \leq_{19} are unary predicates respectively interpreted as the rational numbers greater equal 10 and smaller equal 19. In a p-admissible concrete domain, satisfiability of conjunctions of atomic formulas and validity of implications between such conjunctions must be tractable. In addition,

¹Actually, GCIs were introduced (with a different name) at about the same time as concrete domains [4, 78].

the concrete domain must be *convex*, which roughly speaking means that a conjunction cannot imply a true disjunction. For example, the concrete domain $(\mathbb{Q}; <, =, >)$ is ω -admissible, but it is not convex since $x < y \wedge x < z$ implies $y < z \vee y = z \vee y > z$, but none of the disjuncts. In [3], two p-admissible concrete domains were exhibited, one of them based on \mathbb{Q} with unary predicates $=_p, >_p$ and binary predicates $+_p, =$. To the best of our knowledge, no other p-admissible concrete domains have been described in the literature before our work in [12]. Similarly, after the publication of [70] and before our work in [10], no new ω -admissible concrete domains were exhibited. We believe that the reason for this is that it is quite hard to prove ω -admissibility or p-admissibility of a concrete domain without appropriate mathematical tools.

The main contribution of this thesis is to develop such tools based on a model-theoretic analysis of the conditions required by these two notions of admissibility. It is based on the conference publications [10] and [12], but differs from them w.r.t. some details and also presents additional results. On the one hand, we show that there is a close relationship between ω -admissibility and well-known notions from model theory. In particular, we prove that finitely bounded homogeneous structures yield ω -admissible concrete domains. This allows us to show ω -admissibility of known such concrete domains (like Allen and RCC8 from [70]; see Example 4) and to locate new ω -admissible concrete domains using existing results from model theory (see Examples 5, 6, and 7). We can even show that some of the relevant properties can be algorithmically tested (see Chapter 6). On the other hand, we devise an algebraic characterization of convexity based on the notion of *square embeddings*, which are embeddings of the second power of a relational structure into itself. We investigate the implications of this characterization for so-called ω -categorical structures, finitely bounded structures, and numerical structures. Each of these cases provides us with new examples of p-admissible concrete domains. In particular, we exhibit a new and quite expressive p-admissible concrete domain based on the rational numbers, whose predicates are defined by linear equations over \mathbb{Q} . As it is the case with ω -admissibility, we investigate algorithmic testability of the relevant properties. While testing an already existing concrete domain for convexity could in theory be automatized, we show that defining new p-admissible concrete domains from scratch is an algorithmically unsolvable problem. We also investigate the connection between p-admissibility and ω -admissibility. It turns out that only trivial concrete domains can satisfy both properties. However, we can show that convex finitely bounded homogeneous structures, which are p-admissible, can be integrated into \mathcal{ALC} (even without the length 1 restriction on role paths) without losing decidability. Whereas these structures are not ω -admissible, they can be expressed in an ω -admissible concrete domain.

Preliminaries

In this section, we introduce the algebraic and logical notions that will be used in the rest of the thesis. The set $\{1, \dots, n\}$ is denoted by $[n]$. Given a set A , the *equality* on A is defined as the binary relation $\text{Eq}_A := \{(a, a) \mid a \in A\}$. We use the bar notation for tuples; for a tuple \bar{t} indexed by a set I , the value of \bar{t} at the position $i \in I$ is denoted by $\bar{t}[i]$. For a function $f : A^k \rightarrow B$ and n -tuples $\bar{t}_1, \dots, \bar{t}_k \in A^n$, we use the shortcut $f(\bar{t}_1, \dots, \bar{t}_k) := (f(\bar{t}_1[1], \dots, \bar{t}_k[1]), \dots, f(\bar{t}_1[n], \dots, \bar{t}_k[n]))$.

From a mathematical point of view, concrete domains are relational structures. A *relational signature* τ is a set of *relation symbols*, each with an associated natural number called *arity*. For a relational signature τ , a *relational τ -structure* \mathfrak{A} (or simply τ -structure or structure) consists of a set A (the *domain*) together with the relations $R^{\mathfrak{A}} \subseteq A^k$ for each relation symbol $R \in \tau$ of arity k . Such a structure \mathfrak{A} is *finite* if its domain A is finite. We often describe structures by listing their domain and relations; e.g., we write $\mathfrak{Q} = (\mathbb{Q}; <)$ for the relational structure whose domain is the set of rational numbers \mathbb{Q} , and which has the usual smaller relation $<$ on \mathbb{Q} as its only relation.¹

The *direct product* of a family $(\mathfrak{A}_i)_{i \in I}$ of τ -structures is the τ -structure $\prod_{i \in I} \mathfrak{A}_i$ over $\prod_{i \in I} A_i$ such that, for each $R \in \tau$ of arity k , we have $(\bar{a}_1, \dots, \bar{a}_k) \in R^{\prod_{i \in I} \mathfrak{A}_i}$ if and only if $(\bar{a}_1[i], \dots, \bar{a}_k[i]) \in R^{\mathfrak{A}_i}$ for every $i \in I$. We denote the *square*, i.e., the binary product of a structure \mathfrak{A} with itself by \mathfrak{A}^2 . An *expansion* of a τ -structure \mathfrak{A} is a σ -structure \mathfrak{B} with $A = B$ such that $\tau \subseteq \sigma$ and $R^{\mathfrak{B}} = R^{\mathfrak{A}}$ for each relation symbol $R \in \tau$. Conversely, we call \mathfrak{A} a *reduct* of \mathfrak{B} . We use the notation $(\mathfrak{A}, R_1, \dots, R_n)$ to describe an expansion of \mathfrak{A} by the relations R_1, \dots, R_n over A ; e.g., (\mathfrak{Q}, \neq) stands for $(\mathbb{Q}; <, \neq)$.

One possibility to obtain an expansion of a τ -structure is to use formulas of *first-order* (FO) logic over the signature τ to define new predicates, where a formula with k free variables defines a k -ary predicate in the obvious way. We say that a first-order formula is *k-ary* if it has k free variables. For a first-order formula ϕ , we use the notation $\phi(\bar{x})$ to indicate that the free variables of ϕ are among \bar{x} . This does not necessarily mean that the truth value of ϕ depends on each variable in \bar{x} . We assume that equality $=$ as well as the nullary predicate symbol `false` for falsity are always available when building these formulas. Thus, atomic formulas are of the form `false`, $x_i = x_j$, and $R(x_1, \dots, x_k)$ for some k -ary $R \in \tau$ and variables x_1, \dots, x_k . For a structure \mathfrak{A} we denote its *first-order theory*, i.e., the set of all first-order sentences that hold in \mathfrak{A} , with $\text{Th}(\mathfrak{A})$. For a first-order τ -sentence Φ , we denote the class of all finite models of Φ by $\text{Mod}_{\text{fin}}(\Phi)$.

¹By an abuse of notation, we use $<$ instead of $<^{\mathfrak{Q}}$ to denote the interpretation of the predicate symbol $<$ in \mathfrak{Q} .

2. Preliminaries

In addition to full first-order logic, we also use standard fragments such as the *existential positive* (EP), the *primitive positive* (PP), and the *quantifier-free* (QF) fragment. The existential positive fragment consists of formulas built using conjunction, disjunction, and existential quantification only. The primitive positive fragment consists of existentially quantified conjunctions of atomic formulas, and the quantifier-free fragment consists of Boolean combinations of atomic formulas. A formula is called *equality-free* if it does not contain any occurrence of the default equality predicate $=$.² Let Σ be a set of first-order formulas and \mathfrak{D} a structure. We say that a relation over D has a Σ -*definition* in \mathfrak{D} if it is of the form $\{\bar{t} \in D^k \mid \mathfrak{D} \models \phi(\bar{t})\}$ for some $\phi \in \Sigma$. We refer to this relation by $\phi^{\mathfrak{D}}$. For example, the formula $\phi(x, y) := y < x \vee y = x$ is existential positive and quantifier-free. Interpreted in the structure \mathbb{Q} , it clearly defines the binary relation \geq on \mathbb{Q} . This shows that \geq is EP and QF definable in \mathbb{Q} . An example of a PP formula is the formula $\phi(x) := \exists y(x = y)$, which defines the unary relation interpreted as the whole domain \mathbb{Q} . An *implication* is of the form $\forall \bar{x}(\phi \Rightarrow \psi)$ where ϕ is a conjunction of atomic τ -formulas other than `false`, ψ is a disjunction of atomic τ -formulas, and the tuple \bar{x} consists of all the variables occurring in ϕ or ψ . We refer to ϕ as the *premise*, and to ψ as the *conclusion*. An implication over a signature τ is called a *tautology* if it holds in all τ -structures, i.e., if one disjunct of the conclusion equals one of the conjuncts of the premise. An implication $\forall \bar{x}(\phi \Rightarrow \psi)$ is a *Horn implication* if ψ is a single atomic τ -formula. A universal sentence is called *Horn* if it is a conjunction of Horn implications.

For a fixed τ -structure \mathfrak{B} , the *constraint satisfaction problem* (CSP) for \mathfrak{B} [17] asks whether a given PP τ -sentence is satisfiable in \mathfrak{B} . Typically, CSPs are only defined for structures with a finite signature (for technical reasons). In the present thesis, we do *not* follow this convention. The CSP for \mathfrak{A} can be reduced in polynomial time to the validity problem for Horn implications since ϕ is satisfiable in \mathfrak{A} if and only if $\forall \bar{x}(\phi \Rightarrow \text{false})$ is not valid in \mathfrak{A} . Conversely, validity of Horn implications in a structure \mathfrak{A} can be reduced in polynomial time to $\text{CSP}(\mathfrak{A}^\neg, \neq)$ where \mathfrak{A}^\neg is the expansion of \mathfrak{A} by the complements of all relations. In fact, the Horn implication $\forall \bar{x}(\phi \Rightarrow \psi)$ is valid in \mathfrak{A} if and only if $\phi \wedge \neg\psi$ is not satisfiable in $(\mathfrak{A}^\neg, \neq)$. Note that, in the signature of $(\mathfrak{A}^\neg, \neq)$, $\neg\psi$ can be expressed by an atomic formula.

A *homomorphism* $h: \mathfrak{A} \rightarrow \mathfrak{B}$ for τ -structures \mathfrak{A} and \mathfrak{B} is a mapping $h: A \rightarrow B$ that *preserves* each relation of \mathfrak{A} , i.e., if $\bar{t} \in R^{\mathfrak{A}}$ for some k -ary relation symbol $R \in \tau$, then $h(\bar{t}) \in R^{\mathfrak{B}}$. The homomorphism $h: \mathfrak{A} \rightarrow \mathfrak{B}$ is *strong* if it additionally satisfies the inverse condition: for every k -ary relation symbol $R \in \tau$ and $\bar{t} \in A^k$ we have $h(\bar{t}) \in R^{\mathfrak{B}}$ only if $\bar{t} \in R^{\mathfrak{A}}$. An *embedding* is an injective strong homomorphism. We write $\mathfrak{A} \rightarrow \mathfrak{B}$ ($\mathfrak{A} \hookrightarrow \mathfrak{B}$) if there is a homomorphism (embedding) from \mathfrak{A} to \mathfrak{B} . A *self-embedding* is an embedding of a structure into itself. A *substructure* of \mathfrak{B} is a structure \mathfrak{A} over the domain $A \subseteq B$ such that the inclusion map $i: A \rightarrow B$ is an embedding. Conversely, we call \mathfrak{B} an *extension* of \mathfrak{A} . The *age* of a structure \mathfrak{B} , denoted by $\text{Age}(\mathfrak{B})$, is the class of all *finite* structures \mathfrak{A} with $\mathfrak{A} \hookrightarrow \mathfrak{B}$. An *isomorphism* is a surjective embedding. Two structures \mathfrak{A} and \mathfrak{B} are *isomorphic* (written $\mathfrak{A} \cong \mathfrak{B}$) if there exists an isomorphism from \mathfrak{A} to \mathfrak{B} . An *automorphism* is an isomorphism from \mathfrak{A} to \mathfrak{A} .

²In case the signature τ of a structure contains a symbol that is interpreted as equality in that structure, an equality-free formula can, of course, still use that symbol.

If the signature τ of \mathfrak{B} is finite, the constraint satisfaction problem for \mathfrak{B} can also be conveniently formulated using homomorphisms: given a finite τ -structure \mathfrak{A} , decide whether $\mathfrak{A} \rightarrow \mathfrak{B}$. A *solution* for such an instance \mathfrak{A} of the CSP is then simply a homomorphism $h: \mathfrak{A} \rightarrow \mathfrak{B}$ and $\text{CSP}(\mathfrak{B})$ is the class of all finite τ -structures that homomorphically map to \mathfrak{B} . It is easy to see that this definition of the CSP coincides with the one given above. Indeed, deciding whether a CSP instance \mathfrak{A} admits a solution amounts to evaluating a PP sentences in \mathfrak{B} and vice versa [17]. For example, verifying whether the structure $\mathfrak{A} = (\{a_1, a_2, a_3\}; <^{\mathfrak{A}})$ with $<^{\mathfrak{A}} := \{(a_1, a_2), (a_2, a_3), (a_3, a_1)\}$ homomorphically maps into \mathfrak{Q} is the same as checking whether the PP formula $x_1 < x_2 \wedge x_2 < x_3 \wedge x_3 < x_1$ is satisfiable in \mathfrak{Q} .

The CSP for \mathfrak{Q} is tractable since a structure $\mathfrak{A} = (A; <^{\mathfrak{A}})$ can homomorphically be mapped into \mathfrak{Q} if and only if it does not contain a $<$ -cycle, i.e., there are no $n \geq 1$ and elements $a_1, \dots, a_n \in A$ such that $a_1 <^{\mathfrak{A}} \dots <^{\mathfrak{A}} a_n <^{\mathfrak{A}} a_1$. Testing whether such a cycle exists can be done in nondeterministic logarithmic space since it requires solving the reachability problem in a directed graph (digraph). In the example above, we obviously have a cycle, and thus this instance of $\text{CSP}(\mathfrak{Q})$ has no solution.

The definition of admissibility of a concrete domain in [6] requires that the constraint satisfaction problem for this structure is decidable, the predicates are closed under negation, and there is a predicate for the whole domain. We have already seen that the negation \geq of $<$ is EP definable in \mathfrak{Q} and that the predicate for the whole domain is PP definable. The negation of this predicate has the PP definition $x < x$. The following lemma implies that the expansion of \mathfrak{Q} by these predicates still has a decidable CSP.³

Lemma 1 ([17]). *Let $\mathfrak{C}, \mathfrak{D}$ be structures over the same domain with finite signatures.*

1. *If the relations of \mathfrak{C} have a PP definition in \mathfrak{D} , then $\text{CSP}(\mathfrak{C}) \leq_{\text{PTIME}} \text{CSP}(\mathfrak{D})$.*
2. *If the relations of \mathfrak{C} have an EP definition in \mathfrak{D} , then $\text{CSP}(\mathfrak{C}) \leq_{\text{NPTIME}} \text{CSP}(\mathfrak{D})$.*

³The lemma actually only yields an NP decision procedure for this CSP, but it is easy to see that the above polynomial-time cycle-checking algorithm can be adapted such that it also works for the expanded structure.

Description Logics with Concrete Domains

We assume that the reader is familiar with the basic definitions and results in DL [5, 9], but nevertheless briefly recall the definitions of the two DLs \mathcal{ALC} and \mathcal{EL} relevant for this thesis. Then we describe how these DLs can be extended with concrete domains. The integrations of concrete domains into DLs described in the literature [6, 67, 3, 70, 36, 65] differ in some details. The approaches described below for \mathcal{ALC} and \mathcal{EL} are close to the ones in [70] and [3], respectively, but not identical, mainly as a matter of convenience of presentation. Reasoning in DLs obtained this way may easily become undecidable, and thus one needs to find conditions that guarantee decidability, and even tractability for the case of \mathcal{EL} .

3.1 Basic definitions and undecidability results

Given countably infinite disjoint sets N_C and N_R of concept and role names, \mathcal{ALC} concepts are built using the concept constructors top concept (\top), bottom concept (\perp), negation ($\neg C$), conjunction ($C \sqcap D$), disjunction ($C \sqcup D$), existential restriction ($\exists r.C$), and universal restriction ($\forall r.C$). The semantics of the constructors is defined in the usual way (see, e.g., [5, 9]). It assigns to every \mathcal{ALC} concept C a set $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, where $\Delta^{\mathcal{I}}$ is the interpretation domain of the given interpretation \mathcal{I} . The set of \mathcal{EL} concepts is obtained by restricting the available constructors to \top , $C \sqcap D$, and $\exists r.C$. As usual, a TBox is defined to be a finite set of *general concept inclusions* (GCIs) $C \sqsubseteq D$, where C, D are concepts. The interpretation \mathcal{I} is a *model* of such a TBox if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for all GCIs $C \sqsubseteq D$ occurring in it. Given a concept description C and a TBox \mathcal{T} , we say that C is *satisfiable* w.r.t. \mathcal{T} if $C^{\mathcal{I}}$ is non-empty for some model \mathcal{I} of \mathcal{T} . Concept satisfiability w.r.t. GCIs is EXPTIME-complete in \mathcal{ALC} [78], but trivial in \mathcal{EL} since \mathcal{EL} concepts are always satisfiable. Given concept descriptions C, D and a TBox \mathcal{T} , we say that C is *subsumed* by D w.r.t. \mathcal{T} (written $C \sqsubseteq_{\mathcal{T}} D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for all models of \mathcal{T} . Subsumption w.r.t. TBoxes in \mathcal{ALC} is also EXPTIME-complete since it is interreducible with concept satisfiability, but tractable (i.e., decidable in polynomial time) in \mathcal{EL} [32, 3].

From an algebraic point of view, a concrete domain is a relational structure \mathfrak{D} . To integrate such a structure into \mathcal{ALC} and \mathcal{EL} , we introduce a countably infinite set N_F of *feature names* disjoint from N_C and N_R . Their purpose is to facilitate the connection between the abstract domain $\Delta^{\mathcal{I}}$ and the concrete domain D . A (role-feature) *path* is of the form $r f$ or f where $r \in N_R$ and $f \in N_F$. In this thesis, we do not allow chains of role or feature names in paths.

3. Description Logics with Concrete Domains

The reason is that two cases above already enable us to capture the essence of the expressive power of concrete domains: the ability to relate the data values of different abstract domain elements. This means that paths are always of length 1 or 2. In our example in the introduction, *age* is both a feature name and a path of length 1, and *childage* is a path of length 2.

Definition 1. *Concrete domain restrictions* for a relational τ -structure \mathfrak{D} are concept constructors of the form $\exists p_1, \dots, p_k. \phi(x_1, \dots, x_k)$ or $\forall p_1, \dots, p_k. \phi(x_1, \dots, x_k)$, where p_1, \dots, p_k are paths and ϕ is a first-order τ -formula with free variables x_1, \dots, x_k . The DL $\mathcal{ALC}(\mathfrak{D})$ extends \mathcal{ALC} with concrete domain restrictions where ϕ is allowed to be an arbitrary atomic τ -formula. The DL $\mathcal{EL}(\mathfrak{D})$ is the sublanguage of $\mathcal{ALC}(\mathfrak{D})$ where only the concept constructors of \mathcal{EL} together with existential concrete domain restrictions can be used. Let Σ be a set of first-order τ -formulas and n a natural number. The DL $\mathcal{ALC}_{\Sigma}^n(\mathfrak{D})$ extends \mathcal{ALC} with concrete domain restrictions where ϕ is allowed to be an at most n -ary formula from Σ .

In contrast to previous works on concrete domains [6, 70], we generally allow the use of the equality predicate in concrete domain restrictions, even if it is not explicitly contained in the signature of the concrete domain. This assumption will turn out to be useful later on, and it is basically without loss of generality since virtually all concrete domains considered in the literature can express equality in a way that does not impact on the complexity of reasoning. Our assumption that `false` is an atom implies that $\mathcal{EL}(\mathfrak{D})$ can express the bottom concept \perp by the concrete domain restriction $\exists. \text{false}$. A third difference is that, while features pointing into the concrete domain are functional, we do not allow the use of functional roles in paths. In [6], only functional roles are allowed to occur in paths whereas in [70] both functional and other roles can occur there. For \mathcal{ALC} , this does not really make a difference due to the availability of universal concrete domain restrictions. For \mathcal{EL} , the presence of functional roles would destroy tractability even without concrete domains [3], and thus needs to be avoided anyway.

To define the semantics of concrete domain restrictions, we assume that an interpretation \mathcal{I} assigns functional binary relations $f^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times D$ to feature names $f \in N_F$, where *functional* means that $(a, d) \in f^{\mathcal{I}}$ and $(a, d') \in f^{\mathcal{I}}$ imply $d = d'$. We extend the interpretation function to paths of the form $p = r f$ by setting

$$(r f)^{\mathcal{I}} = \{(a, d) \in \Delta^{\mathcal{I}} \times D \mid \text{there is } b \in \Delta^{\mathcal{I}} \text{ such that } (a, b) \in r^{\mathcal{I}} \text{ and } (b, d) \in f^{\mathcal{I}}\}.$$

The semantics of concrete domain restrictions is now defined as follows:

$$\begin{aligned} (\exists p_1, \dots, p_k. \phi(x_1, \dots, x_k))^{\mathcal{I}} &= \{a \in \Delta^{\mathcal{I}} \mid \text{there are } d_1, \dots, d_k \in D \text{ such that} \\ &\quad (a, d_i) \in p_i^{\mathcal{I}} \text{ for all } i \in [k] \text{ and } \mathfrak{D} \models \phi(d_1, \dots, d_k)\}, \\ (\forall p_1, \dots, p_k. \phi(x_1, \dots, x_k))^{\mathcal{I}} &= \{a \in \Delta^{\mathcal{I}} \mid \text{for all } d_1, \dots, d_k \in D \text{ such that} \\ &\quad (a, d_i) \in p_i^{\mathcal{I}} \text{ for all } i \in [k] \text{ we have } \mathfrak{D} \models \phi(d_1, \dots, d_k)\}. \end{aligned}$$

As already mentioned above, the concrete domain restriction $\exists. \text{false}$ is unsatisfiable, and thus equivalent to \perp . The restriction $\exists f, f. (x_1 = x_2)$ expresses that the value of the feature f

must be defined, without putting any constraint on this value.

Adding a concrete domain to a DL can easily lead to undecidability. Clearly, if the CSP of the concrete domain is undecidable, then this transfers to the DL it is integrated in. If the concrete domain is admissible, i.e., its CSP is decidable and its relations are closed under complements, then concept satisfiability without GCIs is decidable in a variant of \mathcal{ALC} with concrete domains that uses functional roles in paths [6]. But even for very simple concrete domains with decidable CSPs, the presence of GCIs may cause undecidability. For instance, $\mathcal{ALC}(\mathcal{D})$ is undecidable already when \mathcal{D} is a structure over \mathbb{N} that has access to the unary predicate $=_0$, which is interpreted as the singleton set $\{0\}$, and the binary predicate $+_1$, which is interpreted as incrementation (i.e., it consists of the tuples $(m, m+1)$ for $m \in \mathbb{N}$) [9]. We can improve on this result by showing undecidability for even less expressive concrete domains without the predicate $=_0$. Our proof is an adaptation of the undecidability proof in [9] to the case where no zero test $=_0$ is available. In contrast to the setting in [9], our version of \mathcal{ALC} with concrete domains does not allow functional roles, which play an essential role in the original proof. However, we can circumvent this issue by using additional universal quantification (i.e., value restrictions and universal concrete domain restrictions) to ensure that all the successors of an individual w.r.t. a given role behave the same. In this section, we use $\exists \& \forall r. C$ and $\exists \& \forall p_1, \dots, p_k. \phi$ as shortcuts for $\exists r. C \sqcap \forall r. C$ and $\exists p_1, \dots, p_k. \phi \sqcap \forall p_1, \dots, p_k. \phi$, respectively.

A (deterministic) *two-register machine* (2RM) is a pair (Q, P) with *states* $Q = \{q_0, \dots, q_\ell\}$ and *instructions* $P = \{I_0, \dots, I_{\ell-1}\}$. By definition, q_0 is the initial state and q_ℓ the halting state. In state q_i ($i < \ell$) the instruction I_i must be applied. Instructions come in two varieties. An *incrementation instruction* is of the form $I = +(r, q)$ where $r \in \{1, 2\}$ is the register number and q is a state. This instruction increments (the content of) register r and then goes to state q . A *decrementation instruction* is of the form $I = -(r, q, q')$ where $r \in \{1, 2\}$ and q, q' are states. This instruction decrements register r and goes to state q if the content of register r is not zero; otherwise, it leaves register r as it is and goes to state q' . It is well-known that the problem of deciding whether a given 2RM halts on input $(0, 0)$ is undecidable [72].

Proposition 1. *If \mathcal{D} is of the form $(D; +_1)$ for $D \in \{\mathbb{Q}, \mathbb{Z}, \mathbb{N}\}$, then concept satisfiability in $\mathcal{ALC}(\mathcal{D})$ w.r.t. TBoxes is undecidable.*

Proof. Let (Q, P) be an arbitrary 2RM. We define a concept C and a TBox \mathcal{T} in such a way that every model of \mathcal{T} in which C is non-empty represents the computation of (Q, P) on the input $(0, 0)$. For every state q_i we introduce a concept name Q_i . We also introduce two concept names Z_1, Z_2 to indicate a positive zero test for the first and second register, respectively. In addition, we introduce a role name $g \in \mathbf{N}_R$ representing the transitions between configurations of the 2RM. For $p \in \{1, 2\}$, we have features $r_p \in \mathbf{N}_F$ representing the content of register p . However, since our concrete domain does not have the predicate $=_0$, we cannot enforce that, in our representation of the initial configuration, r_1 and r_2 have value zero. What we can ensure, though, is that their value is the same number, which we can store in a concrete feature $z \in \mathbf{N}_F$. The idea is now that register p of the machine actually contain the value of r_p offset with the

3. Description Logics with Concrete Domains

value of z . We also need auxiliary concrete features $s_1, s_2, s \in \mathbb{N}_F$, which respectively refer to the successor values of r_1, r_2, z . They are needed to express equality using $+_1$.

The following GCI ensures that the elements of C represent the initial configuration together with appropriate values for the auxiliary features:

$$C \sqsubseteq Q_0 \sqcap \exists z, s. +_1(x_1, x_2) \sqcap \exists z, s_1. +_1(x_1, x_2) \sqcap \exists r_1, s_1. +_1(x_1, x_2) \\ \sqcap \exists z, s_2. +_1(x_1, x_2) \sqcap \exists r_2, s_2. +_1(x_1, x_2).$$

Next, the GCI $\top \sqsubseteq \exists \&\forall gz, s. +_1(x_1, x_2) \sqcap \exists \&\forall gz, gs. +_1(x_1, x_2)$ guarantees that the value z of an individual carries over to its g -successor. We denote the second value in $\{1, 2\}$ beside p by \hat{p} , i.e., $\hat{p} = 3 - p$. To enforce that the incrementation instructions are executed correctly, for every instruction $I_i = +(p, q_j)$, we include in \mathcal{T} the GCI

$$Q_i \sqsubseteq \exists \&\forall g. Q_j \sqcap \exists \&\forall r_p, gr_p. +_1(x_1, x_2) \sqcap \exists \&\forall gr_{\hat{p}}, s_{\hat{p}}. +_1(x_1, x_2) \\ \sqcap \exists \&\forall s_p, gs_p. +_1(x_1, x_2) \sqcap \exists \&\forall r_{\hat{p}}, gs_{\hat{p}}. +_1(x_1, x_2)$$

The GCIs $Z_p \sqsubseteq \exists z, s_p. +_1(x_1, x_2)$ and $\exists z, s_p. +_1(x_1, x_2) \sqsubseteq Z_p$ ensure that Z_p represents a positive zero test for register p . Note that, for individuals for which values for s, z, s_p, r_p are defined, the negation of Z_p is semantically equivalent to a negative zero test for register p . To enforce that decrementation is executed correctly, for every instruction $I_i = -(p, q_j, q_k)$, we include in \mathcal{T} the GCIs

$$Q_i \sqcap Z_p \sqsubseteq \exists \&\forall g. Q_k \sqcap \exists \&\forall gr_p, s_p. +_1(x_1, x_2) \sqcap \exists \&\forall gr_{\hat{p}}, s_{\hat{p}}. +_1(x_1, x_2) \\ \sqcap \exists \&\forall r_p, gs_p. +_1(x_1, x_2) \sqcap \exists \&\forall r_{\hat{p}}, gs_{\hat{p}}. +_1(x_1, x_2) \\ Q_i \sqcap \neg Z_p \sqsubseteq \exists \&\forall g. Q_j \sqcap \exists \&\forall gr_p, r_p. +_1(x_1, x_2) \sqcap \exists \&\forall gr_{\hat{p}}, s_{\hat{p}}. +_1(x_1, x_2) \\ \sqcap \exists \&\forall gs_p, s_p. +_1(x_1, x_2) \sqcap \exists \&\forall r_{\hat{p}}, gs_{\hat{p}}. +_1(x_1, x_2)$$

Finally, we include the GCI $Q_\ell \sqsubseteq \perp$, which states that the halting state is never reached. It is now easy to see that the computation of (Q, P) on $(0, 0)$ does not reach the halting state if and only if C is satisfiable w.r.t. \mathcal{T} . \square

Note that this undecidability result also holds without our assumption that equality is always available.

It turns out that undecidability also holds if we use the ternary predicate $+$ rather than the binary predicate $+_1$. Intuitively, with $+$ we can easily test for 0 since m is 0 if and only if $m + m = m$. Instead of incrementation by 1, we can then use addition of a fixed non-zero number.

Proposition 2. *If \mathcal{D} is of the form $(D; +)$ for $D \in \{\mathbb{Q}, \mathbb{Z}, \mathbb{N}\}$, then concept satisfiability in $\mathcal{ALC}(\mathcal{D})$ w.r.t. $TBoxes$ is undecidable.*

Proof. Similarly as in the proof of Proposition 1, we reduce the halting problem of two-register

machines to concept satisfiability in $\mathcal{ALC}(\mathfrak{D})$. This time we closely follow the proof of Theorem 5.25 in [9]. For this reason, we only provide the GCIs that encode the run of an arbitrary 2RM on the input $(0, 0)$, the rest is obvious. As before, $g \in \mathbb{N}_R$ represents the transition function, and $r_1, r_2 \in \mathbb{N}_F$ represent the contents of the two registers initialized with the value 0. Additionally, $z \in \mathbb{N}_F$ is an auxiliary feature that assumes the value 0, and $u \in \mathbb{N}_F$ is an auxiliary feature that assumes the value of some non-zero number. The initial configuration is represented by the following GCI which, in particular, prevents u from assuming the value 0:

$$\begin{aligned} C \sqsubseteq Q_0 \sqcap \exists r_1, r_1. r_1. + (x_1, x_2, x_3) \sqcap \exists r_2, r_2. r_2. + (x_1, x_2, x_3) \\ \sqcap \exists z, z. z. + (x_1, x_2, x_3) \sqcap \neg(\exists u, u. u. + (x_1, x_2, x_3)) \end{aligned}$$

The GCI $T \sqsubseteq \exists z, z. z. + (x_1, x_2, x_3) \sqcap \exists \forall u, z. g u. + (x_1, x_2, x_3)$ ensures that z has the value 0 everywhere, and it simultaneously transfers the value of u to g -successors. Consequently, u has a fixed non-zero value on the g -paths starting with our initial element of C .

The incrementation instruction $I_i = +(p, q_j)$ is represented by the GCI

$$Q_i \sqsubseteq \exists \forall g. Q_j \sqcap \exists \forall r_p, u, g r_p. + (x_1, x_2, x_3) \sqcap \exists \forall r_{\hat{p}}, z, g r_{\hat{p}}. + (x_1, x_2, x_3),$$

and the decrementation instruction $I_i = -(p, q_j, q_k)$ is represented by the GCIs

$$Q_i \sqcap Z_p \sqsubseteq \exists \forall g. Q_k \sqcap \exists \forall r_p, z, g r_p. + (x_1, x_2, x_3) \sqcap \exists \forall r_{\hat{p}}, z, g r_{\hat{p}}. + (x_1, x_2, x_3),$$

$$Q_i \sqcap \neg Z_p \sqsubseteq \exists \forall g. Q_j \sqcap \exists \forall g r_p, u, r_p. + (x_1, x_2, x_3) \sqcap \exists \forall r_{\hat{p}}, z, g r_{\hat{p}}. + (x_1, x_2, x_3),$$

where the GCIs $Z_p \sqsubseteq Q_i \sqcap \exists r_p, z. z. + (x_1, x_2, x_3)$ and $Q_i \sqcap \exists r_p, z. z. + (x_1, x_2, x_3) \sqsubseteq Z_p$ ensure that Z_p represents a positive zero test for register p .

The non-termination is, again, represented by the GCI $Q_\ell \sqsubseteq \perp$. □

Even for \mathcal{EL} , integrating a decidable concrete domain may cause undecidability if we allow for paths of length 2. Proving this is, however, more challenging, not only due to the fact that not all Boolean operations are available, but also since the absence of functional roles cannot be compensated by the use of universal quantification. To illustrate the latter point, assume we have a concrete domain with binary predicates P and P' that are disjoint. If g is assumed to be a functional role, then the concept $\exists f, g f. P(x_1, x_2) \sqcap \exists f, g f. P'(x_1, x_2)$ is unsatisfiable, but if g is just an arbitrary role, then it is satisfiable since a given individual belonging to the concept may have two different g -successors, one satisfying the P -constraint and the other satisfying the P' -constraint. However, conjoining this concept with the corresponding universal concrete domain restrictions $\forall f, g f. P(x_1, x_2) \sqcap \forall f, g f. P'(x_1, x_2)$ yields an unsatisfiable concept again.

To show undecidability for a concrete domain extension of \mathcal{EL} without functional roles, we consider the relational structure $\mathfrak{D}_{2\text{-aff}}$ over \mathbb{Q}^2 , which has, for every affine transformation $\mathbb{Q}^2 \rightarrow \mathbb{Q}^2 : \bar{x} \mapsto A\bar{x} + \bar{b}$, the binary relation $R_{A, \bar{b}} := \{(\bar{x}, \bar{y}) \in (\mathbb{Q}^2)^2 \mid \bar{y} = A\bar{x} + \bar{b}\}$ as a predicate. We will show in Corollary 9 that the CSP for this structure is decidable in polynomial time.

3. Description Logics with Concrete Domains

Undecidability of subsumption w.r.t. TBoxes in $\mathcal{EL}(\mathcal{D}_{2\text{-aff}})$ can be shown by a reduction from *2-Dimensional Affine Reachability*, which is undecidable by Corollary 4 in [15]. For this problem, one is given vectors $\bar{v}, \bar{w} \in \mathbb{Q}^2$ and a finite set S of affine transformations from \mathbb{Q}^2 to \mathbb{Q}^2 . The question is then whether \bar{w} can be obtained from \bar{v} by repeated application of transformations from S .

Proposition 3. *Subsumption w.r.t. TBoxes is undecidable in $\mathcal{EL}(\mathcal{D}_{2\text{-aff}})$.*

Proof. We define the reduction of 2-dimensional Affine Reachability to subsumption w.r.t. general TBoxes in $\mathcal{EL}(\mathcal{D}_{2\text{-aff}})$ as follows. For given vectors $\bar{v}, \bar{w} \in \mathbb{Q}^2$ and affine transformations $S = \{\bar{x} \mapsto M_1\bar{x} + \bar{v}_1, \dots, \bar{x} \mapsto M_k\bar{x} + \bar{v}_k\}$, the TBox \mathcal{T} contains, for every $i \in [k]$, the GCI $\top \sqsubseteq \exists f, g f.R_{M_i, \bar{v}_i}(x_1, x_2)$. Additionally, \mathcal{T} contains the GCIs $\exists g. L \sqsubseteq L$ and $\exists f, f.R_{Z, \bar{w}}(x_1, x_2) \sqsubseteq L$, where L is a fresh concept name and Z is the 2×2 zero matrix. Note that $(\bar{x}, \bar{x}) \in R_{Z, \bar{w}}$ if and only if $\bar{x} = \bar{w}$. Each involved concept is either \top , a concept name, or an existential (concrete domain) restriction, and thus definable in $\mathcal{EL}(\mathcal{D}_{2\text{-aff}})$. We claim that $\exists f, f.R_{Z, \bar{v}}(x_1, x_2)$ is subsumed by L w.r.t. \mathcal{T} if and only if \bar{w} can be obtained from \bar{v} through an application of a composition of affine transformations from S .

“ \Leftarrow ”: Suppose that there exists such a composition and let \mathcal{I} be a model of \mathcal{T} . Let a be an arbitrary element of $(\exists f, f.R_{Z, \bar{v}}(x_1, x_2))^{\mathcal{I}}$, i.e., satisfying $f^{\mathcal{I}}(a) = \bar{v}$. Since \mathcal{T} contains $\top \sqsubseteq \exists f, g f.R_{M_i, \bar{v}_i}(x_1, x_2)$ for every $i \in [k]$ and \bar{w} is reachable from \bar{v} through an application of a composition of affine transformations from S , there exists a role path $a \rightarrow_{g^{\mathcal{I}}} \dots \rightarrow_{g^{\mathcal{I}}} b$ to some element b with $f^{\mathcal{I}}(b) = \bar{w}$. Since \mathcal{T} contains the GCI $\exists f, f.R_{Z, \bar{w}}(x_1, x_2) \sqsubseteq L$, we have $b \in L^{\mathcal{I}}$. The GCI $\exists g. L \sqsubseteq L$ then yields $a \in L^{\mathcal{I}}$.

“ \Rightarrow ”: Suppose that $\exists f, f.R_{Z, \bar{v}}(x_1, x_2)$ is subsumed by L w.r.t. \mathcal{T} . Consider the following interpretation \mathcal{I} . The domain of \mathcal{I} is \mathbb{Q}^2 . We define $f^{\mathcal{I}}$ as the identity map on \mathbb{Q}^2 and set $g^{\mathcal{I}} := \{(\bar{x}, \bar{y}) \in (\mathbb{Q}^2)^2 \mid \exists i \in [k] \text{ such that } \bar{y} = M_i\bar{x} + \bar{v}_i\}$. Finally, we set $L^{\mathcal{I}} := \{\bar{w}\} \cup \{\bar{x} \in \mathbb{Q}^2 \mid \text{there exists a role path } \bar{x} \rightarrow_{g^{\mathcal{I}}} \dots \rightarrow_{g^{\mathcal{I}}} \bar{w}\}$. It is easy to check that \mathcal{I} is a model of \mathcal{T} . Since $\bar{v} \in (\exists f, f.R_{Z, \bar{v}}(x_1, x_2))^{\mathcal{I}}$ and $\exists f, f.R_{Z, \bar{v}}(x_1, x_2)$ is subsumed by L w.r.t. \mathcal{T} , we have $\bar{v} \in L^{\mathcal{I}}$. The definition of $L^{\mathcal{I}}$ thus implies that \bar{w} is reachable from \bar{v} through an application of a composition of affine transformations from S . \square

Note that the signature of $\mathcal{D}_{2\text{-aff}}$ is infinite since there are infinitely many affine transformations on \mathbb{Q}^2 . One might think that this is important for our undecidability proof. We can show, however, that this is not the case: a fixed finite set of affine transformations is sufficient.

Corollary 1. *There exists a finite-signature reduct \mathcal{D} of $\mathcal{D}_{2\text{-aff}}$ such that subsumption w.r.t. TBoxes is undecidable in $\mathcal{EL}(\mathcal{D})$.*

Proof. In the proof of Proposition 3, we use concepts of the form $\exists f, g f.R_{M, \bar{v}}(x_1, x_2)$ where $\bar{x} \mapsto M\bar{x} + \bar{v}$ is an arbitrary affine transformation from \mathbb{Q}^2 to \mathbb{Q}^2 . We show that every such concept can be expressed as a conjunction of concepts built using only those affine transformations $\bar{x} \mapsto M\bar{x} + \bar{v}$ where $M \in \{-1, 0, 1\}^{2 \times 2}$ and $\bar{v} \in \{-1, 0, 1\}^2$. This gives us a conservative extension of the concept and the TBox used in the proof of Proposition 3. Thus the statement then follows from Proposition 3.

Note that we can clearly express $\exists f, gf.R_{M,\bar{v}}(x_1, x_2)$ as

$$\exists f, f'.R_{M,\bar{0}}(x_1, x_2) \cap \exists f', f''.R_{E_2,\bar{v}}(x_1, x_2) \cap \exists f'', gf.(x_1 = x_2)$$

where E_2 is the 2×2 unit matrix and f', f'' are fresh features. Thus, we only really need to express concepts of the form $\exists f, g.R_{E_2,\bar{v}}(x_1, x_2)$ and $\exists f, g.R_{M,\bar{0}}(x_1, x_2)$ where M, \bar{v} are elements of our selected finite set of matrices and vectors.

Consider an arbitrary matrix $M = (m_{i,j})_{i,j \in [2]}$ where, w.l.o.g., $m_{i,j} = p_{i,j}/q_{i,j}$ for an integer $p_{i,j}$ and a positive integer $q_{i,j}$. Then $(\bar{x}, \bar{y}) \in R_{M,\bar{0}}$ if and only if

$$q_{i,2}p_{i,1}\bar{x}[1] + q_{i,1}p_{i,2}\bar{x}[2] = q_{i,1}q_{i,2}\bar{y}[i] \text{ for } i \in \{1, 2\}. \quad (3.1)$$

We claim that, for every $n \in \mathbb{Z}$, there exists a concept constructed using our selected set of matrices and vectors that expresses the concept $\exists f, g.R_{A_{n,0},\bar{0}}(x_1, x_2)$ where the affine transformation $\bar{x} \mapsto A_{n,0}\bar{x}$ multiplies the first component by n and the second component by 0, i.e.,

$$A_{n,0} = \begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}.$$

W.l.o.g., $n > 1$, the case $n < 0$ is similar and the case $n \in \{0, 1\}$ is trivial. For every $i \in [n]$, we introduce a fresh feature f_i . Then $\exists f, g.R_{A_{n,0},\bar{0}}(x_1, x_2)$ can be expressed by

$$\begin{aligned} C_{n,0}^{f,g} := & \exists f, f_1.R_{A_{1,0},\bar{0}}(x_1, x_2) \cap \exists f_1, f_2.R_{A_{2,0},\bar{0}}(x_1, x_2) \cap \\ & \cdots \cap \exists f_{n-1}, f_n.R_{A_{n,0},\bar{0}}(x_1, x_2) \cap \exists f_n, g.R_{A_{n+1,0},\bar{0}}(x_1, x_2) \end{aligned}$$

where

$$A_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_i = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ for } i \in \{2, \dots, n\}, \quad A_{n+1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

To see this, note that $A_{n+1} \cdots A_1 = A_{n,0}$. We assume that the features f_1, \dots, f_n are unique for $C_{n,0}^{f,g}$, i.e., they do not appear in any other concept description.

Analogously, there exists a concept $C_{0,n}^{f,g}$ constructed using our selected set of matrices and vectors which expresses $\exists f, g.R_{A_{0,n},\bar{0}}(x_1, x_2)$ where the affine transformation $\bar{x} \mapsto A_{0,n}\bar{x}$ multiplies the first component by 0 and the second component by n . Again, we assume that each feature in $C_{0,n}^{f,g}$ beside f and g does not appear in any other concept description.

Now, guided by (3.1), we can express the original concept $\exists f, g.R_{M,\bar{0}}(x_1, x_2)$ by

$$\begin{aligned} & C_{q_{1,2}p_{1,1},0}^{f,f_{1,1}} \cap C_{0,q_{1,1}p_{1,2}}^{f,f_{1,2}} \cap C_{q_{1,1}q_{1,2},0}^{g,g_1} \cap C_{1,0}^{h_1,f_{1,1}} \cap C_{0,1}^{h_1,f_{1,2}} \cap \exists h_1, g_1.R_{B_1,\bar{0}}(x_1, x_2) \\ & \cap C_{q_{2,2}p_{2,1},0}^{f,f_{2,1}} \cap C_{0,q_{2,1}p_{2,2}}^{f,f_{2,2}} \cap C_{0,q_{2,1}q_{2,2}}^{g,g_2} \cap C_{1,0}^{h_2,f_{2,1}} \cap C_{0,1}^{h_2,f_{2,2}} \cap \exists h_2, g_2.R_{B_2,\bar{0}}(x_1, x_2) \end{aligned}$$

where $f_{1,1}, f_{1,2}, f_{2,1}, f_{2,2}, g_1, g_2, h_1, h_2$ are fresh features and

$$B_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

3. Description Logics with Concrete Domains

Now consider an arbitrary tuple \bar{v} where, w.l.o.g., $\bar{v}[i] = p_i/q_i$ for an integer p_i and a positive integer q_i . We have that $(\bar{x}, \bar{y}) \in R_{E_2, \bar{v}}$ if and only if

$$q_i \bar{x}[i] + p_i = q_i \bar{y}[i] \text{ for } i \in \{1, 2\}. \quad (3.2)$$

For every $n \in \mathbb{Z}$, we can show similarly as above that there exist concepts $D_{n,0}^{f,f}$ and $D_{0,n}^{f,f}$ constructed using our selected finite set of matrices and vectors which express the concepts $\exists f, f. R_{N,(n,0)}(x_1, x_2)$ and $\exists f, f. R_{N,(0,n)}(x_1, x_2)$, respectively, where N is the 2×2 matrix of zeros. One can then express the concept $\exists f, g. R_{E_2, \bar{v}}(x_1, x_2)$ using the concepts $C_{n,0}^{f,g}, C_{0,n}^{f,g}, D_{n,0}^{f,f}$, and $D_{0,n}^{f,f}$ while being guided by (3.2). \square

3.2 Decidable and tractable DLs with concrete domains

There are two strategies for regaining decidability of DLs with concrete domains in the presence of GCIs: syntactically restricting the interaction of the DL with the concrete domain or limiting the expressiveness of the concrete domain itself. Typically, the former is realized by restricting the length of paths in concrete domain restrictions to 1. We indicate this restriction by writing square brackets around the concrete domain instead of round brackets.

Definition 2. The restriction of $\mathcal{EL}(\mathfrak{D})$ and $\mathcal{ALC}(\mathfrak{D})$ to paths of length 1 in concrete domain restrictions is respectively denoted by $\mathcal{EL}[\mathfrak{D}]$ and $\mathcal{ALC}[\mathfrak{D}]$.

For \mathcal{ALC} , this restriction results in decidability [49, 76] for concrete domains that are *admissible* in the sense introduced in [6], i.e., whose predicates are closed under negation and whose CSP is decidable. In the case of \mathcal{EL} , the expectations are a bit higher: the aim there is to regain tractability. To obtain tractability of $\mathcal{EL}[\mathfrak{D}]$, the notion of *p-admissible* concrete domains was introduced in [3], and it was shown that subsumption in $\mathcal{EL}[\mathfrak{D}]$ is decidable in polynomial time if and only if \mathfrak{D} is *p-admissible*. Before defining this condition below, we introduce a condition, called ω -admissibility, which ensures decidability of $\mathcal{ALC}(\mathfrak{D})$ in the presence of GCIs and paths of length > 1 .

ω -Admissible concrete domains

The notion of ω -admissibility was introduced in [70] to regain decidability of \mathcal{ALC} with concrete domains in the presence of GCIs. Motivated by binary constraint calculi like Allen's interval algebra [2] and the region connection calculus [77], only concrete domains where all predicates are binary were considered in [70]. In [10], the notion and the corresponding decidability result were generalized to concrete domains with predicates of arbitrary arity.

We say that a relational τ -structure \mathfrak{D} has *homomorphism ω -compactness* if the following holds for every countable τ -structure \mathfrak{B} : $\mathfrak{B} \rightarrow \mathfrak{D}$ if and only if $\mathfrak{A} \rightarrow \mathfrak{D}$ for every $\mathfrak{A} \in \text{Age}(\mathfrak{B})$. This condition is needed in the correctness proof for the tableau algorithm in [70]. Intuitively, the algorithm computes, from an input concept and TBox, a finite premodel, that can then

be turned into an actual model using a certain abstract unraveling procedure. However, in the setting of DLs with concrete domains, it is often the case that all models witnessing the satisfiability of a concept w.r.t. a TBox are infinite, see Example 1. Thus, the verification of the unraveled premodel might involve a test of an infinite structure for a homomorphism to the concrete domain; homomorphism ω -compactness reduces this difficult task to proving that every finite unraveling step yields a structure that has a homomorphism to the concrete domain. In [70], the inputs to this condition were not formally restricted to countable structures. However, it is clear that this is what the authors meant because (i) the structures produced by the original tableau algorithm that need to be tested for a homomorphism to the concrete domain are always countable, and (ii) the examples of ω -admissible concrete domains presented in [70] are not homomorphism compact for input structures with arbitrarily large cardinalities.

A relational τ -structure \mathfrak{D} satisfies:

- JE if, for every $k \geq 1$, either \mathfrak{D} has no k -ary relation, or $\bigcup\{R^{\mathfrak{D}} \mid R \in \tau, R^{\mathfrak{D}} \subseteq D^k\} = D^k$;
- PD if, for every pair R, \tilde{R} of distinct symbols from τ , we have $R^{\mathfrak{D}} \cap \tilde{R}^{\mathfrak{D}} = \emptyset$;
- JD if the equality Eq_D has a (quantifier and equality)-free definition in \mathfrak{D} .

Here JE stands for “jointly exhaustive,” PD for “pairwise disjoint,” and JD for “jointly diagonal.” In [10], JD was defined in a more restricted way as $\bigcup\{R^{\mathfrak{D}} \mid R \in \tau, R^{\mathfrak{D}} \subseteq \text{Eq}_D\} = \text{Eq}_D$, which explains the name. The combination JEPD is needed in [70] for obtaining a normal form for a given input concept and TBox that eliminates negations in front of concrete domain restrictions. The condition JD was not considered in [70]. We include it here mainly because it makes the comparison with known notions from model theory easier. In the setting considered in this thesis, where concrete domain restrictions always have access to equality, JD is actually needed to ensure decidability. If the equality predicate is dropped from concrete domain restrictions, then the decidability results in [70, 11] do not depend on JD. However, all examples of ω -admissible concrete domains presented in [70] satisfy JD since equality is contained in the signature. In [41], k -ary structures, i.e., structures \mathfrak{D} that have only k -ary predicates, are considered that have the k -ary equality relation $k\text{-Eq}_D = \{(d, \dots, d) \in D^k \mid d \in D\}$. For $k \geq 2$, such a structure satisfies JD in the sense introduced above, since binary equality $x = y$ can be defined as $k\text{-Eq}_D(x, y, \dots, y)$.

A relational τ -structure \mathfrak{D} is a *patchwork* (or has the *patchwork property*) if it is JDJEPD, and for all finite JEPD τ -structures $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ with $e_1: \mathfrak{A} \hookrightarrow \mathfrak{B}_1$, $e_2: \mathfrak{A} \hookrightarrow \mathfrak{B}_2$, $\mathfrak{B}_1 \rightarrow \mathfrak{D}$, and $\mathfrak{B}_2 \rightarrow \mathfrak{D}$, there exist $f_1: \mathfrak{B}_1 \rightarrow \mathfrak{D}$ and $f_2: \mathfrak{B}_2 \rightarrow \mathfrak{D}$ with $f_1 \circ e_1 = f_2 \circ e_2$. In plain words, this condition ensures that two overlapping satisfiable instances of $\text{CSP}(\mathfrak{D})$ with jointly exhaustive and pairwise disjoint relations are always also satisfiable as a whole. It is needed in [70] to enable the unraveling procedure mentioned above.

Definition 3. The relational structure \mathfrak{D} is *ω -admissible* if it has a finite signature, $\text{CSP}(\mathfrak{D})$ is decidable, \mathfrak{D} has homomorphism ω -compactness, and \mathfrak{D} is a patchwork.

By AD we denote the set of all finite disjunctions of atomic τ -formulas. The following theorem is shown in [10, 11] by extending the tableau-based decision procedure given in [70] to our

3. Description Logics with Concrete Domains

more general definition of ω -admissibility.

Theorem 1. *Let \mathcal{D} be an ω -admissible τ -structure with at most d -ary relations for some $d \geq 2$. Then concept satisfiability in $\mathcal{ALC}_{AD}^d(\mathcal{D})$ w.r.t. TBoxes is decidable.*

The main motivation for the definition of ω -admissible concrete domains in [70] was that they can capture qualitative calculi of time and space. In particular, it was shown in [70] that Allen’s interval algebra [2] as well as the region connection calculus RCC8 [77] can be represented as ω -admissible concrete domains. To the best of our knowledge, no other ω -admissible concrete domains have been exhibited in the literature before our investigations in [10], which we will describe in detail in the next section. Among other things, we prove that the structure $(\mathbb{Q}; <, =, >)$ is ω -admissible. The “discrete” version $(\mathbb{Z}; <, =, >)$, on the other hand, is not ω -admissible because it lacks homomorphism ω -compactness (see Example 1 below). By the results in [65], $(\mathbb{Z}; <, =, >)$ nevertheless yields a decidable concrete domain extension of \mathcal{ALC} , but proving this requires a more specialized approach than the tableau algorithm provided by the original paper of Lutz and Milićić [70]. This shows that ω -admissibility is not necessary for decidable reasoning in \mathcal{ALC} with concrete domains in the presence of GCIs.

Example 1. The concept $A \in \mathcal{N}_C$ is satisfiable w.r.t. the TBox

$$\mathcal{T} := \{A \sqsubseteq (\exists f, g. <(x_1, x_2)) \sqcap (\exists r. A) \sqcap (\forall f, r f. <(x_1, x_2)) \sqcap (\forall r g, g. <(x_1, x_2))\}$$

in $\mathcal{ALC}(\mathbb{Q}; <, =, >)$, and this can be tested using the tableau algorithm from [70] because $(\mathbb{Q}; <, =, >)$ is ω -admissible by Theorem 6. However, A is not satisfiable w.r.t. \mathcal{T} in $\mathcal{ALC}(\mathbb{Z}; <, =, >)$ because its satisfiability would imply the existence of a homomorphism to $(\mathbb{Z}; <, =, >)$ from a structure \mathfrak{B} with domain $B = \{f_n, g_n \mid n \in \mathbb{N}\}$ and relations given by $f_n <^{\mathfrak{B}} f_{n+1} <^{\mathfrak{B}} g_{n+1} <^{\mathfrak{B}} g_n$ for every $n \in \mathbb{N}$. Such a homomorphism cannot exist because the ordering of the integers is not dense. Note that $\mathfrak{A} \rightarrow (\mathbb{Z}; <, =, >)$ for every $\mathfrak{A} \in \text{Age}(\mathfrak{B})$, which shows that $(\mathbb{Z}; <, =, >)$ is not ω -homomorphism compact.

p-Admissible concrete domains

The notion of p -admissibility was introduced in [3] to capture precisely those structures \mathcal{D} for which subsumption in $\mathcal{EL}[\mathcal{D}]$ is tractable. Clearly, this requires the CSP of \mathcal{D} to be decidable in polynomial time. However, this is not sufficient since even for a concrete domain \mathcal{D} with tractable CSP, disjunction may be expressible in $\mathcal{EL}[\mathcal{D}]$, which then leads to intractability [3]. To avoid this source of intractability, the concrete domain must be convex. Unfortunately, the definition of convexity given in [3] was ambiguous, and what is really needed in the setting considered in [3] is what we call guarded convexity in [12]. In the setting considered in this thesis, where equality is always available in concrete domain restrictions, we will see that convexity rather than guarded convexity is the adequate notion.

We say that a τ -structure \mathcal{D} is *convex* if the following holds: whenever a conjunction of atomic τ -formulas implies a disjunction of atomic τ -formulas in \mathcal{D} , then it already implies one

of the disjuncts. Note that this definition does not say anything about which variables may occur in the left- and right-hand sides of such implications. *Guarded convexity* requires this condition to hold only for *guarded implications*, where all variables occurring on the right-hand side must also occur on the left-hand side.

To illustrate the difference between convexity and guarded convexity, let us consider the structure $\mathfrak{N} = (\mathbb{N}; E, O)$ in which the unary predicates E and O are respectively interpreted as the even and odd natural numbers. This structure is not convex since $\forall x, y (E(x) \Rightarrow E(y) \vee O(y))$ holds in \mathfrak{N} , but neither $\forall x, y (E(x) \Rightarrow E(y))$ nor $\forall x, y (E(x) \Rightarrow O(y))$ does. However, the first implication is not guarded, and it is easy to see that \mathfrak{N} is in fact guarded convex. Note that, whereas $\forall x, y (E(x) \Rightarrow E(y) \vee O(y))$ holds in \mathfrak{N} , the subsumption $\exists f. E(x_1) \sqsubseteq_{\emptyset} \exists g. E(x_1) \sqcup \exists g. O(x_1)$ does not hold in the extension of $\mathcal{EL}[\mathfrak{N}]$ with disjunction since the feature g need not have a value. However, as we have pointed out above, $\exists g, g. (x_1 = x_2)$ expresses that the value of g must be defined. Thus, $\exists g, g. (x_1 = x_2) \sqsubseteq_{\emptyset} \exists g. E(x_1) \sqcup \exists g. O(x_1)$ is a valid subsumption in $\mathcal{EL}[\mathfrak{N}]$ (although disjunctions are formally not allowed in $\mathcal{EL}[\mathfrak{N}]$). This can be used to show that this DL is not tractable [3], but only under the assumption that equality can be used in concrete domain restrictions. Consequently, in the setting of this thesis, convexity should be used in the definition of p-admissibility.

Definition 4. A relational structure \mathfrak{D} is *p-admissible* if it is convex and validity of Horn implications in \mathfrak{D} is decidable in polynomial time.

The main result of [3] concerning concrete domains can now be stated as follows.

Theorem 2 (Baader, Brandt, and Lutz [3]). *Let \mathfrak{D} be a relational structure. Then subsumption in $\mathcal{EL}[\mathfrak{D}]$ is decidable in polynomial time if and only if \mathfrak{D} is p-admissible.*

Note that the theorem above does not hold for the more expressive logic $\mathcal{EL}(\mathfrak{D})$ where paths of length 2 are allowed in concrete domain constructors. This is because we can show that the concrete domain $\mathfrak{D}_{2\text{-aff}}$ from Proposition 3 is p-admissible (see Corollary 9).

In Section 5, we provide an algebraic characterization of convexity. Regarding the tractability condition in the definition of p-admissibility, we have seen in Section 2 that it is closely related to the constraint satisfaction problem for \mathfrak{D} and $(\mathfrak{D}^{\neg}, \neq)$. In fact, a convex structure \mathfrak{D} is p-admissible if and only if $\text{CSP}(\mathfrak{D}^{\neg}, \neq)$ is decidable in polynomial time. Characterizing tractability of the CSP for a given structure is a very hard problem. Whereas the Feder-Vardi conjecture [47] has recently been confirmed after 25 years of intensive research in the field by giving an algebraic criterion that can distinguish between *finite* structures with tractable and with NP-complete CSP [79, 34], finding comprehensive criteria that ensure tractability for the case of infinite structures is a wide open problem, though first results for special cases have been found (see, e.g., [24, 26, 28, 73, 63, 23, 27]).

ω -Admissibility versus p-admissibility

From an application point of view it would be desirable to have concrete domains \mathfrak{D} that preserve tractability if used in $\mathcal{EL}[\mathfrak{D}]$ and decidability if used in $\mathcal{ALC}(\mathfrak{D})$. This would be the

3. Description Logics with Concrete Domains

case for concrete domains that are both ω -admissible and p-admissible. Unfortunately, for structures over a finite signature, JEPD (required for ω -admissibility) and convexity (required for p-admissibility) do not go well together. More specifically, when combined, these properties trivialize the CSP.

Proposition 4. *Let τ be a finite signature and \mathfrak{D} a relational τ -structure that is both JEPD and convex. Then $R^{\mathfrak{D}} \in \{\emptyset, D^k\}$ for all k -ary relation symbols $R \in \tau$.*

Proof. Assume that $R \in \tau$ is a relation symbol of arity k such that $R^{\mathfrak{D}} \neq \emptyset$. Then JE yields $D^k = \bigcup_{i=1}^m R_i^{\mathfrak{D}}$, where R_1, \dots, R_m are all the relation symbols of arity k in τ . Consequently, the implication $\forall x_1, \dots, x_k (\bigwedge_{i \in [k]} x_i = x_i \Rightarrow \bigvee_{i \in [m]} R_i(x_1, \dots, x_k))$ holds in \mathfrak{D} , and thus convexity implies that there is an i , $1 \leq i \leq m$, such that $\forall x_1, \dots, x_k (\bigwedge_{i \in [k]} x_i = x_i \Rightarrow R_i(x_1, \dots, x_k))$ holds in \mathfrak{D} . This means that $R_i^{\mathfrak{D}} = D^k$. Since we have assumed that $R^{\mathfrak{D}} \neq \emptyset$ and R is of arity k , PD yields that $R = R_i$, and thus we are done. \square

If τ contains a symbol R that is interpreted as equality Eq_D on D , then this proposition implies that $\text{Eq}_D = R^{\mathfrak{D}} = D \times D$, which can only be the case if $|D| \leq 1$. The proof of Proposition 4 makes use of our assumption that equality is always available when building formulas. But even without that assumption, concrete domains \mathfrak{D} that are both p- and ω -admissible would have a rather restricted form. In that case, there always exists a finite partition $\{V_1, \dots, V_m\}$ of D such that the only non-empty k -ary relations of \mathfrak{D} are of the form $V_{i_1} \times \dots \times V_{i_k}$ for $i_1, \dots, i_k \in [m]$. For more details, see the appendix.

One apparent downside of having the equality available as an atomic formula is that, for finite structures, it restricts p-admissibility to those whose domain size is at most 1. To see this, suppose that \mathfrak{D} is finite. By the pigeonhole principle, the implication $\forall x_1, \dots, x_n (\bigwedge_{i \in [n]} x_i = x_i \Rightarrow \bigvee_{i \neq j \in [n]} x_i = x_j)$ holds in \mathfrak{D} for every $n > |D|$. If $|D| > 1$, then it is clearly not uniquely determined which elements among x_1, \dots, x_n must be equal, and therefore \mathfrak{D} is not convex. In the case of ω -admissibility, we essentially have the opposite situation. Every finite structure can be made ω -admissible by first expanding its signature by unary relations for each element of the domain and then decomposing its relations into orbits as described in Section 4.3 (see the last part of Example 4). However, from theoretical perspective, extensions of reasonably expressive DLs such as \mathcal{ALC} or \mathcal{EL} by finite concrete domains are not very interesting because all definable concepts can already be modeled within the original formalism.

Finally, let us point out another notable difference between the two conditions, namely that p-admissibility permits infinite signatures whereas ω -admissibility does not. It turns out that finiteness of the signature is a necessary part of ω -admissibility for attaining decidable reasoning with concrete domains. If we allowed the signature of \mathfrak{D} to be infinite, then we would have the following counterexample. Let \mathfrak{D} be the structure over \mathbb{Z} with the relations $+_k = \{(x, y) \in \mathbb{Z}^2 \mid y = x + k\}$ for every $k \in \mathbb{Z}$. It is easy to see that $\text{CSP}(\mathfrak{D})$ can be solved in polynomial time and that \mathfrak{D} has homomorphism ω -compactness. Moreover, one can show, using the results in Section 4 (Proposition 5 and Theorem 5), that \mathfrak{D} is a patchwork. However,

3.2. Decidable and tractable DLs with concrete domains

we have seen in Proposition 1 that concept satisfiability w.r.t. GCIs is undecidable already in $\mathcal{ALC}(\mathbb{Z}; +_1)$. For more details, see Example 3.

A Model-Theoretic Analysis of ω -Admissibility

We introduce several model-theoretic properties of relational structures and show their connection with ω -admissibility. This allows us to formulate sufficient conditions for ω -admissibility using well-known notions from model theory, and thus to use existing model-theoretic results to find new ω -admissible concrete domains. We start with the notion of ω -categoricity in a countable signature, which is sufficient to obtain homomorphism ω -compactness. Next, we consider homogeneous structures with a finite relational signature, which induce ω -categorical patchworks with a finite signature. This provides us with patchworks with a finite signature that also have homomorphism ω -compactness. What is still missing is decidability of the CSP. This can be achieved by restricting the attention to finitely bounded structures since their CSP is always in NP. Thus, finitely bounded homogeneous structures yield ω -admissible concrete domains. Alternatively, we consider homogeneous structures with a finite relational signature for which we can show by some other means that the CSP is decidable. In this setting, the induced patchwork has a decidable CSP if the structure is a so-called core. Conversely, we prove that every ω -admissible structure is equivalent to a particular homogeneous core in the sense that they both provide the same concrete domain extension of \mathcal{ALC} . The last part of this section investigates closure properties for homogeneity and finite boundedness.

4.1 Homomorphism ω -compactness via ω -categoricity

We start by introducing ω -categoricity since it gives us homomorphism ω -compactness “for free.” A structure is ω -categorical if its first-order theory has exactly one countable model up to isomorphism. For example, it is well-known that \mathcal{Q} is, up to isomorphism, the only countable dense linear order without lower or upper bound. This result, which clearly implies that \mathcal{Q} is ω -categorical, is due to Cantor.

For every structure \mathfrak{A} , its automorphisms form a permutation group with composition as binary operation, which we denote by $\text{Aut}(\mathfrak{A})$ (see Theorem 1.2.1 in [55]). Every relation R with a first-order definition in \mathfrak{A} is easily seen to be *preserved* by $\text{Aut}(\mathfrak{A})$, i.e., $\bar{t} \in R$ implies $h(\bar{t}) \in R$ for every $h \in \text{Aut}(\mathfrak{A})$. For ω -categorical structures, the other direction holds as well.

Theorem 3 (Engeler, Ryll-Nardzewski and Svenonius [55]). *For a countable structure \mathfrak{A} with a countable signature, the following are equivalent:*

1. \mathfrak{A} is ω -categorical.

4. A Model-Theoretic Analysis of ω -Admissibility

2. For every k , only finitely many k -ary relations are first-order definable in \mathfrak{A} .
3. Every relation over A preserved by $\text{Aut}(\mathfrak{A})$ is first-order definable in \mathfrak{A} .

The following corollary to Theorem 3 establishes the first important link between model theory and ω -admissibility.

Corollary 2 (Lemma 3.1.5 in [17]). *Every countable ω -categorical structure with a countable signature has homomorphism ω -compactness.*

In Example 2 below, we show using Theorem 3 that the converse direction in Corollary 2 is not true. In its current form, Theorem 3 is particularly handy for proving that a given structure is not ω -categorical, but it is not well-suited for proving ω -categoricity. We will fix this in Section 4.3 by including a fourth equivalent condition formulated using the notion of an orbit.

Example 2. The structure $(\mathbb{Z}; +_1)$ is homomorphism ω -compact. Given a countable $\{+_1\}$ -structure \mathfrak{B} , we have $\mathfrak{B} \rightarrow (\mathbb{Z}; +_1)$ if and only if there exist no two finite $+_1$ -paths with different lengths but identical endpoints in \mathfrak{B} . Every such pair of paths must already be contained in a finite substructure of \mathfrak{B} which does not have a homomorphism to $(\mathbb{Z}; +_1)$.

However, we can show using Theorem 3 that $(\mathbb{Z}; +_1)$ is not ω -categorical. One way is to find, for some k , infinitely many distinct first-order definable relations. And indeed, for $j \in \mathbb{Z}$, the binary relation $+_j = \{(x, y) \in \mathbb{Z}^2 \mid x + j = y\}$ is definable in $(\mathbb{Z}; +_1)$ by the formula $\exists z_1, \dots, z_j (x = z_1 \wedge y = z_j \wedge \bigwedge_{i \in [j-1]} +_1(x_i, x_{i+1}))$ if $j \geq 0$, and by the formula $\exists z_1, \dots, z_j (x = z_j \wedge y = z_1 \wedge \bigwedge_{i \in [j-1]} +_1(x_i, x_{i+1}))$ otherwise. Another way is to show that there exists a relation over \mathbb{Z} preserved by $\text{Aut}(\mathbb{Z}; +_1)$ which is not first-order definable in $(\mathbb{Z}; +_1)$. For every $S \subseteq \mathbb{Z}$, we define the binary relation $R_S := \{(x, y) \in \mathbb{Z}^2 \mid y - x \in S\}$. We claim that R_S is preserved by $\text{Aut}(\mathbb{Z}; +_1)$. It is easy to see that every automorphism of $(\mathbb{Z}; +_1)$ is a translation by an integer number. Thus, if $\bar{t} \in R_S$ and $h \in \text{Aut}(\mathbb{Z}; +_1)$ are chosen arbitrarily, then $h(\bar{t})[2] - h(\bar{t})[1] = \bar{t}[2] - \bar{t}[1]$, which implies that $\bar{t} \in R_S$ if and only if $h(\bar{t}) \in R_S$. This confirms the claim. Since there are uncountably many different subsets of \mathbb{Z} but only countably many different first-order formulas over the signature $\{+_1\}$, there exists $S \subseteq \mathbb{Z}$ such that R_S is not first-order definable in $(\mathbb{Z}; +_1)$.

4.2 Patchworks via homogeneity

We show that, in order to obtain patchworks with homomorphism ω -compactness, it is sufficient to consider homogeneous structures with a finite relational signature. A structure \mathfrak{A} is *homogeneous* if every isomorphism between finite substructures of \mathfrak{A} extends to an automorphism of \mathfrak{A} . The structure \mathfrak{Q} is homogeneous. Given finite substructures \mathfrak{B} and \mathfrak{C} of \mathfrak{Q} and an isomorphism between them, we know that B consists of finitely many elements p_1, \dots, p_n and C of the same number of elements q_1, \dots, q_n such that $p_1 < \dots < p_n$, $q_1 < \dots < q_n$, and the isomorphism maps p_i to q_i (for $i = 1, \dots, n$). It is now easy to see that $<$ is also a dense linear order without lower or upper bound on the sets $\{p \mid p < p_1\}$ and $\{q \mid q < q_1\}$, and thus there is an order isomorphism between these sets. The same is true for the pairs of sets $\{p \mid p_i < p < p_{i+1}\}$ and

$\{q \mid q_i < q < q_{i+1}\}$, and for the pair $\{p \mid p_n < p\}$ and $\{q \mid q < q_n\}$. Using the isomorphisms between these pairs, we can clearly put together an isomorphism from \mathfrak{Q} to \mathfrak{Q} that extends the original isomorphism from \mathfrak{B} to \mathfrak{C} .

In the case of finite relational signatures, homogeneity can be viewed as a particularly strong case of ω -categoricity where relations preserved by all automorphisms even have quantifier-free definitions. We say that a τ -structure admits *quantifier elimination* if, for every first-order τ -formula, there is a quantifier-free τ -formula that defines the same relation over this structure.

Theorem 4 ([55]). *A countable relational structure with a finite signature is homogeneous if and only if it is ω -categorical and admits quantifier elimination.*

Countable homogeneous structures can be obtained as Fraïssé limits of so-called amalgamation classes. A class \mathcal{K} of relational τ -structures has the *amalgamation property* (AP) if, for all $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{K}$ with $e_1: \mathfrak{A} \hookrightarrow \mathfrak{B}_1$ and $e_2: \mathfrak{A} \hookrightarrow \mathfrak{B}_2$ there exists $\mathfrak{C} \in \mathcal{K}$ with $f_1: \mathfrak{B}_1 \hookrightarrow \mathfrak{C}$ and $f_2: \mathfrak{B}_2 \hookrightarrow \mathfrak{C}$ such that $f_1 \circ e_1 = f_2 \circ e_2$. We call \mathfrak{C} an *amalgam* for the triple $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$.

Theorem 5 (Fraïssé [55]). *For a class \mathcal{K} of finite τ -structures over a countable signature τ , the following are equivalent:*

1. $\mathcal{K} = \text{Age}(\mathfrak{D})$ for a countable homogeneous structure \mathfrak{D} .
2. \mathcal{K} contains countably many structures up to isomorphism, is closed under isomorphisms and taking substructures, and has the AP

The structure \mathfrak{D} in item 1 is unique up to isomorphism and called the Fraïssé limit of \mathcal{K}

If \mathcal{K} satisfies item 2 in Theorem 5, then we call it an *amalgamation class*. In general, amalgamation classes are required to satisfy one additional condition called the *joint embedding property* (JEP) [55], which we will introduce in Section 5. However, since in our case the signature does not contain function symbols, the JEP is actually implied by the AP and closure under taking substructures.

For our running example $\mathfrak{Q} = (\mathbb{Q}; <)$, the class $\text{Age}(\mathfrak{Q})$ consists of all finite linear orders, and thus by Fraïssé's theorem this class of structures is an amalgamation class. In addition, \mathfrak{Q} is the Fraïssé limit of this class.

Proposition 5 below shows that there is a close connection between the AP and the patchwork property. Its proof uses the following lemma.

Lemma 2. *Let $\mathfrak{A}, \mathfrak{B}$ be relational τ -structures.*

1. *If $f: \mathfrak{A} \rightarrow \mathfrak{B}$ is a strong homomorphism and ϕ a k -ary (quantifier and equality)-free τ -formula, then, for every $\bar{a} \in A^k$, $\mathfrak{A} \models \phi(\bar{a})$ if and only if $\mathfrak{B} \models \phi(f(\bar{a}))$.*
2. *If $\mathfrak{A}, \mathfrak{B}$ are JEPD, then every homomorphism from \mathfrak{A} to \mathfrak{B} is strong.*

Proof. For the first part, we assume, without loss of generality, that ϕ is in DNF, i.e., ϕ is of the form $\phi_1 \vee \dots \vee \phi_n$ where each ϕ_i is a conjunction of atomic formulas of the form $R(\bar{x})$ for $R \in \tau$. Since f preserves such atomic formulas and their negations, it also preserves their conjunctions, and thus also disjunctions of such conjunctions.

4. A Model-Theoretic Analysis of ω -Admissibility

For the second part, we show that every homomorphism $f : \mathfrak{A} \rightarrow \mathfrak{B}$ preserves complements of relations of \mathfrak{A} . Let R be an ℓ -ary relation symbol in τ . Since \mathfrak{A} is JEPD, for every $\bar{a} \in A^\ell$ with $\bar{a} \notin R^{\mathfrak{B}}$, there exists exactly one $\tilde{R} \in \tau \setminus \{R\}$ with $\bar{a} \in \tilde{R}^{\mathfrak{A}}$. This implies $f(\bar{a}) \in \tilde{R}^{\mathfrak{B}}$ since f is a homomorphism. It follows that $f(\bar{a}) \notin R^{\mathfrak{B}}$ because \mathfrak{B} is PD. \square

Proposition 5. *A structure \mathfrak{D} is a patchwork if and only if \mathfrak{D} is JDJEPD and $\text{Age}(\mathfrak{D})$ has the AP*

Proof. For simplicity, every statement indexed by i is supposed to hold for both $i \in \{1, 2\}$. Let τ be the signature of \mathfrak{D} .

“ \Leftarrow ”: Suppose that \mathfrak{D} is JDJEPD and $\text{Age}(\mathfrak{D})$ has the AP. Let $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ be finite JEPD τ -structures with $e_i : \mathfrak{A} \hookrightarrow \mathfrak{B}_i$ and $h_i : \mathfrak{B}_i \rightarrow \mathfrak{D}$. We must show that there exist $f_i : \mathfrak{B}_i \rightarrow \mathfrak{D}$ with $f_1 \circ e_1 = f_2 \circ e_2$. Let $\tilde{\mathfrak{A}}_1$ and $\tilde{\mathfrak{A}}_2$ be the substructures of \mathfrak{D} on $(h_1 \circ e_1)(A)$ and $(h_2 \circ e_2)(A)$, respectively. Clearly both $\tilde{\mathfrak{A}}_1$ and $\tilde{\mathfrak{A}}_2$ are JDJEPD, because they are substructures of \mathfrak{D} . Note that JD is witnessed in both $\tilde{\mathfrak{A}}_1$ and $\tilde{\mathfrak{A}}_2$ by an identical formula $\phi(x, y)$ inherited from \mathfrak{D} . We claim that there exists an isomorphism from $\tilde{\mathfrak{A}}_1$ to $\tilde{\mathfrak{A}}_2$ which commutes with $h_1 \circ e_1$ and $h_2 \circ e_2$. Consider the map $g : \tilde{\mathfrak{A}}_1 \rightarrow \tilde{\mathfrak{A}}_2$ given by $g((h_1 \circ e_1)(a)) := (h_2 \circ e_2)(a)$. By Lemma 2, for every pair $a_1, a_2 \in A$, we have

$$\begin{aligned} (h_1 \circ e_1)(a_1) = (h_1 \circ e_1)(a_2) & \text{ iff } \mathfrak{D} \models \phi((h_1 \circ e_1)(a_1), (h_1 \circ e_1)(a_2)) \\ & \text{ iff } \mathfrak{D} \models \phi(a_1, a_2) \\ & \text{ iff } \mathfrak{D} \models \phi((h_2 \circ e_2)(a_1), (h_2 \circ e_2)(a_2)) \\ & \text{ iff } (h_2 \circ e_2)(a_1) = (h_2 \circ e_2)(a_2). \end{aligned}$$

This means that g is well-defined and injective. Let $R \in \tau$ be an arbitrary symbol and ℓ its arity. Since \mathfrak{A} and $\tilde{\mathfrak{A}}_1$ are JEPD, by Lemma 2, $h_1 \circ e_1$ preserves the complements of all relations of \mathfrak{A} . Thus, for every $\bar{t} \in A^\ell$, if $(h_1 \circ e_1)(\bar{t}) \in R^{\tilde{\mathfrak{A}}_1}$, then $\bar{t} \in R^{\mathfrak{A}}$ and consequently $(h_2 \circ e_2)(\bar{t}) \in R^{\tilde{\mathfrak{A}}_2}$. This means that g is a homomorphism from $\tilde{\mathfrak{A}}_1$ to $\tilde{\mathfrak{A}}_2$. Since $\tilde{\mathfrak{A}}_1$ and $\tilde{\mathfrak{A}}_2$ are JEPD, by Lemma 2, g also preserves the complements of all relations of $\tilde{\mathfrak{A}}_1$. Hence g is an isomorphism that additionally satisfies $g \circ h_1 \circ e_1 = h_2 \circ e_2$ by its definition. Let $\tilde{\mathfrak{B}}_1$ and $\tilde{\mathfrak{B}}_2$ be the substructures of \mathfrak{D} on $h_1(B_1)$ and $h_2(B_2)$, respectively. Now consider the inclusions $\tilde{e}_i : \tilde{\mathfrak{A}}_i \hookrightarrow \tilde{\mathfrak{B}}_i$. Since $\text{Age}(\mathfrak{D})$ has the AP, there exists $\mathfrak{C} \in \text{Age}(\mathfrak{D})$ together with $\tilde{f}_i : \tilde{\mathfrak{B}}_i \hookrightarrow \mathfrak{C}$ and $e : \mathfrak{C} \hookrightarrow \mathfrak{D}$ such that $\tilde{f}_1 \circ \tilde{e}_1 = \tilde{f}_2 \circ \tilde{e}_2 \circ g$. We define the homomorphisms $f_i : \mathfrak{B}_i \rightarrow \mathfrak{D}$ by $f_i := e \circ \tilde{f}_i \circ h_i$. Then, for every $a \in A$, we have

$$\begin{aligned} (f_1 \circ e_1)(a) &= (e \circ \tilde{f}_1 \circ h_1 \circ e_1)(a) \\ &= (e \circ \tilde{f}_1 \circ \tilde{e}_1 \circ h_1 \circ e_1)(a) \\ &= (e \circ \tilde{f}_2 \circ \tilde{e}_2 \circ g \circ h_1 \circ e_1)(a) \\ &= (e \circ \tilde{f}_2 \circ \tilde{e}_2 \circ h_2 \circ e_2)(a) \\ &= (e \circ \tilde{f}_2 \circ h_2 \circ e_2)(a) \\ &= (f_2 \circ e_2)(a). \end{aligned}$$

Note that, as inclusions, the mappings \tilde{e}_i are the identity on the elements for which they are

defined. The above identities show that \mathfrak{D} is a patchwork.

“ \Rightarrow ”: Suppose that \mathfrak{D} is a patchwork. Then, by definition, \mathfrak{D} is JDJEPD. Let $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ be finite τ -structures with $e_i: \mathfrak{A} \hookrightarrow \mathfrak{B}_i$ and $h_i: \mathfrak{B}_i \hookrightarrow \mathfrak{D}$. Since \mathfrak{B}_1 and \mathfrak{B}_2 are isomorphic to substructures of \mathfrak{D} , they are clearly JDJEPD. Thus, as \mathfrak{D} is a patchwork, there exist homomorphisms $f_i: \mathfrak{B}_i \rightarrow \mathfrak{D}$ with $f_1 \circ e_1 = f_2 \circ e_2$. Let $\phi(x, y)$ be a formula witnessing JD in both \mathfrak{B}_1 and \mathfrak{B}_2 that is inherited from \mathfrak{D} . By Lemma 2, the operations f_i preserve the complements of all relations of \mathfrak{B}_i , and, for all $b_1, b_2 \in B_i$, we have

$$f_i(b_1) = f_i(b_2) \quad \text{iff} \quad \mathfrak{D} \models \phi(f_i(b_1), f_i(b_2)) \quad \text{iff} \quad \mathfrak{D} \models \phi(b_1, b_2) \quad \text{iff} \quad b_1 = b_2.$$

This means that the operations f_i are embeddings. We obtain the AP for $\text{Age}(\mathfrak{D})$ by choosing \mathfrak{C} to be the substructure of \mathfrak{D} on $f_2(B_1) \cup f_1(B_2)$. \square

We finish this section with an example demonstrating the use of Proposition 5 but also the fact that finiteness of the signature is an indispensable component of ω -admissibility. Together with Example 2, it shows that Theorem 4 does not hold for countably infinite signatures.

Example 3. Consider the relational structure \mathfrak{D} with domain \mathbb{Z} and relations $+_k = \{(x, y) \in \mathbb{Z}^2 \mid x + k = y\}$ for all $k \in \mathbb{Z}$. We claim that \mathfrak{D} is homogeneous. Let $\mathfrak{A}_1, \mathfrak{A}_2$ be two finite substructures of \mathfrak{D} and $f: \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ an isomorphism. Then we have $A_1 = \{x_1, \dots, x_k\}$ and $A_2 = \{y_1, \dots, y_k\}$ for some $k \geq 0$, where $f(x_i) = y_i$ for every $i \in [k]$. We may assume that $x_i < x_{i+1}$ for every $i \in [k-1]$. Since \mathfrak{D} has relations for all possible integer distances between two numbers and f is an isomorphism, we have that $x_{i+1} - x_i = y_{i+1} - y_i$ for every $i \in [k-1]$. But then there must exist $\ell \in \mathbb{Z}$ such that $y_i = f(x_i) = x_i + \ell$ for every $i \in [k]$, in which case $\tilde{f}(x) := x + \ell$ is an automorphism of \mathfrak{D} extending f . Thus \mathfrak{D} is homogeneous.

By Theorem 5, $\text{Age}(\mathfrak{D})$ has the AP. Clearly, \mathfrak{D} is JDJEPD. Thus, by Proposition 5, \mathfrak{D} is a patchwork. We claim that \mathfrak{D} also has homomorphism ω -compactness. Let \mathfrak{B} be a countable structure in the signature of \mathfrak{D} . Then \mathfrak{B} does not have a homomorphism to \mathfrak{D} if and only if there are $x_1, \dots, x_k, y_1, \dots, y_\ell \in B$ with $x_1 = y_1, x_k = y_\ell, (x_i, x_{i+1}) \in +_{m_i}^{\mathfrak{B}}$ for every $i \in [k-1]$, and $(y_i, y_{i+1}) \in +_{n_i}^{\mathfrak{B}}$ for every $i \in [\ell-1]$, but $\sum_{i \in [k]} m_i \neq \sum_{i \in [\ell]} n_i$. In every such case, already the finite substructure of \mathfrak{B} on $\{x_1, \dots, x_k, y_1, \dots, y_\ell\}$ does not have a homomorphism to \mathfrak{D} . Since \mathfrak{B} was chosen arbitrarily, this confirms our claim that \mathfrak{D} has homomorphism ω -compactness. It is also easy to see that $\text{CSP}(\mathfrak{D})$ is decidable in PTIME, e.g., using the Gaussian elimination method. Thus, \mathfrak{D} satisfies all subconditions of ω -admissibility except for finiteness of the signature. However, by Proposition 1, concept satisfiability w.r.t. GCIs is undecidable in $\mathcal{ALC}(\mathfrak{D})$.

4.3 JDJEPD via decomposition into orbits

To apply Proposition 5, we need the structure to be JDJEPD. Given an ω -categorical τ -structure \mathfrak{A} , we can obtain JDJEPD by replacing the original relations with appropriate first-order definable ones, using the results of Theorem 3. The *orbit* of a tuple $\bar{a} \in A^k$ under the natural action of $\text{Aut}(\mathfrak{A})$ on A^k is the set $\{g(\bar{a}) \mid g \in \text{Aut}(\mathfrak{A})\}$. By Theorem 3, the set of all at most

4. A Model-Theoretic Analysis of ω -Admissibility

k -ary relations definable in \mathfrak{A} is finite for every $k \in \mathbb{N}$. Since every such set is closed under intersections, it contains finitely many minimal non-empty relations. Since every relation over A that is preserved by all automorphisms of \mathfrak{A} is first-order definable in \mathfrak{A} , these minimal elements are precisely the orbits of tuples over A under the natural action of $\text{Aut}(\mathfrak{A})$. We have just formulated a well-known fourth equivalent condition in Theorem 3:

4. For every k , there are only finitely many orbits of k -tuples over A under $\text{Aut}(\mathfrak{A})$.

Definition 5. For a given arity bound $d \geq 2$, the d -ary decomposition of the τ -structure \mathfrak{A} , denoted by $\mathfrak{A}^{\leq d}$, is the relational structure over A whose relations are all orbits of at most d -ary tuples over A under $\text{Aut}(\mathfrak{A})$. We denote the signature of $\mathfrak{A}^{\leq d}$ by $\tau^{\leq d}$.

It is easy to see that $\mathfrak{A}^{\leq d}$ is JDJEPD, and that every at most d -ary relation over A first-order definable in \mathfrak{A} can be obtained as a disjunction of atomic $\tau^{\leq d}$ -formulas.

As an example, consider the ω -categorical structure \mathfrak{Q} . The orbits of k -tuples of elements of \mathbb{Q} can be defined by quantifier-free formulas that are conjunctions of atomic formulas of the form $x_i = x_j$ or $x_i < x_j$. For example, the orbit of the tuple $(2, 3, 2, 5)$ consists of all tuples $(q_1, q_2, q_3, q_4) \in \mathbb{Q}^4$ that satisfy the formula $x_1 < x_2 \wedge x_1 = x_3 \wedge x_2 < x_4$ if x_i is replaced by q_i for $i = 1, \dots, 4$. The first-order definable k -ary relations in \mathfrak{Q} are obtained as unions of these orbits, where the defining formula is then the disjunction of the formulas defining the respective orbits. Since these formulas are quantifier-free, this also shows that \mathfrak{Q} admits quantifier elimination.

We have seen that, to obtain JDJEPD, we actually need to take the d -ary decomposition of a given ω -categorical structure, rather than the structure itself. Fortunately, homogeneity transfers from \mathfrak{D} to $\mathfrak{D}^{\leq d}$.

Proposition 6. Let \mathfrak{D} be a countable homogeneous structure with a finite relational signature τ , and let d be a natural number that exceeds or is equal to the maximal arity of a symbol from τ . Then $\mathfrak{D}^{\leq d}$ is homogeneous.

Proof. By Theorem 4, \mathfrak{D} has quantifier elimination. Note that the relations of $\mathfrak{D}^{\leq d}$ and \mathfrak{D} are first-order interdefinable, which implies $\text{Aut}(\mathfrak{D}^{\leq d}) = \text{Aut}(\mathfrak{D})$ by Theorem 3. This shows in particular that $\mathfrak{D}^{\leq d}$ is ω -categorical. Every first-order $\tau^{\leq d}$ -formula ϕ defines a relation in $\mathfrak{D}^{\leq d}$ that has a first-order definition ϕ' in \mathfrak{D} . We can assume that ϕ' is quantifier-free due to Theorem 4. We replace every atomic formula $\psi(\bar{x})$ in ϕ' by $\bigvee_{i=1}^n R_i(\bar{x})$ with $R_1, \dots, R_n \in \tau^{\leq d}$, where $R_1^{\mathfrak{D}^{\leq d}} \cup \dots \cup R_n^{\mathfrak{D}^{\leq d}}$ is the unique decomposition of $\psi^{\mathfrak{D}}$ into orbits of k -tuples over D under $\text{Aut}(\mathfrak{D})$. The resulting formula is a quantifier-free definition of $\phi^{\mathfrak{D}^{\leq d}}$ in $\mathfrak{D}^{\leq d}$. Thus $\mathfrak{D}^{\leq d}$ has quantifier elimination as well, which means that it is homogeneous due to Theorem 4. \square

4.4 Upper bounds via finite boundedness

In the three previous sections, we have described model-theoretic properties that provide us with all the ingredients needed for ω -admissibility, except for decidability of the CSP. This is quite literally true because there exist homogeneous structures with a finite relational signature

but an undecidable CSP [30]. Finding model-theoretic conditions that guarantee decidability of the CSP for infinite structures is a very broad topic with many open questions. Here we focus on a well-known condition that ensures that the CSP is decidable in NP and the first-order theory in PSPACE.

For a class \mathcal{N} of τ -structures (called *bounds* or *forbidden patterns*), we denote by $\text{Forb}_e(\mathcal{N})$ the class of all finite τ -structures not embedding any member of \mathcal{N} . Following the terminology in [30], we say that a relational structure \mathfrak{A} is *finitely bounded* if its signature is finite and $\text{Age}(\mathfrak{A}) = \text{Forb}_e(\mathcal{N})$ for a finite set of bounds \mathcal{N} . Note that if \mathfrak{A} is additionally homogeneous, then, by Theorem 5, it is described by \mathcal{N} up to isomorphism. The above definition of finite boundedness is popular among mathematicians, especially in the context of the classification program for homogeneous structures in a finite relational signature initiated by [66]. The notion arose naturally as a tool for describing more general amalgamation classes than the standard examples such as the class of all finite triangle-free graphs. There is a second, arguably more practical, definition of finite boundedness.

Lemma 3. *Let \mathfrak{A} be a structure with a finite relational signature. Then the following are equivalent:*

- \mathfrak{A} is finitely bounded;
- there exists a universal sentence $\Phi(\mathfrak{A})$ such that $\text{Age}(\mathfrak{A}) = \text{Mod}_{\text{fin}}(\Phi(\mathfrak{A}))$.

Proof. Let τ be the signature of \mathfrak{A} .

“ \Rightarrow ”: Let $\text{Age}(\mathfrak{A}) = \text{Forb}_e(\mathcal{N})$ for $\mathcal{N} = \{\mathfrak{C}_1, \dots, \mathfrak{C}_k\}$. For every $i \in [k]$, we can write down a quantifier-free τ -formula $\phi_{\mathfrak{C}_i}$ with free variables c_1, \dots, c_{n_i} , where $\{c_1, \dots, c_{n_i}\}$ is the domain of \mathfrak{C}_i , that describes \mathfrak{C}_i up to isomorphism. Then we set

$$\Phi(\mathfrak{A}) := \bigwedge_{i \in [k]} (\forall c_1, \dots, c_{n_i}. \neg \phi_{\mathfrak{C}_i}(c_1, \dots, c_{n_i})).$$

“ \Leftarrow ”: Given a universal τ -sentence $\Phi(\mathfrak{A})$, we define \mathcal{N} as the set of all finite τ -structures \mathfrak{C} of size at most n that do not satisfy $\Phi(\mathfrak{A})$, where n is the number of variables in $\Phi(\mathfrak{A})$. Then $\text{Age}(\mathfrak{A}) = \text{Forb}_e(\mathcal{N})$ clearly holds. \square

The structure \mathfrak{Q} is finitely bounded. To show this using the original definition, we can use the set \mathcal{N} consisting of the following four structures: the self loop, the 2-cycle, the 3-cycle, and two isolated vertices. We must show that $\text{Age}(\mathfrak{Q}) = \text{Forb}_e(\mathcal{N})$. Clearly, none of the structures in \mathcal{N} embeds into a linear order, which shows $\text{Age}(\mathfrak{Q}) \subseteq \text{Forb}_e(\mathcal{N})$. Conversely, assume that \mathfrak{A} is an element of $\text{Forb}_e(\mathcal{N})$. We must show that $<^{\mathfrak{A}}$ is a linear order. Since \mathcal{N} contains the self loop, we have $(a, a) \notin <^{\mathfrak{A}}$ for all $a \in A$, which shows that $<^{\mathfrak{A}}$ is irreflexive. For distinct elements $a, b \in A$, we must have $a <^{\mathfrak{A}} b$ or $b <^{\mathfrak{A}} a$ since otherwise the structure consisting of two isolated vertices could be embedded into \mathfrak{A} . This shows that any two distinct elements are comparable w.r.t. $<^{\mathfrak{A}}$. To show that $<^{\mathfrak{A}}$ is transitive, assume that $a <^{\mathfrak{A}} b$ and $b <^{\mathfrak{A}} c$ holds. Since the 2-cycle does not embed into \mathfrak{A} , a and c must be distinct, and are thus comparable. We cannot have $c <^{\mathfrak{A}} a$ since then we could embed the 3-cycle into \mathfrak{A} . Consequently, we must have $a <^{\mathfrak{A}} c$, which proves transitivity. This shows that \mathfrak{A} is a linear order. As formula $\Phi(\mathfrak{Q})$ we can take the conjunction of the usual axioms defining linear orders.

4. A Model-Theoretic Analysis of ω -Admissibility

Finitely bounded structures are useful in the context of this thesis due to the following proposition.

Proposition 7. *Let \mathcal{D} be a relational structure.*

1. *If \mathcal{D} is finitely bounded, then $\text{CSP}(\mathcal{D})$ is in NP*
2. *If \mathcal{D} is finitely bounded and homogeneous, then $\text{Th}(\mathcal{D})$ is in PSPACE.*

The first result is stated in [29, 17], and the second result is stated in [61, 64]. We include a full proof of both for the sake of completeness.

Proof. Let τ be the signature of \mathcal{D} . Recall that, since \mathcal{D} is finitely bounded, by Lemma 3, there exists a universal first-order sentence $\Phi(\mathcal{D})$ that defines $\text{Age}(\mathcal{D})$, i.e., a finite τ -structure can be embedded into \mathcal{D} if and only if it satisfies $\Phi(\mathcal{D})$. Since the structure \mathcal{D} is fixed, this sentence is also fixed, which means that it has constant size. We refer to $\Phi(\mathcal{D})$ simply by Φ .

For item 1, we show that $\text{CSP}(\mathcal{D})$ is definable in existential second-order logic. Then the statement follows from Fagin's theorem [46] (see also [59]). Let R_1, \dots, R_ℓ be an enumeration of the symbols in τ . For every $i \in [\ell]$, we introduce a second-order variable S_i of the same arity as R_i . Moreover, we introduce a binary second-order variable \sim . We obtain Φ' from Φ by replacing each atomic formula of the form $R_i(\bar{x})$ for $i \in [\ell]$ in Φ by $S_i(\bar{x})$, and each atomic formula of the form $(x = y)$ in Φ by $(x \sim y)$. For every $i \in [\ell]$, let n_i be the arity of R_i , and let Θ_i be the sentence

$$\Theta_i := \bigwedge_{j \in [n_i]} \forall x_1, \dots, x_{n_i}, y (S_i(x_1, \dots, x_{n_i}) \wedge (x_j \sim y) \Rightarrow S_i(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_{n_i})).$$

Now consider the existential second-order sentence Ψ defined as follows:

$$\begin{aligned} \Psi := \exists \sim \exists S_1 \dots \exists S_\ell & \left(\Phi' \wedge \bigwedge_{i \in [\ell]} \Theta_i \wedge \forall \bar{x} (R_i(\bar{x}) \Rightarrow S_i(\bar{x})) \right. \\ & \left. \wedge \forall x, y, z ((x \sim y \wedge y \sim z \Rightarrow x \sim z) \wedge (x \sim y \Rightarrow y \sim x) \wedge x \sim x) \right). \end{aligned}$$

Let \mathfrak{A} be an instance of $\text{CSP}(\mathcal{D})$. Suppose that \mathfrak{A} satisfies Ψ . By the definition of Ψ , \sim is an equivalence relation on A and also compatible with the relations S_i for $i \in [\ell]$ (due to the sentences Θ_i). This means that the structure \mathfrak{A}/\sim on the equivalence classes of \sim and with the relations

$$R_i^{\mathfrak{A}/\sim} = \{([x_1]_\sim, \dots, [x_{n_i}]_\sim) \in (A/\sim)^{n_i} \mid (x_1, \dots, x_{n_i}) \in S_i\}$$

is well-defined. By the definition of Ψ , we have that $\mathfrak{A} \rightarrow \mathfrak{A}/\sim$ and that $\mathfrak{A}/\sim \models \Phi$. Since Φ defines $\text{Age}(\mathcal{D})$, we conclude that $\mathfrak{A} \rightarrow \mathcal{D}$. On the other hand, if there exists a homomorphism $h: \mathfrak{A} \rightarrow \mathcal{D}$, then $S_i := \{\bar{x} \in A^{n_i} \mid h(\bar{x}) \in R_i^{\mathcal{D}}\}$ for $i \in [\ell]$ and $\sim := \{(x, y) \in A^2 \mid h(x) = h(y)\}$ witness that Ψ is satisfied in \mathfrak{A} .

For item 2, we describe a PSPACE algorithm that decides the first-order theory of \mathcal{D} . It is based on the algorithm from the proof of Proposition 3.5 in [64], for which an exponential time complexity is shown in [64]. Note that, since D is possibly infinite, we cannot simply substitute all elements from D , one after the other, for a particular quantified variable.

Now, let b_1, b_2, \dots be a countably infinite sequence of pairwise distinct symbols. For a first-order τ -formula ϕ with free variables x_1, \dots, x_n , let $[\phi]_{\mathfrak{D}}$ denote the set of all τ -structures \mathfrak{B} with domain $\{b_1, \dots, b_n\}$ for which there exists an embedding $h: \mathfrak{B} \hookrightarrow \mathfrak{D}$ such that $\mathfrak{D} \models \phi(h(b_1), \dots, h(b_n))$. Every such embedding $h: \mathfrak{B} \hookrightarrow \mathfrak{D}$ represents an injective¹ substitution of elements from D for the variables x_1, \dots, x_n . We claim that $[\phi]_{\mathfrak{D}}$ does not depend on the choice of h . To see this, consider two embeddings $h_1, h_2: \mathfrak{B} \hookrightarrow \mathfrak{D}$ such that $\mathfrak{D} \models \phi(h_1(b_1), \dots, h_1(b_n))$. For each $i \in [2]$, let \mathfrak{B}_i be the substructure of \mathfrak{D} on the image of $\{b_1, \dots, b_n\}$ under h_i . Consider the map $\tilde{f}: B_1 \rightarrow B_2$ that sends, for every $j \in [n]$, $h_1(b_j)$ to $h_2(b_j)$. Using the definition of an embedding, it is easy to show that \tilde{f} is an isomorphism from \mathfrak{B}_1 to \mathfrak{B}_2 . By assumption, \mathfrak{D} is homogeneous. By homogeneity of \mathfrak{D} , there exists an automorphism f of \mathfrak{D} that extends \tilde{f} . Since ϕ is a first-order formula, $\phi^{\mathfrak{D}}$ is preserved by f , which shows that $\mathfrak{D} \models \phi(h_2(b_1), \dots, h_2(b_n))$ holds as well.

We show by *induction* on the structure of a first-order τ -formula ϕ with free variables x_1, \dots, x_n that, given a τ -structure \mathfrak{B} with domain $\{b_1, \dots, b_n\}$, it can be decided in PSPACE in the size of ϕ whether $\mathfrak{B} \in [\phi]_{\mathfrak{D}}$. This proves the PSPACE upper bound claimed in the proposition because, if ϕ has no free variables, then testing whether the empty structure is contained in $[\phi]_{\mathfrak{D}}$ is equivalent to answering $\mathfrak{D} \models \phi$.

In the *base case*, we consider an atomic formula $\phi(x_1, \dots, x_n)$. Suppose that \mathfrak{B} is a τ -structure with domain $\{b_1, \dots, b_n\}$. If $\mathfrak{B} \models \neg\phi(b_1, \dots, b_n)$, then clearly $\mathfrak{B} \notin [\phi(x_1, \dots, x_n)]_{\mathfrak{D}}$ because embeddings are injective and preserve complements of relations. If $\mathfrak{B} \models \phi(b_1, \dots, b_n)$, then $\mathfrak{D} \models \phi(h(b_1), \dots, h(b_n))$ holds for every embedding $h: \mathfrak{B} \rightarrow \mathfrak{D}$. Consequently, testing whether $\mathfrak{B} \in [\phi(x_1, \dots, x_n)]_{\mathfrak{D}}$ boils down to testing whether $\mathfrak{B} \hookrightarrow \mathfrak{D}$, which is the case if and only if $\mathfrak{B} \models \Phi$. This can be done in PSPACE in the size of ϕ because it is well-known that first-order model checking with a fixed first-order sentence can be done in polynomial time in the size of the input structure.

For the *induction step*, we can restrict the attention to formulas ϕ of the form $\psi_1 \vee \psi_2$, $\neg\psi$ and $\exists x. \psi$. Suppose that ϕ is of the form $\psi_1 \vee \psi_2$ such that the induction hypothesis applies to both ψ_1 and ψ_2 . For each $i \in [2]$, let \mathfrak{B}_i be the substructure of \mathfrak{B} on those b_j s that correspond to the free variables of ψ_i . We claim that $\mathfrak{B} \in [\phi]_{\mathfrak{D}}$ if and only if $\mathfrak{B} \models \Phi$ and $\mathfrak{B}_i \in [\psi_i]_{\mathfrak{D}}$ for $i = 1$ or $i = 2$. The forward direction is trivial. Now suppose that $\mathfrak{B}_i \in [\psi_i]_{\mathfrak{D}}$ for $i = 1$ or $i = 2$ and $\mathfrak{B} \models \Phi$. Then we have an embedding $h_i: \mathfrak{B}_i \hookrightarrow \mathfrak{D}$ witnessing $\mathfrak{B}_i \in [\psi_i]_{\mathfrak{D}}$, and we also have an embedding $h: \mathfrak{B} \hookrightarrow \mathfrak{D}$. But then $\mathfrak{B}_i \in [\psi_i]_{\mathfrak{D}}$ is also witnessed by $h|_{B_i}$ because $[\psi_i]_{\mathfrak{D}}$ does not depend on the choice of the embedding. This shows that $\mathfrak{B} \in [\phi]_{\mathfrak{D}}$ is witnessed by h . Testing whether $\mathfrak{B}_i \in [\psi_i]_{\mathfrak{D}}$ can be done in PSPACE in the size of ψ_i by the induction hypothesis, and we have already seen in the base case that testing whether $\mathfrak{B} \models \Phi$ can be done in polynomial time in the size of ϕ .

Suppose that ϕ is of the form $\neg\psi$ such that the induction hypothesis applies to ψ . We claim that $\mathfrak{B} \in [\phi]_{\mathfrak{D}}$ if and only if $\mathfrak{B} \notin [\psi]_{\mathfrak{D}}$. Suppose that there exists $h: \mathfrak{B} \hookrightarrow \mathfrak{D}$ such that $\mathfrak{D} \models \neg\psi(h(b_1), \dots, h(b_n))$. Then there cannot be an embedding $h': \mathfrak{B} \hookrightarrow \mathfrak{D}$ such that

¹In our proof we will ensure that injective substitutions are sufficient, by appropriately identifying variables.

4. A Model-Theoretic Analysis of ω -Admissibility

$\mathfrak{D} \models \psi(h'(b_1), \dots, h'(b_n))$ because containment in $[\phi]_{\mathfrak{D}}$ does not depend on the choice of the embedding. The backward direction is analogous. By the induction hypothesis, testing whether $\mathfrak{B} \in [\psi]_{\mathfrak{D}}$ can be done in PSPACE in the size of ψ and thus also in the size of ϕ .

Now suppose that ϕ is of the form $\phi(x_1, \dots, x_n) = \exists x_{n+1} \cdot \psi(x_1, \dots, x_{n+1})$ such that the induction hypothesis applies to ψ . We claim that $\mathfrak{B} \in [\phi]_{\mathfrak{D}}$ if and only if one of the following is true

1. there exists an extension \mathfrak{B}' of \mathfrak{B} by b_{n+1} such that $\mathfrak{B}' \in [\psi]_{\mathfrak{D}}$,
2. there exists $i \in [n]$ such that $\mathfrak{B} \in [\psi_i]_{\mathfrak{D}}$ holds for the formula ψ_i obtained from ψ by replacing each occurrence of the variable x_{n+1} in ψ by x_i .

First, suppose that $\mathfrak{B} \in [\phi]_{\mathfrak{D}}$ is witnessed by some embedding $h: \mathfrak{B} \hookrightarrow \mathfrak{D}$. Then there exists $d \in D$ such that $\mathfrak{D} \models \psi(h(b_1), \dots, h(b_n), d)$. If d is distinct from $h(b_1), \dots, h(b_n)$, then we are in the case (1) and consider the extension h' of h that maps b_{n+1} to d . We define \mathfrak{B}' as the τ -structure with the domain $\{b_1, \dots, b_{n+1}\}$ such that, for every k -ary symbol $R \in \tau$, we have $t \in R^{\mathfrak{B}'}$ if and only if $h'(t) \in R^{\mathfrak{D}}$. Clearly h' is an embedding that witnesses $\mathfrak{B}' \in [\psi]_{\mathfrak{D}}$. Otherwise we have $d = h(b_i)$ for some $i \in [n]$. We consider the formula ψ_i from (2). Then h is an embedding that witnesses $\mathfrak{B} \in [\psi_i]_{\mathfrak{D}}$. Since the backward direction is obvious, it remains to show that the tests required by (1) and (2) can be performed in PSPACE.

In case (1), we generate all extensions \mathfrak{B}' of \mathfrak{B} by b_{n+1} and test, using the induction hypothesis, whether $\mathfrak{B}' \in [\psi]_{\mathfrak{D}}$ for some such extension. This can clearly be done in PSPACE because τ is fixed and finite, and for each extension \mathfrak{B}' we can test $\mathfrak{B}' \in [\psi]_{\mathfrak{D}}$ within PSPACE due to the induction hypothesis. In case (2) we guess any such $i \in [n]$ and test, using the induction hypothesis, whether $\mathfrak{B} \in [\psi_i]_{\mathfrak{D}}$. This completes the proof. \square

Proposition 7 applies not only to a given finitely bounded homogeneous structure \mathfrak{D} , but also to its d -ary decomposition $\mathfrak{D}^{\leq d}$. This is a direct consequence of the following result.

Proposition 8. *Let \mathfrak{A} be a finitely bounded homogeneous structure and \mathfrak{B} a structure with the same domain and finitely many relations that are first-order definable in \mathfrak{A} . Then the expansion of \mathfrak{A} by the relations of \mathfrak{B} is finitely bounded homogeneous.*

Proof. Let $\tilde{\mathfrak{A}}$ be the expansion of \mathfrak{A} by the relations of \mathfrak{B} , where we assume that the signatures of \mathfrak{A} and \mathfrak{B} are disjoint. By Theorem 4, each of the new relations has a quantifier-free definition in \mathfrak{A} . Consequently, we can choose any universal sentence $\Phi(\mathfrak{A})$ for $\text{Age}(\mathfrak{A})$ and extend it with universal sentences defining the relations of \mathfrak{B} , which yields a universal sentence that shows finite boundedness of $\tilde{\mathfrak{A}}$. The structure $\tilde{\mathfrak{A}}$ is homogeneous since an isomorphism between two finite substructures of $\tilde{\mathfrak{A}}$ induces an isomorphism between their reducts to the signature of \mathfrak{A} , which extends to an automorphism of \mathfrak{A} by homogeneity of \mathfrak{A} . This is also an automorphism of $\tilde{\mathfrak{A}}$ since automorphisms preserve first-order definable relations. \square

4.5 ω -admissible finitely bounded homogeneous structures

We are now ready to formulate the main results of this section.

Theorem 6. *Let \mathfrak{D} be a finitely bounded homogeneous relational structure with at most d -ary relations for some $d \geq 2$. Then $\mathfrak{D}^{\leq d}$ is ω -admissible.*

Proof. It follows directly from the definition of d -ary decompositions that $\mathfrak{D}^{\leq d}$ is JDJEPD. By Proposition 6, $\mathfrak{D}^{\leq d}$ is homogeneous. By Theorem 4, $\mathfrak{D}^{\leq d}$ is ω -categorical. Thus \mathfrak{D} has homomorphism ω -compactness by Corollary 2. By Theorem 5, $\text{Age}(\mathfrak{D}^{\leq d})$ has the AP. Thus $\mathfrak{D}^{\leq d}$ is a patchwork by Proposition 5. By Proposition 8, Lemma 1, and Proposition 7, $\text{CSP}(\mathfrak{D}^{\leq d})$ is in NP. Hence $\mathfrak{D}^{\leq d}$ is ω -admissible. \square

This theorem, together with Theorem 1, immediately yields decidability for concept satisfiability in $\mathcal{ALC}_{\text{AD}}^d(\mathfrak{D}^{\leq d})$. The following corollary shows that we can even allow for arbitrary first-order definable relations with arity bounded by d in the concrete domain. The idea for proving this result is to reduce concept satisfiability in $\mathcal{ALC}_{\text{FO}}^d(\mathfrak{D})$ to concept satisfiability in $\mathcal{ALC}_{\text{AD}}^d(\mathfrak{D}^{\leq d})$. We know that every at most d -ary relation over D first-order definable in \mathfrak{D} can be obtained as a disjunction of atomic formulas built using the signature of $\mathfrak{D}^{\leq d}$. What still needs to be shown is that, given a first-order formula in the signature of \mathfrak{D} with at most d free variables, this disjunction can effectively be computed.

Corollary 3. *Let \mathfrak{D} be a reduct of a finitely bounded homogeneous relational structure with at most d -ary relations for some $d \geq 2$. Then concept satisfiability in $\mathcal{ALC}_{\text{FO}}^d(\mathfrak{D})$ w.r.t. TBoxes is decidable.*

Proof. Let τ be the signature of \mathfrak{D} . We claim that satisfiability of $\mathcal{ALC}_{\text{FO}}^d(\mathfrak{D})$ concepts w.r.t. TBoxes can be reduced to satisfiability of $\mathcal{ALC}_{\text{AD}}^d(\mathfrak{D}^{\leq d})$ concepts w.r.t. TBoxes. For this purpose, we need to replace first-order τ -formulas ϕ in concrete domain constructors $\forall p_1, \dots, p_k. \phi$ or $\exists p_1, \dots, p_k. \phi$ with disjunctions ψ of atomic formulas in the signature $\tau^{\leq d}$ of $\mathfrak{D}^{\leq d}$. By Theorem 4 together with Theorem 3, the (finitely many) relations in $\tau^{\leq d}$ have quantifier-free definitions in \mathfrak{D} . Since d and \mathfrak{D} are fixed, we can make a list consisting of the quantifier-free definitions for each of them in constant time. Given a first-order τ -formulas ϕ with k free variables, let ψ_1, \dots, ψ_m be the quantifier-free definitions in \mathfrak{D} for all the k -ary relations of $\tau^{\leq d}$ that we have listed before. We test, for every $i \in [m]$, whether $\mathfrak{D} \models \exists \bar{y} (\phi(\bar{y}) \wedge \psi_i(\bar{y}))$, which is possible in PSPACE by Proposition 7. By selecting those $\psi_{i_1}, \dots, \psi_{i_s}$ that tested positively, we know that, for every $\bar{a} \in D^k$, $\mathfrak{D} \models \phi(\bar{a})$ if and only if $\mathfrak{D} \models \bigvee_{r=1}^s \psi_{i_r}(\bar{a})$. We replace each $\psi_{i_r}(\bar{y})$ with $R(\bar{y})$, where R is the unique k -ary relation symbol from $\tau^{\leq d}$ for which $\mathfrak{D} \models \psi_{i_r}(\bar{a})$ if and only if $\mathfrak{D}^{\leq d} \models R(\bar{a})$. This yields the desired formula ψ that replaces ϕ . Now the claim follows from Theorem 6 and Theorem 1. \square

Example 4. The examples for ω -admissible concrete domains given in [70] were RCC8 and Allen's interval algebra, for which the patchwork property is proved “by hand” in [70]. Given our Theorem 6, we obtain these results as a consequence of known results from model theory.

It was shown in [31] that RCC8 has a representation by a homogeneous structure $\mathfrak{D}_{\text{RCC8}}$ with a finite relational signature (see Theorem 2 in [31]). Since $\text{Age}(\mathfrak{D}_{\text{RCC8}})$ has a finite universal axiomatization (see Definition 3 in [31]), $\mathfrak{D}_{\text{RCC8}}$ is finitely bounded.

Relation (Allen)	Illustration	Relation (RCC8)	Illustration
$X < Y$ $Y > X$		$X \text{ DC } Y$	
$X \text{ m } Y$ $Y \text{ mi } X$		$X \text{ EC } Y$	
$X \text{ o } Y$ $Y \text{ oi } X$		$X \text{ PO } Y$	
$X \text{ s } Y$ $Y \text{ si } X$		$X \text{ TPP } Y$ $Y \text{ TPPi } X$	
$X \text{ d } Y$ $Y \text{ di } X$		$X \text{ NTPP } Y$ $Y \text{ NTPPi } X$	
$X \text{ f } Y$ $Y \text{ fi } X$			
$X = Y$		$X = Y$	

Figure 4.1.: The basic relations of Allen and RCC8.

For Allen's interval algebra, it was shown in [53] that it has a representation by a homogeneous structure $\mathfrak{D}_{\text{Allen}}$ with a finite relational signature (see the second example on page 270 in [53]). Since $\text{Age}(\mathfrak{D}_{\text{Allen}})$ has a finite universal axiomatization (see the composition table from Figure 4 in [2]), $\mathfrak{D}_{\text{Allen}}$ is finitely bounded.

The structure $\mathfrak{Q} = (\mathbb{Q}; <)$ we used as our running example also satisfies the preconditions of Theorem 6, and thus Corollary 3 yields decidability of $\mathcal{ALC}_{\text{FO}}^d(\mathfrak{Q})$ with TBoxes. For \mathfrak{Q} expanded just by the relations $>, \leq, \geq, =, \neq$, decidability was proved in [67], using an automata-based procedure. Our results show that there is also a tableau-based decision procedure for this logic.

It also follows from our results that every finite structure can be made ω -admissible. Let \mathfrak{D} be any finite structure with a finite relational signature τ , $d \geq 2$ an arity bound on its relations. We define $\tilde{\mathfrak{D}}$ as the expansion of \mathfrak{D} by a unary relation for each domain element. Then $\tilde{\mathfrak{D}}$ is homogeneous because two substructures of $\tilde{\mathfrak{D}}$ are isomorphic if and only if they are identical. Also, $\tilde{\mathfrak{D}}$ is clearly finitely bounded because $\text{Age}(\tilde{\mathfrak{D}})$ is finite. We can take as \mathcal{N} the set of all τ -structures of size $\leq |D|$ which do not embed into $\tilde{\mathfrak{D}}$ together with all τ -structures of size $|D| + 1$. It follows from Theorem 6 that $\tilde{\mathfrak{D}}^{\leq d}$ is ω -admissible.

4.6 ω -admissible homogeneous cores with a decidable CSP

Here, we consider the situation where we have a homogeneous relational structure \mathfrak{D} with finitely many at most d -ary relations that is not necessarily finitely bounded, but which we can show (by some other means) to have a decidable CSP. In this setting, we obtain decidability for $\mathcal{ALC}_{\text{EP}}^d(\mathfrak{D})$ under the additional assumption that \mathfrak{D} is a core. A structure \mathfrak{D} is a *core* if every endomorphism of \mathfrak{D} is a self-embedding of \mathfrak{D} . It is easy to see that this applies to our

running example $\mathfrak{Q} = (\mathbb{Q}; <)$. The structure $\tilde{\mathfrak{Q}} := (\mathbb{Q}; \leq)$, on the other hand, is not a core because it has the trivial endomorphism $x \mapsto 0$ that is not a self-embedding of $\tilde{\mathfrak{Q}}$. Among ω -categorical structures, cores are characterized by the following condition. A countable ω -categorical structure \mathfrak{D} with a countable signature is a core if and only if every relation with an existential definition in \mathfrak{D} has an existential positive definition in \mathfrak{D} [19]. Consider the binary inequality relation \neq over \mathbb{Q} which clearly has an existential definition in both \mathfrak{Q} and $\tilde{\mathfrak{Q}}$. Since the complements of the basic relations defined by $=$ and $<$ in \mathfrak{Q} have a positive quantifier-free definition in \mathfrak{Q} , every relation with an existential definition in \mathfrak{Q} has an EP definition in \mathfrak{Q} . Thus, \mathfrak{Q} is a core according to the characterization of ω -categorical cores from above. However, there can be no EP definition of \neq in $\tilde{\mathfrak{Q}}$ because relations with an EP definition are always preserved by all endomorphisms and $x \mapsto 0$ does not preserve \neq .

If \mathfrak{D} is a homogeneous core, then the orbits of tuples over D under $\text{Aut}(\mathfrak{D})$ are PP definable in \mathfrak{D} [16]. As an easy consequence of this fact, we obtain the following sufficient condition for ω -admissibility.

Theorem 7. *Let \mathfrak{D} be a homogeneous core with finitely many at most d -ary relations for some $d \geq 2$ and decidable CSP. Then $\mathfrak{D}^{\leq d}$ is ω -admissible.*

Proof. It follows directly from the definition of d -ary decompositions that $\mathfrak{D}^{\leq d}$ is JDJEPD. By Proposition 6, $\mathfrak{D}^{\leq d}$ is homogeneous. By Theorem 4, $\mathfrak{D}^{\leq d}$ is ω -categorical. Thus \mathfrak{D} has homomorphism ω -compactness by Lemma 2. By Theorem 5, $\text{Age}(\mathfrak{D}^{\leq d})$ has the AP. Thus, $\mathfrak{D}^{\leq d}$ is a patchwork by Proposition 5. By the results of [16], orbits of tuples over D under $\text{Aut}(\mathfrak{D})$ (i.e., the relations of $\mathfrak{D}^{\leq d}$) are PP definable in \mathfrak{D} . Thus, Lemma 1 yields $\text{CSP}(\mathfrak{D}^{\leq d}) \leq_{\text{PTIME}} \text{CSP}(\mathfrak{D})$. Hence, $\mathfrak{D}^{\leq d}$ is ω -admissible. \square

Let \mathfrak{D} be a structure as in the above theorem. By showing that concept satisfiability in $\mathcal{ALC}_{\text{EP}}^d(\mathfrak{D})$ can be reduced to concept satisfiability in $\mathcal{ALC}_{\text{AD}}^d(\mathfrak{D}^{\leq d})$, we obtain the following decidability result.

Corollary 4. *Let \mathfrak{D} be a homogeneous core with finitely many at most d -ary relations for some $d \geq 2$ and a decidable CSP. Then concept satisfiability in $\mathcal{ALC}_{\text{EP}}^d(\mathfrak{D})$ w.r.t. TBoxes is decidable.*

Proof. Since satisfiability of $\mathcal{ALC}_{\text{AD}}^d(\mathfrak{D}^{\leq d})$ concepts w.r.t. TBoxes is decidable by Theorems 7 and 1, it is sufficient to reduce concept satisfiability w.r.t. TBoxes in $\mathcal{ALC}_{\text{EP}}^d(\mathfrak{D})$ to this problem. As in the proof of Corollary 3, we do this by showing how EP formulas ϕ occurring in concrete domain constructors can be replaced by disjunctions ψ of atomic formulas in the signature of $\mathfrak{D}^{\leq d}$. By the results of [16], the relations of $\mathfrak{D}^{\leq d}$ have PP definitions in \mathfrak{D} . Since d and \mathfrak{D} are fixed, we can make a list consisting of the PP definitions for each of them in constant time. Given an EP τ -formula ϕ with $k \leq d$ free variables, let ψ_1, \dots, ψ_m be the PP definitions in \mathfrak{D} for all the k -ary relations of $\mathfrak{D}^{\leq d}$ that we have listed before. Since $\text{CSP}(\mathfrak{D})$ is decidable, we can decide for $i \in [m]$ whether $\mathfrak{D} \models \exists \bar{y}(\psi_i(\bar{y}) \wedge \phi(\bar{y}))$. In fact, deciding whether an EP sentence is true in \mathfrak{D} only differs from solving $\text{CSP}(\mathfrak{D})$ in a non-deterministic step that deals with disjunction. By selecting those $\psi_{i_1}, \dots, \psi_{i_s}$ that tested positively, we know that, $\mathfrak{D} \models \phi(\bar{a})$

if and only if $\mathfrak{D} \models \bigvee_{r=1}^s \psi_{i_r}(\bar{a})$ holds for every $\bar{a} \in D^k$. Now we replace each $\psi_{i_r}(\bar{y})$ with $R(\bar{y})$, where R is the unique k -ary relation symbol from the signature of $\mathfrak{D}^{\leq d}$ that satisfies $\mathfrak{D} \models \psi_{i_r}(\bar{a})$ if and only if $\mathfrak{D}^{\leq d} \models R(\bar{a})$. This yields the desired formula ψ , which completes the reduction. \square

4.7 Coverage of the developed sufficient conditions

The next example demonstrates that Theorem 7 and Corollary 4 cover structures to which Theorem 6 and Corollary 3 do not apply. In fact, since the latter consider finitely bounded structures, whose CSP is in NP by Proposition 7, they cannot provide us with ω -admissible concrete domains whose CSP has a higher complexity. Theorem 7 and Corollary 4 make no assumption on the complexity of the CSP: they only require that the CSP is decidable. However, for these results to apply, the structure needs to be a homogeneous core.

Example 5. The paper [48] provides us with examples of structures that are homogeneous cores and whose CSP is considerably more complex than NP. Such structures are called CSP monsters in [48]. To be more precise, Theorem 8 in [48] shows that, for every complexity class \mathcal{C} for which there exist $\text{coNP}^{\mathcal{C}}$ -complete problems, there exists a homogeneous structure $\mathfrak{H}_{\mathcal{C}}$ with a finite signature such that $\text{CSP}(\mathfrak{H}_{\mathcal{C}})$ is $\text{coNP}^{\mathcal{C}}$ -complete. By Theorem 4 together with Theorem 3.6.23 and Proposition 3.6.24 from [17], for every such structure $\mathfrak{H}_{\mathcal{C}}$, there exists an up to isomorphism unique homogeneous core $\mathfrak{C}_{\mathcal{C}}$ with the property that $\mathfrak{H}_{\mathcal{C}}$ maps homomorphically to $\mathfrak{C}_{\mathcal{C}}$ and vice versa. In particular, this implies that $\text{CSP}(\mathfrak{H}_{\mathcal{C}}) = \text{CSP}(\mathfrak{C}_{\mathcal{C}})$. It follows from Theorem 7 that these structures yield ω -admissible concrete domains whose CSPs have arbitrarily high complexity. Recall that all previously known examples of ω -admissible concrete domains were finitely bounded (see Example 4), and thus their CSPs are in NP by Proposition 7. However, already $\mathfrak{C}_{\text{NEXPTIME}}$ cannot possibly be even a reduct of a finitely bounded structure due to Proposition 7 because $\text{NP} \subsetneq \text{NEXPTIME} \subseteq \text{coNP}^{\text{NEXPTIME}}$. Consequently, the homogeneous cores induced by the CSP monsters of [48] provide us with previously unknown ω -admissible concrete domains that are not covered by Theorem 6 and Corollary 3.

Next, we investigate the coverage of Theorem 7. This theorem states that every homogeneous core with a finite signature and a decidable CSP yields an ω -admissible structure via its d -ary decomposition. The following two results show that, if we are interested in extensions of \mathcal{ALC} of the form $\mathcal{ALC}_{\text{AD}}^d(\mathfrak{D})$, then ω -admissible structures yields the same extensions of \mathcal{ALC} as homogeneous cores with decidable CSPs.

Theorem 8. *Let \mathfrak{B} be an ω -admissible τ -structure. Then there exists an (up to isomorphism) unique countable homogeneous τ -structure \mathfrak{A} that is a core with decidable CSP and embeds the same countable structures as \mathfrak{B} , i.e., $\mathfrak{C} \hookrightarrow \mathfrak{A}$ if and only if $\mathfrak{C} \hookrightarrow \mathfrak{B}$ for every countable structure \mathfrak{C} .*

Proof. Since \mathfrak{B} is JDJEPD and a patchwork, $\text{Age}(\mathfrak{B})$ has the AP by Proposition 5. Since τ is finite, $\text{Age}(\mathfrak{B})$ contains only countably many structures up to isomorphism, and thus is an amalgamation class. By Theorem 5, there exists a countable homogeneous structure \mathfrak{A} with

$\text{Age}(\mathfrak{A}) = \text{Age}(\mathfrak{B})$. Next, we show that $\mathfrak{C} \hookrightarrow \mathfrak{A}$ if and only if $\mathfrak{C} \hookrightarrow \mathfrak{B}$ holds for every countable structure \mathfrak{C}

“ \Leftarrow ”: Let \mathfrak{C} be a countable τ -structure that embeds into \mathfrak{B} . By Theorem 4, \mathfrak{A} is ω -categorical. It is known that ω -categorical structures satisfy an even stronger property than homomorphism ω -compactness, which we refer to as *embedding ω -compactness* (Lemma 3.1.5 in [17]). This property guarantees an embedding from a given countable structure if there exists an embedding from every structure in its age. Since $\text{Age}(\mathfrak{C}) \subseteq \text{Age}(\mathfrak{B}) = \text{Age}(\mathfrak{A})$, we conclude that $\mathfrak{C} \hookrightarrow \mathfrak{A}$.

“ \Rightarrow ”: Let \mathfrak{C} be a countable τ -structure and $e : \mathfrak{C} \hookrightarrow \mathfrak{A}$ be an embedding. If we can show that there is an embedding $f : \mathfrak{A} \hookrightarrow \mathfrak{B}$, then we are done since we can use the composition of e and f as embedding from \mathfrak{C} to \mathfrak{B} . Since \mathfrak{A} is countable, \mathfrak{B} has homomorphism ω -compactness, and $\text{Age}(\mathfrak{B}) = \text{Age}(\mathfrak{A})$, there exists a homomorphism $f : \mathfrak{A} \rightarrow \mathfrak{B}$. We show that f is in fact an embedding.

We claim that \mathfrak{A} is JEPD since \mathfrak{B} is so. In fact, assume the \mathfrak{A} is not PD. Then there are distinct k -ary relations R_1, R_2 and a k -tuple \bar{a} such that $\bar{a} \in R_1^{\mathfrak{A}} \cap R_2^{\mathfrak{A}}$. Thus, the substructure of \mathfrak{A} consisting of the elements of \bar{a} is an element of $\text{Age}(\mathfrak{A})$ that is not PD. But then $\text{Age}(\mathfrak{B}) = \text{Age}(\mathfrak{A})$ contains a structure that is not PD, which yields a contradiction since \mathfrak{B} is PD. The fact that JE transfers from \mathfrak{B} to \mathfrak{A} can be shown similarly. If $\phi(x, y)$ is the formula witnessing that \mathfrak{B} is JD, then one can also show in a similar way that this formula witnesses JD of \mathfrak{A} as well.

Since we now know that both \mathfrak{A} and \mathfrak{B} are JEPD, we can apply Lemma 2, which yields that the homomorphism f preserves also the complements of all relations of \mathfrak{A} . In addition, it preserves the formula ϕ witnessing JD. Thus, the following holds for all $a_1, a_2 \in A$: $a_1 = a_2$ if and only if $\mathfrak{A} \models \phi(a_1, a_2)$ if and only if $\mathfrak{B} \models \phi(f(a_1), f(a_2))$ if and only if $f(a_1) = f(a_2)$. Thus f is an embedding, which concludes the proof of “ \Rightarrow ”.

Since \mathfrak{A} is JDJEPD, every endomorphism of \mathfrak{A} is a self-embedding of \mathfrak{A} , which can be shown by a similar argument as above. Thus \mathfrak{A} is a core.

Decidability of the CSP transfers from \mathfrak{B} to \mathfrak{A} since the two CSPs coincide. If \mathfrak{F} is a finite structure with $\mathfrak{F} \rightarrow \mathfrak{A}$, then the image \mathfrak{C} of \mathfrak{F} in \mathfrak{A} is a finite (and thus countable) structure such that $\mathfrak{F} \rightarrow \mathfrak{C} \hookrightarrow \mathfrak{A}$. But then $\mathfrak{C} \hookrightarrow \mathfrak{B}$, and thus $\mathfrak{F} \rightarrow \mathfrak{B}$. The inclusion in the other direction can be shown in the same way. \square

Since the structures \mathfrak{A} and \mathfrak{B} in the theorem have the same signature, the DLs $\mathcal{ALC}_{\text{AD}}^d(\mathfrak{A})$ and $\mathcal{ALC}_{\text{AD}}^d(\mathfrak{B})$ have the same syntax. We show that they also have the same semantics when it comes to concept satisfiability.

Corollary 5. *Let \mathfrak{A} and \mathfrak{B} be as in Theorem 8 and let d be the largest arity of an atomic τ -formula. Then a concept C is satisfiable w.r.t. a TBox \mathcal{T} in $\mathcal{ALC}_{\text{AD}}^d(\mathfrak{A})$ if and only if it is satisfiable in $\mathcal{ALC}_{\text{AD}}^d(\mathfrak{B})$ w.r.t. \mathcal{T} .*

Proof. First note that, since \mathfrak{B} is ω -admissible, the DL $\mathcal{ALC}_{\text{AD}}^d(\mathfrak{B})$ has the *countable model property*, i.e., a concept C is satisfiable in this logic w.r.t. a TBox \mathcal{T} if and only if there is a finite model of \mathcal{T} in which C is interpreted as a non-empty set. This is a direct consequence of the proof of Theorem 1 in [11] because the model constructed in this proof is countable. Now

4. A Model-Theoretic Analysis of ω -Admissibility

suppose that \mathcal{I} is a countable interpretation witnessing that C is satisfiable w.r.t. \mathcal{T} in $\mathcal{ALC}_{AD}^d(\mathfrak{B})$. Let \mathfrak{C} be the substructure of \mathfrak{B} on $\{b \in B \mid \text{there is } f \in N_F \text{ and } a \in \Delta^{\mathcal{I}} \text{ such that } (a, b) \in f^{\mathcal{I}}\}$. By Theorem 8, there exists an embedding $e: \mathfrak{C} \hookrightarrow \mathfrak{A}$. Since e is an embedding, we can obtain an interpretation witnessing that C is satisfiable w.r.t. \mathcal{T} in $\mathcal{ALC}_{AD}^d(\mathfrak{A})$ from \mathcal{I} by replacing every $(a, d) \in f^{\mathcal{I}}$ with $(a, e(d))$.

The argument used above also works the other way around. Here \mathfrak{A} is a homogeneous core with decidable CSP. The proof of Corollary 4 shows that satisfiability of concepts w.r.t. TBoxes in $\mathcal{ALC}_{AD}^d(\mathfrak{A})$ can be reduced to satisfiability of concepts w.r.t. TBoxes in $\mathcal{ALC}_{AD}^d(\mathfrak{D})$ for an ω -admissible concrete domain \mathfrak{D} . As above, we can show that this yields the countable model property for $\mathcal{ALC}_{AD}^d(\mathfrak{A})$. The rest of the proof is exactly as for the other direction. \square

The following example shows that equi-satisfiability no longer holds if we replace AD with FO in Corollary 5, i.e., the logics $\mathcal{ALC}_{FO}^d(\mathfrak{A})$ and $\mathcal{ALC}_{FO}^d(\mathfrak{B})$ may have a different semantics.

Example 6. The *random graph* is the unique countably infinite homogeneous undirected graph $\mathfrak{G} = (G; E^{\mathfrak{G}})$ such that $\text{Age}(\mathfrak{G})$ consists of all finite undirected graphs [55]. Note that $\text{Age}(\mathfrak{G})$ is defined by $\forall x, y (E(x, y) \Rightarrow E(y, x)) \wedge \forall x (E(x, x) \Rightarrow \text{false})$. Thus, by Lemma 3, \mathfrak{G} is finitely bounded. It also has the *extension property*: if X and Y are disjoint finite subsets of G , then there exists a vertex $v \in G \setminus (X \cup Y)$ that has an edge in \mathfrak{G} to each vertex from X and to none from Y . To see this, let \mathfrak{A} be the extension of the substructure of \mathfrak{G} on $X \cup Y$ by a vertex u that has an edge to each vertex from X and to none from Y . Then there exists an embedding $e: \mathfrak{A} \hookrightarrow \mathfrak{G}$. Since \mathfrak{G} is homogeneous and the map $f: X \cup Y \rightarrow e(X \cup Y)$, $a \mapsto e(a)$ is an isomorphism between its finite substructures, there exists $\tilde{f} \in \text{Aut}(\mathfrak{G})$ extending f . Then $v := \tilde{f}^{-1}(e(u))$ has the desired property.

Consider the direct product \mathfrak{H} of \mathfrak{G} with itself. It is easy to see that $\text{Age}(\mathfrak{G}) = \text{Age}(\mathfrak{H})$. The inclusion from left to right holds since \mathfrak{H} contains an isomorphic copy of \mathfrak{G} , and the one from right to left since $\text{Age}(\mathfrak{G})$ contains all finite undirected graphs. The equality of the two ages implies that $\text{Age}(\mathfrak{H})$ has the AP by Theorem 5. Also, by Theorem 3 and Theorem 4, \mathfrak{H} is ω -categorical because its relations are first-order definable in the algebraic product of \mathfrak{G} with itself and homogeneous structures are closed under building algebraic products. We will introduce the algebraic product and show that it preserves homogeneity in Section 4.8. However, \mathfrak{H} does not have the extension property. To see this, let a, b, c be three distinct vertices in G and set $X := \{(a, b), (b, c)\}$, and $Y := \{(a, c)\}$. Suppose that there exists $(u, v) \in H$ that has an edge in \mathfrak{H} to each vertex from X and to none from Y . By the definition of \mathfrak{H} as the direct product of \mathfrak{G} with itself, there is an edge in \mathfrak{G} from u to a and from v to c . But then there is an edge from (u, v) to (a, c) in \mathfrak{H} , which contradicts to our previous assumption. This implies that \mathfrak{G} and \mathfrak{H} are not isomorphic since the extension property is clearly preserved under isomorphism. We conclude that \mathfrak{H} is not homogeneous since homogeneous structures are uniquely determined up to isomorphism by their age due to Theorem 5. Note that we have just shown with this example that homogeneous structures are not closed under building direct products.

Now consider the expansion \mathfrak{A} of \mathfrak{G} with two new relation symbols R_1, R_2 , where R_1 is interpreted as the equality Eq_G and R_2 as $G^2 \setminus (\text{Eq}_G \cup E^{\mathfrak{G}})$. Likewise we construct the expansion

\mathfrak{B} of \mathfrak{H} with R_1, R_2 . Let \mathfrak{C} be a substructure of \mathfrak{A} and $\tilde{\mathfrak{C}}$ its $\{E\}$ -reduct. Since $\text{Age}(\mathfrak{C}) = \text{Age}(\mathfrak{H})$, there exists an isomorphism f from $\tilde{\mathfrak{C}}$ to some substructure $\tilde{\mathfrak{D}}$ of \mathfrak{H} . Let \mathfrak{D} be the substructure of \mathfrak{B} on the domain $\tilde{\mathfrak{D}}$ of $\tilde{\mathfrak{D}}$. We claim that f is also an isomorphism from \mathfrak{C} to \mathfrak{D} . We have $\bar{x} \in R_1^{\mathfrak{C}} = \text{Eq}_{\mathfrak{C}}$ if and only if $f(\bar{x}) \in R_1^{\mathfrak{D}} = \text{Eq}_{\mathfrak{D}}$ because f is bijective. Moreover, we have $\bar{x} \in R_2^{\mathfrak{C}}$ if and only if $\bar{x} \notin (E^{\tilde{\mathfrak{C}}} \cup \text{Eq}_{\mathfrak{C}})$ if and only if $f(\bar{x}) \notin (E^{\tilde{\mathfrak{D}}} \cup \text{Eq}_{\mathfrak{D}})$ if and only if $f(\bar{x}) \in R_2^{\mathfrak{D}}$. We conclude that $\text{Age}(\mathfrak{A}) \subseteq \text{Age}(\mathfrak{B})$. Using an analogous argument, we can show $\text{Age}(\mathfrak{A}) \supseteq \text{Age}(\mathfrak{B})$, and thus $\text{Age}(\mathfrak{A}) = \text{Age}(\mathfrak{B})$. Since every homomorphism from a finite structure has a finite range, $\text{Age}(\mathfrak{A}) = \text{Age}(\mathfrak{B})$ implies $\text{CSP}(\mathfrak{A}) = \text{CSP}(\mathfrak{B})$ (see the last paragraph in the proof of Theorem 8).

The following two facts are direct consequences of $R_1^{\mathfrak{A}}$ and $R_2^{\mathfrak{A}}$ being first-order definable in \mathfrak{C} . First, \mathfrak{A} is homogeneous since \mathfrak{C} is homogeneous and first-order definable relations are preserved by automorphisms. Thus, $\text{Age}(\mathfrak{A})$ has the AP by Theorem 5. Second, \mathfrak{A} is a reduct of a finitely bounded structure by Proposition 8, and thus $\text{CSP}(\mathfrak{A})$ is in NP by Proposition 7.

By definition, \mathfrak{B} is JDJEPD. Since $\text{Age}(\mathfrak{B}) = \text{Age}(\mathfrak{A})$ has the AP, the structure \mathfrak{B} is a patchwork by Proposition 5. By Lemma 2, \mathfrak{B} has homomorphism ω -compactness since it is ω -categorical. This is the case since \mathfrak{H} is ω -categorical and expansions by first-order definable relations do not change the automorphism group. Since \mathfrak{A} and \mathfrak{B} are both countable but not isomorphic, we conclude using Theorem 5 that \mathfrak{B} is ω -admissible but not homogeneous. Since $\text{Age}(\mathfrak{A}) = \text{Age}(\mathfrak{B})$ and \mathfrak{A} is countable and homogeneous, it must be the homogeneous core of \mathfrak{B} from Theorem 8.

It follows from our proof that \mathfrak{H} does not have the extension property that the concept A is satisfiable w.r.t. the TBox

$$\mathcal{T} := \{A \sqsubseteq \exists f. (x_1 = x_1 \wedge \forall x, y, z. \exists u (E(u, x) \wedge E(u, y) \wedge \neg E(u, z)))\}$$

in $\mathcal{ALC}_{\text{FO}}^2(\mathfrak{A})$, but not in $\mathcal{ALC}_{\text{FO}}^2(\mathfrak{B})$.

4.8 Closure properties: homogeneity & finite boundedness

We have seen above that finitely bounded homogeneous structures provide us with ω -admissible concrete domains. Closure properties allow us to construct new ω -admissible concrete domains from ones satisfying these properties.

For instance, when modelling concepts in a DL with a concrete domain, it is often useful to be able to refer to specific elements c of the domain, i.e., to have unary predicate symbols $=_c$ that are interpreted as $\{c\}$. For example, when using the ω -admissible concrete domain \mathfrak{Q} of our running example, one can compare two numbers (e.g., describing the ages of two individuals), but one cannot state that the value of a feature must be equal to some fixed number (e.g., that a person's age is 17). For a finitely bounded homogeneous structure (such as \mathfrak{Q}), adding *finitely many* such singleton predicates is harmless since the class of reducts of finitely bounded homogeneous structures is closed under expansion by finitely many predicates of the form $=_c$.

4. A Model-Theoretic Analysis of ω -Admissibility

Proposition 9 ([29]). *Let \mathfrak{A} be a finitely bounded homogeneous structure. Any expansion of \mathfrak{A} by a relation of the form $\{c\}$ for $c \in A$ is a reduct of a finitely bounded homogeneous structure.*

We have seen in Proposition 8 that this class is also closed under taking expansions by first-order definable relations.

It would also be useful to be able to refer to predicates of different concrete domains (say RCC8 and Allen) when defining concepts. This can sometimes be achieved by using the disjoint union. The *union* of a family $(\mathfrak{A}_i)_{i \in I}$ of τ -structures is the τ -structure $\bigcup_{i \in I} \mathfrak{A}_i$ over $\bigcup_{i \in I} A_i$ such that $R^{\bigcup_{i \in I} \mathfrak{A}_i} = \bigcup_{i \in I} R^{\mathfrak{A}_i}$ for each $R \in \tau$. This union is called *disjoint* if $A_i \cap A_j = \emptyset$ for all distinct $i, j \in I$.

In [7], it was shown that admissible concrete domains are closed under disjoint union. We can show the corresponding result for finitely bounded homogeneous structures. In our definition of the disjoint union, we have assumed that the component structures $\mathfrak{A}_1, \dots, \mathfrak{A}_k$ have the same signature, but disjoint domains. In [7], the signatures of the structures are assumed to be disjoint as well (as is, e.g., the case for RCC8 and Allen). The case of disjoint signatures can, however, be reduced to the case of a common signature: we simply expand the structures to the union of their signatures by interpreting relation symbols not belonging to the respective signature as the empty set. Since empty relations can be defined by first-order formulas, such an expansion by empty relations leaves homogeneity and finite boundedness intact (see Proposition 8).

Proposition 10. *Let $\mathfrak{A}_1, \dots, \mathfrak{A}_k$ be finitely bounded homogeneous structures over a common signature τ , but with disjoint domains. Then their disjoint union $\bigcup_{i=1}^k \mathfrak{A}_i$ is a reduct of a finitely bounded homogeneous structure.*

Proof. For brevity we write \mathfrak{A} for the disjoint union $\bigcup_{i=1}^k \mathfrak{A}_i$. Let σ be the signature τ extended by a unary symbol D_i for each $i \in [k]$. Consider the σ -expansion \mathfrak{A}' of \mathfrak{A} where $D_i^{\mathfrak{A}'} = A_i$ for each $i \in [k]$.

To show that \mathfrak{A}' is homogeneous, we first observe the following. If, for each $i \in [k]$, f_i is an automorphism of \mathfrak{A}_i , then the map $f : A \rightarrow A$ satisfying $f|_{A_i} := f_i$ is an automorphism of \mathfrak{A}' since it additionally preserves $D_i^{\mathfrak{A}'}$ for each $i \in [k]$. Conversely, if f is an automorphism of \mathfrak{A}' , then $f|_{A_i}$ is an automorphism of \mathfrak{A}_i for each $i \in [k]$. Now, let $f : \mathfrak{B}_1 \rightarrow \mathfrak{B}_2$ be an isomorphism between two finite substructures of \mathfrak{A}' . Since f preserves $D_i^{\mathfrak{B}_1} = B_1 \cap A_i$ for each $i \in [k]$, the restrictions $f|_{B_1 \cap A_i}$ are isomorphisms, and thus extend to automorphism of \mathfrak{A}_i for each $i \in [k]$ by homogeneity of the structures \mathfrak{A}_i . By the observation about automorphisms above, this implies that f itself extends to an automorphism of \mathfrak{A}' .

Next we show that \mathfrak{A}' is finitely bounded. For each $i \in [k]$, let

$$\Phi(\mathfrak{A}_i) = \forall x_{i_1}, \dots, x_{i_{n_i}}. \phi_i(x_{i_1}, \dots, x_{i_{n_i}}) \text{ with } \phi_i \text{ quantifier-free}$$

be a universal sentence that defines $\text{Age}(\mathfrak{A}_i)$. Now consider the universal sentence $\Phi(\mathfrak{A}') :=$

$\Phi_1 \wedge \Phi_2$, where

$$\begin{aligned}\Phi_1 &:= \forall x \left(\left(\bigwedge_{i,j \in [k], i \neq j} \neg (D_i(x) \wedge D_j(x)) \right) \wedge \left(\bigvee_{i \in [k]} D_i(x) \right) \right), \\ \Phi_2 &:= \bigwedge_{i \in [k]} \forall x_{i_1}, \dots, x_{i_{n_i}} \left(\bigwedge_{j=1}^{n_i} D_i(x_{i_j}) \Rightarrow \phi_i(x_{i_1}, \dots, x_{i_{n_i}}) \right).\end{aligned}$$

Let \mathfrak{B} be a finite σ -structure that satisfies $\Phi(\mathfrak{A}')$. By Φ_1 , the unary relations $D_i^{\mathfrak{B}}$ are pairwise disjoint and exhaustive. By Φ_2 , the τ -reduct of the substructure of \mathfrak{B} on $D_i^{\mathfrak{B}}$ is contained in $\text{Age}(\mathfrak{A}_i)$ for each $i \in [k]$. Hence \mathfrak{B} is a substructure of \mathfrak{A}' . Conversely, every finite substructure of \mathfrak{A}' must satisfy $\Phi(\mathfrak{A}')$. This completes the proof as \mathfrak{A} is the τ -reduct of \mathfrak{A}' . \square

Using disjoint union to refer to several concrete domain works well if the paths employed in concrete domain constructors contain only functional roles, which is the case considered in [7], but it is not appropriate if non-functional roles occur in paths, as in this thesis. This is illustrated by the following example.

Example 7. If we want to refer to time and location of an event, we can use the disjoint union of RCC8 and Allen, employing two feature names *time* and *location*. If *succ* is a functional role, then the concept description

$$\text{Event} \sqcap \exists \text{succ}. \text{Event} \sqcap \exists \text{time}, \text{succ time}. < (x_1, x_2) \sqcap \exists \text{location}, \text{succ location}. \text{EC}(x_1, x_2)$$

describes an event e that takes place before its unique successor event e' , which happens in a region that is externally connected to e . However, if *succ* is not functional, then the above concept description does not express that e has a successor event e' that satisfies both the temporal and the spatial constraint. Instead, there could be two different successor events, one satisfying the temporal constraint and the other the spatial one.

To overcome this problem, we propose to use the algebraic product [17].² Let $\mathfrak{A}_1, \dots, \mathfrak{A}_k$ be relational structures with disjoint signatures τ_1, \dots, τ_k . Furthermore, let $=_1, \dots, =_k$ be fresh binary symbols such that, for every $i \in [k]$, $=_i$ is interpreted as $\text{Eq}_{\mathfrak{A}_i}$ over \mathfrak{A}_i . We assume in the following that the relation $=_i$ is part of the signature of \mathfrak{A}_i . This assumption is without loss of generality since the equality predicate is first-order definable, and thus extending a homogeneous structure with an explicit relation symbol for it leaves the structure finitely bounded and homogeneous (see Proposition 8).

The *algebraic product* of $\mathfrak{A}_1, \dots, \mathfrak{A}_k$, denoted by $\mathfrak{A}_1 \boxtimes \dots \boxtimes \mathfrak{A}_k$, has as its domain the Cartesian product $A := A_1 \times \dots \times A_k$ and as its signature the union of the signatures τ_i . The relations of $\mathfrak{A} := \mathfrak{A}_1 \boxtimes \dots \boxtimes \mathfrak{A}_k$ are defined by $R^{\mathfrak{A}} := \{(\bar{a}_1, \dots, \bar{a}_n) \in A^n \mid (\bar{a}_1[i], \dots, \bar{a}_n[i]) \in R^{\mathfrak{A}_i}\}$ for every $i \in [k]$ and every n -ary relation $R \in \tau_i$.

Taking the algebraic product of structures preserves homogeneity and finite boundedness, and thus the prerequisites for Theorem 6 and Corollary 3 to apply.

²We have seen in Example 6 that employing the usual direct product of the structures does not work since it does not preserve homogeneity.

4. A Model-Theoretic Analysis of ω -Admissibility

Proposition 11. *Let $\mathfrak{A}_1, \dots, \mathfrak{A}_k$ be structures with disjoint relational signatures τ_1, \dots, τ_k such that, for $i \in [k]$, τ_i contains the symbol $=_i$, which is defined in \mathfrak{A}_i as Eq_{A_i} .*

1. *If $\mathfrak{A}_1, \dots, \mathfrak{A}_k$ are homogeneous, then $\mathfrak{A}_1 \boxtimes \dots \boxtimes \mathfrak{A}_k$ is also homogeneous.*
2. *If $\mathfrak{A}_1, \dots, \mathfrak{A}_k$ are finitely bounded, then $\mathfrak{A}_1 \boxtimes \dots \boxtimes \mathfrak{A}_k$ is also finitely bounded.*

Proof. By proj_i we denote the usual projection function $\text{proj}_i: A_1 \times \dots \times A_k \rightarrow A_i$ with $\text{proj}_i(\bar{t}) = \bar{t}[i]$. We use the abbreviation $\mathfrak{A} := \mathfrak{A}_1 \boxtimes \dots \boxtimes \mathfrak{A}_k$ and denote the signature of \mathfrak{A} by τ .

For item 1, let $f: \mathfrak{B}_1 \rightarrow \mathfrak{B}_2$ be an isomorphism between two finite substructures \mathfrak{B}_1 and \mathfrak{B}_2 of \mathfrak{A} . For every $i \in [k]$, we define $\mathfrak{B}_{1,i}$ and $\mathfrak{B}_{2,i}$ as the substructure of \mathfrak{A}_i on $\text{proj}_i(B_1)$ and $\text{proj}_i(B_2)$, respectively. For every $i \in [k]$ and $R \in \tau_i \cup \{=_i\}$, the relation $R^{\mathfrak{B}_1}$ is preserved by f . Consider the map $f_i: \mathfrak{B}_{1,i} \rightarrow \mathfrak{B}_{2,i}$ given by $f_i(\bar{t}[i]) := f(\bar{t})[i]$. It is well defined, since for any $\bar{t}, \bar{t}' \in B_1$ with $\bar{t}[i] = \bar{t}'[i]$, we have $(\bar{t}, \bar{t}') \in =_i^{\mathfrak{B}_1}$, which implies $f(\bar{t})[i] = f(\bar{t}')[i]$, because $=_i$ is preserved by f . Since f is an isomorphism, the previous argument can also be read backwards, which implies that f_i is injective. It follows directly from the definition of f_i that it is surjective, because f is surjective. Finally, f_i is an isomorphism since, for every $R \in \tau_i \cup \{=_i\}$, we have

$$\begin{aligned}
 (\bar{t}_1[i], \dots, \bar{t}_k[i]) \in R^{\mathfrak{B}_{1,i}} & \text{ iff } (\bar{t}_1[i], \dots, \bar{t}_k[i]) \in R^{\mathfrak{A}_i} \cap \text{proj}_i(B_1)^k \\
 & \text{ iff } (\bar{t}_1, \dots, \bar{t}_k) \in R^{\mathfrak{A}} \cap B_1^k \\
 & \text{ iff } (\bar{t}_1, \dots, \bar{t}_k) \in R^{\mathfrak{B}_1} \\
 & \text{ iff } (f(\bar{t}_1), \dots, f(\bar{t}_k)) \in R^{\mathfrak{B}_2} \\
 & \text{ iff } (f(\bar{t}_1), \dots, f(\bar{t}_k)) \in R^{\mathfrak{A}} \cap B_2^k \\
 & \text{ iff } (f(\bar{t}_1)[i], \dots, f(\bar{t}_k)[i]) \in R^{\mathfrak{A}_i} \cap \text{proj}_i(B_2)^k \\
 & \text{ iff } (f(\bar{t}_1)[i], \dots, f(\bar{t}_k)[i]) \in R^{\mathfrak{B}_{2,i}} \\
 & \text{ iff } (f_i(\bar{t}_1[i]), \dots, f_i(\bar{t}_k[i])) \in R^{\mathfrak{B}_{2,i}}.
 \end{aligned}$$

Each f_i extends to an automorphism f'_i of \mathfrak{A}_i , because \mathfrak{A}_i is homogeneous. Let f' be the map from A to A defined by $f'(\bar{t}) := f'_1(\bar{t}[1]), \dots, f'_k(\bar{t}[k])$. Clearly, f' is bijective because each f'_i is bijective. Let $R \in \tau$ be an arbitrary and n its arity. Then $R \in \tau_i \cup \{=_i\}$ for some $i \in [k]$. Since f'_i is an automorphism of \mathfrak{A}_i , for every $(\bar{t}_1, \dots, \bar{t}_n) \in A^n$, we have

$$\begin{aligned}
 (\bar{t}_1, \dots, \bar{t}_n) \in R^{\mathfrak{A}} & \text{ iff } (\bar{t}_1[i], \dots, \bar{t}_n[i]) \in R^{\mathfrak{A}_i} \\
 & \text{ iff } (f'_i(\bar{t}_1[i]), \dots, f'_i(\bar{t}_n[i])) \in R^{\mathfrak{A}_i} \\
 & \text{ iff } (f'(\bar{t}_1)[i], \dots, f'(\bar{t}_n)[i]) \in R^{\mathfrak{A}_i} \\
 & \text{ iff } (f'(\bar{t}_1), \dots, f'(\bar{t}_n)) \in R^{\mathfrak{A}}.
 \end{aligned}$$

Hence, f' is an automorphism of \mathfrak{A} . It follows from the definition of f_i that f' extends f .

For item 2 let, for each $i \in [k]$, $\Phi(\mathfrak{A}_i)$ be the universal sentence that defines $\text{Age}(\mathfrak{A}_i)$. Let $\Phi'(\mathfrak{A}_i)$ be the sentence obtained from $\Phi(\mathfrak{A}_i)$ by replacing each occurrence of an atomic formula of the form $(x = y)$ in $\Phi(\mathfrak{A}_i)$ by $(x =_i y)$. Furthermore, for each symbol $R \in \tau_i$ of arity n other

than $=_i$, let Φ_R be the sentence

$$\forall x_1, \dots, x_n, y_1, \dots, y_n \left(\bigwedge_{j=1}^n x_j =_i y_j \Rightarrow (R(x_1, \dots, x_n) \Leftrightarrow R(y_1, \dots, y_n)) \right).$$

Now consider the τ -sentence $\Phi(\mathfrak{A}) := \Phi_1 \wedge \Phi_2 \wedge \bigwedge_{i=1}^k \Phi'(\mathfrak{A}_i) \wedge \bigwedge_{R \in \tau \setminus \{=_1, \dots, =_k\}} \Phi_R$, where

$$\begin{aligned} \Phi_1 &:= \bigwedge_{i=1}^k \forall x, y, z (x =_i x \wedge (x =_i y \Leftrightarrow y =_i x) \wedge (x =_i y \wedge y =_i z \Rightarrow x =_i z)) \\ \Phi_2 &:= \forall x, y (x = y \Leftrightarrow \bigwedge_{i=1}^k x =_i y) \end{aligned}$$

We claim that $\Phi(\mathfrak{A})$ defines $\text{Age}(\mathfrak{A})$.

For the forward direction, let \mathfrak{B} be a finite substructure of $\mathfrak{A} := \mathfrak{A}_1 \boxtimes \dots \boxtimes \mathfrak{A}_k$. By the definition of \boxtimes , the relation $=_i^{\mathfrak{A}}$ is an equivalence relation for each $i \in [k]$, because $=_i^{\mathfrak{A}_i}$ is an equivalence relation. Since \mathfrak{B} is a substructure of \mathfrak{A} , $=_i^{\mathfrak{B}}$ is an equivalence relation for each $i \in [k]$ as well. Thus $\mathfrak{B} \models \Phi_1$. For all $\bar{t}, \bar{s} \in B$ we have $\bar{t} = \bar{s}$ if and only if $\bar{t} =_i^{\mathfrak{B}} \bar{s}$ for each $i \in [k]$, because $=_i^{\mathfrak{B}}$ stands for the equality in the i -th coordinate. Thus $\mathfrak{B} \models \Phi_2$. Let \mathfrak{B}_i be the substructure of \mathfrak{A}_i on $\text{proj}_i(B)$. As a substructure of \mathfrak{A}_i , \mathfrak{B}_i satisfies $\Phi(\mathfrak{A}_i)$ because $\Phi(\mathfrak{A}_i)$ defines $\text{Age}(\mathfrak{A}_i)$. But then \mathfrak{B}_i must also satisfy $\Phi'(\mathfrak{A}_i)$ because $=_i$ interprets as the binary equality predicate in \mathfrak{B}_i . We claim that \mathfrak{B} satisfies $\Phi'(\mathfrak{A}_i)$ for each $i \in [k]$. Let $\bar{t}_1, \dots, \bar{t}_m \in B$ be any tuples to be substituted for the universally quantified variables x_1, \dots, x_m of $\Phi'(\mathfrak{A}_i)$. Let $\psi'(x_1, \dots, x_m)$ be a formula in DNF equivalent to the quantifier-free part of $\Phi'(\mathfrak{A}_i)$. Let ψ^* be a disjunct in ψ' such that $\mathfrak{B}_i \models \psi^*(\bar{t}_{i_1[i]}, \dots, \bar{t}_{i_n[i]})$. Recall that $\Phi'(\mathfrak{A}_i)$ contains no atomic formulas of the form $(x = y)$. Also recall that, for every n -ary symbol $R \in \tau_i$, we have $(\bar{t}_{i_1}, \dots, \bar{t}_{i_n}) \in R^{\mathfrak{B}}$ if and only if $(\bar{t}_{i_1[i]}, \dots, \bar{t}_{i_n[i]}) \in R^{\mathfrak{B}_i}$ by the definition of \boxtimes . This means that, if ψ^* contains an atomic formula of the form $R(x_{i_1}, \dots, x_{i_n})$ for some n -ary symbol $R \in \tau_i$, then we have $\mathfrak{B}_i \models R(\bar{t}_{i_1[i]}, \dots, \bar{t}_{i_n[i]})$ if and only if $\mathfrak{B} \models R(\bar{t}_{i_1}, \dots, \bar{t}_{i_n})$. Likewise we have $\mathfrak{B}_i \models \neg R(\bar{t}_{i_1[i]}, \dots, \bar{t}_{i_n[i]})$ if and only if $\mathfrak{B} \models \neg R(\bar{t}_{i_1}, \dots, \bar{t}_{i_n})$. Since $\mathfrak{B} \models \psi^*(\bar{t}_1, \dots, \bar{t}_m)$ and $\bar{t}_1, \dots, \bar{t}_n$ were chosen arbitrarily, we conclude that $\mathfrak{B} \models \Phi'(\mathfrak{A}_i)$. It follows directly from the argumentation above and the fact that $=_i$ interprets as the binary equality predicate in \mathfrak{A}_i that $\mathfrak{B} \models \Phi_R$ for each $R \in \tau \setminus \{=_1, \dots, =_k\}$. Hence $\mathfrak{B} \models \Phi(\mathfrak{A})$.

For the backward direction, let \mathfrak{B} be a finite τ -structure that satisfies $\Phi(\mathfrak{A})$. Then $=_i^{\mathfrak{B}}$ is an equivalence relation for each $i \in [k]$. For each $i \in [k]$, consider the following τ_i -structure \mathfrak{B}_i . The domain of \mathfrak{B}_i consists of the equivalence classes w.r.t. $=_i^{\mathfrak{B}}$. Moreover, for each n -ary symbol $R \in \tau_i$, we have $(X_1, \dots, X_n) \in R^{\mathfrak{B}_i}$ if and only if $(b_1, \dots, b_n) \in R^{\mathfrak{B}}$ for some representatives $b_i \in X_i$. The relations of \mathfrak{B}_i are well-defined because $\mathfrak{B} \models \Phi_R$ for each $R \in \tau \setminus \{=_1, \dots, =_k\}$. We claim that $\mathfrak{B}_i \models \Phi'(\mathfrak{A}_i)$ for each $i \in [k]$. Recall that $\Phi'(\mathfrak{A}_i)$ contains no atomic formulas of the form $(x = y)$. Let X_1, \dots, X_m be any equivalence classes of elements from B w.r.t. $=_i^{\mathfrak{B}}$ to be substituted for the universally quantified variables x_1, \dots, x_m of $\Phi'(\mathfrak{A}_i)$, and b_1, \dots, b_m any representatives of these equivalence classes, respectively. Let $\psi'(x_1, \dots, x_m)$ be a formula in DNF equivalent to the quantifier-free part of $\Phi'(\mathfrak{A}_i)$. Since $\mathfrak{B} \models \Phi'(\mathfrak{A}_i)$, we have that $\mathfrak{B} \models \psi'(b_1, \dots, b_m)$. Let ψ^* be a disjunct in ψ' such that $\mathfrak{B} \models \psi^*(b_1, \dots, b_m)$.

4. A Model-Theoretic Analysis of ω -Admissibility

If ψ^* contains an atomic formula of the form $(x_{i_1} =_i x_{i_2})$, then we have $\mathfrak{B} \models (b_{i_1} =_i b_{i_2})$. This means that b_{i_1} and b_{i_2} are contained in the same equivalence class w.r.t. $=_i^{\mathfrak{B}}$, that is, $X_{i_1} = X_{i_2}$. We conclude that $\mathfrak{B}_i \models (X_{i_1} =_i X_{i_2})$ because the symbol $=_i$ interprets in \mathfrak{B}_i as the binary equality predicate. If ψ^* contains the negation of an atomic formula of the form $(x_{i_1} =_i x_{i_2})$, then we have $\mathfrak{B} \models \neg(b_{i_1} =_i b_{i_2})$ which means that b_{i_1} and b_{i_2} are contained in distinct equivalence classes. Then clearly $\mathfrak{B}_i \models \neg(X_{i_1} =_i X_{i_2})$.

If ψ^* contains an atomic formula of the form $R(x_{i_1}, \dots, x_{i_n})$ for some n -ary symbol $R \in \tau_i \setminus \{=_i\}$, then we have $\mathfrak{B} \models R(b_{i_1}, \dots, b_{i_n})$. It follows directly from the definition of \mathfrak{B}_i that $\mathfrak{B}_i \models R(X_{i_1}, \dots, X_{i_n})$. If ψ^* contains the negation of an atomic formula of the form $R(x_{i_1}, \dots, x_{i_n})$ for some n -ary symbol $R \in \tau_i$, then we have $\mathfrak{B} \models \neg R(b_{i_1}, \dots, b_{i_n})$. Suppose that $(X_{i_1}, \dots, X_{i_n}) \in R^{\mathfrak{B}_i}$. Then $(b'_{i_1}, \dots, b'_{i_n}) \in R^{\mathfrak{B}}$ for some representatives b'_{i_ℓ} of X_{i_ℓ} . But then $(b_{i_1}, \dots, b_{i_n}) \in R^{\mathfrak{B}}$ because $\mathfrak{B} \models \Phi_R$, a contradiction. Thus $\mathfrak{B}_i \models \neg R(X_{i_1}, \dots, X_{i_n})$.

Since $\mathfrak{B}_i \models \psi'(X_1, \dots, X_m)$ and X_1, \dots, X_m were chosen arbitrarily, we conclude that $\mathfrak{B}_i \models \Phi'(\mathfrak{A}_i)$. Since the symbol $=_i$ interprets in \mathfrak{B}_i as the binary equality predicate, we have that $\mathfrak{B}_i \models \Phi(\mathfrak{A}_i)$. Thus $\mathfrak{B}_i \in \text{Age}(\mathfrak{A}_i)$ for each $i \in [k]$. For each $i \in [k]$, let e_i be an embedding from \mathfrak{B}_i into \mathfrak{A}_i . For each $b \in B$ and each $i \in [k]$, we denote by $[b]_{=_i^{\mathfrak{B}}}$ the equivalence class of $b \in B$ w.r.t. $=_i^{\mathfrak{B}}$. Now consider the map $e: B \rightarrow A_1 \times \dots \times A_k$ that sends b to $(e_1([b]_{=_1^{\mathfrak{B}}}), \dots, e_k([b]_{=_k^{\mathfrak{B}}}))$. Note that e is well-defined because we map from elements to their equivalence classes and not the other way around. By the first clause on the second line in $\Phi(\mathfrak{A})$, for all $x, y \in B$, we have $x = y$ if and only if $x =_i^{\mathfrak{B}} y$ for each $i \in [k]$. This means that e is injective. For every $i \in [k]$ and every n -ary symbol $R \in \tau_i$, we have

$$\begin{aligned} (b_1, \dots, b_n) \in R^{\mathfrak{B}} & \text{ iff } ([b_1]_{=_1^{\mathfrak{B}}}, \dots, [b_n]_{=_1^{\mathfrak{B}}}) \in R^{\mathfrak{B}_1} \\ & \text{ iff } (e_1([b_1]_{=_1^{\mathfrak{B}}}), \dots, e_1([b_n]_{=_1^{\mathfrak{B}}})) \in R^{\mathfrak{A}_1} \\ & \text{ iff } (e(b_1)[1], \dots, e(b_n)[1]) \in R^{\mathfrak{A}_1} \\ & \text{ iff } (e(b_1), \dots, e(b_n)) \in R^{\mathfrak{A}}. \end{aligned}$$

Hence e is an embedding from \mathfrak{B} into \mathfrak{A} . This completes the proof. \square

Coming back to Example 7, we can use a feature *time&location* that maps into the algebraic product of Allen and RCC8 to describe an event e that has some successor event e' (among possibly others) such that e takes place before e' and the regions where e and e' happen are externally connected:

$$\text{Event} \sqcap \exists \text{succ. Event} \sqcap \exists \text{time\&location, succ time\&location. } (<(x_1, x_2) \wedge \text{EC}(x_1, x_2)).$$

The algebraic product also allows us to transfer ω -admissibility to some other well-known formalisms instead of having to prove the condition by hand.

Example 8. The *Cardinal Direction Calculus* is a formalism used to relate pairs of points in the plane with respect to the eight cardinal directions. It can be represented by a structure $\mathfrak{D}_{\text{CDC}}$ with domain \mathbb{Q}^2 and relations of the form $\{(\bar{x}, \bar{y}) \mid (\bar{x}[i], \bar{y}[i]) \in R_i \text{ for both } i \in [2]\}$, where

=	N	E	S	W	NE	SE	SW	NW
(=,=)	(=,>)	(>,=)	(=,<)	(<,=)	(>,>)	(>,<)	(<,<)	(<,>)

Figure 4.2.: The basic relations of the Cardinal Direction Calculus.

$R_1, R_2 \in \{<, =, >\}$, see Figure 4.2. Clearly, all relations of \mathcal{D}_{CDC} have a first-order definition in $\mathcal{Q} \boxtimes \mathcal{Q}$, which is finitely bounded and homogeneous by Proposition 11. By Proposition 8, \mathcal{D}_{CDC} is a reduct of a finitely homogeneous structure \mathcal{D} , namely $\mathcal{Q} \boxtimes \mathcal{Q}$ expanded by the relations of \mathcal{D}_{CDC} . We claim that \mathcal{D}_{CDC} itself is homogeneous. By Theorem 4, \mathcal{D} is ω -categorical. Then it follows from Theorem 3 that \mathcal{D}_{CDC} is ω -categorical as well because $\text{Aut}(\mathcal{D}) \subseteq \text{Aut}(\mathcal{D}_{\text{CDC}})$. Note that every relation of \mathcal{D} has a quantifier-free definition in \mathcal{D}_{CDC} . The idea is that we can recover the relation of $\mathcal{Q} \boxtimes \mathcal{Q}$ that acts as a particular relation R of \mathcal{Q} in the i -th component by taking the disjunction of all atomic formulas in the signature of \mathcal{D}_{CDC} for relations that behave as R in the i -th component. For instance, the formula $S(x, y) \vee SE(x, y) \vee SW(x, y)$ defines the relation of $\mathcal{Q} \boxtimes \mathcal{Q}$ that acts as $<$ in the second component. Now, since every relation of \mathcal{D} has a quantifier-free first-order definition in \mathcal{D}_{CDC} , the homogeneity of \mathcal{D}_{CDC} follows from Theorem 4. By Theorem 5, $\text{Age}(\mathcal{D}_{\text{CDC}})$ has the AP. Since \mathcal{D}_{CDC} is JDJEPD and $\text{Age}(\mathcal{D}_{\text{CDC}})$ has the AP, by Proposition 5, it is a patchwork. Since \mathcal{D}_{CDC} is a reduct of a finitely bounded structure, by Proposition 7, its CSP is in NP. Since \mathcal{D}_{CDC} is ω -categorical, by Corollary 2, it has homomorphism ω -compactness. It follows that \mathcal{D}_{CDC} is ω -admissible.

Example 9. For $n \geq 1$, the n -dimensional block algebra, introduced in [13], is a formalism used to represent all possible configurations of pairs of n -dimensional blocks whose sides are parallel to the axes of some orthogonal basis of the n -dimensional Euclidean space. It can be represented by a structure $\mathcal{D}_{n\text{-block}}$ whose domain consists of all n -tuples of ordered pairs of rational numbers, and whose relations are of the form $\{(\bar{x}, \bar{y}) \mid (\bar{x}[i], \bar{y}[i]) \in R_i \text{ for every } i \in [n]\}$, where R_1, \dots, R_n are any relations of Allen. Clearly, all relations of $\mathcal{D}_{n\text{-block}}$ have a first-order definition in $\text{Allen} \boxtimes \dots \boxtimes \text{Allen}$, the n -fold algebraic product of Allen with itself. Based on this fact, we can prove ω -admissibility for $\mathcal{D}_{n\text{-block}}$ similarly as in Example 8.

One might think that it should be possible to simulate $\mathcal{ALC}(\mathcal{D}_{\text{CDC}})$ within $\mathcal{ALC}(\mathcal{Q})$ and $\mathcal{ALC}(\mathcal{D}_{n\text{-block}})$ within $\mathcal{ALC}(\text{Allen})$ simply by using additional features. However, this is not the case because we allow non-functional roles in concrete domain restrictions, see Example 7.

A Model-Theoretic Analysis of p-Admissibility

Recall that a structure \mathfrak{D} is p-admissible if it is convex and validity of Horn implications in \mathfrak{D} is tractable. As argued at the end of Section 3.2, developing algebraic conditions that characterize tractability is way beyond the scope of this thesis. For this reason, we will concentrate on algebraic conditions that ensure convexity. We will see, however, that for finitely bounded convex structures we obtain tractability for free.

5.1 Convexity via square embeddings

Convex structures can be characterized using the square embedding condition introduced in Definition 6. Basically, this condition says that the square of every finite substructure of \mathfrak{B} embeds into \mathfrak{B} . However, since we allow the signature to be infinite, the exact formulation of the property is a bit more complicated.

Definition 6. A class \mathcal{K} of relational τ -structures has the *square embedding property* (SEP) if, for every finite $\sigma \subseteq \tau$ and every $\mathfrak{A} \in \mathcal{K}$, there is $\mathfrak{C} \in \mathcal{K}$ such that the σ -reducts of \mathfrak{A}^2 and \mathfrak{C} coincide.

Theorem 9. For a structure \mathfrak{B} with a (not necessarily finite) relational signature τ , the following are equivalent:

1. \mathfrak{B} is convex.
2. $\mathcal{K} = \text{Age}(\mathfrak{B})$ has the SEP.

Note that the direction “2 \Rightarrow 1” is a slightly more general version of a result commonly known as McKinsey’s lemma [54]. We cannot use McKinsey’s lemma directly since we also want to cover structures whose ages are not closed under taking squares, e.g., the second structure in Theorem 11.

Proof. “2 \Rightarrow 1”: Suppose to the contrary that the implication $\forall x_1, \dots, x_n (\phi \Rightarrow \psi)$ is valid in \mathfrak{B} , where ϕ is a conjunction of atomic formulas and ψ is a disjunction of atomic formulas ψ_1, \dots, ψ_k , but we also have $\mathfrak{B} \not\models \forall x_1, \dots, x_n (\phi \Rightarrow \psi_i)$ for every $i \in [k]$. Without loss of generality, we assume that $\phi, \psi_1, \dots, \psi_k$ all have the same free variables x_1, \dots, x_n , some of which might not influence their truth value. For every $i \in [k]$, there exists a tuple $\bar{t}_i \in B^n$ such

5. A Model-Theoretic Analysis of p -Admissibility

that

$$\mathfrak{B} \models \phi(\bar{t}_i) \wedge \neg \psi_i(\bar{t}_i). \quad (5.1)$$

We show by induction on i that, for every $i \in [k]$, there exists a tuple $\bar{s}_i \in B^n$ that satisfies the *induction hypothesis*

$$\mathfrak{B} \models \phi(\bar{s}_i) \wedge \neg \left(\bigvee_{\ell \in [i]} \psi_\ell(\bar{s}_i) \right). \quad (5.2)$$

In the *base case* ($i = 1$), it follows from (5.1) that $\bar{s}_1 := \bar{t}_1$ satisfies (5.2).

In the *induction step* ($i \rightarrow i + 1$), let $\bar{s}_i \in B^n$ be any tuple that satisfies (5.2). Let $\sigma \subseteq \tau$ be the finite set of relation symbols occurring in the implication $\forall x_1, \dots, x_n (\phi \Rightarrow \psi)$, and let \mathfrak{A}_i be the substructure of \mathfrak{B} on the set $\{\bar{s}_i[1], \bar{t}_{i+1}[1], \dots, \bar{s}_i[n], \bar{t}_{i+1}[n]\}$. Then $\mathfrak{A}_i \models \phi(\bar{s}_i)$ and $\mathfrak{A}_i \models \phi(\bar{t}_{i+1})$, and thus $\mathfrak{A}_i^2 \models \phi(\bar{s}_i \times \bar{t}_{i+1})$ where

$$\bar{s}_i \times \bar{t}_{i+1} := ((\bar{s}_i[1], \bar{t}_{i+1}[1]), \dots, (\bar{s}_i[n], \bar{t}_{i+1}[n])).$$

By 2., there exists a structure $\mathfrak{C}_i \in \text{Age}(\mathfrak{B})$ whose σ -reduct coincides with \mathfrak{A}_i^2 , which implies that $\mathfrak{C}_i \models \phi(\bar{s}_i \times \bar{t}_{i+1})$. Let f_i be the embedding of \mathfrak{C}_i into \mathfrak{B} . Since ϕ is a conjunction of atomic σ -formulas and f_i is a homomorphism, we have that $\mathfrak{B} \models \phi(f_i(\bar{s}_i \times \bar{t}_{i+1}))$. Suppose that $\mathfrak{B} \models \psi_{i+1}(f_i(\bar{s}_i \times \bar{t}_{i+1}))$. Since f_i is an embedding, we obtain $\mathfrak{C}_i \models \psi_{i+1}(\bar{s}_i \times \bar{t}_{i+1})$, and thus $\mathfrak{A}_i \models \psi_{i+1}(\bar{t}_{i+1})$. This implies $\mathfrak{B} \models \psi_{i+1}(\bar{t}_{i+1})$, which contradicts (5.1). Similarly, we can show that assuming $\mathfrak{B} \models \psi_j(f_i(\bar{s}_i \times \bar{t}_{i+1}))$ for some $j \leq i$ leads to a contradiction with (5.2). We conclude that $\bar{s}_{i+1} := f_i(\bar{s}_i \times \bar{t}_{i+1})$ satisfies (5.2).

Since $\mathfrak{B} \models \forall x_1, \dots, x_n (\phi \Rightarrow \psi)$, the existence of a tuple $\bar{s}_i \in B^n$ that satisfies (5.2) for $i = k$ leads to a contradiction. This completes the proof of “ $2 \Rightarrow 1$ ” of our theorem.

Before we proceed with the proof of “ $1 \Rightarrow 2$ ”, let us take a closer look at the contraposition of the convexity condition. Suppose that we have a conjunction ϕ of atomic formulas and tuples \bar{r} and \bar{s} over B together with disjunctions $\psi_{\bar{r}}$ and $\psi_{\bar{s}}$ of atomic formulas such that $\mathfrak{B} \models (\phi \wedge \neg \psi_{\bar{r}})(\bar{r})$ and $\mathfrak{B} \models (\phi \wedge \neg \psi_{\bar{s}})(\bar{s})$. Then clearly there must exist a tuple \bar{t} over B such that $\mathfrak{B} \models (\phi \wedge \neg \psi_{\bar{r}} \wedge \neg \psi_{\bar{s}})(\bar{t})$ as otherwise $\mathfrak{B} \models \forall x_1, \dots, x_n (\phi \Rightarrow \psi_{\bar{r}} \vee \psi_{\bar{s}})$, but neither $\mathfrak{B} \models \forall x_1, \dots, x_n (\phi \Rightarrow \psi_{\bar{r}})$ nor $\mathfrak{B} \models \forall x_1, \dots, x_n (\phi \Rightarrow \psi_{\bar{s}})$ is true (which contradicts convexity).

We are now ready to prove “ $1 \Rightarrow 2$ ”. Let σ be a finite subset of τ and $\mathfrak{A} \in \text{Age}(\mathfrak{B})$. In addition, let $\{(r_1, s_1), \dots, (r_n, s_n)\}$ be the domain of \mathfrak{A}^2 . Consider the tuples $\bar{r} := (r_1, \dots, r_n)$ and $\bar{s} := (s_1, \dots, s_n)$. Let $\phi(x_1, \dots, x_n)$ be the conjunction of all atomic σ -formulas such that $\mathfrak{A}^2 \models \phi((r_1, s_1), \dots, (r_n, s_n))$, i.e., we consider all atomic σ -formulas built using a relation symbol from σ (or the equality predicate) and containing variables from $\{x_1, \dots, x_n\}$, assign (r_i, s_i) to the variable x_i , and take those atomic σ -formulas for which the corresponding tuple of elements of \mathfrak{A}^2 belongs to the respective relation in \mathfrak{A}^2 .

Clearly, the tuples \bar{r} and \bar{s} both satisfy ϕ in \mathfrak{B} since the projection to a single coordinate is a homomorphism from \mathfrak{A}^2 to \mathfrak{B} . Now let $\psi_{\bar{r}}$ be the disjunction of all atomic σ -formulas

that do not hold on the tuple \bar{r} in \mathfrak{B} . Analogously, let $\psi_{\bar{s}}$ be the disjunction of all atomic σ -formulas that do not hold on the tuple \bar{s} in \mathfrak{B} . Without loss of generality $|A| > 1$, and thus both disjunctions are non-empty.

We have that $\mathfrak{B} \models \phi \wedge \neg\psi_{\bar{r}}(\bar{r})$ and $\mathfrak{B} \models \phi \wedge \neg\psi_{\bar{s}}(\bar{s})$. Since \mathfrak{B} is convex, there must exist a tuple \bar{t} such that $\mathfrak{B} \models \phi \wedge \neg\psi_{\bar{r}}(\bar{t}) \wedge \neg\psi_{\bar{s}}(\bar{t})$. Now consider the map f that sends, for every $i \in [n]$, the tuple (r_i, s_i) to $\bar{t}[i]$. Clearly f is a homomorphism from the σ -reduct of \mathfrak{A} to the σ -reduct of \mathfrak{B} because $\mathfrak{B} \models \phi(\bar{t})$. Moreover, f is an embedding because, if ψ is a single atomic σ -formula, then $\mathfrak{B} \models \psi(\bar{t})$ only if $\mathfrak{B} \models \psi(\bar{r})$ and $\mathfrak{B} \models \psi(\bar{s})$. We define \mathfrak{C} as the substructure of \mathfrak{B} on $f(A^2)$. \square

As a consequence of Theorem 9, every CSP in PTIME gives rise to a p-admissible structure, see Corollary 6. This further substantiates our remark at the end of Section 3.2 that characterizing all p-admissible concrete domains is at least as hard as characterizing all tractable CSPs.

Definition 7. The *canonical database* $\text{DB}(\exists\bar{x}. \phi)$ in the signature τ for a satisfiable equality-free PP τ -sentence $\exists\bar{x}. \phi$ is the τ -structure whose domain consists of the quantified variables \bar{x} and whose relations are specified by the quantifier-free part ϕ .

Corollary 6. For every structure \mathfrak{B} , there exists a convex structure \mathfrak{D} such that

1. $\text{CSP}(\mathfrak{D}) = \text{CSP}(\mathfrak{B})$;
2. \mathfrak{D} is p-admissible if and only if $\text{CSP}(\mathfrak{B})$ is in PTIME.

Proof. Let \mathfrak{D} be the disjoint union of all finite structures which homomorphically map to \mathfrak{B} . Clearly, $\text{CSP}(\mathfrak{D}) = \text{CSP}(\mathfrak{B})$, because a PP sentence is satisfied in \mathfrak{B} if and only if it is satisfied in a finite structure which homomorphically maps to \mathfrak{B} . Moreover, $\text{Age}(\mathfrak{D})$ has the SEP, because it is even closed under taking second powers. Thus, by Theorem 9, \mathfrak{D} is convex.

To show the second claim of the corollary, let $\forall\bar{x}(\phi \Rightarrow \psi)$ be a Horn implication. Without loss of generality, we assume that ϕ does not contain equality atoms since we can remove them by identifying variables in such equality atoms in ϕ and ψ . We claim that $\mathfrak{D} \models \exists\bar{x}(\phi \wedge \neg\psi)$ if and only if $\mathfrak{D} \models \exists\bar{x}. \phi$ and ψ does not occur as a conjunct in ϕ . This can be tested in polynomial time if and only if $\text{CSP}(\mathfrak{B})$ is in PTIME.

The only-if direction is trivial. For the if direction, note that, by a standard result in database theory, $\mathfrak{D} \models \exists\bar{x}. \phi$ if and only if the canonical database $\text{DB}(\exists\bar{x}. \phi)$ homomorphically maps to \mathfrak{D} [37]. Since ψ does not occur as a conjunct in ϕ , this atomic formula does not hold in $\text{DB}(\exists\bar{x}. \phi)$. Suppose that there exists a homomorphism from $\text{DB}(\exists\bar{x}. \phi)$ to \mathfrak{D} . Then there also exists a homomorphism from $\text{DB}(\exists\bar{x}. \phi)$ to \mathfrak{B} . By the definition of \mathfrak{D} , there exists an embedding $e: \text{DB}(\exists\bar{x}. \phi) \hookrightarrow \mathfrak{D}$. Since e is an embedding, $\mathfrak{D} \models \exists\bar{x}(\phi \wedge \neg\psi)$. \square

Using Theorem 9, we can also obtain a statement similar to that of Theorem 5, where convexity replaces homogeneity and the square embedding property together with the joint embedding property replaces the AP. A class \mathcal{K} of relational τ -structures has the *joint embedding property* (JEP) if, for every $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{K}$ there exists $\mathfrak{C} \in \mathcal{K}$ such that $\mathfrak{B}_i \hookrightarrow \mathfrak{C}$ for $i \in \{1, 2\}$. Recall our definition of the square embedding property from Theorem 9.

Corollary 7. *For a class \mathcal{K} of finite τ -structures, the following are equivalent:*

1. *There exists a countable convex structure \mathfrak{D} with $\mathcal{K} = \text{Age}(\mathfrak{D})$.*
2. *\mathcal{K} contains countably many structures up to isomorphism, is closed under isomorphisms and building substructures, has the JEP and the SEP*

Proof. The direction “ $1 \Rightarrow 2$ ” is a direct consequence of Theorem 9 since classes of the form $\text{Age}(\mathfrak{D})$ for a relational structure \mathfrak{D} trivially satisfy the JEP. The direction “ $2 \Rightarrow 1$ ” follows from Theorem 6.1.1 in [55] and Theorem 9. In fact, Theorem 6.1.1 in [55] implies that a class \mathcal{K} of up to isomorphism countably many finite relational structures that is closed under building substructures and has the JEP is of the form $\mathcal{K} = \text{Age}(\mathfrak{D})$ for a countable structure \mathfrak{D} . An application of Theorem 9 then yields convexity of \mathfrak{D} . \square

In contrast to countable homogeneous structures, countable convex structures are in general *not* uniquely determined up to isomorphism by their age. The random graph can again serve as a counterexample.

Example 10. The random graph \mathfrak{G} introduced in Example 6 is convex since $\text{Age}(\mathfrak{G})$ satisfies the square embedding condition. In fact, since \mathfrak{G} embeds every finite undirected graph, it also embeds \mathfrak{A}^2 for any undirected graph \mathfrak{A} . The direct product \mathfrak{H} of \mathfrak{G} with itself is thus also convex since $\text{Age}(\mathfrak{H}) = \text{Age}(\mathfrak{G})$. However, we have seen in Example 6 that \mathfrak{G} and \mathfrak{H} are not isomorphic. It is not hard to see that \mathfrak{G} is actually p -admissible. Instead of proving this directly, we will show it as a consequence of Theorem 12 below.

5.2 Convex ω -categorical structures

For countably infinite ω -categorical structures, the characterization of convexity of Theorem 9 can be improved to the following simpler statement.

Theorem 10. *For a countably infinite ω -categorical relational structure \mathfrak{B} with a countable signature τ , the following are equivalent:*

1. *\mathfrak{B} is convex.*
2. *\mathfrak{B}^2 embeds into \mathfrak{B} .*

Proof. The direction “ $2 \Rightarrow 1$ ” follows immediately from Theorem 9 since $\mathfrak{B}^2 \hookrightarrow \mathfrak{B}$ obviously implies that $\text{Age}(\mathfrak{B})$ satisfies the square embedding property. Note that for this direction, ω -categoricity of \mathfrak{B} is not required.

The proof of “ $1 \Rightarrow 2$ ” combines the proof of this direction for Theorem 9 with the following two facts, which are implied by ω -categoricity of \mathfrak{B} . First, there exists an embedding from \mathfrak{B}^2 to \mathfrak{B} if and only if there exists an embedding from \mathfrak{A} to \mathfrak{B} for every $\mathfrak{A} \in \text{Age}(\mathfrak{B}^2)$ (see, e.g., Lemma 3.1.5 in [17]). Second, to deal with the fact that τ may be infinite we can use Theorem 3, which ensures that, for every $k \geq 1$, there are only finitely many inequivalent k -ary formulas over \mathfrak{B} consisting of a single atomic τ -formula. \square

Besides this simplification, ω -categoricity does not contribute to p-admissibility in any useful way. The reason is that, in contrast to ω -admissibility, the question whether a given structure is p-admissible only depends on its age, i.e., if $\text{Age}(\mathfrak{A}) = \text{Age}(\mathfrak{B})$, then \mathfrak{A} is p-admissible if and only if \mathfrak{B} is p-admissible. Nevertheless, convex ω -categorical structures often appear in the infinite-domain CSP literature. Here we mention two interesting examples: atomless Boolean algebras and countably infinite-dimensional vector spaces over finite fields. Since the CSP for atomless Boolean algebras is NP-complete [14], this example does not provide us with a p-admissible concrete domain; but the vector space example does.

As shown in [20], the relational representation $\mathfrak{V}_q = (V_q; R^+, R^{s_0}, \dots, R^{s_{q-1}})$ of the countably infinite-dimensional vector space over a finite field $\text{GF}(q)$ is ω -categorical, satisfies $\mathfrak{V}_q^2 \cong \mathfrak{V}_q$, and its CSP is decidable in polynomial time, even if the complements of all predicates are added. Here R^+ is a ternary predicate corresponding to addition of vectors, and the R^{s_i} are binary predicates corresponding to scalar multiplication of a vector with the element s_i of $\text{GF}(q)$. These properties are preserved if we add finitely many unary predicates R^{e_i} that correspond to unit vectors e_1, \dots, e_k .

Corollary 8. *The structure \mathfrak{V}_q expanded with predicates R^{e_1}, \dots, R^{e_k} for unit vectors e_1, \dots, e_k is p-admissible.*

Proof. We have $\mathfrak{V}_q^2 \cong \mathfrak{V}_q$, and thus both structures are vector spaces over $\text{GF}(q)$ of countably infinite dimension. Now if we fix finitely many unit vectors $e_1, \dots, e_k \in V_q$ by expanding \mathfrak{V}_q with the unary predicates R^{e_1}, \dots, R^{e_k} , we can still extend the map which sends (e_i, e_i) to e_i for each $i \in [k]$ to a bijection between bases of both vector spaces. This bijection then naturally extends to an isomorphism from $(\mathfrak{V}_q, R^{e_1}, \dots, R^{e_k})^2$ to $(\mathfrak{V}_q, R^{e_1}, \dots, R^{e_k})$. Thus, Theorem 10 yields convexity of $(\mathfrak{V}_q, R^{e_1}, \dots, R^{e_k})$. The CSP in its expansion by inequality and the complements of all relations can be solved, similarly as in the Gaussian elimination algorithm, by iterated elimination of variables from equations and subsequent search for unsatisfiable equalities and/or inequalities between unit vectors (e.g., $e_1 \neq e_1$ or $e_1 = e_2$) (see [20] for details). This implies that testing validity of Horn implications in $(\mathfrak{V}_q, R^{e_1}, \dots, R^{e_k})$ is tractable. We conclude that $(\mathfrak{V}_q, R^{e_1}, \dots, R^{e_k})$ is p-admissible. \square

For the case $q = 2$, the vectors in V_q are one-sided infinite tuples of zeros and ones containing only finitely many ones, which can be viewed as representing finite subsets of \mathbb{N} . For example, $(0, 1, 1, 0, 1, 0, 0, \dots)$ represents the set $\{1, 2, 4\}$. Thus, if we use \mathfrak{V}_2 as concrete domain, the features assign finite sets of natural numbers to individuals. For example, assume that the feature *dages* assigns the set of ages of daughters to a person, and *sages* the set of ages of sons. Then $\exists \text{dages}, \text{sages}, \text{zero}. R^+(x_1, x_2, x_3)$ describes persons that, for every age, have either both a son and a daughter of this age, or no child at all of this age. The feature *zero* is supposed to point to the zero vector, which can, e.g., be enforced using the GCI $\top \sqsubseteq \exists \text{zero}, \text{zero}, \text{zero}. R^+(x_1, x_2, x_3)$. If e_1 is the unit vector $(0, 1, 0, 0, \dots)$ and e_4 is the unit vector $(0, 0, 0, 0, 1, 0, 0, \dots)$, then the concept $\exists \text{one}, \text{four}, \text{dages}. R^+(x_1, x_2, x_3)$ describes humans that have daughters of age one and four, and of no other age, if the TBox contains the GCI $\top \sqsubseteq \exists \text{one}. R^{e_1}(x_1) \sqcap \exists \text{four}. R^{e_4}(x_1)$.

5.3 Convex numerical structures

Outside of the scope of ω -categoricity, we exhibit two new p -admissible concrete domain that are respectively based on the real and the rational numbers, and whose predicates are defined by linear equations.

Let $\tilde{\mathcal{D}}_{\text{lin}}$ be the relational structure over \mathbb{R} that has, for every linear equation system $A\bar{x} = \bar{b}$ over \mathbb{Q} , a relation consisting of all its solutions in \mathbb{R} . We define \mathcal{D}_{lin} as the substructure of $\tilde{\mathcal{D}}_{\text{lin}}$ on \mathbb{Q} . For example, using the matrix $A = \begin{pmatrix} 2 & 1 & -1 \end{pmatrix}$ and the vector $\bar{b} = (0)$ one obtains the ternary relation $\{(p, q, r) \in \mathbb{Q}^3 \mid 2p + q = r\}$ in \mathcal{D}_{lin} . Our proof of the fact these two structures are p -admissible uses the following simple observation about PP definable relations.

Lemma 4. *Let \mathcal{D} be a relational structure and $\{R_i \mid i \in I\}$ a set of relations that are PP definable in \mathcal{D} . Then every isomorphism from \mathcal{D}^2 to \mathcal{D} is also an isomorphism from $(\mathcal{D}, \{R_i \mid i \in I\})^2$ to $(\mathcal{D}, \{R_i \mid i \in I\})$.*

Proof. Let $f : \mathcal{D}^2 \rightarrow \mathcal{D}$ be an isomorphism. By a standard result in model theory, if R is PP definable in \mathcal{D} , then f is also a homomorphism from $(\mathcal{D}, R)^2$ to (\mathcal{D}, R) (e.g., Proposition 5.2.2 in [17]). Since f is bijective, it only remains to show that f is even an embedding from $(\mathcal{D}, R)^2$ to (\mathcal{D}, R) . Let $\phi(x_1, \dots, x_k) := \exists x_{k+1}, \dots, x_\ell. \psi(x_1, \dots, x_\ell)$ be the PP formula that defines R in \mathcal{D} , where ψ is the quantifier-free part of ϕ . Let $\bar{r} \in R$ be an arbitrary tuple of the form $\bar{r} = f(\bar{r}_1, \bar{r}_2)$ for some $\bar{r}_1, \bar{r}_2 \in D^k$. Then there exists $\bar{s} \in D^{\ell-k}$ such that $\mathcal{D} \models \psi(\bar{r}[1], \dots, \bar{r}[k], \bar{s}[1], \dots, \bar{s}[\ell-k])$. Since f is surjective, there exist $\bar{s}_1, \bar{s}_2 \in D^{\ell-k}$ such that $\bar{s} = f(\bar{s}_1, \bar{s}_2)$. Since f is an embedding from \mathcal{D}^2 to \mathcal{D} , we have $\mathcal{D} \models \psi(\bar{r}_i[1], \dots, \bar{r}_i[k], \bar{s}_i[1], \dots, \bar{s}_i[\ell-k])$ for both $i \in \{1, 2\}$. This means that $\bar{r}_1, \bar{r}_2 \in R$, which confirms our claim. \square

Theorem 11. *The relational structures $\tilde{\mathcal{D}}_{\text{lin}}$ and \mathcal{D}_{lin} are p -admissible.*

Proof. To prove this theorem for \mathbb{R} , we start with the well-known fact that $(\mathbb{R}; +, 0)^2$ and $(\mathbb{R}; +, 0)$ are isomorphic [60]. Such an isomorphism exists because $(\mathbb{R}; +, 0)^2$ and $(\mathbb{R}; +, 0)$ are both vector spaces over \mathbb{Q} whose dimensions are uncountably infinite and of the same cardinality. Thus every bijective map from a basis of $(\mathbb{R}; +, 0)^2$ to a basis of $(\mathbb{R}; +, 0)$ extends to an isomorphism. Now we simply choose any two bases of $(\mathbb{R}; +, 0)^2$ and $(\mathbb{R}; +, 0)$ such that the first basis contains $(1, 1)$ and the second basis contains 1. Then we choose an arbitrary bijection from the first basis to the second basis that sends $(1, 1)$ to 1. This bijection extends to an isomorphism $f : (\mathbb{R}; +, 0, 1)^2 \rightarrow (\mathbb{R}; +, 0, 1)$. It is easy to see that every relation of $\tilde{\mathcal{D}}_{\text{lin}}$ can be defined in $(\mathbb{R}; +, 0, 1)$ using a PP formula. By Lemma 4, f is an isomorphism from $\tilde{\mathcal{D}}_{\text{lin}}^2$ to $\tilde{\mathcal{D}}_{\text{lin}}$, which implies that $\tilde{\mathcal{D}}_{\text{lin}}$ is convex by Theorem 9.

Now recall that validity of Horn implications in $\tilde{\mathcal{D}}_{\text{lin}}$ can be tested in polynomial time if the CSP for $(\tilde{\mathcal{D}}_{\text{lin}}^\neg, \neq)$ is in PTIME. It is easy to see that f is a homomorphism from $(\tilde{\mathcal{D}}_{\text{lin}}^\neg, \neq)^2$ to $(\tilde{\mathcal{D}}_{\text{lin}}^\neg, \neq)$. It follows from Corollary 5.10 in [22] that both the CSP and validity of Horn implications in $\tilde{\mathcal{D}}_{\text{lin}}$ are decidable in polynomial time. We conclude that $\tilde{\mathcal{D}}_{\text{lin}}$ is p -admissible.

For \mathbb{Q} , we cannot employ the same argument since $(\mathbb{Q}; +, 0)^2$ does not even embed into $(\mathbb{Q}; +, 0)$. Instead, we use the well-known fact that $\text{Th}(\mathbb{Q}; +, 0) = \text{Th}(\mathbb{R}; +, 0)$ [60] to show that convexity of $\tilde{\mathfrak{D}}_{\text{lin}}$ implies convexity of $\mathfrak{D}_{\text{lin}}$.

We claim that a stronger statement is true, namely, that $\text{Th}(\mathbb{Q}; +, 0, 1) = \text{Th}(\mathbb{R}; +, 0, 1)$. Let ϕ be an arbitrary first-order sentence in the signature of $(\mathbb{R}; +, 0, 1)$. We obtain the formula $\psi(x)$ in the signature of $(\mathbb{R}; +, 0)$ by replacing the constant 1 in ϕ by a fresh free variable x , i.e., we have $(\mathbb{R}; +, 0, 1) \models \phi$ if and only if $(\mathbb{R}; +, 0) \models \psi(1)$. For every $c \in \mathbb{R} \setminus \{0\}$, the map $x \mapsto cx$ is an automorphism of $(\mathbb{R}; +, 0)$ that sends 1 to c . Since $\{x \in \mathbb{R} \mid (\mathbb{R}; +, 0) \models \psi(x)\}$ has a first-order definition in $(\mathbb{R}; +, 0)$, it is preserved by all automorphisms of $(\mathbb{R}; +, 0)$ [55]. Now we distinguish the following two cases. If $(\mathbb{R}; +, 0) \models \psi(0)$, then $(\mathbb{R}; +, 0, 1) \models \phi$ if and only if $(\mathbb{R}; +, 0) \models \exists x. \psi(x)$. Otherwise $(\mathbb{R}; +, 0, 1) \models \phi$ if and only if $(\mathbb{R}; +, 0) \models \exists x (\neg(x = 0) \wedge \psi(x))$. Using an analogous argument we have either $(\mathbb{Q}; +, 0, 1) \models \phi$ if and only if $(\mathbb{Q}; +, 0) \models \exists x. \psi(x)$ in the case where $(\mathbb{Q}; +, 0) \models \psi(0)$, or $(\mathbb{Q}; +, 0, 1) \models \phi$ if and only if $(\mathbb{Q}; +, 0) \models \exists x (\neg(x = 0) \wedge \psi(x))$. Since ϕ was chosen arbitrarily, we conclude that indeed $\text{Th}(\mathbb{Q}; +, 0, 1) = \text{Th}(\mathbb{R}; +, 0, 1)$.

Since the relations of $\mathfrak{D}_{\text{lin}}$ are definable in $(\mathbb{Q}; +, 0, 1)$ using the same PP formulas as for their counterparts in $\tilde{\mathfrak{D}}_{\text{lin}}$, and $\text{Th}(\mathbb{Q}; +, 0, 1) = \text{Th}(\mathbb{R}; +, 0, 1)$, we conclude that $\mathfrak{D}_{\text{lin}}$ is p-admissible as well. \square

In Section 3 we have introduced the structure $\mathfrak{D}_{2\text{-aff}}$ and have shown in Proposition 3 that subsumption w.r.t. TBoxes is undecidable in $\mathcal{EL}(\mathfrak{D}_{2\text{-aff}})$. Using Theorems 2 and 11 we can now show that subsumption w.r.t. TBoxes is tractable in $\mathcal{EL}[\mathfrak{D}_{2\text{-aff}}]$ since $\mathfrak{D}_{2\text{-aff}}$ is p-admissible.

Corollary 9. *The relational structure $\mathfrak{D}_{2\text{-aff}}$ is p-admissible.*

Proof. First, note that the CSP and validity of Horn implications in $\mathfrak{D}_{2\text{-aff}}$ can be reduced in linear time to the same problems for $\mathfrak{D}_{\text{lin}}$.

It remains to show that $\mathfrak{D}_{2\text{-aff}}$ is convex. Let σ be a finite subset of the signature of $\mathfrak{D}_{2\text{-aff}}$ and let $\mathfrak{A} \in \text{Age}(\mathfrak{D}_{2\text{-aff}})$. We may assume without loss of generality that \mathfrak{A} is a substructure of $\mathfrak{D}_{2\text{-aff}}$. It is sufficient to show that the σ -reduct of \mathfrak{A}^2 embeds into the σ -reduct of $\mathfrak{D}_{2\text{-aff}}$.

For every relation $R_{M, \bar{v}}$ of $\mathfrak{D}_{2\text{-aff}}$ we consider the 4-ary relation $\{(\bar{x}[1], \bar{x}[2], \bar{y}[1], \bar{y}[2]) \in \mathbb{Q}^4 \mid \bar{y} = M\bar{x} + \bar{v}\}$ of $\mathfrak{D}_{\text{lin}}$ which we denote by $\tilde{R}_{M, \bar{v}}$. Consider the substructure $\tilde{\mathfrak{A}}$ of $\mathfrak{D}_{\text{lin}}$ on the set $\tilde{A} := \{x \in \mathbb{Q} \mid \text{there is } \bar{x} \in A \text{ such that } x \in \{\bar{x}[1], \bar{x}[2]\}\}$. Let $\tilde{\sigma}$ be the finite subset of the signature of $\mathfrak{D}_{\text{lin}}$ that contains a symbol for every relation $\tilde{R}_{M, \bar{v}}$ for which there exists a symbol in σ interpreted as $R_{M, \bar{v}}$ in $\mathfrak{D}_{2\text{-aff}}$. Since $\mathfrak{D}_{\text{lin}}$ is convex, Theorem 9 yields an embedding \tilde{f} from the $\tilde{\sigma}$ -reduct of $\tilde{\mathfrak{A}}^2$ to the $\tilde{\sigma}$ -reduct of $\mathfrak{D}_{\text{lin}}$. Let f be the mapping from A^2 to \mathbb{Q}^2 defined as $f(\bar{x}_1, \bar{x}_2) := (\tilde{f}(\bar{x}_1[1], \bar{x}_2[1]), \tilde{f}(\bar{x}_1[2], \bar{x}_2[2]))$. It is well-defined by the definition of $\tilde{\mathfrak{A}}$. Let $(\bar{x}_1, \bar{x}_2), (\bar{y}_1, \bar{y}_2) \in A^2$ be arbitrary. Then, by the definition of \tilde{A} , $(\bar{x}_1[1], \bar{x}_2[1]), (\bar{x}_1[2], \bar{x}_2[2]), (\bar{y}_1[1], \bar{y}_2[1]), (\bar{y}_1[2], \bar{y}_2[2]) \in \tilde{A}^2$ and, for every affine transformation

$\bar{x} \mapsto M\bar{x} + \bar{v}$, we have the following chain of equivalent statements.

$$\underbrace{\begin{pmatrix} \bar{x}_1 \\ \bar{y}_1 \end{pmatrix}, \begin{pmatrix} \bar{x}_2 \\ \bar{y}_2 \end{pmatrix}}_{\in R_{M,\bar{v}}} \text{ iff } \underbrace{\begin{pmatrix} \bar{x}_1[1] \\ \bar{x}_1[2] \\ \bar{y}_1[1] \\ \bar{y}_1[2] \end{pmatrix}, \begin{pmatrix} \bar{x}_2[1] \\ \bar{x}_2[2] \\ \bar{y}_2[1] \\ \bar{y}_2[2] \end{pmatrix}}_{\in \tilde{R}_{M,\bar{v}}} \text{ iff } \underbrace{\begin{pmatrix} \tilde{f}(\bar{x}_1[1], \bar{x}_2[1]) \\ \tilde{f}(\bar{x}_1[2], \bar{x}_2[2]) \\ \tilde{f}(\bar{y}_1[1], \bar{y}_2[1]) \\ \tilde{f}(\bar{y}_1[2], \bar{y}_2[2]) \end{pmatrix}}_{\in \tilde{R}_{M,\bar{v}}} \text{ iff } \underbrace{\begin{pmatrix} f(\bar{x}_1, \bar{x}_2) \\ f(\bar{y}_1, \bar{y}_2) \end{pmatrix}}_{\in R_{M,\bar{v}}}$$

The first equivalence follows from the definition of $\tilde{R}_{M,\bar{v}}$, the second from the fact that \tilde{f} is an embedding, and the third from the definition of f . These equivalences also hold for the equality predicate which can be written as $R_{E,\bar{z}}$ for E the 2×2 identity matrix and $\bar{z} = (0, 0)$. It follows that f is an embedding from the σ -reduct of \mathfrak{A}^2 to the σ -reduct of $\mathfrak{D}_{2\text{-aff}}$. \square

5.4 Ages defined by forbidden substructures

Finitely bounded structures also provide us with interesting examples of convex structures. In this setting, convexity already implies tractability.

Theorem 12. *For a finitely bounded structure \mathfrak{B} , the following are equivalent:*

1. \mathfrak{B} is convex,
2. $\text{Age}(\mathfrak{B})$ is defined by a universal Horn sentence,
3. \mathfrak{B} is p -admissible.

Proof. “1 \Rightarrow 2”: Using the logical reformulation of finite boundedness in Lemma 3, we know that \mathfrak{B} is finitely bounded if its signature is finite and there is a universal first-order sentence Φ such that $\text{Age}(\mathfrak{B})$ consists precisely of the finite models of Φ . We bring Φ into prenex normal form, and then transform its quantifier-free part in conjunctive normal form. This shows that we can assume that Φ is a conjunction of implications (in the sense defined in Section 2). Note that a universal sentence holds in a relational structure if and only if it holds in each of its finite substructures. In particular, we have $\mathfrak{B} \models \Phi$. For every implication in Φ where the conclusion consists of at least two atomic formulas we apply the definition of convexity and reduce Φ to a universal Horn sentence Φ' such that $\mathfrak{B} \models \Phi'$. This implies that Φ' holds in all elements of $\text{Age}(\mathfrak{B})$. In addition, by the construction of Φ' , the original formula Φ is a logical consequence of Φ' . Thus, if a finite τ -structure satisfies Φ' , it also satisfies Φ , and thus belongs to $\text{Age}(\mathfrak{B})$. This shows that Φ' defines $\text{Age}(\mathfrak{B})$.

“2 \Rightarrow 3”: We first show that \mathfrak{B} is convex using Theorem 9. We set $\sigma := \tau$ and select an arbitrary finite substructure \mathfrak{A} of \mathfrak{B} . Let $\forall \bar{x}(\phi_i \Rightarrow \psi_i)$ be one of the Horn implications whose conjunction Φ over $i \in [\ell]$ defines $\text{Age}(\mathfrak{B})$. Let \bar{t} be a tuple over A^2 such that $\mathfrak{A}^2 \models \phi_i(\bar{t})$ for some $i \in [\ell]$ and let k be its arity. Then \bar{t} is of the form $((x_1, y_1), \dots, (x_k, y_k))$ such that $\mathfrak{B} \models \phi_i(x_1, \dots, x_k)$ and $\mathfrak{B} \models \phi_i(y_1, \dots, y_k)$. Since the substructure of \mathfrak{B} on $\{x_1, \dots, x_k, y_1, \dots, y_k\}$ satisfies $\forall \bar{x}(\phi_i \Rightarrow \psi_i)$, we have $\mathfrak{B} \models \psi_i(x_1, \dots, x_k) \wedge \psi_i(y_1, \dots, y_k)$, and thus $\mathfrak{A}^2 \models \psi_i(\bar{t})$.

Since the tuple \bar{t} and the index $i \in [\ell]$ were chosen arbitrarily, we know that $\mathfrak{A}^2 \models \forall \bar{x}(\phi_i \Rightarrow \psi_i)$ for all $i \in [\ell]$. Thus, we have $\mathfrak{A}^2 \models \Phi$, which implies $\mathfrak{A}^2 \in \text{Age}(\mathfrak{B})$. We have shown that $\text{Age}(\mathfrak{B})$ is closed under taking squares, which is a strong form of the square embedding property from Theorem 9.

Regarding tractability, note that the structure \mathfrak{B} satisfies a given Horn implication $\forall \bar{x}(\phi \Rightarrow \psi)$ if and only if this implication is satisfied by all elements of $\text{Age}(\mathfrak{B})$. This is the case if and only if the universal Horn sentence Φ that defines $\text{Age}(\mathfrak{B})$ implies the Horn implication $\forall \bar{x}(\phi \Rightarrow \psi)$. It is well-known that the entailment problem is decidable in polynomial time for Horn implications [43].

“ $3 \Rightarrow 1$ ”: This direction is trivial. □

This theorem yields the following two examples of p-admissible concrete domains.

Example 11. The random graph \mathfrak{G} is p-admissible since its age can be defined by the universal Horn sentence $\forall x(E(x, x) \Rightarrow \text{false}) \wedge \forall x, y(E(x, y) \Rightarrow E(y, x))$.

The structure \mathfrak{Q} is not convex, as otherwise Theorem 9 would imply that it contains incomparable elements since the square of this linear order is not linear. In the universal sentence defining $\text{Age}(\mathfrak{Q})$ (see Lemma 3), the totality axiom $\forall x, y(x < y \vee x = y \vee x > y)$ is the culprit since it is not Horn. If we remove this axiom, we obtain the theory of strict partial orders.

It is well-known that there exists a unique countable homogeneous strict partial order \mathfrak{P} [75], whose age is defined by the universal Horn sentence

$$\forall x, y, z(x < y \wedge y < z \Rightarrow x < z) \wedge \forall x(x < x \Rightarrow \text{false}). \quad (5.3)$$

Thus, \mathfrak{P} is finitely bounded and convex. Using \mathfrak{P} as a concrete domain means that the feature values satisfy the theory of strict partial orders, but not more. One can, for instance, use this concrete domain to model preferences of people; e.g., *Italian* $\sqcap \exists \text{pizzapref}, \text{pastapref}. (x_1 > x_2)$ is a concept describing Italians that like pizza more than pasta. Using \mathfrak{P} here means that preferences may be incomparable. As we have seen above, adding totality would break convexity and thus p-admissibility.

By combining Corollary 3 with Theorem 12, we can obtain non-trivial p-admissible concrete domains \mathfrak{D} for which subsumption in $\mathcal{ALC}(\mathfrak{D})$ is decidable. Note that, according to Proposition 4, such a non-trivial structure \mathfrak{D} cannot be ω -admissible, but it is the reduct of the ω -admissible structure $\mathfrak{D}^{\leq d}$.

Corollary 10. *Let \mathfrak{D} be a finitely bounded convex structure that is a reduct of a finitely bounded homogeneous structure. Then subsumption w.r.t. TBoxes is tractable in $\mathcal{EL}[\mathfrak{D}]$ and decidable in $\mathcal{ALC}(\mathfrak{D})$.*

Examples of infinitely many non-trivial structures satisfying the condition stated in this corollary will be given in the next subsection.

5.5 Ages defined by forbidden homomorphic images

Beside finitely bounded structures, the literature also considers structures whose age can be described by a finite set of forbidden homomorphic images [38, 58]. For a class \mathcal{N} of τ -structures, $\text{Forb}_h(\mathcal{N})$ stands for the class of all finite τ -structures that do not contain a homomorphic image of any member of \mathcal{N} . A structure is *connected* if its so-called Gaifman graph is connected. The *Gaifman graph* of a structure \mathfrak{A} is the undirected graph $(A; E)$ such that there is an edge in E between two elements $a, a' \in A$ if and only if they occur together in a tuple from a relation of \mathfrak{A} .

Theorem 13 (Cherlin, Shelah, and Shi [38, 58]). *Let \mathcal{N} be a finite set of connected relational structures with a finite signature τ . Then there exists an ω -categorical τ -structure $\mathfrak{D}_{\mathcal{N}}$ that is a reduct of a finitely bounded homogeneous structure and $\text{Age}(\mathfrak{D}_{\mathcal{N}}) = \text{Forb}_h(\mathcal{N})$.*

We can show that the structures of the form $\mathfrak{D}_{\mathcal{N}}$ provided by this theorem are always p -admissible.

Proposition 12. *Let \mathcal{N} be a finite family of connected relational structures with a finite signature τ . Then $\mathfrak{D}_{\mathcal{N}}$ is p -admissible.*

Proof. By Theorem 13, we have $\mathfrak{A} \in \text{Age}(\mathfrak{D}_{\mathcal{N}})$ if and only if \mathfrak{A} does not contain a homomorphic image of any $\mathfrak{F} \in \mathcal{N}$ as a substructure. If we can show $\text{Age}(\mathfrak{D}_{\mathcal{N}}^2) \subseteq \text{Age}(\mathfrak{D}_{\mathcal{N}})$, then it follows from Theorem 9 that $\mathfrak{D}_{\mathcal{N}}$ is convex. Suppose that there exists $\mathfrak{C} \in \text{Age}(\mathfrak{D}_{\mathcal{N}}^2)$ such that $\mathfrak{C} \notin \text{Age}(\mathfrak{D}_{\mathcal{N}})$. Then there exists $\mathfrak{F} \in \mathcal{N}$ such that $\mathfrak{F} \rightarrow \mathfrak{C}$. Since the projection to a single component is a homomorphism, this shows that there is a homomorphism $\mathfrak{F} \rightarrow \mathfrak{D}_{\mathcal{N}}$. But then the image of \mathfrak{F} under this homomorphism is a finite substructure of $\mathfrak{D}_{\mathcal{N}}$ that does not belong to $\text{Forb}_h(\mathcal{N})$, which contradicts the fact that $\text{Age}(\mathfrak{D}_{\mathcal{N}}) = \text{Forb}_h(\mathcal{N})$. Thus indeed $\text{Age}(\mathfrak{D}_{\mathcal{N}}^2) \subseteq \text{Age}(\mathfrak{D}_{\mathcal{N}})$ and $\mathfrak{D}_{\mathcal{N}}$ is convex.

Since there are, up to isomorphisms, only finitely many homomorphic images of each $\mathfrak{F} \in \mathcal{N}$ in $\mathfrak{D}_{\mathcal{N}}$, there exists a finite set \mathcal{N}' of finite structures such that $\text{Age}(\mathfrak{D}_{\mathcal{N}}) = \text{Forb}_e(\mathcal{N}')$, which means that $\mathfrak{D}_{\mathcal{N}}$ is finitely bounded. Since $\mathfrak{D}_{\mathcal{N}}$ is convex, its p -admissibility follows from Theorem 12. \square

Proposition 12 together with the next example provides us with infinitely many countable p -admissible concrete domains satisfying the preconditions of Corollary 10, which all yield a different extension of \mathcal{EL} . The usefulness of these concrete domains for defining interesting concepts is, however, as yet unclear.

Example 12. A directed graph is a *tournament* if every two distinct vertices in it are connected by exactly one directed edge. Henson [52] proved that there are uncountably many homogeneous directed graphs by showing that, for any (not necessarily finite) set \mathcal{N} of finite tournaments (plus the loop and the 2-cycle) such that no member of \mathcal{N} is embeddable into any other member of \mathcal{N} , $\text{Forb}_e(\mathcal{N})$ is an amalgamation class whose Fraïssé limit is a homogeneous directed graph.

Furthermore, the Fraïssé limits for two distinct sets of such tournaments are distinct as well. In the literature, such directed graphs are often called *Henson digraphs* [71].

An important observation about Henson digraphs is that $\text{Forb}_e(\mathcal{N}) = \text{Forb}_h(\mathcal{N})$ holds for any set \mathcal{N} of finite tournaments plus the loop and the 2-cycle. The inclusion $\text{Forb}_h(\mathcal{N}) \subseteq \text{Forb}_e(\mathcal{N})$ holds since every embedding is a homomorphism. To show the other inclusion, suppose that $\mathfrak{A} \in \text{Forb}_e(\mathcal{N})$. The loop clearly does not homomorphically map to \mathfrak{A} because every homomorphism from the loop to \mathfrak{A} is an embedding. Since the loop does not homomorphically map to \mathfrak{A} , every homomorphism from the 2-cycle to \mathfrak{A} is an embedding. Thus, the 2-cycle does not homomorphically map to \mathfrak{A} . Since the loop and the 2-cycle do not homomorphically map to \mathfrak{A} , every homomorphism from a tournament to \mathfrak{A} is an embedding. Thus, \mathfrak{A} does not admit any homomorphic image of a structure from \mathcal{N} . We conclude that $\text{Forb}_e(\mathcal{N}) \subseteq \text{Forb}_h(\mathcal{N})$.

For every selection \mathcal{N} of finitely many tournaments that do not embed into each other, the set \mathcal{N} consists of connected structures since tournaments as well as the loop and the 2-cycle are connected. Moreover, if $\mathcal{N}_1, \mathcal{N}_2$ are two distinct such sets, then $\text{Forb}_h(\mathcal{N}_1) \neq \text{Forb}_h(\mathcal{N}_2)$ [71]. Since there are infinitely many such families \mathcal{N} , Theorem 13 and Proposition 12 yield infinitely many non-isomorphic p-admissible and finitely bounded concrete domains that have different ages. Consequently, the ages of these structures are defined by universal Horn sentences that are not equivalent. This implies that, in the extension of \mathcal{EL} with these concrete domains, different subsumptions hold.

To make this more precise, assume that $\forall \bar{x}(\phi \Rightarrow \psi)$ is a Horn implication that is satisfied by all elements of $\text{Forb}_h(\mathcal{N}_1) = \text{Age}(\mathfrak{D}_{\mathcal{N}_1})$, but for which there is an element \mathfrak{G} of $\text{Forb}_h(\mathcal{N}_2) = \text{Age}(\mathfrak{D}_{\mathcal{N}_2})$ that does not satisfy it. We can easily turn the conjunction of atomic formulas ϕ and the atomic formulas ψ into concepts C_ϕ and C_ψ of the DLs $\mathcal{EL}[\mathfrak{D}_{\mathcal{N}_1}]$ and $\mathcal{EL}[\mathfrak{D}_{\mathcal{N}_2}]$ by viewing the variables in \bar{x} as features and replacing the conjunct operators \wedge in ϕ by DL conjunction \sqcap . If we additionally ensure that all these features are defined (using GCIs $\top \sqsubseteq \exists x, x.(x_1 = x_2)$ for all x occurring in \bar{x}), then C_ϕ is subsumed by C_ψ w.r.t. these GCIs in $\mathcal{EL}[\mathfrak{D}_{\mathcal{N}_1}]$, but not in $\mathcal{EL}[\mathfrak{D}_{\mathcal{N}_2}]$ since one can use $\mathfrak{G} \in \text{Age}(\mathfrak{D}_{\mathcal{N}_2})$ to construct a counterexample to the subsumption.

A more general class of p-admissible structures can be obtained from connected MMSNP (for monotone monadic strict NP) sentences. Recall the notion of a canonical database from Definition 7.

Definition 8. A *connected (equality-free) MMSNP* sentence Φ over a finite relational signature τ is of the form $\Phi = \exists P_1, \dots, P_n \forall \bar{x}(\bigwedge_i \neg(\alpha_i \wedge \beta_i))$ where P_1, \dots, P_n are unary relation symbols not in τ , each α_i is a conjunction of atomic formulas of the form $R(\bar{x})$ for $R \in \tau$ with free variables \bar{x}_i such that $\text{DB}(\exists \bar{x}_i. \alpha_i)$ is connected, and each β_i is a conjunction of atomic formulas of the form $P_i(x)$ for $i \in [n]$ and their negations.

Note that, for every family \mathcal{N} as in Theorem 13, the class $\text{Age}(\mathfrak{D}_{\mathcal{N}})$ consists of all finite models of a particular MMSNP sentence of the form $\forall \bar{x}(\bigwedge_i \neg \alpha_i)$ where each α_i encodes a single structure $\mathfrak{F} \in \mathcal{N}$ up to homomorphic equivalence. The following result is thus as a generalization of Theorem 13 to more complicated forbidden patterns involving existentially quantified unary predicates.

5. A Model-Theoretic Analysis of p -Admissibility

Theorem 14 (Theorem 7 in [21]). *For every connected MMSNP sentence Φ over a finite signature τ , there exists an ω -categorical τ -structure \mathfrak{D}_Φ that is a reduct of a finitely bounded homogeneous structure and such that $\text{Age}(\mathfrak{D}_\Phi)$ consists of all finite models of Φ .*

Like Theorem 13, this theorem can be used to produce p -admissible concrete domains. However, in contrast to the setting considered in Theorem 13, connected MMSNP is known to exhibit a complexity dichotomy between PTIME and NP-complete [26]. The following proposition shows that, already within the class of *reducts* of finitely bounded homogeneous structures, p -admissibility does not only depend on convexity, in contrast to what holds for finitely bounded structures (see Theorem 12).

Proposition 13. *Let Φ be a connected MMSNP sentence over a finite relational signature τ and \mathfrak{D}_Φ any τ -structure as in Theorem 14. Then \mathfrak{D}_Φ is convex. Moreover, \mathfrak{D}_Φ is p -admissible if and only if satisfiability of Φ in finite τ -structures can be tested in polynomial time.*

Proof. We show convexity using Theorem 9. Let \mathfrak{A} be a finite substructure of \mathfrak{D}_Φ . Then $\mathfrak{A} \models \Phi$ and this is witnessed by some sets $P_1, \dots, P_n \subseteq A$. Assume that $\mathfrak{A}^2 \not\models \Phi$. For every $i \in [n]$, we set $P'_i := P_i \times A$. Since $\mathfrak{A}^2 \not\models \Phi$, there exists a tuple \bar{s} over A^2 such that $(\mathfrak{A}^2, P'_1, \dots, P'_n) \models (\alpha_i \wedge \beta_i)(\bar{s})$ for some i . Let \bar{r} be the tuple over A obtained from \bar{s} by taking the projection of each entry in \bar{s} to the first coordinate. By the definition of the product of structures and of the sets P'_i , we obtain $(\mathfrak{A}, P_1, \dots, P_n) \models (\alpha_i \wedge \beta_i)(\bar{r})$, which contradicts our assumption that $\mathfrak{A} \models \Phi$ is witnessed by P_1, \dots, P_n . Thus $\mathfrak{A}^2 \models \Phi$, which shows $\mathfrak{A}^2 \in \text{Age}(\mathfrak{D}_\Phi)$. An application of Theorem 9 thus yields convexity of \mathfrak{D}_Φ .

It remains to determine in which cases we can test validity of Horn implications in \mathfrak{D}_Φ in polynomial time. The proof of Theorem 7 in [21] yields $\text{CSP}(\mathfrak{D}_\Phi) = \text{Age}(\mathfrak{D}_\Phi)$. It can be shown as in the proof of Corollary 6 that testing satisfiability of Horn implications in \mathfrak{D}_Φ reduces in polynomial time to $\text{CSP}(\mathfrak{D}_\Phi)$, which amounts to testing satisfiability of Φ by Theorem 14 because $\text{CSP}(\mathfrak{D}_\Phi) = \text{Age}(\mathfrak{D}_\Phi)$. Hence, testing satisfiability of Horn implications in \mathfrak{D}_Φ can be done in polynomial time if and only if testing satisfiability of Φ in finite structures can be done in polynomial time. \square

Example 13. Consider the following two connected MMSNP sentences:

$$\begin{aligned}\Phi_1 &:= \exists P \forall x, y \left(\neg(E(x, y) \wedge P(x) \wedge P(y)) \wedge \neg(E(x, y) \wedge \neg P(x) \wedge \neg P(y)) \right) \\ \Phi_2 &:= \exists P \forall x, y, z \left(\neg(E(x, y) \wedge E(y, z) \wedge E(z, x) \wedge P(x) \wedge P(y) \wedge P(z)) \right. \\ &\quad \left. \wedge \neg(E(x, y) \wedge E(y, z) \wedge E(z, x) \wedge \neg P(x) \wedge \neg P(y) \wedge \neg P(z)) \right)\end{aligned}$$

It is easy to see that testing satisfiability of Φ_1 in finite structures corresponds to solving the well-known 2-colorability problem, which is known to be tractable. Thus, the structure \mathfrak{B}_{Φ_1} is a p -admissible concrete domain. Satisfiability of Φ_2 in finite structures corresponds to the problem *No-Mono-Tri* (for “no mono-chromatic triangle”), which is known to be NP-complete [19]. Thus, the structure \mathfrak{B}_{Φ_2} is convex, but it is not p -admissible (unless $P=NP$). More examples of connected MMSNP sentences can be found in [19].

5.6 (Non-)closure properties of convexity

In contrast to homogeneity, convexity is quite fragile. For example, it is in general not preserved under adding predicates of the form $=_c$, even under the assumption of finite boundedness.

Proposition 14. *Convex (finitely bounded) structures are not closed under adding singleton predicates $=_c$.*

Proof. The unique countable homogeneous strict partial order \mathfrak{P} was introduced and shown to be convex in Example 11. Consider the extension \mathfrak{P}_c of \mathfrak{P} by a smallest element $c \notin P$, i.e., $c < p$ for every $p \in P$. It is easy to see that $\text{Age}(\mathfrak{P}_c) = \text{Age}(\mathfrak{P})$, which means that \mathfrak{P}_c is still finitely bounded and convex. Now consider its expansion \mathfrak{P}'_c by the unary relation $=_c$, which is interpreted as $\{c\}$. The structure \mathfrak{P}'_c is not convex since $\forall x, y (x = x \wedge =_c(y) \Rightarrow y < x \vee =_c(x))$ holds in it, but neither $\forall x, y (x = x \wedge =_c(y) \Rightarrow y < x)$ nor $\forall x, y (x = x \wedge =_c(y) \Rightarrow =_c(x))$. \square

When it comes to expansions by first-order definable relations, we clearly run into problems if we allow definitions containing disjunctions of atomic formulas. However, except for very specific situations as in Lemma 4, convexity is not even preserved under taking expansions by PP definable relations.

Proposition 15. *Convex (finitely bounded) structures are not closed under taking expansions by PP definable relations.*

Proof. As shown in [51], there exists a unique countable homogeneous undirected graph \mathfrak{H} that embeds precisely those finite undirected graphs not containing the complete graph on three vertices \mathfrak{K}_3 as an induced subgraph. By Lemma 3, \mathfrak{H} is finitely bounded because $\text{Age}(\mathfrak{H})$ is defined by the following universal Horn sentence:

$$\begin{aligned} & \forall x, y, z (E(x, y) \wedge E(y, z) \wedge E(z, x) \Rightarrow \text{false}) \\ & \wedge \forall x, y (E(x, y) \Rightarrow E(y, x)) \\ & \wedge \forall x (E(x, x) \Rightarrow \text{false}) \end{aligned} .$$

By Theorem 12, \mathfrak{H} is also convex. However, the expansion (\mathfrak{H}, \neq) is not convex since

$$(\mathfrak{H}, \neq) \models \forall x_1, x_2, x_3, x_4 (x_1 \neq x_2 \wedge x_3 \neq x_4 \Rightarrow x_1 \neq x_3 \vee x_1 \neq x_4),$$

but both $x_1 = x_3 \neq x_4$ and $x_1 = x_4 \neq x_3$ is possible in (\mathfrak{H}, \neq) . We claim that \neq can be primitively positively defined in \mathfrak{H} by the formula

$$\phi(x_1, x_4) = \exists x_2, x_3 (E(x_1, x_2) \wedge E(x_2, x_3) \wedge E(x_3, x_4)).$$

First, suppose that $\mathfrak{H} \models \phi(h_1, h_4)$ for some $h_1, h_4 \in H$. Then clearly $h_1 \neq h_4$ as otherwise \mathfrak{H} would embed \mathfrak{K}_3 . Second, let h_1, h_4 be arbitrary distinct elements of H . Consider the undirected path \mathfrak{P}_4 with four vertices v_1, v_2, v_3, v_4 . Since \mathfrak{P}_4 does not embed \mathfrak{K}_3 , there exists an embedding

5. A Model-Theoretic Analysis of p -Admissibility

$e: \mathfrak{P}_4 \hookrightarrow \mathfrak{H}$. If there is an edge between h_1 and h_4 , then we can take $x_2 = h_4$ and $x_3 = h_1$ to show that $\mathfrak{H} \models \phi(h_1, h_4)$. Otherwise, the substructures of \mathfrak{H} on $\{h_1, h_4\}$ and on $\{e(v_1), e(v_4)\}$ are isomorphic. Since \mathfrak{H} is homogeneous, there exists $\alpha \in \text{Aut}(\mathfrak{H})$ which sends $e(v_1)$ to h_1 and $e(v_4)$ to h_4 . Since $\alpha \circ e$ is a homomorphism, it follows that $(x_1, \dots, x_4) := (\alpha \circ e(v_1), \dots, \alpha \circ e(v_4))$ satisfies the quantifier-free part of ϕ in \mathfrak{H} , and thus $\mathfrak{H} \models \phi(h_1, h_4)$ also in this case. \square

Also, convexity is not preserved under taking disjoint unions.

Proposition 16. *Convex (finitely bounded) structures are not closed under disjoint union.*

Proof. Consider a signature with a single unary predicate symbol and a structure $(S; R)$ where S is countably infinite and R is interpreted as the whole domain S . This structure is finitely bounded and convex by Lemma 3 and Theorem 12 since its age is defined by the universal Horn sentence $\forall x. R(x)$. If we build the union of $(S; R)$ with an isomorphic copy of itself over a domain disjoint with S , then we obtain a structure isomorphic to the structure $\mathfrak{N} = (\mathbb{N}; E, O)$, of which we have seen in Section 3.2 that it is not convex. \square

However, convexity is preserved under taking the algebraic product. This is an easy consequence of Theorem 9 combined with the fact that the mapping

$$((x_1, x_2), (y_1, y_2)) \mapsto ((x_1, y_1), (x_2, y_2))$$

is an isomorphism between $\mathfrak{D}_1^2 \boxtimes \mathfrak{D}_2^2$ and $(\mathfrak{D}_1 \boxtimes \mathfrak{D}_2)^2$.

Proposition 17. *Convex structures are closed under taking the algebraic product.*

Towards user-definable concrete domains

DL systems that can handle concrete domains allow their users to employ a fixed set of predicates of one or more fixed concrete domains when modelling concepts. They do not provide their users with means for defining new predicates, let alone new concrete domains. Our results in Section 4 alleviate the first restriction since Corollary 3 allows the use of first-order definable predicates and Corollary 4 of predicates definable by EP formulas. To overcome the second restriction, one would need to provide the user with (i) a mechanism for defining a concrete domain; (ii) an algorithm that checks whether this concrete domain is ω - or p-admissible; and (iii) an automated way of generating the required reasoning procedures for this concrete domain. The present chapter is devoted to this topic.

Suppose that we successfully defined a concrete domain \mathfrak{D} , which subsequently passed a test for ω - or p-admissibility. If \mathfrak{D} was confirmed to be ω -admissible, then, by the combination of Theorem 8, Theorem 5, and Corollary 5, the complexity of reasoning in $\mathcal{ALC}(\mathfrak{D})$ only depends on $\text{Age}(\mathfrak{D})$ and not \mathfrak{D} itself. If \mathfrak{D} was confirmed to be p-admissible, then we are only interested in the language $\mathcal{EL}[\mathfrak{D}]$ because, by Corollary 1 and Corollary 9, the full logic $\mathcal{EL}(\mathfrak{D})$ might be undecidable. The fragment $\mathcal{EL}[\mathfrak{D}]$ has the finite model property, and the complexity of reasoning again only depends on $\text{Age}(\mathfrak{D})$. Based on these two observations, the user-definability problem for ω - or p-admissible concrete domains can be vaguely stated as follows:

INPUT: A finite description of a class \mathcal{K} of finite structures in a finite relational signature τ .

QUESTION: Does there exist an ω - or p-admissible τ -structure \mathfrak{D} with $\text{Age}(\mathfrak{D}) = \mathcal{K}$?

For the case of ω -admissible concrete domains, one might think that Theorem 6 provides us with the correct ingredients. To define a concrete domain satisfying the preconditions of this theorem, one could start with selecting a finite set \mathcal{N} of bounds (or equivalently, by Lemma 3, a universal sentence). The first question is then whether $\text{Forb}_e(\mathcal{N})$ really describes the age of a structure. The bad news is that this question is in general undecidable.

Proposition 18. *Let τ be a finite relational signature containing at least one binary symbol. The question whether, for a given finite set \mathcal{N} of finite τ -structures, there is a τ -structure \mathfrak{D} such that $\text{Age}(\mathfrak{D}) = \text{Forb}_e(\mathcal{N})$ is in general undecidable.*

Proof. It is shown in [33] that the JEP is undecidable for classes of undirected graphs definable by finitely many bounds. In addition, it is known that a class of finite structures definable by finitely many bounds has the JEP if and only if this class is the age of some countable structure (Theorem 6.1.1 in [55]). \square

6. Towards user-definable concrete domains

However, to apply Theorem 6, we need the AP rather than just the JEP. In contrast to the JEP, the amalgamation property (AP) is decidable for classes over finite binary signatures defined by finitely many forbidden finite substructures [62]. In Theorem 15, we prove an even stronger statement, namely decidability in Π_2^P . The proof of Theorem 15 uses the following proposition which is well-known [66]; for a proof see, e.g., Proposition 2.3.17 in [18].

Definition 9. Let \mathcal{K} be a class of relational τ -structures. A *one-point amalgamation diagram* for \mathcal{K} is a triple $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ of relational structures from \mathcal{K} together with identity maps $e_i: \mathfrak{A} \hookrightarrow \mathfrak{B}_i$ such that $B_i \setminus A = \{b_i\}$ for $i \in [2]$. We say that \mathcal{K} has the *one-point amalgamation property* if it has the AP restricted to one-point amalgamation diagrams.

Proposition 19. A class of finite relational τ -structures that is closed under isomorphisms and substructures has the AP if and only if it has the AP restricted to one-point amalgamation diagrams.

Theorem 15. Let \mathcal{N} be a finite set of finite τ -structures over a finite relational signature τ consisting of binary symbols. If $\text{Forb}_e(\mathcal{N})$ does not have the AP, then the size of a smallest counterexample to the AP is polynomial in the size of \mathcal{N} . Consequently, the question whether there exists a homogeneous τ -structure \mathfrak{D} such that $\text{Age}(\mathfrak{D}) = \text{Forb}_e(\mathcal{N})$ is decidable in Π_2^P .

Proof. According to [62], it is decidable whether $\text{Forb}_e(\mathcal{N})$ has the AP. By Theorem 5, this is the case if and only if there is a homogeneous structure \mathfrak{D} such that $\text{Age}(\mathfrak{D}) = \text{Forb}_e(\mathcal{N})$. A decision procedure for this problem was also described recently in [25]. By Proposition 19, it is enough to show that $\text{Forb}_e(\mathcal{N})$ has the one-point amalgamation property. Let R_1, \dots, R_n be an enumeration of τ . Suppose that there exists $S \subseteq [n]$ such that the formula

$$\phi_S(y_1, y_2) := \left(\bigwedge_{i \in S} R_i(y_1, y_2) \right) \wedge \left(\bigwedge_{i \in [n] \setminus S} \neg R_i(y_1, y_2) \right) \wedge \left(\bigwedge_{i \in [n]} \neg R_i(y_2, y_1) \right)$$

is not satisfiable in any $\mathfrak{F} \in \mathcal{N}$. Then $\text{Forb}_e(\mathcal{N})$ has the AP because, for every one-point amalgamation diagram $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$, we can choose \mathfrak{C} with domain $B_1 \cup B_2$ and relations

$$R_i^{\mathfrak{C}} := \begin{cases} R_i^{\mathfrak{B}_1} \cup R_i^{\mathfrak{B}_2} \cup \{(b_1, b_2)\} & \text{if } i \in S \\ R_i^{\mathfrak{B}_1} \cup R_i^{\mathfrak{B}_2} & \text{if } i \in [n] \setminus S. \end{cases}$$

It is easy to see that $\mathfrak{C} \in \text{Forb}_e(\mathcal{N})$: since $\mathfrak{F} \in \mathcal{N}$ cannot embed into \mathfrak{B}_1 or \mathfrak{B}_2 , the image of an embedding of \mathfrak{F} into \mathfrak{C} would need to contain b_1 and b_2 , but then the formula $\phi_S(y_1, y_2)$ would be satisfiable in \mathfrak{F} .

Now suppose that $\text{Forb}_e(\mathcal{N})$ does not have the AP. We define the size of \mathcal{N} as the sum of the sizes of all structures in \mathcal{N} , where the size of a structure is the sum of the cardinalities of the domain and all relations. By the argument in the previous paragraph, for every $S \subseteq [n]$, the formula $\phi_S(y_1, y_2)$ must be satisfiable in some $\mathfrak{F} \in \mathcal{N}$. Consequently, this structure contains a tuple (a_1, a_2) such that $\phi_S(a_1, a_2)$ holds, but $\phi_{S'}(a_1, a_2)$ does not hold for any $S' \neq S$. This shows that, over all structures in \mathcal{N} , there are at least $2^{|[n]|}$ tuples. Since all of them except for one belongs to at least one relation and the cardinality of the structures in \mathcal{N} is at least 1, this shows that the size of \mathcal{N} is at least $2^{|[n]|}$.

To prove that the original problem is in Π_2^P , it is sufficient to show that the complement can be decided by an NP procedure that uses a coNP oracle. Given a finite set of bounds \mathcal{N} , we guess a one-point amalgamation diagram $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ and check whether it witnesses that $\text{Forb}_e(\mathcal{N})$ does not have the AP. According to the proof of Theorem 4 in [25], the size of a smallest counterexample to the one-point amalgamation property for $\text{Forb}_e(\mathcal{N})$ is bounded by a polynomial in $m \cdot \ell$ where $m := \max_{\mathfrak{B} \in \mathcal{N}} |F|$ and $\ell := 2^{|\tau|}$. Thus, by what we have shown above for the size of \mathcal{N} , we may assume that the size of $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2$ is polynomial in the size of the input \mathcal{N} , which shows that this triple can be guessed within NP. To verify that it is a counterexample to the AP, we need to check that

1. $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \text{Forb}_e(\mathcal{N})$, and
2. there exists no $\mathfrak{C} \in \text{Forb}_e(\mathcal{N})$ with embeddings $f_i: \mathfrak{B}_i \hookrightarrow \mathfrak{C}$, $i \in [2]$, such that $f_1|_A = f_2|_A$.

The test in item 1 can be performed by a coNP oracle. In fact, to check whether a finite structure does *not* belong to $\text{Forb}_e(\mathcal{N})$, it is sufficient to guess an embedding from an element of \mathcal{N} into this structure. Clearly, this can be done by an NP procedure.

For item 2, first note that it is clearly sufficient to consider structures \mathfrak{C} such that $C = B_1 \cup B_2$ and where the embeddings f_i are the identity.¹ There are only polynomially many structures of this kind. In fact, to determine such a structure, we need to decide for the tuples (b_1, b_2) and (b_2, b_1) to which of the binary relations in τ they belong. There are $2^{|\tau|}$ possibilities for each tuple, and we already know that $2^{|\tau|}$ is polynomial in the size of the input. The test whether $\mathfrak{C} \in \text{Forb}_e(\mathcal{N})$ can again be solved by a coNP oracle.

Thus, we have shown that the complement of the problem of deciding the AP can be solved by an NP procedure that uses a coNP oracle, which finishes the proof of the lemma. \square

Assume that, in the binary case, the test whether $\text{Forb}_e(\mathcal{N})$ has the AP was successful, and let \mathfrak{D} be the corresponding homogeneous structure. Using the results from Chapter 4, we can then transform \mathfrak{D} into an ω -admissible concrete domain through a decomposition of its relations into orbits under $\text{Aut}(\mathfrak{D})$. The required decision procedure for the CSP can then be obtained from the proof of Proposition 7. Thus, for the case of binary signatures, Theorem 6 together with related results in Section 4 provides us with the necessary ingredients for enabling user-definable ω -admissible concrete domains.

Another option would be to directly test whether $\text{Forb}_e(\mathcal{N})$ defines the age of an ω -admissible structure. We show in Corollary 11 that this problem is decidable in Π_2^P as well.

Corollary 11. *The question whether, for a given finite set \mathcal{N} of finite τ -structures over a finite relational signature τ consisting of at most binary symbols, there exists an ω -admissible τ -structure \mathfrak{D} such that $\text{Age}(\mathfrak{D}) = \text{Forb}_e(\mathcal{N})$ is decidable in Π_2^P .*

The proof of Corollary 11 uses the following lemma.

Lemma 5. *Let \mathfrak{D} be a structure with a finite relational signature. Then the following are equivalent:*

¹The case where $f_1(b_1) = f_2(b_2)$ can only yield a counterexample if \mathfrak{B}_1 and \mathfrak{B}_2 are equal up to renaming of b_1 with b_2 , which can easily be checked.

6. Towards user-definable concrete domains

- \mathcal{D} is JD;
- for every $\mathfrak{A} \in \text{Age}(\mathcal{D})$, every strong endomorphism of \mathfrak{A} is an embedding of \mathfrak{A} .

Proof. Let τ be the signature of \mathcal{D} .

“ \Rightarrow ”: Let $\phi(x, y)$ be a formula witnessing JD for \mathcal{D} . Moreover, let \mathfrak{A} be an arbitrary finite structure that embeds to \mathcal{D} and f a strong endomorphism of \mathfrak{A} . Since $\mathfrak{A} \hookrightarrow \mathcal{D}$, the formula $\phi(x, y)$ also witnesses JD for \mathfrak{A} . Then, by Lemma 2, for every pair $a_1, a_2 \in A$, we have $a_1 = a_2$ if and only if $\mathfrak{A} \models \phi(a_1, a_2)$ if and only if $\mathfrak{A} \models \phi(f(a_1), f(a_2))$ if and only if $f(a_1) = f(a_2)$. We conclude that f is an embedding.

“ \Leftarrow ”: Let $\phi(x, y)$ be the disjunction of all τ -formulas $\psi(x, y)$ such that (i) $\psi(x, y)$ is a conjunction of atomic formulas with a symbol from τ and negations of such formulas, (ii) for every atomic formula with a symbol from τ and free variables among x, y , either the formula itself or its negation appears in $\psi(x, y)$, and (iii) there exists $d \in D$ with $\mathcal{D} \models \psi(d, d)$. Clearly, for every $d \in D$, we have $\mathcal{D} \models \phi(d, d)$. Now assume, towards a contradiction, that there is a pair $d_1, d_2 \in D$ of distinct elements such that $\mathcal{D} \models \phi(d_1, d_2)$. Then $\mathcal{D} \models \psi(d_1, d_2)$ for some disjunct ψ of ϕ . By (i) and (ii), ψ completely describes the relations of the substructure \mathfrak{A} of \mathcal{D} on $\{d_1, d_2\}$. By (iii), there exists $d \in D$ such that $\mathcal{D} \models \psi(d, d)$. Thus, the map that sends both d_1 and d_2 to d is a strong homomorphism from \mathfrak{A} to \mathcal{D} . Now it is easy to see that the map sending both d_1 and d_2 to d_1 is strong endomorphism of \mathfrak{A} but not an embedding, a contradiction to our assumption. Thus, $\mathcal{D} \not\models \phi(d_1, d_2)$. Since d_1, d_2 were chosen arbitrarily, we conclude that $\phi(x, y)$ witnesses JD for \mathcal{D} . \square

Proof of Corollary 11. By Proposition 5, for every structure \mathcal{D} with $\text{Age}(\mathcal{D}) = \text{Forb}_e(\mathcal{N})$, we have that \mathcal{D} is a patchwork if and only if $\text{Forb}_e(\mathcal{N})$ has the AP and \mathcal{D} is JDJEPD. If $\text{Forb}_e(\mathcal{N})$ does not have the AP, then no structure \mathcal{D} with $\text{Age}(\mathcal{D}) = \text{Forb}_e(\mathcal{N})$ can be ω -admissible. If, on the other hand, $\text{Forb}_e(\mathcal{N})$ does have the AP, then, by Theorem 5, there exists even a homogeneous structure \mathcal{D} with $\text{Age}(\mathcal{D}) = \text{Forb}_e(\mathcal{N})$. In that case, by Theorem 4 and Corollary 2, \mathcal{D} has homomorphism ω -compactness because it is homogeneous in a finite relational signature. Moreover, by Proposition 7, $\text{CSP}(\mathcal{D})$ is in NP because \mathcal{D} is finitely bounded. Thus, the only subcondition of ω -admissibility that might not be satisfied by \mathcal{D} is JDJEPD. We have seen in Theorem 15 that the question whether $\text{Forb}_e(\mathcal{N})$ has the AP is decidable in Π_2^P . We show that the question whether every structure \mathcal{D} with $\text{Age}(\mathcal{D}) = \text{Forb}_e(\mathcal{N})$ is JDJEPD can be decided in Π_2^P as well. Then we are done.

Let \mathcal{D} be an arbitrary structure with $\text{Age}(\mathcal{D}) = \text{Forb}_e(\mathcal{N})$. Clearly, \mathcal{D} is JEPD if and only if every structure in $\text{Age}(\mathcal{D}) = \text{Forb}_e(\mathcal{N})$ is JEPD. Suppose that some structure in $\text{Forb}_e(\mathcal{N})$ is not JEPD. Since $\text{Forb}_e(\mathcal{N})$ is preserved under taking substructures, the size of a smallest counterexample \mathfrak{A} is bounded by the largest arity of a symbol in τ , which is polynomial in the size of \mathcal{N} . The fact that \mathfrak{A} is not JEPD can be confirmed in NP and $\mathfrak{A} \in \text{Forb}_e(\mathcal{N})$ can be verified using a coNP oracle as in the proof of Theorem 15. Thus, testing JEPD can be done in Π_2^P . Finally, by Lemma 5, \mathcal{D} is not JD if and only if there exists $\mathfrak{A} \in \text{Age}(\mathcal{D}) = \text{Forb}_e(\mathcal{N})$ which has a strong endomorphism that is not an embedding. Since $\text{Forb}_e(\mathcal{N})$ is preserved under taking

substructures, we may assume that $|A| = 2$. Whether such a structure \mathfrak{A} exists can, again, be tested in NP using a coNP oracle. \square

This automated approach can be used to identify RCC8 and Allen as ω -admissible concrete domains because they are both finitely bounded (see Example 4).

It is an open question whether the decidability result for the AP in [62] can be extended to signatures containing symbols of higher arities. In Sections 6.3 and 6.4, we demonstrate that going past binary signatures allows us to obtain polynomial-time reductions from decision problems that are complete for PSPACE or even EXPSpace. However, we were not able to obtain any lower bound stronger than EXPSpace-hardness, let alone a proof of undecidability.

In the setting relevant for p-admissibility, we can show that the analogous problem is undecidable already for signatures containing at most binary symbols, see Theorem 16. This is an easy consequence of Theorem 19 from Section 6.2, which asserts that deciding the JEP for the class of all finite models of a given universal Horn sentence in a finite signature is undecidable even if τ is limited to binary symbols.

Theorem 16. *The question whether, for a given finite set \mathcal{N} of finite τ -structures over a finite relational signature τ , there exists a p-admissible τ -structure \mathfrak{D} such that $\text{Age}(\mathfrak{D}) = \text{Forb}_e(\mathcal{N})$ is undecidable even if τ is limited to binary symbols.*

Proof. By Lemma 3, for every universal Horn sentence Φ in the signature τ , there exists a finite set \mathcal{N}_Φ of finite τ -structures such that $\text{Forb}_e(\mathcal{N}_\Phi)$ is the set of all finite models of Φ . By Corollary 7, there exists a structure \mathfrak{D} with $\text{Age}(\mathfrak{D}) = \text{Forb}_e(\mathcal{N}_\Phi)$ if and only if $\text{Forb}_e(\mathcal{N}_\Phi)$ has the JEP. Moreover, whenever there exists a structure \mathfrak{D} with $\text{Age}(\mathfrak{D}) = \text{Forb}_e(\mathcal{N}_\Phi)$, then \mathfrak{D} is p-admissible by Theorem 12. Thus undecidability follows directly from Theorem 19. \square

Section 6.2 also contains a proof of Π_2^P -completeness of the question whether a given universal sentence in a finite relational signature is equivalent to a universal Horn sentence. Since every finite set of bounds can be translated to a universal sentence of identical size that defines the same class of structures, the problem in Theorem 16 becomes decidable in Π_2^P under the assumption that $\text{Forb}_e(\mathcal{N})$ has the JEP.

6.1 A proof-theoretic perspective

From now on, we will restrict our attention to (sometimes even equality-free) Horn sentences. This will not lead to less general results, quite the opposite, it will allow us to obtain strong lower bounds for the JEP and the AP.

We say that a Horn implication $\forall \bar{x}(\phi \Rightarrow \psi)$ can be applied to a structure \mathfrak{A} if $\exists \bar{x}(\phi \wedge \neg \psi)$ is satisfiable in \mathfrak{A} . Let Φ be a Horn sentence. We say that \mathfrak{A} is closed under application of Horn implications from Φ if no conjunct of Φ is applicable to \mathfrak{A} , i.e., if \mathfrak{A} is a model of Φ .

It will be handy to take a proof-theoretic perspective on the JEP and the AP using the following notion.

Definition 10. Let Φ be an equality-free universal Horn sentence over the relational signature τ , and let $\phi(\bar{x})$ and $\psi(\bar{x}, \bar{y})$ be equality-free conjunctions of atomic τ -formulas. We say that $\phi(\bar{x})$ *dominates* $\psi(\bar{x}, \bar{y})$ *modulo* Φ and write $\psi(\bar{x}, \bar{y}) \leq_{\Phi} \phi(\bar{x})$ if, for every atomic τ -formula $\chi(\bar{x})$ other than equality, $\Phi \models \forall \bar{x}, \bar{y} (\psi(\bar{x}, \bar{y}) \Rightarrow \chi(\bar{x}))$ implies $\Phi \models \forall \bar{x} (\phi(\bar{x}) \Rightarrow \chi(\bar{x}))$.

For brevity, we will sometimes omit universal quantifiers in universal sentences, such as in Definition 11, all first-order variables are then implicitly universally quantified.

Definition 11. Let Φ be an equality-free universal Horn sentence and ψ a Horn implication, both in a fixed relational signature τ . An *SLD-derivation* of ψ from Φ of length s is a finite sequence of Horn implications $\psi_0, \dots, \psi_s = \psi$ such that ψ_0 is a conjunct in Φ and each ψ_i ($i \in [s]$) is a (binary) *resolvent* of ψ_{i-1} and a conjunct ϕ_i from Φ , i.e., for some atomic formula ψ_{i-1}^j , we have

$$\frac{\overbrace{(\psi_{i-1}^1 \wedge \dots \wedge \psi_{i-1}^j \wedge \dots \wedge \psi_{i-1}^{n_{i-1}}) \Rightarrow \psi_{i-1}^0}^{\psi_{i-1}} \quad \overbrace{(\phi_i^1 \wedge \dots \wedge \phi_i^{m_i}) \Rightarrow \psi_{i-1}^j}^{\phi_i}}{\underbrace{(\psi_{i-1}^1 \wedge \dots \wedge \psi_{i-1}^{j-1} \wedge \phi_i^1 \wedge \dots \wedge \phi_i^{m_i} \wedge \psi_{i-1}^{j+1} \wedge \dots \wedge \psi_{i-1}^{n_{i-1}}) \Rightarrow \psi_{i-1}^0}_{\psi_i}}$$

We say that ψ is a *weakening* of a Horn implication ψ' if ψ' can be obtained from ψ by removing any amount of atoms from the premise of ψ and/or replacing the conclusion of ψ by false. Note that ψ is a weakening of itself. There exists an *SLD-deduction* of ψ from Φ , written as $\Phi \vdash \psi$, if ψ is a tautology or a weakening of a Horn implication ψ' that has an SLD-derivation from Φ up to renaming of variables. We will sometimes omit mentioning that variables might have been renamed if this is clear from the context.

The following theorem presents a fundamental property of equality-free universal Horn sentences: that SLD-deduction is a sound and complete calculus for entailment of equality-free Horn implications by equality-free Horn sentences.

Theorem 17 (Theorem 7.10 in [74]). *Let Φ be an equality-free universal Horn sentence and ψ an equality-free Horn implication, both in a fixed signature τ . Then $\Phi \models \psi$ if and only if $\Phi \vdash \psi$.*

6.2 Universal Horn sentences and the JEP

This section deals with two separate topics. First, we discuss the semantical difference between universal sentences and universal Horn sentences. We prove that the the problem of deciding whether a universal sentence is equivalent to a universal Horn sentence is Π_2^P -complete. Second, we prove undecidability of the JEP for classes defined by universal Horn sentences.

Let τ be a relational signature. In an analogy to our definition for relational structures, we call a universal τ -sentence Φ *convex* if whenever Φ entails a τ -implication $\bigwedge_{i \in [n]} \phi_i \Rightarrow \bigvee_{j \in [k]} \psi_j$, then there exists $j \in [k]$ such that Φ already entails $\bigwedge_{i \in [n]} \phi_i \Rightarrow \psi_j$.

We say that Φ is *preserved in products* if for every non-empty family $(\mathfrak{A}_i)_{i \in I}$ of models of Φ , the direct product $\prod_{i \in I} \mathfrak{A}_i$ is also a model of Φ . The following is well known; e.g., the direction “3 \Rightarrow 1” is Corollary 9.1.7 in [54]. Since we also need item 4 of this theorem, we provide a proof for the convenience of the reader.

Theorem 18 (McKinsey). *Let Φ be a universal sentence with relational signature τ . Then the following are equivalent.*

1. Φ is convex;
2. Φ is equivalent to a universal Horn sentence;
3. Φ is preserved in products;
4. Φ is preserved in binary products of finite structures.

Proof. “1 \Rightarrow 2”: We may assume that Φ is in prenex normal form and that its quantifier-free part ϕ is in conjunctive normal form. Every conjunct in ϕ is equivalent to an implication of the form $\bigwedge_{i \in [n]} \phi_i \Rightarrow \bigvee_{j \in [k]} \psi_j$. Since Φ is convex, we can replace the conjunct by $\bigwedge_{i \in [n]} \phi_i \Rightarrow \psi_j$ for some $j \in [k]$. In this way, Φ can be rewritten into an equivalent universal Horn sentence.

“2 \Rightarrow 3”: Corollary 9.1.6 in [54].

“3 \Rightarrow 4”: This direction is trivial.

“4 \Rightarrow 1”: Suppose that Φ has m variables and is not convex, i.e., $\Phi \models \bigwedge_{i \in [n]} \phi_i \Rightarrow \bigvee_{j \in [k]} \psi_j$ but, for every $j \in [k]$, there exists a model \mathfrak{A}_j of Φ such that $\mathfrak{A}_j \models (\bigwedge_{i \in [n]} \phi_i \wedge \neg \psi_j)(\bar{t}_j)$ for some tuple $\bar{t}_j \in A_j^m$. We may assume that each \mathfrak{A}_j is finite; otherwise we replace it with its substructure on the coordinates of \bar{t}_j while preserving the desired properties. For $\bar{s}_j := ((\bar{t}_1[1], \dots, \bar{t}_j[1]), \dots, (\bar{t}_1[m], \dots, \bar{t}_j[m]))$ we have

$$\prod_{i \in [j]} \mathfrak{A}_i \models (\bigwedge_{i \in [n]} \phi_i \wedge \bigwedge_{i \in [j]} \neg \psi_i)(\bar{s}_j).$$

It follows by induction on $j \in [k]$ that if Φ is preserved in binary products of finite structures, then $\prod_{i \in [j]} \mathfrak{A}_i \models \Phi$. We then obtain a contradiction for $j = k$. \square

Proposition 20. *Deciding whether a given universal sentence Φ is equivalent to a universal Horn sentence is Π_2^P -complete. The problem is Π_2^P -hard even when the signature is limited to unary relation symbols.*

Proof. We first prove containment in Π_2^P . For a given universal sentence Φ , let $\phi(x_1, \dots, x_n)$ be the quantifier-free part of Φ . If Φ is not equivalent to a universal Horn sentence, then, by Theorem 18, Φ has two finite models $\mathfrak{A}, \mathfrak{B}$ such that $\mathfrak{A} \times \mathfrak{B}$ is not a model of Φ . This means that there exists $\bar{t} \in (A \times B)^n$ such that $\mathfrak{A} \times \mathfrak{B} \not\models \phi(\bar{t})$. But then, by the definition of product of structures, there exist substructures \mathfrak{A}' of \mathfrak{A} and \mathfrak{B}' of \mathfrak{B} of size at most n with $\bar{t} \in (A' \times B')^n$ and $\mathfrak{A}' \times \mathfrak{B}' \not\models \phi(\bar{t})$. Since models of Φ are preserved under taking substructures, we have $\mathfrak{A}' \models \Phi$ and $\mathfrak{B}' \models \Phi$. Conversely, if there exist two models of Φ of size at most n whose product is not a model of Φ , then clearly Φ is not equivalent to a universal Horn sentence by Theorem 18.

The argument above shows that the following algorithm is sound and complete for the complement of the original problem. We first guess two models \mathfrak{A} and \mathfrak{B} of Φ of size at most

6. Towards user-definable concrete domains

n such that $\mathfrak{A} \times \mathfrak{B}$ is not a model of Φ . The latter can be verified in time polynomial in n by guessing a tuple $\bar{t} \in (A \times B)^n$ such that $\mathfrak{A} \times \mathfrak{B} \not\models \phi(\bar{t})$. Verifying $\mathfrak{A} \models \Phi$ and $\mathfrak{B} \models \Phi$ iteratively would require a loop over up to n^n many tuples, which would not yield an efficient procedure. Instead, we deal with the verification using a coNP oracle that guesses any potential tuple witnessing that $\mathfrak{B} \not\models \Phi$ or $\mathfrak{A} \not\models \Phi$.

Since this shows that the complement of the original problem is in Σ_2^P , the original problem itself is in Π_2^P .

The Π_2^P -hardness can be shown by a reduction from the complement of the propositional $\exists\forall$ SAT problem. Consider an instance

$$\exists X_1, \dots, X_k \forall X_{k+1}, \dots, X_\ell. \Psi(X_1, \dots, X_\ell) \quad (6.1)$$

of propositional $\exists\forall$ SAT. We first obtain the signature $\tau = \{C_1, \dots, C_k, C, L, R\}$ consisting of unary symbols only. Let $\psi(x, x_{k+1}, \dots, x_\ell)$ be the quantifier-free τ -sentence obtained from $\Psi(X_1, \dots, X_\ell)$ by replacing each propositional variable X_i with $C_i(x)$ if $i \in [k]$, and with $C(x_i)$ if $i \in [\ell] \setminus [k]$. Now we set

$$\Phi := \forall x, y, x_{k+1}, \dots, x_\ell (\neg C(x) \wedge C(y) \Rightarrow \psi(x, x_{k+1}, \dots, x_\ell) \wedge (L(x) \vee R(x))).$$

The idea is to show that: Φ is equivalent to $\forall x, y (C(y) \Rightarrow C(x))$ if (6.1) is not satisfiable, and otherwise Φ is not equivalent to any universal Horn sentence.

“ \Rightarrow ”: Suppose that (6.1) is satisfiable, i.e., there exists a map $f : \{X_1, \dots, X_k\} \rightarrow \{0, 1\}$ such that every map $f' : \{X_1, \dots, X_\ell\} \rightarrow \{0, 1\}$ which extends f is a satisfactory assignment for $\Psi(X_1, \dots, X_\ell)$. Let \mathfrak{A}_L be the τ -structure over $\{a_1, a_2\}$ such that

1. for every $i \in [k]$ and $j \in [2]$, $\mathfrak{A}_L \models C_i(a_j)$ if and only if $f(X_i) = 1$,
2. $\mathfrak{A}_L \models \neg C(a_1) \wedge L(a_1)$, and $\mathfrak{A}_L \models C(a_2)$.

We define \mathfrak{A}_R analogously by switching the roles of L and R in item 2 above. It follows directly from our assumption about f , item 1, and item 2 that $\mathfrak{A}_L \models \Phi$. We also clearly get $\mathfrak{A}_R \models \Phi$ since the construction of \mathfrak{A}_L and \mathfrak{A}_R is symmetrical w.r.t. Φ . Now consider the structure $\mathfrak{A}_L \times \mathfrak{A}_R$. We have that $\mathfrak{A}_L \times \mathfrak{A}_R \models \neg C((a_1, a_1)) \wedge C((a_2, a_2))$. However, $\mathfrak{A}_L \times \mathfrak{A}_R \not\models L((a_1, a_1)) \vee R((a_1, a_1))$. Thus, $\text{Mod}_{\text{fin}}(\Phi)$ is not preserved under products, which means that Φ is not equivalent to any universal Horn sentence by Theorem 18.

“ \Leftarrow ”: Suppose that Φ is not equivalent to any universal Horn sentence. By Theorem 18, $\text{Mod}_{\text{fin}}(\Phi)$ is not preserved in binary products of finite structures, i.e., there exist $\mathfrak{A}, \mathfrak{B} \in \text{Mod}_{\text{fin}}(\Phi)$ such that $\mathfrak{A} \times \mathfrak{B} \not\models \Phi$.

First, suppose that either $\mathfrak{A} \models C(a)$ for every $a \in A$ or $\mathfrak{A} \models \neg C(a)$ for every $a \in A$, and either $\mathfrak{B} \models C(b)$ for every $b \in B$ or $\mathfrak{B} \models \neg C(b)$ for every $b \in B$. Then it also holds that either $\mathfrak{A} \times \mathfrak{B} \models C((a, b))$ for every $(a, b) \in A \times B$ or $\mathfrak{A} \times \mathfrak{B} \models \neg C((a, b))$ for every $(a, b) \in A \times B$. But then we clearly have $\mathfrak{A} \times \mathfrak{B} \models \Phi$, a contradiction to our original assumption.

Next suppose that, without loss of generality, there exist $a_1, a_2 \in A$ such that $\mathfrak{A} \models \neg C(a_1) \wedge C(a_2)$. Let $f : \{X_1, \dots, X_k\} \rightarrow \{0, 1\}$ be the map defined by $f(X_i) = 1$ if and only if $\mathfrak{A} \models C_i(a_1)$.

Let $f': \{X_1, \dots, X_\ell\} \rightarrow \{0, 1\}$ be an arbitrary extension of f . We want to show that f' is a satisfactory assignment for $\Psi(X_1, \dots, X_\ell)$. For this, we consider the map $f'': \{x_{k+1}, \dots, x_\ell\} \rightarrow \{a_1, a_2\}$ given by $f''(x_i) = a_2$ if and only if $f'(X_i) = 1$. Since $\mathfrak{A} \models \Phi_1$, we have $\mathfrak{A} \models \psi(a_1, f(x_{k+1}), \dots, f(x_\ell))$. By the definition of ψ and f'' , the map f' is a satisfying assignment for $\Psi(X_1, \dots, X_\ell)$. We conclude that (6.1) is satisfiable. \square

In the context of Theorem 12, the Π_2^p -hardness in Proposition 20 would also be interesting under the assumption that $\text{Mod}_{\text{fin}}(\Phi)$ has the JEP i.e., under the assumption that $\text{Mod}_{\text{fin}}(\Phi)$ is the age of some structure. This can be achieved with a simple trick where we introduce a fresh binary symbol E into τ and replace the conjunction $\neg C(x) \wedge C(y)$ in Φ by the conjunction $\neg C(x) \wedge C(y) \wedge E(x, y) \wedge E(y, x_{k+1}) \wedge \bigwedge_{i=k+1}^{\ell-1} E(x_i, x_{i+1})$. As a consequence, $\text{Mod}_{\text{fin}}(\Phi)$ is even closed under the formation of disjoint unions, which is a strong form of the JEP. This trick is based on a general property of *connected* universal sentences that we later use in our undecidability proof for the JEP. We will only need to formulate this property for equality-free universal Horn sentences, see Proposition 21, but note that it can be formulated for arbitrary universal sentences even if equality atoms are allowed.

Corollary 12. *Deciding whether a given universal sentence Φ such that $\text{Mod}_{\text{fin}}(\Phi)$ has the JEP is equivalent to a universal Horn sentence is Π_2^p -hard, even when the signature is limited to binary relation symbols.*

Definition 12. An equality-free Horn implication $\forall \bar{x}(\phi \Rightarrow \psi)$ is called *connected* if $\text{DB}(\exists \bar{x}. \phi)$ is connected.

Proposition 21. *Let Φ be an equality-free universal Horn sentence such that each conjunct in Φ is connected. Then $\text{Mod}_{\text{fin}}(\Phi)$ is closed under taking disjoint unions and therefore has the JEP.*

Proof. Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \text{Mod}_{\text{fin}}(\Phi)$ be arbitrary. Define \mathfrak{C} as the disjoint union of \mathfrak{B}_1 and \mathfrak{B}_2 . Then there exist embeddings $e_i: \mathfrak{B}_i \hookrightarrow \mathfrak{C}$ ($i \in [2]$) with $e_1(B_1) \cap e_2(B_2) = \emptyset$ and $e_1(B_1) \cup e_2(B_2) = C$. Let $\phi \Rightarrow \psi$ be an arbitrary conjunct in Φ . Suppose that $\mathfrak{C} \models \phi(\bar{c})$ for some tuple \bar{c} over C . Since $\phi \Rightarrow \psi$ is connected and no two elements $c_1, c_2 \in C$ with $c_1 \in e_1(B_1)$ and $c_2 \in e_2(B_2)$ appear in a tuple from a relation of \mathfrak{C} simultaneously, \bar{c} must be a tuple over $e_1(B_1)$ or over $e_2(B_2)$. Without loss of generality, $\bar{c} = e_1(\bar{b})$ for some tuple \bar{b} over B_1 . Since e_1 is an embedding, we have $\mathfrak{B}_1 \models \phi(\bar{b})$. Since $\mathfrak{B}_1 \models (\phi \Rightarrow \psi)$, it follows that $\mathfrak{B}_1 \models \psi(\bar{b})$. Thus, ψ cannot be of the form *false*. Since e_1 is an embedding, we have $\mathfrak{C} \models \psi(\bar{c})$. This shows that $\mathfrak{C} \models (\phi \Rightarrow \psi)$ because \bar{c} was chosen arbitrarily. Since $\phi \Rightarrow \psi$ was also chosen arbitrarily, $\mathfrak{C} \models \Phi$. \square

Example 14. The class \mathcal{P} of all finite strict partial orders, defined by (5.3), is the age of the set of all finite subsets of \mathbb{N} partially ordered by set inclusion. This structure is isomorphic to the disjoint union of the countably many representatives of \mathcal{P} up to isomorphism.

Universal sentences preserved under disjoint unions can be fully characterized in terms of being equivalent to a universally quantified conjunction of connected implications, see, e.g., Theorem 4.4 in [40]. One should not expect any similar normal form for universal sentences

6. Towards user-definable concrete domains

whose finite models have the JEP since, by the results in [33], this property is undecidable. However, there is a useful reformulation of the JEP for equality-free universal Horn sentences. We will use it to simplify the presentation of our undecidability proof.

Lemma 6. *Let Φ be an equality-free universal Horn sentence over the relational signature τ . Then the following are equivalent:*

1. $\text{Mod}_{\text{fin}}(\Phi)$ has the joint embedding property.
2. Suppose that $\phi_1(\bar{x}_1)$ and $\phi_2(\bar{x}_2)$ are equality-free conjunctions of atomic formulas with disjoint sets of variables such that $\phi_i(\bar{x}_i) \wedge \Phi$ is satisfiable for both $i \in [2]$. Then

$$\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \leq_{\Phi} \phi_1(\bar{x}_1).$$

Proof. “1 \Rightarrow 2”: Let $\phi_1(\bar{x}_1)$ and $\phi_2(\bar{x}_2)$ be as in the first part of item 2. Let $\Psi_1(\bar{x}_1)$ and $\Psi_2(\bar{x}_2)$ be the conjunctions of all R -atoms for $R \in \tau$ that are implied by $\Phi \wedge \phi_1(\bar{x}_1)$ and $\Phi \wedge \phi_2(\bar{x}_2)$, respectively. By our assumption, $\Phi \wedge \Psi_1(\bar{x}_1)$ and $\Phi \wedge \Psi_2(\bar{x}_2)$ are both satisfiable. Define \mathfrak{B}_1 and \mathfrak{B}_2 as the structures whose domains consist of the variables $\{\bar{x}_1[1], \dots\}$, and $\{\bar{x}_2[1], \dots\}$, respectively, and where \bar{z} is a tuple of a relation for $R \in \tau$ if the conjunct $R(\bar{z})$ is contained in Ψ_1 or Ψ_2 , respectively. Note that, by construction, \mathfrak{B}_1 and \mathfrak{B}_2 satisfy every Horn implication in Φ . Since Φ is universal Horn, this implies that $\mathfrak{B}_1, \mathfrak{B}_2 \in \text{Mod}_{\text{fin}}(\Phi)$. Since $\text{Mod}_{\text{fin}}(\Phi)$ has the joint embedding property, there exists $\mathfrak{C} \in \text{Mod}_{\text{fin}}(\Phi)$ together with embeddings $f_i: \mathfrak{B}_i \hookrightarrow \mathfrak{C}$ for $i \in \{1, 2\}$. By the construction of \mathfrak{B}_1 and \mathfrak{B}_2 , it follows that $\Phi \not\models \forall \bar{x}_1, \bar{x}_2 (\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \Rightarrow \text{false})$. Let $\chi(\bar{x}_1)$ be an atomic τ -formula other than equality such that $\Phi \models \forall \bar{x}_1, \bar{x}_2 (\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \Rightarrow \chi(\bar{x}_1))$. By the construction of \mathfrak{B}_1 and \mathfrak{B}_2 , and because f_1 and f_2 are homomorphisms, there exist a tuple \bar{z} over B_1 such that $\mathfrak{C} \models \chi(f_1(\bar{z}))$. Since f_1 is an embedding, we must also have $\mathfrak{B}_1 \models \chi(\bar{z})$. Thus, by the construction of \mathfrak{B}_1 and \mathfrak{B}_2 , it follows that $\Phi \models \forall \bar{x}_1 (\phi_1(\bar{x}_1) \Rightarrow \chi(\bar{x}_1))$.

“2 \Rightarrow 1”: Let $\mathfrak{B}_1, \mathfrak{B}_2 \in \text{Mod}_{\text{fin}}(\Phi)$ be arbitrary. We construct a structure $\mathfrak{C} \in \text{Mod}_{\text{fin}}(\Phi)$ with $f_i: \mathfrak{B}_i \hookrightarrow \mathfrak{C}$ as follows. Without loss of generality we may assume that $B_1 \cap B_2 = \emptyset$. Let $\phi_1(\bar{x}_1)$ and $\phi_2(\bar{x}_2)$ be the conjunctions of all R -atoms for $R \in \tau$ which hold in \mathfrak{B}_1 and \mathfrak{B}_2 , respectively. By construction, $\phi_i(\bar{x}_i) \wedge \Phi$ is satisfiable for both $i \in [2]$. Let $\Psi(\bar{x}_1, \bar{x}_2)$ be the conjunction of all atomic formulas implied by $\Phi \wedge \phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2)$. We claim that $\Phi \wedge \Psi$ is satisfiable; otherwise, $\Phi \models \forall \bar{x}_1, \bar{x}_2 (\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \Rightarrow \text{false})$, and then item 2 implies that $\Phi \models \forall \bar{x}_1 (\phi_1(\bar{x}_1) \Rightarrow \text{false})$, which is impossible since $\mathfrak{B}_1 \models \Phi$. Define \mathfrak{C} as the structure with domain $\{\bar{x}_1[1], \dots, \bar{x}_2[1], \dots\}$ and such that $R^{\mathfrak{C}}$ contains a tuple \bar{z} if and only if Ψ contains the conjunct $R(\bar{z})$. For $i \in [2]$, let f_i be the identity map. We claim that f_i is an embedding from \mathfrak{B}_i to \mathfrak{C} . It is clear from the construction of \mathfrak{C} that f_i is a homomorphism. Suppose for contradiction that there exists $R \in \tau$ and a tuple \bar{z} over B_i such that $\bar{z} \notin R_i^{\mathfrak{B}_i}$ while $f_i(\bar{z}) \in R^{\mathfrak{C}}$. For the sake of notation, we assume that $i = 1$; the case that $i = 2$ can be shown analogously. Note that the construction of \mathfrak{C} implies that $\Phi \models \forall \bar{x}_1, \bar{x}_2 (\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \Rightarrow R(\bar{z}))$. Then item 2 implies that $\Phi \models \forall \bar{x}_1 (\phi_1(\bar{x}_1) \Rightarrow R(\bar{z}))$, a contradiction to $\mathfrak{B}_1 \in \text{Mod}_{\text{fin}}(\Phi)$. Thus, f_i is an embedding from \mathfrak{B}_i to \mathfrak{C} . This concludes the proof of the joint embedding property. \square

Now, we are ready to prove that the problem of deciding whether the class of all finite models of a given universal Horn sentence Φ has the joint embedding property is undecidable.

Theorem 19. *For a given equality-free universal Horn sentence Φ the question whether $\text{Mod}_{\text{fin}}(\Phi)$ has the JEP is undecidable even if the signature is limited to at most binary relation symbols.*

Our proof is based on a reduction from the problem of deciding the universality of a given context-free grammar. As usual, the Kleene plus of Σ , denoted by Σ^+ , is the set of all finite words over Σ of length ≥ 1 . A context-free grammar (CFG) is a 4-tuple $G = (N, \Sigma, P, S)$ where

- N is a finite set of *non-terminal symbols*,
- Σ is a finite set of *terminal symbols*,
- P is a finite set of *production rules* of the form $A \rightarrow w$ for $A \in N$ and $w \in (N \cup \Sigma)^+$,
- $S \in N$ is the *start symbol*.

For $u, v \in (N \cup \Sigma)^+$ we write $u \rightarrow_G v$ if there are $x, y \in (N \cup \Sigma)^+$ and $(A \rightarrow w) \in P$ such that $u = xAy$ and $v = xwy$. The transitive closure of \rightarrow_G is denoted by \rightarrow_G^* . The language of G is $L(G) := \{w \in \Sigma^+ \mid S \rightarrow_G^* w\}$. Note that with this definition the *empty word*, i.e., the word ϵ of length 0, can never be an element of $L(G)$; some authors use a modified definition that also allows rules that derive ϵ , but for our purposes the difference is not essential.

Example 15. Let $G := (\{S\}, \{a, b\}, \{S \rightarrow aSb, S \rightarrow ab\}, S)$. Then it follows by a simple induction that $L(G) = \{a^n b^n \mid n \geq 1\}$ because every accepting derivation path in \rightarrow_G is of the form $S \rightarrow_G \dots \rightarrow_G a^{n-1} S b^{n-1} \rightarrow_G a^n b^n$.

The idea of the reduction is to compute from a given context-free grammar G a universal Horn sentence which consists of two parts, Φ_1 and Φ_2 : the sentence Φ_2 only depends on Σ and entails many Horn implications witnessing failure of the JEP via Lemma 6; the sentence Φ_1 can be computed efficiently from G and is such that $\text{Mod}_{\text{fin}}(\Phi_1)$ is closed under the formation of disjoint unions and prevents all the failures of the JEP of $\text{Mod}_{\text{fin}}(\Phi_1 \wedge \Phi_2)$ if and only if G is universal, i.e., $L(G) = \Sigma^+$.

Here, we assume that $(A \rightarrow A) \in P$ for every $A \in N$. Note that this assumption does not influence $L(G)$ at all.

Encoding context-free grammars into ages of relational structures

Let τ_1 be the signature that contains the unary symbols I and T , and the binary relation symbol R_a for every element $a \in N \cup \Sigma$. Let Φ_1 be the universal Horn sentence that contains, for every $(A \rightarrow a_1 \dots a_n) \in P$, the Horn implication

$$\bigwedge_{i \in [n]} R_{a_i}(x_i, x_{i+1}) \Rightarrow R_A(x_1, x_{n+1}), \quad (6.2)$$

and additionally the Horn implication

$$I(x_1) \wedge T(x_2) \wedge R_S(x_1, x_2) \Rightarrow \text{false}. \quad (6.3)$$

6. Towards user-definable concrete domains

Note that each conjunct of Φ_1 is connected, which means that $\text{Mod}_{\text{fin}}(\Phi_1)$ has the JEP by Proposition 21. The following correspondence can be shown via a straightforward induction.

Lemma 7. *For every $w = a_1 \dots a_n \in (N \cup \Sigma)^+$, we have $A \rightarrow_G^* w$ if and only if*

$$\Phi_1 \models \forall x_1, \dots, x_{n+1} \left(\bigwedge_{i \in [n]} R_{a_i}(x_i, x_{i+1}) \Rightarrow R_A(x_1, x_{n+1}) \right). \quad (6.4)$$

Proof. “ \Rightarrow ”: Suppose that $A \rightarrow_G^* a_1 \dots a_n$ for $A \in N$. Then there is a path in \rightarrow_G from A to $a_1 \dots a_n$ of length $\lambda \geq 1$. We prove the statements by induction on λ .

In the *induction base* $\lambda = 1$ we have $(A \rightarrow a) \in P$ in which case (6.4) is a conjunct of Φ_1 .

In the *induction step* $\lambda \rightarrow \lambda + 1$, we assume that the claim holds for all paths of length $\leq \lambda$, and that there exists a path of length $\lambda + 1$ from A to $a_1 \dots a_n$, i.e., there exists $(A \rightarrow a_1 \dots a_{k-1} B a_{\ell+1} \dots a_n) \in P$ and a path of length λ from B to $a_k \dots a_\ell$. By the induction hypothesis, after renaming of variables we have that

$$\Phi_1 \models \forall x_k, \dots, x_{\ell+1} \left(\bigwedge_{i \in [\ell] \setminus [k-1]} R_{a_i}(x_i, x_{i+1}) \Rightarrow R_B(x_k, x_{\ell+1}) \right). \quad (6.5)$$

By the construction of Φ_1 , after renaming of variables we also have that

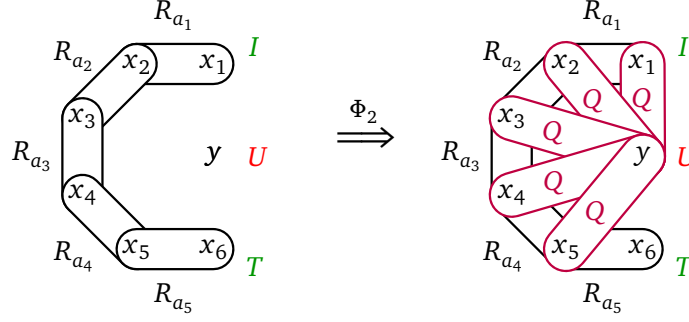
$$\begin{aligned} \Phi_1 \models \forall x_1, \dots, x_k, x_{\ell+1}, \dots, x_{n+1} \left(\bigwedge_{i \in [k-1]} R_{a_i}(x_i, x_{i+1}) \right. \\ \left. \wedge R_B(x_k, x_{\ell+1}) \wedge \bigwedge_{i \in [n] \setminus [\ell]} R_{a_i}(x_i, x_{i+1}) \Rightarrow R_A(x_1, x_{n+1}) \right). \end{aligned} \quad (6.6)$$

We can now apply an SLD derivation step to (6.6) with (6.5) to obtain (6.4).

“ \Leftarrow ”: Suppose that $\Phi_1 \models (6.4)$. By Theorem 17, there is an SLD-derivation of (6.4) from Φ_1 . If (6.4) is a tautology, then $n = 1$ and $a_1 = A$ in which case the statement is true because $(A \rightarrow A) \in P$ for every $A \in N$. Otherwise, (6.4) is a weakening of a Horn implication ψ that has an SLD-derivation from Φ_1 modulo renaming variables. Note that the removal of any atom from the premise of (6.4) would make it disconnected, the same also applies to ψ since its atomic subformulas are among the atomic subformulas of (6.4). Since every Horn implication in Φ_1 is connected, and SLD-derivations modulo renaming variables clearly preserve connectedness, it cannot be the case that (6.4) was obtained from ψ by adding an atom to the premise. Next, note that $\Phi_1 \wedge \bigwedge_{i \in [n]} R_{a_i}(x_i, x_{i+1})$ is satisfiable for every $a_1 \dots a_n \in (N \cup \Sigma)^+$. Thus, since (6.4) is not a tautology, it also cannot be the case that (6.4) was obtained from ψ by adding an atom to the conclusion. Hence, we may assume that (6.4) and ψ are equal. We prove the claim by induction on the length λ of a shortest possible SLD-derivation for ψ .

In the *base case* $\lambda = 0$, ψ must be a conjunct of Φ_1 . By the construction of Φ_1 , we get that $(A \rightarrow a_1 \dots a_n) \in P$ and thus $A \rightarrow_G^* a_1 \dots a_n$.

In the *induction step* $\lambda \rightarrow \lambda + 1$, we assume that the claim holds if ψ has an SLD-derivation of length $\leq \lambda$. Suppose that ψ requires an SLD-derivation of length $\lambda + 1$. By the construction of Φ_1 , there must exist $(B, a_k \dots a_\ell) \in P$ such that Φ_1 contains a conjunct of the form (6.2) that is used in the last step in a shortest possible SLD-derivation of ψ . Moreover, there exists an

Figure 6.1.: An illustration of the situation in Lemma 8 for $n = 6$.

SLD-derivation of length $\leq \lambda$ of

$$\bigwedge_{i \in [k-1]} R_{a_i}(x_i, x_{i+1}) \wedge R_B(x_k, x_{\ell+1}) \wedge \bigwedge_{i \in [n] \setminus [\ell]} R_{a_i}(x_i, x_{i+1}) \Rightarrow R_A(x_1, x_{n+1}) \quad (6.7)$$

from Φ_1 . By the induction hypothesis, (6.7) is equivalent to $A \rightarrow_G^* a_1 \dots a_{k-1} B a_{\ell+1} \dots a_{n-1}$. Therefore, $A \rightarrow_G^* a_1, \dots, a_n$. \square

We remark that context-sensitive grammars *cannot* be encoded into ages of relational structures in a similar fashion, not even under the assumption that all production rules are length-increasing.

Creating candidates for failure of the JEP

Let τ_2 be the signature which contains all symbols from τ_1 except for the ones coming from N and additionally the unary symbol U and the binary symbol Q . The sentence Φ_2 consists of the following Horn implications for every $a \in \Sigma$:

$$U(y) \wedge I(x_1) \Rightarrow Q(y, x_1) \quad (6.8)$$

$$U(y) \wedge Q(y, x_1) \wedge R_a(x_1, x_2) \Rightarrow Q(y, x_2) \quad (6.9)$$

$$U(y) \wedge Q(y, x_1) \wedge R_a(x_1, x_2) \wedge T(x_2) \Rightarrow \text{false} \quad (6.10)$$

The proof of the following claim is left to the reader (a straightforward consequence of Theorem 17), see Figure 6.1 for an illustration of Lemma 8.

Lemma 8. *Let $\phi(\bar{x})$ be a conjunction of $(\tau_2 \setminus \{Q\})$ -atoms. Then $\Phi_2 \models \forall \bar{x} (\phi(\bar{x}) \Rightarrow \text{false})$ if and only if there exists $a_1 \dots a_n \in \Sigma^+$ such that ϕ has a subformula of the form*

$$U(y) \wedge I(x_1) \wedge T(x_{n+1}) \wedge \bigwedge_{i \in [n]} R_{a_i}(x_i, x_{i+1}) \quad (6.11)$$

where the variables need not be distinct.

Now we are ready for the proof of Theorem 19.

6. Towards user-definable concrete domains

Proof of Theorem 19. The universality problem for context-free grammars is known to be undecidable [50] (Lemma 8.4.2, page 259). We set $\Phi := \Phi_1 \wedge \Phi_2$, and show that $\text{Mod}_{\text{fin}}(\Phi)$ has the JEP if and only if $L(G) = \Sigma^+$.

“ \Rightarrow ”: Suppose that $\text{Mod}_{\text{fin}}(\Phi)$ has the JEP. Let $a_1 \dots a_n \in \Sigma^+$. Consider the formulas $\phi_1(y)$ and $\phi_2(x_1, \dots, x_{n+1})$ given by

$$\begin{aligned} \phi_1 &:= U(y) \\ \text{and } \phi_2 &:= I(x_1) \wedge T(x_{n+1}) \wedge \bigwedge_{i \in [n]} R_{a_i}(x_i, x_{i+1}). \end{aligned}$$

By Lemma 8, we have

$$\Phi_2 \models \forall x_1, \dots, x_{n+1}, y (\phi_1 \wedge \phi_2 \Rightarrow \text{false}). \quad (6.12)$$

Since $\text{Mod}_{\text{fin}}(\Phi)$ has the JEP, by Lemma 6(2), $\Phi \wedge \phi_1$ or $\Phi \wedge \phi_2$ is not satisfiable.

Note that every Horn implication in Φ_1 contains an R_a -atom for some $a \in N \cup \Sigma$ in its premise, and ϕ_1 contains none. Also note that every Horn implication in Φ_2 contains an I -atom or an R_a -atom for some $a \in N \cup \Sigma$ in its premise, and ϕ_1 contains none. Hence, the τ -structure with domain $\{y\}$ and whose relations are described by the atomic formula $U(y)$ is closed under application of Horn implications from Φ , i.e., $\Phi \wedge \phi_1$ is satisfiable. Since one of the formulas $\Phi \wedge \phi_1$ and $\Phi \wedge \phi_2$ is not satisfiable, we must have

$$\Phi \models \forall x_1, \dots, x_{n+1} (\phi_2 \Rightarrow \text{false}). \quad (6.13)$$

Next, note that every Horn implication in Φ_2 contains a U -atom in its premise, ϕ_2 contains no U -atom, and no Horn implication in Φ_1 contains an U -atom in its conclusion. We claim that then

$$\Phi_1 \models \forall x_1, \dots, x_{n+1} (\phi_2 \Rightarrow \text{false}). \quad (6.14)$$

Suppose, on the contrary, that $\Phi_1 \wedge \phi_2$ is satisfiable. Let \mathfrak{A} be any finite model of $\Phi_1 \wedge \phi_2$. We assume that $U^{\mathfrak{A}} = \emptyset$, otherwise we remove all elements from $U^{\mathfrak{A}}$. Then \mathfrak{A} still satisfies ϕ_2 because ϕ_2 does not contain any U -atoms, and \mathfrak{A} remains closed under application of Horn implications from Φ_1 because no Horn implication in Φ_1 contains an U -atom in its conclusion. But then \mathfrak{A} is also closed under application of Horn implications from Φ_2 because every Horn implication in Φ_2 contains an U -atom in its premise. We conclude that $\mathfrak{A} \models \Phi \wedge \phi_2$, a contradiction to (6.13). Hence (6.14) holds.

Finally, note that (6.3) is the only Horn implication in Φ_1 which contains an I - or T -atom in its premise, and that no Horn implication in Φ_1 contains an I - or T -atom in its conclusion. We claim that then

$$\Phi_1 \models \forall x_1, \dots, x_{n+1} \left(\bigwedge_{i \in [n]} R_{a_i}(x_i, x_{i+1}) \Rightarrow R_S(x_1, x_{n+1}) \right). \quad (6.15)$$

Suppose, on the contrary, that (6.15) does not hold. Then there exists a finite τ -structure \mathfrak{A} satisfying $\Phi_1 \wedge \bigwedge_{i \in [n]} R_{a_i}(x_i, x_{i+1}) \wedge \neg R_S(x_1, x_{n+1})$. We may assume that $A = \{x_1, \dots, x_{n+1}\}$ because the models of a universal sentence are always closed under taking substructures. We may also assume that $I^{\mathfrak{A}} = T^{\mathfrak{A}} = \emptyset$, otherwise we remove all elements from these relations. Then \mathfrak{A} remains closed under application of Horn implications from Φ_1 because no Horn implication in Φ_1 contains an I - or T -atom in its conclusion. Now consider the structure \mathfrak{A}' obtained from \mathfrak{A} by adding x_1 to $I^{\mathfrak{A}'}$ and x_{n+1} to $T^{\mathfrak{A}'}$. Then \mathfrak{A}' is closed under application of Horn implications from Φ_1 because (6.3) is the only Horn implication in Φ_1 that has an I - or T -atom in its premise and it cannot be applied to \mathfrak{A}' since $(x_1, x_{n+1}) \notin R_S^{\mathfrak{A}'}$. But then $\mathfrak{A}' \models \Phi_1 \wedge \phi_2$, a contradiction to (6.14). Hence (6.15) holds.

Now it follows from Lemma 7 and (6.15) that $a_1 \dots a_n \in L(G)$ and we are done.

“ \Leftarrow ”: We prove the contrapositive and assume that $\text{Mod}_{\text{fin}}(\Phi)$ does not have the JEP. Then there exists a counterexample to Lemma 6(2), i.e., there exists a Horn implication ψ of the following form

$$\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2) \Rightarrow \chi$$

where χ is either false or an atomic τ -formula with free variables among \bar{x}_1 such that

$$\Phi \models \forall \bar{x}_1, \bar{x}_2 (\phi_1 \wedge \phi_2 \Rightarrow \chi), \quad (6.16)$$

$$\Phi \not\models \forall \bar{x}_1 (\phi_1 \Rightarrow \chi), \quad (6.17)$$

and, for both $i \in \{1, 2\}$,

$$\Phi \not\models \forall \bar{x}_i (\phi_i \Rightarrow \text{false}). \quad (6.18)$$

We choose ψ minimal with respect to the number of its atomic subformulas.

Our proof strategy is as follows. First we show that ψ encodes a single word $w \in \Sigma^+$ in the sense of Lemma 8. Then we show that the word w may not be contained in $L(G)$, because otherwise a part of the counterexample would encode w in the sense of Lemma 7 which would lead to a contradiction.

Observation 1. ψ has an SLD-derivation from Φ_2 , and only contains symbols from τ_2 .

Proof of Claim 1. First, we claim that $\Phi_1 \vdash \psi$ or $\Phi_2 \vdash \psi$. By Theorem 17, we have $\Phi \vdash \psi$. Note that $\chi(\bar{x}_1)$ cannot be a subformula of $\phi_1(\bar{x}_1)$, by (6.17). Also note that $\chi(\bar{x}_1)$ cannot be a subformula of $\phi_2(\bar{x}_2)$ as these two formulas have no common variables. Hence, $\chi(\bar{x}_1)$ is not a subformula of $\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2)$, i.e., ψ is not a tautology. Let ψ' be a Horn implication such that ψ is a weakening of ψ' and ψ' has an SLD-derivation $\psi'_0, \dots, \psi'_s = \psi'$ from Φ . Note that the Horn implications in Φ have the property that, depending on whether they come from Φ_1 or from Φ_2 , they either contain no Q -atoms or no R_A -atoms for $A \in N$. This applies in particular to ψ'_0 which is a conjunct from Φ . Since the conclusion of each Horn implication in Φ_1 is an R_A -atom for $A \in N$ and the conclusion of each Horn implication in Φ_2 is a Q -atom, the property of ψ'_0 from above propagates inductively to every ψ'_i for $i \in [k]$. But this means that ψ' has

6. Towards user-definable concrete domains

an SLD-derivation from Φ_1 or from Φ_2 . Hence, ψ has an SLD-deduction from Φ_1 or from Φ_2 , which concludes the claim.

Next, we claim that $\Phi_2 \vdash \psi$. Suppose, on the contrary, that $\Phi_1 \vdash \psi$. Let ϕ'_1 and ϕ'_2 be the formulas obtained from ϕ_1 and ϕ_2 , respectively, by removing all Q -atoms. Since $\Phi_1 \vdash \psi$, the SLD-derivation sequence ψ'_0, \dots, ψ'_s from the paragraph above contains no Q -atoms. Thus, all Q -atoms occurring in ψ come from the weakening step, which means that

$$\Phi_1 \vdash \forall \bar{x}_1, \bar{x}_2 (\phi'_1 \wedge \phi'_2 \Rightarrow \chi). \quad (6.19)$$

Since ψ'_0 is a Horn implication from Φ_1 , it follows from the minimality assumption for ψ that either χ is an R_A -atom for some $A \in N$, or χ is **false**. In both cases, (6.18), (6.19), and (6.17) witness that $\text{Mod}_{\text{fin}}(\Phi_1)$ does not have JEP through an application of Lemma 6. But this is in contradiction to Proposition 21. Thus, $\Phi_1 \vdash \psi$ does not hold, and $\Phi_2 \vdash \psi$ holds instead. This concludes the claim.

Since $\Phi_2 \vdash \psi$, the premise $\phi_1 \wedge \phi_2$ of ψ can only contain symbols from τ_2 , otherwise we could remove all $(\tau_1 \setminus \tau_2)$ -atoms and get a contradiction to the minimality of ψ . Since ψ'_0 is a Horn implication from Φ_2 , it also follows from the minimality assumption for ψ that either χ is an Q -atom, or χ is **false**. Thus, ψ only contains symbols from τ_2 . \square

Observation 2. $\phi_1(\bar{x}_1) \wedge \phi_2(\bar{x}_2)$ does not contain any Q -atom, and χ equals **false**.

Proof of Claim 2. By Claim 1, ψ has an SLD-deduction from Φ_2 . Recall from the proof of Observation 2 that ψ cannot be a tautology. By the minimality of ψ , we may assume that there exists an SLD-derivation of ψ from Φ_2 . Consider any SLD-derivation ψ_0, \dots, ψ_s of ψ from Φ_2 . Note that, by the construction of Φ_2 , for every $i \in [k]$, if there exists a variable y in ψ_{i-1} such that

$$\text{every } Q\text{-atom contains } y \text{ in its first argument,} \quad (*)$$

then ψ_i also satisfies $(*)$ for the same variable y . Since every possible choice of ψ_0 from Φ_2 initially satisfies these two conditions, it follows via induction that $(*)$ must hold for $\psi = \psi_s$ for some y . Also note that (6.8) is the only Horn implication in Φ_2 that is not connected, but the lack of connectivity is only because the variable y satisfying $(*)$ is isolated from the remaining variables. It follows by induction that this is also true for ψ .

We claim that ψ_0 is of the form (6.10). Suppose, on the contrary, that ψ_0 is of the form (6.8) or (6.9). Then the conclusion of ψ is a Q -atom. By our assumption, the conclusion of ψ may only contain variables from \bar{x}_1 . Thus, also the variable y that satisfies $(*)$ for ψ is contained in \bar{x}_1 . Since the second variable in the conclusion of ψ is connected to all remaining variables and the variables of ϕ_1 and ϕ_2 are disjoint, ϕ_2 must be the empty conjunction. This leads to a contradiction to (6.17). Thus our claim holds. The claim implies that χ equals **false**.

Since y satisfies $(*)$ for ψ , if ψ contains any Q -atom in the premise, then ψ is connected. But then, since the variables of ϕ_1 and ϕ_2 are disjoint while ψ is connected, either ϕ_1 or ϕ_2 must be the empty conjunction. Since χ equals **false**, this leads to a contradiction to (6.18).

Thus, ψ does not contain any Q -atoms at all. \square

As a consequence of Claim 1 and Claim 2 we have that $\Phi_2 \models \psi$ where ψ is of the form $\phi_1 \wedge \phi_2 \Rightarrow \text{false}$ and ϕ_1, ϕ_2 are conjunctions of atomic $\tau_2 \setminus \{Q\}$ -formulas. Therefore, Lemma 8 implies that there exists $a_1 \dots a_n \in \Sigma^+$ such that $\phi_1 \wedge \phi_2$ is of the form

$$U(y) \wedge I(x_1) \wedge T(x_{n+1}) \wedge \bigwedge_{i \in [n]} R_{a_i}(x_i, x_{i+1}) \quad (6.20)$$

where the variables need not all be distinct. More specifically,

- $\phi_1(\bar{x}_1)$ equals $U(y)$, and
- $\phi_2(\bar{x}_2)$ equals $I(x_1) \wedge T(x_{n+1}) \wedge \bigwedge_{i \in [n]} R_{a_i}(x_i, x_{i+1})$.

Note that, if $L(G) = \Sigma^+$, then Lemma 7 together with (6.3) implies that

$$\Phi_1 \models \forall x_1, \dots, x_{n+1} (\phi_2 \Rightarrow \text{false}). \quad (6.21)$$

If some variables among x_1, \dots, x_{n+1} are identified in (6.20), then we still have (6.21) even if we perform the same identification of variables. But then we get a contradiction to (6.18). Thus, $L(G) \neq \Sigma^+$.

We have thus found a reduction from the undecidable universality problem for G to the decidability problem of the JEP for $\text{Mod}_{\text{fin}}(\Phi)$; note that Φ is universal Horn and can be computed from G in polynomial time. \square

6.3 Universal sentences and the AP: the Horn case

The contributions of this section are twofold as well. First, we show that the variant of the question in Theorem 19 where we consider the AP instead of the JEP is decidable in Π_2^P . Second, we prove, using a similar technique as in the previous section, that the problem becomes PSPACE-hard if we allow even a single ternary relation symbol in the signature. We invite the reader to identify why the *exact same* strategy as before, using context-free grammars, would fail in showing undecidability for the AP.

Suppose that we are given an equality-free universal Horn sentence Φ in a finite binary signature τ such that $\text{Mod}_{\text{fin}}(\Phi)$ does not have the AP. Then this fact can be verified algorithmically by translating Φ into a set of bounds and applying Theorem 15. However, Theorem 15 only gives us an exponential upper bound for the size of a minimal counterexample because the translation of a universal sentence into a set of bounds might result in a single exponential blowup in size of the input. And indeed, for general universal sentences in binary signatures, a smallest counterexample to the AP can be of exponential size, see Example 16.

Example 16. Consider the sentence $\Phi := \Phi_1 \wedge \Phi_2$ in the binary relational signature $\tau =$

6. Towards user-definable concrete domains

$\{E_1, E_2\} \cup \{L_k, R_k \mid k \in [n]\}$, where

$$\begin{aligned}\Phi_1 &:= \forall x, y_1, y_2 (E_1(y_1, y_1) \wedge E_1(x, y_1) \wedge E_2(y_2, y_2) \wedge E_2(x, y_2) \\ &\quad \Rightarrow \bigwedge_{k \in [n]} (L_k(y_1, y_2) \vee R_k(y_1, y_2)) \\ &\quad \wedge (\bigvee_{k \in [n]} \neg(L_k(x, x) \wedge L_k(y_1, y_2)) \wedge \neg(R_k(x, x) \wedge R_k(y_1, y_2))))), \\ \Phi_2 &:= \forall x (\neg(E_1(x, x) \wedge E_2(x, x)) \wedge \bigwedge_{k \in [n]} \neg(L_k(x, x) \wedge R_k(x, x))).\end{aligned}$$

We claim that $\text{Mod}_{\text{fin}}(\Phi)$ does not have the AP, but a smallest counterexample is of size at least 2^n . By Proposition 19, it is enough to look for a smallest counterexample among one-point amalgamation diagrams. Let $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ be a one-point amalgamation diagram for $\text{Mod}_{\text{fin}}(\Phi)$. Note that $\mathfrak{B}_1 \cup \mathfrak{B}_2 \models \Phi_2$ because Φ_2 has only one free variable. If

$$\mathfrak{B}_1 \cup \mathfrak{B}_2 \not\models E_1(b_1, b_1) \wedge E_2(b_2, b_2) \quad \text{and} \quad \mathfrak{B}_1 \cup \mathfrak{B}_2 \not\models E_2(b_1, b_1) \wedge E_1(b_2, b_2),$$

then also $\mathfrak{B}_1 \cup \mathfrak{B}_2 \models \Phi_1$ because the only way how one could possibly obtain $\mathfrak{B}_1 \cup \mathfrak{B}_2 \not\models \Phi_1$ is by substituting b_1 for y_1 and b_2 for y_2 or vice versa in the quantifier-free part of Φ_1 . But then $\mathfrak{C} := \mathfrak{B}_1 \cup \mathfrak{B}_2$ is an amalgam for $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$. Thus, without loss of generality, suppose that $\mathfrak{B}_1 \cup \mathfrak{B}_2 \models E_1(b_1, b_1) \wedge E_2(b_2, b_2)$. If, for every $k \in [n]$, we can define $\phi_k(x) := L_k(x, x)$ or $\phi_k(x) := R_k(x, x)$ in such a way that there exists no $a \in A$ with

$$\mathfrak{A} \models E_1(a, b_1) \wedge E_2(a, b_2) \wedge \bigwedge_{k \in [n]} \phi_k(a),$$

then we can choose, as an amalgam, the structure \mathfrak{C} with domain $C := B_1 \cup B_2$ and relations $E_1^{\mathfrak{C}} := E_1^{\mathfrak{B}_1} \cup E_1^{\mathfrak{B}_2}$, $E_2^{\mathfrak{C}} := E_2^{\mathfrak{B}_1} \cup E_2^{\mathfrak{B}_2}$, and, for every $k \in [n]$,

$$\begin{aligned}L_k^{\mathfrak{C}} &:= \begin{cases} L_k^{\mathfrak{B}_1} \cup L_k^{\mathfrak{B}_2} \cup \{(b_1, b_2)\} & \text{if } \phi_k(x) \text{ equals } L_k(x, x), \\ L_k^{\mathfrak{B}_1} \cup L_k^{\mathfrak{B}_2} & \text{if } \phi_k(x) \text{ equals } R_k(x, x), \end{cases} \\ R_k^{\mathfrak{C}} &:= \begin{cases} R_k^{\mathfrak{B}_1} \cup R_k^{\mathfrak{B}_2} \cup \{(b_1, b_2)\} & \text{if } \phi_k(x) \text{ equals } R_k(x, x), \\ R_k^{\mathfrak{B}_1} \cup R_k^{\mathfrak{B}_2} & \text{if } \phi_k(x) \text{ equals } L_k(x, x). \end{cases}\end{aligned}$$

If, on the other hand, it is not possible to define ϕ_1, \dots, ϕ_n in this way, then no amalgam of $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ can satisfy Φ_1 , i.e., $\text{Mod}_{\text{fin}}(\Phi)$ does not have the AP. For that to be the case, \mathfrak{A} must contain elements which prevent all 2^n possible choices for ϕ_1, \dots, ϕ_n . Since $\mathfrak{A} \models \Phi_2$, every element of A can only prevent one particular choice, which implies that $|A| \geq 2^n$. It remains to provide an actual counterexample. We define $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ as follows: $A := \{a_S \mid S \subseteq [n]\}$, $B_i := A \cup \{b_i\}$ ($i \in [2]$), and the relations are

- $E_1^{\mathfrak{A}} := \emptyset$, $E_2^{\mathfrak{A}} := \emptyset$, and, for $k \in [n]$, $L_k^{\mathfrak{A}} := \{(a_S, a_S) \mid k \in [n] \setminus S\}$, $R_k^{\mathfrak{A}} := \{(a_S, a_S) \mid k \in S\}$;
- for $i \in [2]$, $E_i^{\mathfrak{B}_i} := \{(b_i, b_i)\} \cup \{(a_S, b_i) \mid S \subseteq [n]\}$, $E_{3-i}^{\mathfrak{B}_i} := \emptyset$, and, for $k \in [n]$, $L_k^{\mathfrak{B}_i} := L_k^{\mathfrak{A}}$ and $R_k^{\mathfrak{B}_i} := R_k^{\mathfrak{A}}$.

In the equality-free binary Horn case, however, we can show that a counterexample of

polynomial size always exists, see Proposition 22. We remark that a similar argument would also work in the binary Horn case where equalities are allowed.

Proposition 22. *Let Φ be an equality-free universal Horn sentence over a finite relational signature τ consisting of binary symbols. If $\text{Mod}_{\text{fin}}(\Phi)$ does not have the AP, then the size of a smallest counterexample to the AP is polynomial in the size of Φ . Consequently, the question whether $\text{Mod}_{\text{fin}}(\Phi)$ has the AP is decidable in Π_2^P .*

Proof. By Proposition 19, a smallest counterexample to the AP for $\text{Mod}_{\text{fin}}(\Phi)$ is a one-point amalgamation diagram $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ for which we cannot form an amalgam $\mathfrak{C} \in \text{Mod}_{\text{fin}}(\Phi)$ simply by adding (b_1, b_2) or (b_2, b_1) to some relations of $\mathfrak{B}_1 \cup \mathfrak{B}_2$. We now show how to inductively verify that $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ has no amalgam within $\text{Mod}_{\text{fin}}(\Phi)$ in i steps where $1 \leq i \leq 2|\tau|$. Let m be the number of free variables of Φ . We set $\mathfrak{C}_0 := \mathfrak{B}_1 \cup \mathfrak{B}_2$.

Suppose that $i = 1$, or that the $(i - 1)$ -th step has not yet provided us with a certificate that $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ does not have any amalgam. Since \mathfrak{C}_{i-1} is not an amalgam for $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$, it is not closed under application of Horn implications from Φ . Thus, there is a substructure \mathfrak{F}_{i-1} of \mathfrak{C}_{i-1} with $|F_{i-1}| \leq m$ such that the application of some Horn implication from Φ to \mathfrak{F}_{i-1} either yields **false**, or expands some relation of \mathfrak{F}_{i-1} by a new pair of elements. If **false** is obtained or if the new pair of elements is not among $(b_1, b_2), (b_2, b_1)$, then we get a certificate for the fact that $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ does not have any amalgam satisfying Φ . Otherwise, we obtain \mathfrak{C}_i from \mathfrak{C}_{i-1} by adding (b_1, b_2) or (b_2, b_1) to the particular relation and proceed to the $(i + 1)$ -th step.

Clearly, it takes at most $2|\tau|$ -many steps to arrive at a certificate for the fact that $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ does not have any amalgam satisfying Φ . Since $\text{Mod}_{\text{fin}}(\Phi)$ is closed under taking substructures, we may assume that A only contains elements from $\bigcup_i F_i$. Since $|F_i| \leq m$ for every i , we conclude that the size of a minimal counterexample is polynomial in the size of Φ . Now the decidability in Π_2^P follows similarly as in the proof of Theorem 15. \square

The situation changes drastically when we allow ternary symbols in the signature. In Theorem 20, we show that the involvement of just a single ternary relation symbol makes the problem in Proposition 22 PSPACE-hard. Also, the existence of a polynomial-sized counterexample to the AP can no longer be guaranteed, as we show in Corollary 14. This is the simplest case in which no upper bound for the complexity of testing the AP is known.

Similarly as in the proof of Theorem 19, in the proof of Theorem 20, we rely on a certain trick which gives us the AP “for free.”

Definition 13. An equality-free Horn implication $\forall \bar{x}(\phi \Rightarrow \psi)$ is called *complete* if the Gaifman graph of $\text{DB}(\exists \bar{x}. \phi)$ is complete.

Proposition 23. *Let Φ be an equality-free universal Horn sentence such that each conjunct in Φ is complete. Then $\text{Mod}_{\text{fin}}(\Phi)$ is closed under taking unions and therefore has the AP.*

Proof. By Proposition 19, it is enough to show that $\text{Mod}_{\text{fin}}(\Phi)$ has the one-point amalgamation property. Let $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ be a one-point amalgamation diagram for $\text{Mod}_{\text{fin}}(\Phi)$. If $b_1 = b_2$, then $\mathfrak{B}_1 \cup \mathfrak{B}_2 = \mathfrak{B}_1$ is a trivial amalgam for $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ satisfying Φ . So suppose that instead,

6. Towards user-definable concrete domains

$b_1 \neq b_2$. Let $\phi \Rightarrow \psi$ be an arbitrary conjunct in Φ . Suppose that $\mathfrak{B}_1 \cup \mathfrak{B}_2 \models \phi(\bar{b})$ for some tuple \bar{b} over $B_1 \cup B_2$. Since $\phi \Rightarrow \psi$ is complete and b_1, b_2 never appear in a tuple from a relation of $\mathfrak{B}_1 \cup \mathfrak{B}_2$ simultaneously, we already have $\mathfrak{B}_1 \models \phi(\bar{b})$ or $\mathfrak{B}_2 \models \phi(\bar{b})$. But, then we also have $\mathfrak{B}_1 \cup \mathfrak{B}_2 \models \psi(\bar{b})$ because $\mathfrak{B}_1, \mathfrak{B}_2 \in \text{Mod}_{\text{fin}}(\Phi)$. Since $\phi \Rightarrow \psi$ was chosen arbitrarily, we conclude that $\mathfrak{B}_1 \cup \mathfrak{B}_2 \models \Phi$. \square

A class \mathcal{C} of relational τ -structures has the *strong amalgamation property* if $\mathfrak{C} \in \mathcal{C}$ and $f_i: \mathfrak{B}_i \hookrightarrow \mathfrak{C}$ can be chosen so that $f_1(B_1) \cap f_2(B_2) = f_1(e_1(A)) = f_2(e_2(A))$.

Lemma 9. *Let Φ be an equality-free universal Horn sentence over the relational signature τ . Then the following are equivalent:*

- 1a. $\text{Mod}_{\text{fin}}(\Phi)$ has the amalgamation property.
- 1b. $\text{Mod}_{\text{fin}}(\Phi)$ has the one-point amalgamation property.
- 2a. $\text{Mod}_{\text{fin}}(\Phi)$ has the strong amalgamation property.
- 2b. $\text{Mod}_{\text{fin}}(\Phi)$ has the one-point strong amalgamation property.
- 3. Suppose that $\phi(\bar{x})$, $\phi_1(\bar{x}, y_1)$, and $\phi_2(\bar{x}, y_2)$ are equality-free conjunctions of atomic formulas, where y_1 and y_2 are distinct variables not contained in \bar{x} , such that, for $i \in [2]$, every atom in $\phi_i(\bar{x}, y_i)$ contains the variable y_i , and $\phi(\bar{x}) \wedge \phi_i(\bar{x}, y_i) \leq_{\Phi} \phi(\bar{x})$. Then

$$\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1) \wedge \phi_2(\bar{x}, y_2) \leq_{\Phi} \phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1).$$

Proof. The equivalence of items 1a and 1b follows from Proposition 19. The equivalence of items 2a and 2b is also clear as it follows exactly the same principle, see, e.g., Proposition 2.3.18 in [18].

“2a \Rightarrow 1a”: This direction is trivial.

“1b \Rightarrow 3”: Let $\phi(\bar{x})$, $\phi_1(\bar{x}, y_1)$, and $\phi_2(\bar{x}, y_2)$ be as in the first part of item 3. Let $\Psi(\bar{x})$, $\Psi_1(\bar{x}, y_1)$, and $\Psi_2(\bar{x}, y_2)$ be the conjunctions of all R -atoms for $R \in \tau$ implied by $\Phi \wedge \phi(\bar{x})$, $\Phi \wedge \phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1)$, and $\Phi \wedge \phi(\bar{x}) \wedge \phi_2(\bar{x}, y_2)$, respectively. If $\Phi \wedge \Psi(\bar{x})$, $\Phi \wedge \Psi_1(\bar{x}, y_1)$, or $\Phi \wedge \Psi_2(\bar{x}, y_2)$ is unsatisfiable, then $\Phi \wedge \Psi(\bar{x})$ is unsatisfiable by the domination assumption and we are done. So suppose that all three conjunctions are satisfiable. Define \mathfrak{A} , \mathfrak{B}_1 , and \mathfrak{B}_2 as the structures whose domains consist of the variables $\{\bar{x}[1], \dots\}$, $\{y_1, \bar{x}[1], \dots\}$, and $\{y_2, \bar{x}[1], \dots\}$, respectively, and where \bar{z} is a tuple of a relation for $R \in \tau$ if the conjunct $R(\bar{z})$ is contained in Ψ , Ψ_1 , or Ψ_2 , respectively. Since $\phi(\bar{x}) \wedge \phi_i(\bar{x}, y_i) \leq_{\Phi} \phi(\bar{x})$ for $i \in \{1, 2\}$, there exist embeddings $e_i: \mathfrak{A} \hookrightarrow \mathfrak{B}_i$ for $i \in \{1, 2\}$. Note that, by construction, \mathfrak{A} , \mathfrak{B}_1 , and \mathfrak{B}_2 satisfy every Horn implication in Φ . Since Φ is universal Horn, this implies that $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \text{Mod}_{\text{fin}}(\Phi)$. Since $\text{Mod}_{\text{fin}}(\Phi)$ has the one-point amalgamation property, there exists $\mathfrak{C} \in \text{Mod}_{\text{fin}}(\Phi)$ together with embeddings $f_i: \mathfrak{B}_i \hookrightarrow \mathfrak{C}$ for $i \in \{1, 2\}$ such that $f_1 \circ e_1 = f_2 \circ e_2$. By the construction of $\mathfrak{A}, \mathfrak{B}_1$, and \mathfrak{B}_2 , it follows that $\Phi \not\models \forall \bar{x}, y_1, y_2 (\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1) \wedge \phi_2(\bar{x}, y_2) \Rightarrow \text{false})$. Let $\chi(\bar{x}, y_1)$ be an atomic τ -formula other than equality such that $\Phi \models \forall \bar{x}, y_1, y_2 (\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1) \wedge \phi_2(\bar{x}, y_2) \Rightarrow \chi(\bar{x}, y_1))$. By the construction of $\mathfrak{A}, \mathfrak{B}_1$, and \mathfrak{B}_2 , and because f_1 and f_2 are homomorphisms, there exist a tuple \bar{z} over B_1 such that $\mathfrak{C} \models \chi(f_1(\bar{z}))$. Since f_1 is an

embedding, we must also have $\mathfrak{B}_1 \models \chi(\bar{z})$. Thus, by the construction of $\mathfrak{A}, \mathfrak{B}_1$, and \mathfrak{B}_2 , it follows that $\Phi \models \forall \bar{x}, y_1 (\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1) \Rightarrow \chi(\bar{x}, y_1))$.

“ $3 \Rightarrow 2b$ ”: Let $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \text{Mod}_{\text{fin}}(\Phi)$ be such that $e_i: \mathfrak{A} \hookrightarrow \mathfrak{B}_i$ and $B_i \setminus e_i(A) = \{y_i\}$ for $i \in \{1, 2\}$. We construct a structure $\mathfrak{C} \in \text{Mod}_{\text{fin}}(\Phi)$ with $f_i: \mathfrak{B}_i \hookrightarrow \mathfrak{C}$ and $f_1 \circ e_1 = f_2 \circ e_2$ as follows. Without loss of generality we may assume that $A = e_1(A) = e_2(A)$, i.e., \mathfrak{A} is the intersection of \mathfrak{B}_1 and \mathfrak{B}_2 . Let \bar{x} be a tuple of variables representing the elements of $B_1 \cap B_2$ in some order, and let $\phi(\bar{x})$ be the conjunction of all R -atoms for $R \in \tau$ which hold in \mathfrak{A} . Moreover, for $i \in [2]$, let $\phi_i(\bar{x}, y_i)$ be the conjunction of all R -atoms for $R \in \tau$ which contain the variable y_i and hold in \mathfrak{B}_i . Note that $\phi(\bar{x}) \wedge \phi_i(\bar{x}, y_i) \leq_{\Phi} \phi(\bar{x})$ for both $i \in [2]$, because e_i is an embedding. Let $\Psi(\bar{x}, y_1, y_2)$ be the conjunction of all R -atoms for $R \in \tau$ implied by $\Phi \wedge \phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1) \wedge \phi_2(\bar{x}, y_2)$. We claim that $\Phi \wedge \Psi$ is satisfiable: otherwise, $\Phi \models \forall \bar{x}, y_1, y_2 (\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1) \wedge \phi_2(\bar{x}, y_2) \Rightarrow \text{false})$, and then item 3 implies that $\Phi \models \forall \bar{x}, y_1 (\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1) \Rightarrow \text{false})$, which is impossible since $\mathfrak{B}_1 \models \Phi$. Define \mathfrak{C} as the structure with domain $\{y_1, y_2, \bar{x}[1], \dots\}$ and such that $R^{\mathfrak{C}}$ contains a tuple \bar{z} if and only if Ψ contains the conjunct $R(\bar{z})$. For $i \in [2]$, let f_i be the identity map. We claim that f_i is an embedding from \mathfrak{B}_i to \mathfrak{C} . It is clear from the construction of \mathfrak{C} that f_i is a homomorphism. Suppose for contradiction that there exists $R \in \tau$ and a tuple \bar{z} over B_i such that $\bar{z} \notin R_i^{\mathfrak{B}_i}$ while $f_i(\bar{z}) \in R^{\mathfrak{C}}$. For the sake of notation, we assume that $i = 1$; the case that $i = 2$ can be shown analogously. Note that the construction of \mathfrak{C} implies that $\Phi \models \forall \bar{x}, y_1, y_2 (\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1) \wedge \phi_2(\bar{x}, y_2) \Rightarrow R(\bar{z}))$. Then item 3 implies that $\Phi \models \forall \bar{x}, y_1 (\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1) \Rightarrow R(\bar{z}))$, a contradiction to $\mathfrak{B}_1 \in \text{Mod}_{\text{fin}}(\Phi)$. Thus, f_i is an embedding from \mathfrak{B}_i to \mathfrak{C} . By the construction of \mathfrak{C} we also clearly have that $f_1 \circ e_1 = f_2 \circ e_2$, which concludes the proof of the one-point strong amalgamation property. \square

The equivalence between the AP and the strong AP is essentially due to the fact that equality atoms are not permitted in Φ .

Theorem 20. *For a given universal Horn sentence Φ the question whether $\text{Mod}_{\text{fin}}(\Phi)$ has the AP is PSPACE-hard even if the signature is limited to at most ternary relation symbols.*

Our proof is based on a reduction from the problem of deciding the universality of a given regular grammar. A (left-) regular grammar is a 4-tuple $G = (N, \Sigma, P, S)$ where

- N is a finite set of non-terminal symbols,
- Σ is a finite set of terminal symbols,
- P is a finite set of production rules of the form $A \rightarrow a$ or $A \rightarrow Ba$ for $A, B \in N$ and $a \in \Sigma$,
- $S \in N$ is the start symbol.

For $u \in (N \cup \Sigma)^+$ we write $u \rightarrow_G v$ if there exist $x \in \Sigma^+$ and $(p \rightarrow q) \in P$ such that $u = xp$ and $v = xq$. The transitive closure of \rightarrow_G is denoted by \rightarrow_G^* . The language of G is $L(G) := \{w \in \Sigma^+ \mid S \rightarrow_G^* w\}$. Note that with this definition the empty word, i.e., the word ϵ of length 0, can never be an element of $L(G)$; some authors use a modified definition that also allows rules that derive ϵ , but for our purposes the difference is not essential.

The idea of the reduction is to compute from a given regular grammar G a universal Horn sentence which consists of two parts, Φ_1 and Φ_2 : the sentence Φ_2 does not depend on G and

entails many Horn implications witnessing failure of the AP via Lemma 9; the sentence Φ_1 can be computed efficiently from G and is such that $\text{Mod}_{\text{fin}}(\Phi_1)$ is closed under taking unions and prevents all the failures of the AP of $\text{Mod}_{\text{fin}}(\Phi_1 \wedge \Phi_2)$ if and only if G is universal, i.e., $L(G) = \Sigma^+$.

Encoding regular grammars into amalgamation classes

Let τ_1 be the signature that contains the unary symbols I and T , the binary symbol E_1 , and the binary relation symbol R_a for every element $a \in N \cup \Sigma$. Let Φ_1 be the universal Horn sentence that contains for every $(A \rightarrow Ba) \in P$ the Horn implication

$$I(y_1) \wedge E_1(y_1, x_2) \wedge R_B(y_1, x_1) \wedge R_a(x_1, x_2) \Rightarrow R_A(y_1, x_2), \quad (6.22)$$

for every $(A \rightarrow a) \in P$, the Horn implication

$$I(y_1) \wedge R_a(y_1, x_1) \Rightarrow R_A(y_1, x_1), \quad (6.23)$$

and additionally the Horn implication

$$I(y_1) \wedge T(x_1) \wedge R_S(y_1, x_1) \Rightarrow \text{false}. \quad (6.24)$$

Note that, due to the presence of the E_1 -atom in (6.22), each conjunct of Φ_1 is complete, which means that $\text{Mod}_{\text{fin}}(\Phi_1)$ has the AP by Proposition 23. The following correspondence can be shown via a straightforward induction.

Lemma 10. *For every $w = a_1 \dots a_n \in \Sigma^+$ and $A \in N$, $A \rightarrow_G^* w$ if and only if*

$$\begin{aligned} \Phi_1 \models \forall x_1, \dots, x_n, y_1 & (I(y_1) \wedge R_{a_1}(y_1, x_1) \\ & \wedge \bigwedge_{i \in [n-1]} R_{a_i}(x_i, x_{i+1}) \wedge E_1(y_1, x_{i+1}) \Rightarrow R_A(y_1, x_n)). \end{aligned} \quad (6.25)$$

Proof. “ \Rightarrow ”: Suppose that $A \rightarrow_G^* a_1 \dots a_n$ for $A \in N$. Then there is a path in \rightarrow_G from A to $a_1 \dots a_n$ of length $\lambda \geq 1$. We prove the statements by induction on λ .

In the *induction base* $\lambda = 1$ we must have $(A \rightarrow a) \in P$ in which case (6.25) is a conjunct of Φ_1 .

In the *induction step* $\lambda \rightarrow \lambda + 1$, we assume that the claim holds for all paths of length $\leq \lambda$, and that there exists a path of length $\lambda + 1$ from A to $a_1 \dots a_n$, i.e., there exists $(A \rightarrow Ba_n) \in P$ and a path of length λ from B to $a_1 \dots a_{n-1}$. By the induction hypothesis, we have that

$$\begin{aligned} \Phi_1 \models \forall x_1, \dots, x_{n-1}, y_1 & (I(y_1) \wedge R_{a_1}(y_1, x_1) \\ & \wedge \bigwedge_{i \in [n-2]} R_{a_{i+1}}(x_i, x_{i+1}) \wedge E_1(y_1, x_{i+1}) \Rightarrow R_B(y_1, x_{n-1})). \end{aligned} \quad (6.26)$$

By the construction of Φ_1 , after renaming of variables we also have that

$$\begin{aligned} \Phi_1 \models \forall x_{n-1}, x_n, y_1 (I(y_1) \wedge R_B(y_1, x_{n-1}) \\ \wedge R_{a_n}(x_{n-1}, x_n) \wedge E_1(y_1, x_n) \Rightarrow R_A(y_1, x_n)). \end{aligned} \quad (6.27)$$

We can now apply an SLD derivation step to (6.27) with (6.26) to obtain (6.25).

“ \Leftarrow ”: Suppose that $\Phi_1 \models (6.25)$. By Theorem 17, there is an SLD-deduction of (6.25) from Φ_1 . It cannot be the case that (6.25) is a tautology because each a_i is a terminal symbol and A is a non-terminal symbol. Thus, (6.25) is a weakening of a Horn implication ψ that has an SLD-derivation from Φ_1 modulo renaming variables. Note that, for every Horn implication ψ' in Φ_1 except (6.24), there is a distinguished variable y_1 such that

1. the conclusion of ψ' is a binary atom whose first entry is occupied by y_1 and the second entry contains a different variable,
2. for every other variable x_i , the premise of ψ' contains a binary atom whose first entry is occupied by y_1 and the second entry contains x_i ,
3. the Horn implication obtained by removing the variable y_1 together with all atoms involving this variable from ψ' is connected.

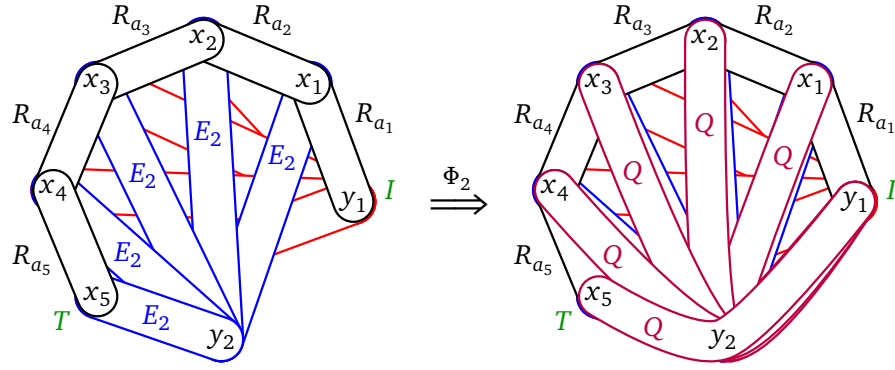
Since the premise of (6.25) does not contain any T -atom, neither does the premise of ψ . Thus, no SLD-derivation for ψ from Φ_1 may involve the Horn implication (6.24). The existence of a variable satisfying item 1, item 2, and item 3 clearly propagates inductively through SLD-derivations modulo renaming variables, and thus also to ψ . Since the removal of any atom from (6.25) would violate item 1, item 2, or item 3, the weakening from above must be trivial, i.e., we may assume that (6.25) and ψ are equal. We prove the claim by induction on the length λ of a shortest possible SLD-derivation for ψ .

In the *base case* $\lambda = 0$, ψ must be a conjunct of Φ_1 . Since each a_i is a terminal symbol, ψ must be of the form (6.23). By the construction of Φ_1 , we get that $(A \rightarrow a_1) \in P$ and thus $A \rightarrow_G^* a_1$.

In the *induction step* $\lambda \rightarrow \lambda + 1$, we assume that the claim holds if ψ has an SLD-derivation of length $\leq \lambda$. Suppose that ψ requires an SLD-derivation of length $\lambda + 1$. By the construction of Φ_1 , there must exist $(A, Ba_n) \in P$ such that Φ_1 contains a conjunct of the form (6.22) that is used in the last step in a shortest possible SLD-derivation of ψ . Moreover, there exists an SLD-deduction of

$$I(y_1) \wedge R_{a_1}(y_1, x_1) \wedge \bigwedge_{i \in [n-2]} R_{a_{i+1}}(x_i, x_{i+1}) \wedge E_1(y_1, x_{i+1}) \Rightarrow R_B(y_1, x_{n-1}) \quad (6.28)$$

from Φ_1 of length $\leq \lambda$. By the induction hypothesis, (6.28) is equivalent to $B \rightarrow_G^* a_1 \dots a_{n-1}$. Therefore, $A \rightarrow_G^* a_1, \dots, a_n$. \square


 Figure 6.2.: An illustration of the situation in Lemma 11 for $n = 5$.

Creating candidates for failure of the AP

Let τ_2 be the signature which contains all symbols from τ_1 except for the ones coming from N and additionally the binary symbol E_2 and the ternary symbol Q . The sentence Φ_2 consists of the following Horn implications for every $a \in \Sigma$:

$$I(y_1) \wedge R_a(y_1, x_1) \wedge E_2(y_2, x_1) \Rightarrow Q(y_1, y_2, x_1) \quad (6.29)$$

$$I(y_1) \wedge Q(y_1, y_2, x_1) \wedge R_a(x_1, x_2) \wedge E_1(y_1, x_2) \wedge E_2(y_2, x_2) \Rightarrow Q(y_1, y_2, x_2) \quad (6.30)$$

$$I(y_1) \wedge Q(y_1, y_2, x_1) \wedge T(x_1) \Rightarrow \text{false} \quad (6.31)$$

The proof of the following claim is straightforward and left to the reader.

Lemma 11. *Let $\phi(\bar{x})$ be a conjunction of $(\tau_2 \setminus \{Q\})$ -atoms. Then $\Phi_2 \models \forall \bar{x}(\phi(\bar{x}) \Rightarrow \text{false})$ if and only if there exists $a_1 \dots a_n \in \Sigma^+$ such that ϕ has a subformula of the form*

$$\begin{aligned} & R_{a_1}(y_1, x_1) \wedge \bigwedge_{i \in [n-1]} R_{a_{i+1}}(x_i, x_{i+1}) \wedge E_1(y_1, x_{i+1}) \\ & \wedge I(y_1) \wedge T(x_n) \wedge \bigwedge_{i \in [n]} E_2(y_2, x_i), \end{aligned} \quad (6.32)$$

where the variables need not be distinct.

Now we are ready for the proof of Theorem 20.

Proof of Theorem 20. The universality problem for regular expressions is known to be PSPACE-complete [1] (Theorem 10.14, page 399). From every regular expression L over a finite alphabet Σ , we can compute in polynomial time a left-regular grammar $G = (N, \Sigma, P, S)$ such that $L(G) = L$. Thus, deciding whether $L(G) = \Sigma^+$ for a given left-regular grammar G is still PSPACE-hard. We set $\Phi := \Phi_1 \wedge \Phi_2$, and show that $\text{Mod}_{\text{fin}}(\Phi)$ has the AP if and only if $L(G) = \Sigma^+$.

“ \Rightarrow ”: Suppose that $\text{Mod}_{\text{fin}}(\Phi)$ has the AP. Let $a_1 \dots a_n \in \Sigma^+$ be arbitrary. Consider the formulas $\phi(x_1, \dots, x_n)$, $\phi_1(x_1, \dots, x_n, y_1)$, and $\phi_2(x_1, \dots, x_n, y_2)$ given by

$$\begin{aligned}\phi &:= T(x_n) \wedge \bigwedge_{i \in [n-1]} R_{a_{i+1}}(x_i, x_{i+1}), \\ \phi_1 &:= I(y_1) \wedge R_{a_1}(y_1, x_1) \wedge \bigwedge_{i \in [n-1]} E_1(y_1, x_{i+1}), \\ \text{and } \phi_2 &:= \bigwedge_{i \in [n]} E_2(y_2, x_i).\end{aligned}$$

By Lemma 11, we have

$$\Phi_2 \models \forall x_1, \dots, x_n, y_1, y_2 (\phi \wedge \phi_1 \wedge \phi_2 \Rightarrow \text{false}). \quad (6.33)$$

We claim that $\Phi \wedge \phi \wedge \phi_2$ is satisfiable and $\phi \wedge \phi_2 \leq_{\Phi} \phi$. Let $\chi(x_1, \dots, x_n)$ be an atomic τ -formula other than equality such that $\Phi \models \forall x_1, \dots, x_n, y_2 (\phi \wedge \phi_2 \Rightarrow \chi)$. Note that every Horn implication in Φ has an I -atom in its premise while $\phi \wedge \phi_2$ contains no I -atoms. As a consequence, $\phi \wedge \phi_2 \Rightarrow \chi$ must be a tautology, i.e., χ is a subformula of $\phi \wedge \phi_2$. In particular, χ cannot be of the form false . Since every atom in ϕ_2 contains the variable y_2 while χ does not, χ is a subformula of ϕ . It follows that $\Phi \models \forall x_1, \dots, x_n (\phi \Rightarrow \chi)$. Since χ was chosen arbitrarily, this confirms our claim.

Since $\text{Mod}_{\text{fin}}(\Phi)$ has the AP and we already have $\phi \wedge \phi_2 \leq_{\Phi} \phi$, by Lemma 9, it cannot be the case that $\phi \wedge \phi_1 \leq_{\Phi} \phi$. Otherwise, (6.33) would lead to a contradiction to the satisfiability of $\Phi \wedge \phi \wedge \phi_2$ via item 3 of Lemma 9. Thus, there must exist an atomic formula $\chi(x_1, \dots, x_n)$ other than equality such that

$$\Phi \models \forall x_1, \dots, x_n, y_1 (\phi \wedge \phi_1 \Rightarrow \chi) \quad \text{and} \quad \Phi \not\models \forall x_1, \dots, x_n (\phi \Rightarrow \chi). \quad (6.34)$$

By Theorem 17, we have $\Phi \vdash \forall x_1, \dots, x_n, y_1 (\phi \wedge \phi_1 \Rightarrow \chi)$. Clearly, by (6.34), $\phi \wedge \phi_1 \Rightarrow \chi$ is not a tautology because every atom in ϕ_1 contains the variable y_1 while χ does not. Thus, $\phi \wedge \phi_1 \Rightarrow \chi$ is a weakening of a Horn implication ψ that has an SLD-derivation $\psi_0, \dots, \psi_s = \psi$ from Φ modulo renaming variables. Since every Horn implication in Φ_2 contains a Q -atom or an E_2 -atom in its premise while Horn implications in Φ_1 , as well as $\phi \wedge \phi_1$, do not contain any $\{E_2, Q\}$ -atoms, Φ_2 cannot contribute anything during the SLD-derivation step. Therefore, the SLD-derivation $\psi_0, \dots, \psi_s = \psi$ is from Φ_1 . Note that, if a variable appears in an I -atom in the premise of a Horn implication from Φ_1 whose conclusion is not false , then this variable also appears in an atom from the conclusion. Moreover, no Horn implication from Φ_1 contains an I -atom in its conclusion. Since y_1 is the only variable which appears in an I -atom in $\phi \wedge \phi_1$ and χ does not contain the variable y_1 , ψ_0 must be of the form (6.24). Thus, we have that

$$\Phi_1 \vdash \forall x_1, \dots, x_n, y_1 (\phi \wedge \phi_1 \Rightarrow \text{false}). \quad (6.35)$$

Let ϕ' be the conjunction of atomic formulas obtained from ϕ by removing the atom $T(x_n)$. Note that x_n is the only variable which appears in a T -atom in $\phi \wedge \phi_1$ and no Horn implication

6. Towards user-definable concrete domains

from Φ_1 contains an T -atom in its conclusion. Since ψ_0 is of the form (6.24), we must have

$$\Phi_1 \models \forall x_1, \dots, x_n, y_1 (\phi' \wedge \phi_1 \Rightarrow R_S(y_1, x_n)),$$

otherwise a counterexample to (6.35) can be easily constructed. But then it follows from Lemma 10 that $a_1 \dots a_n \in L(G)$ and we are done.

“ \Leftarrow ”: We prove the contrapositive. We assume that $\text{Mod}_{\text{fin}}(\Phi)$ does not have the AP. Then there exists a counterexample to item 3 in Lemma 9, i.e., there exists a Horn implication ψ of the form $\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1) \wedge \phi_2(\bar{x}, y_2) \Rightarrow \chi$, where ϕ, ϕ_1 , and ϕ_2 satisfy the prerequisites of item 3 in Lemma 9 and $\chi(\bar{x}, y_1)$ is an atomic τ -formula other than equality, such that

$$\Phi \models \forall \bar{x}, y_1, y_2 (\phi \wedge \phi_1 \wedge \phi_2 \Rightarrow \chi) \quad (6.36)$$

$$\text{and } \Phi' \not\models \forall \bar{x}, y_1 (\phi \wedge \phi_1 \Rightarrow \chi). \quad (6.37)$$

We choose ψ minimal with respect to the number of its atomic subformulas.

Our proof strategy is as follows. First we show that ψ encodes a single word $w \in \Sigma^+$ in the sense of Lemma 11. Then we show that the word w may not be contained in $L(G)$, because otherwise a part of the counterexample would encode w in the sense of Lemma 10 which would lead to a contradiction.

Observation 3. *The formula $\Phi \wedge \phi(\bar{x}) \wedge \phi_i(\bar{x}, y_i)$ is satisfiable for both $i \in \{1, 2\}$.*

Proof of Observation 3. Suppose for contradiction that $\Phi \models \forall x_1, \dots, x_n, y_i (\phi \wedge \phi_i \Rightarrow \text{false})$ for some $i \in \{1, 2\}$. Since the conclusion of $\phi \wedge \phi_i \Rightarrow \text{false}$ does not contain the variable y_i and $\phi(\bar{x}) \wedge \phi_i(\bar{x}, y_i) \leq_\Phi \phi(\bar{x})$ for both $i \in [2]$, it follows that $\Phi \models \forall \bar{x} (\phi(\bar{x}) \Rightarrow \text{false})$. But this yields a contradiction to (6.37). Thus, the statement holds. \square

Observation 4. *The formula ψ*

- *is not a tautology,*
- *has an SLD-derivation from Φ_2 ,*
- *only contains τ_2 -atoms.*

Proof of Observation 4. First, we claim that $\Phi_1 \vdash \psi$ or $\Phi_2 \vdash \psi$. By Theorem 17, we have $\Phi \vdash \psi$. Note that $\chi(\bar{x}, y_1)$ cannot be a subformula of $\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1)$, by (6.37). Also note that $\chi(\bar{x}, y_1)$ cannot be a subformula of $\phi_2(\bar{x}, y_2)$, because every atom in $\phi_2(\bar{x}, y_2)$ contains the variable y_2 . Hence, $\chi(\bar{x}, y_1)$ is not a subformula of $\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1) \wedge \phi_2(\bar{x}, y_2)$, i.e., ψ is not a tautology. Thus, ψ is a weakening of a Horn implication ψ' which has an SLD-derivation $\psi'_0, \dots, \psi'_s = \psi'$ from Φ modulo renaming variables. Note that the Horn implications in Φ have the property that, depending on whether they come from Φ_1 or from Φ_2 , they either contain no Q -atoms or no R_A -atoms for $A \in N$, respectively. This applies in particular to ψ'_0 which is a conjunct of Φ . Since the conclusion of each Horn implication in Φ_1 is an R_A -atom for $A \in N$ and the conclusion of each Horn implication in Φ_2 is a Q -atom, the property of ψ'_0 from above propagates inductively to every ψ'_i for $i \in [k]$. But this means that ψ' has an SLD-derivation

from Φ_1 or from Φ_2 . Hence, ψ has an SLD-deduction from Φ_1 or from Φ_2 , which concludes the claim.

Next, we claim that $\Phi_2 \vdash \psi$. Suppose, on the contrary, that $\Phi_1 \vdash \psi$. Let ϕ' , ϕ'_1 , and ϕ'_2 be the formulas obtained from ϕ , ϕ_1 , and ϕ_2 , respectively, by removing all Q -atoms. Since $\Phi_1 \vdash \psi$, the SLD-derivation sequence ψ'_0, \dots, ψ'_s from the first paragraph contains no Q -atoms. Thus, all Q -atoms occurring in ψ come from the weakening step, which means that

$$\Phi_1 \vdash \forall \bar{x}, y_1, y_2 (\phi' \wedge \phi'_1 \wedge \phi'_2 \Rightarrow \chi). \quad (6.38)$$

We show that, for both $i \in [2]$,

$$\phi'(\bar{x}) \wedge \phi'_i(\bar{x}, y_i) \leq_{\Phi_1} \phi'(\bar{x}). \quad (6.39)$$

Let $\eta(\bar{x})$ be an atomic τ_1 -formula other than equality such that $\Phi_1 \models \forall \bar{x}, y_i (\phi' \wedge \phi'_i \Rightarrow \eta)$ for some $i \in [2]$. Since $\phi(\bar{x}) \wedge \phi_i(\bar{x}, y_i) \leq_{\Phi} \phi(\bar{x})$ for both $i \in [2]$, we have $\Phi \models \forall \bar{x} (\phi \Rightarrow \eta)$. Clearly, for both $i \in [2]$, η cannot be a subformula of ϕ_i because every atom in ϕ_i contains the variable y_i . If η is a subformula of ϕ , then trivially $\Phi_1 \models \forall \bar{x} (\phi' \Rightarrow \eta)$ and the claim holds. So suppose that this is not the case. By Theorem 17, we have $\Phi \vdash \forall \bar{x} (\phi \Rightarrow \eta)$. Since $\phi \Rightarrow \eta$ is not a tautology, it is a weakening of a Horn implication which has an SLD-derivation from Φ modulo renaming variables. Since $\Phi \wedge \phi$ is satisfiable by Observation 3, η neither is of the form `false` nor was it obtained by replacing `false` with a different τ_1 -atom during the weakening step. Consequently, η can only be an R_A -atom for some $A \in N$ because other symbols from τ_1 do not appear in the conclusion of any Horn implication in Φ . But then, since R_A -atoms only appear in the conclusion of Horn implications from Φ_1 , and no Horn implication from Φ_2 contains an atom in its conclusion that would appear in the premise of a Horn implication from Φ_1 , it must be the case that $\Phi_1 \vdash \forall \bar{x} (\phi \Rightarrow \eta)$. Consequently, $\Phi_1 \vdash \forall \bar{x} (\phi' \Rightarrow \eta)$ because $\{Q, E_2\}$ -atoms do not appear in any Horn implication from Φ_1 . Since η was chosen arbitrarily, we conclude that (6.39) indeed holds. Now we come back to the SLD-derivation sequence ψ'_0, \dots, ψ'_s . Since ψ'_0 is a Horn implication from Φ_1 , it follows from the minimality assumption for ψ that either χ is an R_A -atom for some $A \in N$, or χ equals `false`. In both cases, (6.39), (6.38), and (6.37) witness that $\text{Mod}_{\text{fin}}(\Phi_1)$ does not have AP through an application of Lemma 9. But this is in contradiction to Proposition 23. Thus, $\Phi_1 \vdash \psi$ does not hold, and $\Phi_2 \vdash \psi$ holds instead.

Since we have $\Phi_2 \vdash \psi$, the premise $\phi \wedge \phi_1 \wedge \phi_2$ of ψ can only contain symbols from τ_2 , otherwise we could remove all $(\tau_1 \setminus \tau_2)$ -atoms and get a contradiction to the minimality of ψ . Since ψ'_0 is a Horn implication from Φ_2 , it also follows from the minimality assumption for ψ that either χ is a Q -atom, or χ is `false`. Thus, ψ only contains symbols from τ_2 . \square

Let ψ' be a Horn implication such that ψ is a weakening of ψ' and there is an SLD-derivation $\psi'_0, \dots, \psi'_s = \psi'$ from Φ_2 modulo renaming variables. Recall that, since ψ is a weakening of ψ' , every atom from the premise of ψ' also appears in the premise of ψ .

Observation 5. ψ' does not contain any Q -atoms and its conclusion equals `false`.

6. Towards user-definable concrete domains

Proof of Observation 5. Note that, by the construction of Φ_2 , for every $i \in [k]$, if there exist variables z_1, z_2 in ψ'_{i-1} such that

$$\text{every } Q\text{-atom contains } z_1 \text{ in its first and } z_2 \text{ in its second argument, respectively,} \quad (*)$$

then ψ'_i also satisfies $(*)$ for the same variables z_1, z_2 up to renaming. Since every possible choice of ψ'_0 from Φ_2 initially satisfies $(*)$ for some variables, it follows via induction that $(*)$ must hold for $\psi' = \psi'_s$ for some variables. Also note that (6.29) is the only Horn implication in Φ_2 that is not complete, but the incompleteness is only due to one missing edge in the Gaifman graph between the two distinguished variables satisfying $(*)$ for (6.29).

We claim that $\{z_1, z_2\} = \{y_1, y_2\}$ holds for the pair z_1, z_2 satisfying $(*)$ for ψ' . Suppose, on the contrary, that both z_1 and z_2 are among \bar{x}, y_1 or \bar{x}, y_2 . For every $i \in [k]$, ψ'_i is a resolvent of ψ'_{i-1} and a Horn implication from Φ_2 which is almost complete except for one missing edge in the Gaifman graph between a pair of variables which must be substituted for the pair (z_1, z_2) satisfying $(*)$ for ψ'_{i-1} . Since the variables y_1 and y_2 do not appear together in any atom in ψ' and $\{z_1, z_2\} \neq \{y_1, y_2\}$, they also do not appear together in any atom during the SLD-derivation. Then it follows from the fact that ϕ, ϕ_1 , and ϕ_2 satisfy the prerequisites of item 3 in Lemma 9 that we already have $\Phi_2 \vdash \forall \bar{x}, y_1 (\phi \wedge \phi_1 \Rightarrow \chi)$, a contradiction to (6.37). Therefore, $\{z_1, z_2\} = \{y_1, y_2\}$.

We claim that ψ'_0 is of the form (6.31). Otherwise, ψ'_0 is of the form (6.29) or (6.30), in which case the conclusion of ψ' is a Q -atom. But then, since z_1, z_2 with $\{z_1, z_2\} = \{y_1, y_2\}$ satisfy $(*)$ for ψ' , the conclusion of ψ' would be an atom containing both variables y_1 and y_2 . This leads to a contradiction to (6.36) where we assume that the conclusion of ψ' may only contain variables from \bar{x}, y_1 . The claim implies that the conclusion of ψ' equals `false`. Hence, ψ' does not contain any Q -atoms at all. \square

By Lemma 11, there exists $a_1, \dots, a_n \in \Sigma^+$ such that the premise of ψ' has a subformula of the form (6.32) where the variables need not all be distinct. Since ϕ, ϕ_1 , and ϕ_2 satisfy the prerequisites of item 3 in Lemma 9, it must be the case that

- $\phi(\bar{x})$ has $T(x_n) \wedge \bigwedge_{i \in [n-1]} R_{a_{i+1}}(x_i, x_{i+1})$ as a subformula,
- $\phi_1(\bar{x}, y_1)$ has $I(y_1) \wedge R_{a_1}(y_1, x_1) \wedge \bigwedge_{i \in [n-1]} E_1(y_1, x_{i+1})$ as a subformula, and
- $\phi_2(\bar{x}, y_2)$ has $\bigwedge_{i \in [n]} E_2(y_2, x_i)$ as a subformula,

where the variables need not all be distinct and the roles of ϕ_1 and ϕ_2 might be swapped. Otherwise, we get a contradiction to Observation 3. Note that, if $L(G) = \Sigma^+$, then Lemma 10 together with (6.24) implies that

$$\Phi_1 \models \forall x_1, \dots, x_{n+1} (\phi \wedge \phi_1 \Rightarrow \text{false}). \quad (6.40)$$

If some variables among x_1, \dots, x_n are identified in $\phi \wedge \phi_1 \wedge \phi_2$, then we still have (6.40) even if we perform the same identification of variables. But then we get a contradiction to Observation 3. Thus, $L(G) \neq \Sigma^+$.

We have thus found a reduction from the PSPACE-hard universality problem for G to the decidability problem of the AP for $\text{Mod}_{\text{fin}}(\Phi)$; note that Φ is universal Horn and can be computed from G in polynomial time. \square

Recall from Lemma 9 that a universal Horn sentence Φ has the strong AP if and only if it has the AP.

Corollary 13. *For a given equality-free universal sentence Φ the question whether $\text{Mod}_{\text{fin}}(\Phi)$ has the strong AP is PSPACE-hard even if Φ is Horn and the signature is limited to ternary relation symbols.*

Our proof of PSPACE-hardness has an interesting consequence. Namely, it shows that, for ternary signatures, even in the equality-free Horn case there is no subexponential upper bound on the size of a smallest triple without an amalgam.

Corollary 14. *There is a sequence $(\Phi_k)_{k \geq 3}$ of equality-free universal Horn sentences with at most ternary relation symbols such that, for each $k \geq 3$, $\text{Mod}_{\text{fin}}(\Phi_k)$ does not have the AP, but the cardinality of A for a smallest triple $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ witnessing that $\text{Mod}_{\text{fin}}(\Phi_k)$ does not have the AP is in $\Omega(2^{|\Phi_k|})$.*

Proof. By Theorem 33 in [44], there exists a sequence $(G_k)_{k \geq 3}$ of left-regular grammars $G_k = (N_k, \{0, 1\}, P_k, S)$ such that the size of a smallest word rejected by G_k is in $\Omega(2^{|G_k|})$. For every $k \geq 3$, let Φ_{G_k} be the Horn sentence constructed in the proof of Theorem 20 from G_k . By the construction of Φ_{G_k} , there exist $a, b > 0$ such that $|\Phi_{G_k}| = a|G_k| + b$ for every $k \geq 3$.

We claim that the domain size of \mathfrak{A} for a smallest triple $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ without an amalgam is greater than or equal to the size of the smallest word $w \in \{0, 1\}^+ \setminus L(G_k)$. By Lemma 9 combined with the “ \Leftarrow ” direction of the proof of Theorem 20, each smallest counterexample to AP for $\text{Mod}_{\text{fin}}(\Phi_{G_k})$ is represented by a formula of the form (6.32). Let $\phi \wedge \phi_1 \wedge \phi_2$ be such a formula, and let $w := a_1 \dots a_n$ be the word encoded by $\phi \wedge \phi_1$. We must have that $w \notin L(G_k)$. Otherwise, by Lemma 10, $\Phi_{G_k} \wedge \phi \wedge \phi_2$ is unsatisfiable, which contradicts Observation 3.

Suppose, for contradiction, that some variables among $\{y_1, x_1, \dots, x_n\}$ coincide in $\phi \wedge \phi_1 \wedge \phi_2$. Then $\phi \wedge \phi_1$ already encodes a subword $v := a_{i_1} \dots a_{i_m}$ of w along a shortest possible R_{a_i} -path from y_1 to x_n . Again, we must have $v \notin L(G_k)$. Otherwise, by Lemma 10, $\Phi_{G_k} \wedge \phi \wedge \phi_1$ is unsatisfiable, which contradicts Observation 3. But now we can fix an arbitrary shortest possible R_{a_i} -path from y_1 to x_n and remove all subformulas from $\phi \wedge \phi_1 \wedge \phi_2$ which do not contain any variable occurring along this path in order to obtain a strictly smaller counterexample than $\phi \wedge \phi_1 \wedge \phi_2$. This leads to a contradiction to the minimality of $\phi \wedge \phi_1 \wedge \phi_2$. Thus, no variables among $\{y_1, x_1, \dots, x_n\}$ may coincide in $\phi \wedge \phi_1 \wedge \phi_2$.

Now suppose, for contradiction, that there exists $b_1 \dots b_m \in \{0, 1\}^+ \setminus L(G_k)$ with $m < n$. Since all variables in $\phi \wedge \phi_1 \wedge \phi_2$ are distinct, “ \Rightarrow ” of the proof of Theorem 20 applied to $b_1 \dots b_m$ yields us a strictly smaller counterexample than $\phi \wedge \phi_1 \wedge \phi_2$. Again, this leads to a contradiction to the minimality of $\phi \wedge \phi_1 \wedge \phi_2$. We conclude that w is a shortest word in $\{0, 1\}^+ \setminus L(G_k)$, which is what we had to show. \square

6.4 Universal sentences and the AP: the general case

In Theorem 20, we presented our strongest hardness result for deciding the AP in the simplest setting for which no upper bound is known. Namely, when Φ is Horn, equality-free, and the signature contains at most one symbol of arity ≥ 3 . In Theorem 21, we present our strongest hardness result for the AP in the most general setting. Namely, when Φ is an arbitrary universal sentence, there is no arity bound for the symbols in the signature, and we allow equality predicates as atomic formulas. But before we do that, let us take a quick look at the restricted case of binary signatures. Recall from Example 16 that there exist non-amalgamating universal sentences over binary signatures where a smallest counterexample to the AP is always of exponential size. Therefore, the following corollary to Theorem 15 provides a reasonable naïve upper bound for the complexity of deciding the AP in the case of binary signatures.

Corollary 15. *Let Φ be a universal sentence over a finite relational signature τ consisting of binary symbols. If $\text{Mod}_{\text{fin}}(\Phi)$ does not have the AP, then the size of a smallest counterexample to the AP is at most exponential in the size of Φ . Consequently, the question whether $\text{Mod}_{\text{fin}}(\Phi)$ has the AP is decidable in coNEXPTIME .*

Proof. Suppose that $\text{Mod}_{\text{fin}}(\Phi)$ does not have the AP. By the proof of “ \Leftarrow ” in Lemma 3, we have $\text{Mod}_{\text{fin}}(\Phi) = \text{Forb}_e(\mathcal{N})$ for an \mathcal{N} of size at most exponential in the size of Φ . Therefore, it follows directly from Theorem 15 that there exists a counterexample $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ of size at most exponential in the size of Φ . By Proposition 19, we may assume that $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ is a one-point amalgamation diagram. First, we must verify that $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \models \Phi$. This can be done in time exponential in the size of Φ , simply by evaluating the quantifier-free part of Φ on all possible inputs. Subsequently, we must verify that no amalgam $\mathfrak{C} \in \text{Mod}_{\text{fin}}(\Phi)$ of $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ can be obtained by either identifying b_1 and b_2 , or by adding (b_1, b_2) or (b_2, b_1) to some relations of $\mathfrak{B}_1 \cup \mathfrak{B}_2$. This can also be done in time exponential in the size of Φ because there are only exponentially many such structures \mathfrak{C} that need to be checked. In sum, the counterexample $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ can be verified in NEXPTIME , which is what we had to show. \square

Theorem 21. *For a given universal sentence Φ over an arbitrary finite signature, the question whether $\text{Mod}_{\text{fin}}(\Phi)$ has the AP is EXPSPACE-hard .*

The *rectangle tiling problem* asks whether, given natural numbers $k, n > 0$ and relations $C_{\text{horiz}}, C_{\text{vert}} \subseteq [k]^2$, $C_{\text{left}}, C_{\text{right}}, C_{\text{top}}, C_{\text{bot}} \subseteq [k]$, there exists a natural number $m > 0$ and a function $f : [n] \times [m] \rightarrow [k]$ satisfying

- $(f(i, j), f(i + 1, j)) \in C_{\text{horiz}}$ for every $1 \leq i \leq n - 1$ and $1 \leq j \leq m$,
- $(f(i, j), f(i, j + 1)) \in C_{\text{vert}}$ for every $1 \leq i \leq n$ and $1 \leq j \leq m - 1$, and
- $f(\{1\} \times [m]) \subseteq C_{\text{left}}$, $f(\{n\} \times [m]) \subseteq C_{\text{right}}$, $f([n] \times \{1\}) \subseteq C_{\text{bot}}$, $f([n] \times \{m\}) \subseteq C_{\text{top}}$.

Note that, in contrast to the better-known NP-complete square tiling problem, one dimension of the tiling grid is not part of the input and is existentially quantified instead. As a result, the rectangle tiling can encode any polynomial space bounded computation and the problem

becomes PSPACE-complete [45, 39].² We can further blow up the complexity by allowing a succinct encoding of the space bound. The input to the problem remains the same but now we ask for a rectangle tiling with 2^n columns. Analogously to the natural complete problems based on Turing machines, this yields a decision problem that is complete for the class EXPSPACE. We prove Theorem 21 by giving a polynomial-time reduction from the complement of this problem, i.e., we construct a universal sentence Φ such that $\text{Mod}_{\text{fin}}(\Phi)$ has the AP if and only if there exists no exponential rectangle tiling satisfying given parameters. As in the PSPACE-hardness proof, our encoding is based on equality-free Horn sentences, with a little twist. Namely, we use disjunctions and equality atoms to represent exponentially many equality-free Horn implications in a universal sentence of polynomial size. We also again divide Φ into two parts Φ_1 and Φ_2 such that $\Phi = \Phi_1 \wedge \Phi_2$. However, this time, the semantics of this subdivision is different and the correctness proof for the reduction is simpler.

We use a very compact encoding where each row, i.e., 2^n -many ordered tiles, is represented using a constant amount of variables. To achieve this, we store the information about each individual row via binary encoding into $(n+1)$ -ary atoms whose entries always contain at most three variables. We refer to the variables representing rows of the tiling as *path* nodes. In order to check the tiling from bottom to top, i.e., parse a chain of path nodes, we require each pair of subsequent path nodes to be verified by a set of 2^n -many *verifier* nodes. This process ensures the vertical consistency of the tiling as well as the presence of 2^n -many tiles in every row. The precise number of verifier nodes is achieved using Boolean combinations of n pairs of unary atoms, similarly as in Example 16.

To control the occurrence of amalgamation failures, we introduce three different types of auxiliary binary symbols: E , E_1 , and E_2 . Atoms with symbols E , E_1 , or E_2 serve no other purpose than to ensure that each conjunct in Φ except for one defines a class of structures preserved by taking unions. We will use the following shortcut:

$$\phi_{\text{pad}}(y_1, y_2, x_1, \dots, x_q) := \bigwedge_{i \in [q]} E_1(y_1, x_i) \wedge E_2(y_2, x_i) \wedge \bigwedge_{j \in [q]} E(x_i, x_j).$$

The soundness part of our encoding

The first part Φ_1 is a necessary evil. It does not describe how our encoding works, but rather ensures that it does not fall apart through ill-behaved identification of variables.

The signature τ contains two unary relation symbols P and V . We call a variable a *path* or *verifier* node if it appears in a P - or V -atom, respectively. Path and verifier nodes are distinguished using the following sentence, which we include as the first conjunct in Φ_1 :

$$\forall x (P(x) \wedge V(x) \Rightarrow \text{false}).$$

²In [45], this problem is called the corridor tiling problem.

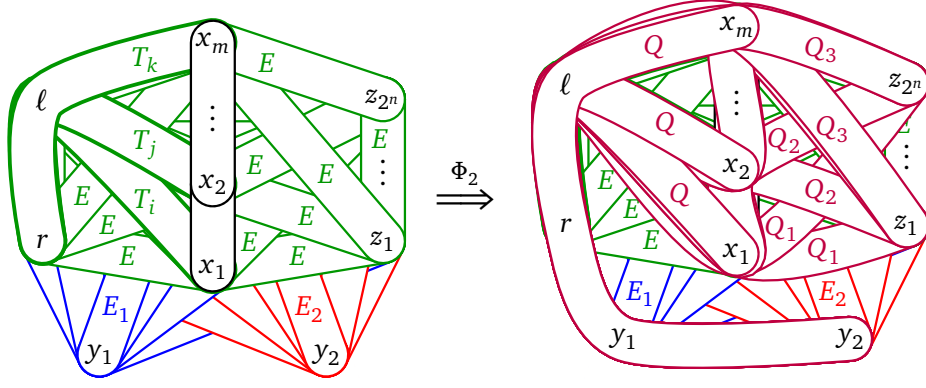


Figure 6.3.: An illustration of the encoding.

For every $i \in [k]$, the signature τ contains an $(n+1)$ -ary symbol T_i . The first n arguments in a T_i -atom serve as binary counters, and the last argument carries a given path node. Suppose that the variables ℓ and r represent the bits 0 and 1, respectively. Then each atomic formula $T_i(b_1, \dots, b_n, x)$ with $b_1, \dots, b_n \in \{\ell, r\}$ represents the situation in which the i -th tile is present in the x -th row and in the $(1 + \sum_{q \in [n]} \mathbf{1}_r(b_q) \cdot 2^{n-q})$ -th column, where $\mathbf{1}_r$ is the Boolean indicator function for the variable r .

We want to ensure horizontal consistency of the tiling, i.e., the j -th tile should not appear right next to the i -th tile in the x -th row unless $(i, j) \in C_{\text{horiz}}$. We can encode the successor relation w.r.t. binary addition using a Boolean combination of equalities because (b_{n+1}, \dots, b_{2n}) is the successor of (b_1, \dots, b_n) if and only if there exists $q \in [n]$ such that (i) both tuples agree up to the $(q-1)$ -th entry, (ii) the q -th entry in (b_1, \dots, b_n) and (b_{n+1}, \dots, b_{2n}) contains ℓ and r , respectively, and (iii) all remaining entries in (b_1, \dots, b_n) and (b_{n+1}, \dots, b_{2n}) contain r and ℓ , respectively. Clearly, this encoding only makes sense if ℓ and r are two different variables. For this reason, we introduce two unary symbols L and R to distinguish between ℓ and r . We include the following sentences as conjuncts in Φ_1 :

$$\forall x (L(x) \wedge R(x) \Rightarrow \text{false}),$$

and, for every pair $(i, j) \in [k]^2 \setminus C_{\text{horiz}}$, the sentence

$$\forall x, \ell, r, b_1, \dots, b_{2n} \left(\begin{aligned} &P(x) \wedge T_i(b_1, \dots, b_n, x) \wedge T_j(b_{n+1}, \dots, b_{2n}, x) \wedge L(\ell) \wedge R(r) \\ &\wedge \left(\bigvee_{q \in [n]} \bigwedge_{p \in [q-1]} (b_p = \ell \wedge b_{p+n} = \ell \vee b_p = r \wedge b_{p+n} = r) \right. \\ &\quad \left. \wedge b_q = \ell \wedge b_{q+n} = r \wedge \bigwedge_{p \in [n] \setminus [q]} b_p = r \wedge b_{p+n} = \ell \right) \end{aligned} \right) \Rightarrow \text{false}.$$

Next, we want to ensure that every position in the x -th row is occupied by at most one tile. For every pair $(i, j) \in [t]^2$ with $i \neq j$, we include the following sentence as a conjunct in Φ_1 :

$$\forall x, b_1, \dots, b_n (P(x) \wedge T_i(b_1, \dots, b_n, x) \wedge T_j(b_1, \dots, b_n, x) \Rightarrow \text{false}).$$

Finally, we want to ensure that each verifier node represents at most one number from $[2^n]$. For every $q \in [n]$, the signature τ contains two unary symbols L_q and R_q which will be used to encode numbers in binary. We include the following sentence as the last conjunct in Φ_1 :

$$\forall x (V(x) \wedge (\bigvee_{q \in [n]} L_q(x) \wedge R_q(x)) \Rightarrow \text{false}).$$

Note that we have not yet added the requirement for rows, represented by path nodes, to be completely tiled from left to right. We have also not yet added the requirement for verifier nodes to represent at least one number from $[2^n]$. It will become clear at the end of the proof of Theorem 21 why this does not have to be stated explicitly.

The completeness part of our encoding

The second part Φ_2 describes how the parsing of a tiling is encoded. We will need the following auxiliary formula that defines the successor relation for verifier nodes:

$$\begin{aligned} \phi_{succ}(z_1, z_2) := & \bigvee_{q \in [n]} \bigwedge_{p \in [q-1]} (L_p(z_1) \wedge L_p(z_2) \vee R_p(z_1) \wedge R_p(z_2)) \\ & \wedge L_q(z_1) \wedge R_q(z_2) \wedge \bigwedge_{p \in [n] \setminus [q]} R_p(z_1) \wedge L_p(z_2). \end{aligned}$$

We will also need an auxiliary formula to test whether a sequence of bits represents the same number as a particular verifier node:

$$\phi_{match}(z, \ell, r, b_1, \dots, b_n) := \bigwedge_{q \in [n]} (L_q(z) \wedge b_q = \ell \vee R_q(z) \wedge b_q = r).$$

The parsing of a tiling starts from a path node x representing a row whose leftmost position contains a tile that can be present in the bottom left corner of a tiling grid. This must be confirmed by a verifier node, in which case a 6-ary Q_1 -atom is derived, representing the fact that the leftmost column of the x -th row has been checked. For every $i \in C_{left} \cap C_{bot}$, we include the following sentence as the first conjunct in Φ_2 :

$$\begin{aligned} & \forall y_1, y_2, x, z, \ell, r, b_1, \dots, b_n \\ & \left(\begin{array}{l} P(x) \wedge V(z) \wedge \phi_{pad}(y_1, y_2, x, z, \ell, r) \\ \wedge \phi_{match}(z, \ell, r, b_1, \dots, b_n) \wedge \bigwedge_{q \in [n]} b_q = \ell \\ \wedge T_i(b_1, \dots, b_n, x) \end{array} \right) \Rightarrow Q_1(y_1, y_2, x, z, \ell, r). \end{aligned} \quad (6.41)$$

Note that, at this point, it is not yet clear why ℓ and r should represent 0 and 1, respectively, and not the other way around. This particular order is achieved implicitly through the use of

6. Towards user-definable concrete domains

ϕ_{match} and ϕ_{succ} . Using 2^n -many verifier nodes and propagation of Q_1 -atoms, the whole x -th row is checked for the presence of tiles. Their horizontal consistency already follows from the conditions imposed on path nodes themselves by Φ_1 and needs not to be checked during this step. For every $i \in C_{bot}$, we include the following sentence as a conjunct in Φ_2 :

$$\forall y_1, y_2, x, z_1, z_2, \ell, r, b_1, \dots, b_n \left(\begin{array}{l} P(x) \wedge V(z_1) \wedge V(z_2) \wedge \phi_{pad}(y_1, y_2, x, z_1, z_2, \ell, r) \\ \wedge \phi_{match}(z_1, \ell, r, b_1, \dots, b_n) \wedge \phi_{succ}(z_1, z_2) \\ \wedge T_i(b_1, \dots, b_n, x) \wedge Q_1(y_1, y_2, x, z_1, \ell, r) \end{array} \right) \Rightarrow Q_1(y_1, y_2, x, z_2, \ell, r).$$

After the x -th row has been checked by a 2^n -th verifier node, we mark the variable x with a Q -atom that indicates that the parsing can progress to a successor path node. For every $i \in C_{bot} \cap C_{right}$, we include the following sentence as a conjunct in Φ_2 :

$$\forall y_1, y_2, x, z, \ell, r, b_1, \dots, b_n \left(\begin{array}{l} P(x) \wedge V(z) \wedge \phi_{pad}(y_1, y_2, x, z, \ell, r) \\ \wedge \phi_{match}(z, \ell, r, b_1, \dots, b_n) \wedge \bigwedge_{q \in [n]} b_q = r \\ \wedge T_i(b_1, \dots, b_n, x) \wedge Q_1(y_1, y_2, x, z, \ell, r) \end{array} \right) \Rightarrow Q(y_1, y_2, x, \ell, r).$$

The successor relation for path nodes is represented by the binary symbol S , and the vertical verification for pairs of path nodes is represented by the 7-ary symbol Q_2 . We include the following sentences as conjuncts in Φ_2 : for every $(i, j) \in C_{vert} \cap C_{left}^2$, the sentence

$$\forall y_1, y_2, x_1, x_2, z, \ell, r, b_1, \dots, b_{2n} \left(\begin{array}{l} P(x_1) \wedge P(x_2) \wedge V(z) \wedge \phi_{pad}(y_1, y_2, x_1, x_2, z, \ell, r) \\ \wedge \phi_{match}(z, \ell, r, b_1, \dots, b_n) \wedge \bigwedge_{q \in [n]} b_q = \ell \\ \wedge \phi_{match}(z, \ell, r, b_{n+1}, \dots, b_{2n}) \wedge \bigwedge_{q \in [n]} b_{q+n} = \ell \\ \wedge Q(y_1, y_2, x_1, \ell, r) \wedge T_i(b_1, \dots, b_n, x_1) \\ \wedge S(x_1, x_2) \wedge T_j(b_{n+1}, \dots, b_{2n}, x_2) \end{array} \right) \Rightarrow Q_2(y_1, y_2, x_1, x_2, z, \ell, r),$$

for every $(i, j) \in C_{vert}$, the sentence

$$\forall y_1, y_2, x_1, x_2, z_1, z_2, \ell, r, b_1, \dots, b_{2n} \left(\begin{array}{l} P(x_1) \wedge P(x_2) \wedge V(z_1) \wedge V(z_2) \\ \wedge \phi_{pad}(y_1, y_2, x_1, x_2, z_1, z_2, \ell, r) \\ \wedge \phi_{match}(z_1, \ell, r, b_1, \dots, b_n) \wedge \phi_{succ}(z_1, z_2) \\ \wedge \phi_{match}(z_1, \ell, r, b_{n+1}, \dots, b_{2n}) \\ \wedge T_i(b_1, \dots, b_n, x_1) \wedge T_j(b_{n+1}, \dots, b_{2n}, x_2) \\ \wedge Q(y_1, y_2, x_1, \ell, r) \wedge Q_2(y_1, y_2, x_1, x_2, z_1, \ell, r) \end{array} \right) \Rightarrow Q_2(y_1, y_2, x_1, x_2, z_2, \ell, r),$$

and, for every $(i, j) \in C_{\text{vert}} \cap C_{\text{right}}^2$, the sentence

$$\forall y_1, y_2, x_1, x_2, z, \ell, r, b_1, \dots, b_{2n} \left(\begin{array}{l} P(x_1) \wedge P(x_2) \wedge V(z) \wedge \phi_{\text{pad}}(y_1, y_2, x_1, x_2, z, \ell, r) \\ \wedge \phi_{\text{match}}(z, \ell, r, b_1, \dots, b_n) \wedge \bigwedge_{q \in [n]} b_q = r \\ \wedge \phi_{\text{match}}(z, \ell, r, b_{n+1}, \dots, b_{2n}) \wedge \bigwedge_{q \in [n]} b_{q+n} = r \\ \wedge T_i(b_1, \dots, b_n, x_1) \wedge T_j(b_{n+1}, \dots, b_{2n}, x_2) \\ \wedge Q(y_1, y_2, x_1, \ell, r) \wedge Q_2(y_1, y_2, x_1, x_2, z, \ell, r) \end{array} \right) \Rightarrow Q(y_1, y_2, x_2, \ell, r).$$

The top row is verified using a 6-ary symbol Q_3 in a similar fashion as the bottom row. We include the following sentences as conjuncts in Φ_2 : for every $i \in C_{\text{left}} \cap C_{\text{top}}$, the sentence

$$\forall y_1, y_2, x, z, \ell, r, b_1, \dots, b_n \left(\begin{array}{l} P(x) \wedge V(z) \wedge \phi_{\text{pad}}(y_1, y_2, x, z, \ell, r) \\ \wedge \phi_{\text{match}}(z, \ell, r, b_1, \dots, b_n) \wedge \bigwedge_{q \in [n]} b_q = \ell \\ \wedge T_i(b_1, \dots, b_n, x) \wedge Q(y_1, y_2, x, \ell, r) \end{array} \right) \Rightarrow Q_3(y_1, y_2, x, z, \ell, r),$$

for every $i \in C_{\text{top}}$, the sentence

$$\forall y_1, y_2, x, z_1, z_2, \ell, r, b_1, \dots, b_n \left(\begin{array}{l} P(x) \wedge V(z_1) \wedge V(z_2) \wedge \phi_{\text{pad}}(y_1, y_2, x, z_1, z_2, \ell, r) \\ \wedge \phi_{\text{match}}(z_1, \ell, r, b_1, \dots, b_n) \wedge \phi_{\text{succ}}(z_1, z_2) \\ \wedge T_i(b_1, \dots, b_n, x) \wedge Q(y_1, y_2, x, \ell, r) \\ \wedge Q_3(y_1, y_2, x, z_1, \ell, r) \end{array} \right) \Rightarrow Q_3(y_1, y_2, x, z_2, \ell, r),$$

and, for every $i \in C_{\text{top}} \cap C_{\text{right}}$, the sentence

$$\forall y_1, y_2, x, z, \ell, r, b_1, \dots, b_n \left(\begin{array}{l} P(x) \wedge V(z) \wedge \phi_{\text{pad}}(y_1, y_2, x, z, \ell, r) \\ \wedge \phi_{\text{match}}(z, \ell, r, b_1, \dots, b_n) \wedge \bigwedge_{q \in [n]} b_q = r \\ \wedge T_i(b_1, \dots, b_n, x) \wedge Q(y_1, y_2, x, \ell, r) \\ Q_3(y_1, y_2, x, z, \ell, r) \end{array} \right) \Rightarrow \text{false}. \quad (6.42)$$

Now we are ready to prove Theorem 21.

Proof of Theorem 21. We first show that Φ is equivalent to a particular equality-free Horn sentence Φ' . Note that each conjunct in Φ has the form of an implication where the premise possibly also contains instances of disjunction, which we normally do not allow in implications, but no instances of negation. Therefore, each conjunct in Φ can be rewritten as a conjunction

of Horn implications by converting the premise into positive DNF and then considering each disjunct as a separate premise. As a result, the size of Φ increases exponentially, but this does not matter for the purpose of the proof. Subsequently, all equality atoms can be eliminated by replacing each variable b_i with either ℓ or r . We denote the resulting equality-free Horn sentence by Φ' and its two parts stemming from Φ_1 and Φ_2 , respectively, by Φ'_1 and Φ'_2 .

“ \Rightarrow ”: Suppose that a tiling $f : [2^n] \times [m] \rightarrow [k]$ satisfying the given input parameters exists. Guided by f , we define a triple $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \text{Mod}_{\text{fin}}(\Phi')$ such that no amalgam \mathfrak{C} for $(\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2)$ can satisfy Φ' . The domains are $A := \{x_1, \dots, x_m, z_1, \dots, z_{2^n}, \ell, r\}$ and $B_i := A \cup \{y_i\}$ ($i \in [2]$), and the relations are given by the following (conjunctions of) atomic formulas:

- $L(\ell) \wedge R(r)$, and $T_i(b_1, \dots, b_n, x_j)$ is satisfied for $b_1, \dots, b_n \in \{\ell, r\}$ if and only if $f(1 + \sum_{q \in [n]} \mathbf{1}_r(b_q) \cdot 2^{n-q}, j) = i$, thus the tiling atoms are placed correctly;
- $\phi_{\text{pad}}(y_1, y_2, x_1, \dots, x_m, z_1, \dots, z_{2^n}, \ell, r)$ enables the premises of the Horn implications;
- $(\bigwedge_{i \in [m]} P(x_i)) \wedge (\bigwedge_{i \in [2^n]} V(z_i))$ distinguishes path and verifier nodes;
- $\bigwedge_{i \in [m-1]} F(x_i, x_{i+1})$ defines a successor chain through path nodes; and
- $L_i(z_j)$ or $R_i(z_j)$ is satisfied if and only if $j = 1 + \sum_{q \in [n]} \lambda_q \cdot 2^{n-q}$ for $\lambda_1, \dots, \lambda_n \in \{0, 1\}$ and $\lambda_i = 0$ or $\lambda_i = 1$, respectively, thus verifier nodes correctly represent values in $[2^n]$.

Since \mathfrak{A} does not satisfy any E_1 - or E_2 -atoms, \mathfrak{B}_1 does not satisfy any E_2 atoms, \mathfrak{B}_2 does not satisfy any E_1 atoms, and f is horizontally consistent, we clearly have $\mathfrak{A}, \mathfrak{B}_1, \mathfrak{B}_2 \in \text{Mod}_{\text{fin}}(\Phi'_1)$. But, since f is also vertically consistent, we have $\mathfrak{B}_1 \cup \mathfrak{B}_2 \not\models \Phi'_2$. Since Φ'_2 is an equality-free universal Horn sentence, this implies that also $\mathfrak{C} \not\models \Phi'_2$ for every τ -structure \mathfrak{C} which admits a homomorphism from $\mathfrak{B}_1 \cup \mathfrak{B}_2$. Thus, $\text{Mod}_{\text{fin}}(\Phi')$ does not have the AP.

“ \Leftarrow ”: Suppose that $\text{Mod}_{\text{fin}}(\Phi')$ does not have the AP. Then there exists a counterexample to item 3 in Lemma 9, i.e., there exists a Horn implication ψ of the form $\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1) \wedge \phi_2(\bar{x}, y_2) \Rightarrow \chi$, where ϕ, ϕ_1 , and ϕ_2 satisfy the prerequisites of item 3 in Lemma 9 and $\chi(\bar{x}, y_1)$ is an atomic τ -formula other than equality, such that

$$\Phi' \models \forall \bar{x}, y_1, y_2 (\phi \wedge \phi_1 \wedge \phi_2 \Rightarrow \chi) \quad (6.43)$$

$$\text{and } \Phi' \not\models \forall \bar{x}, y_1 (\phi \wedge \phi_1 \Rightarrow \chi). \quad (6.44)$$

By Theorem 17, ψ has an SLD-deduction from Φ' . Note that, by (6.44), $\chi(\bar{x}, y_1)$ cannot be a subformula of $\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1)$. Also, $\chi(\bar{x}, y_1)$ cannot be a subformula of $\phi_2(\bar{x}, y_2)$ because every atom in $\phi_2(\bar{x}, y_2)$ contains the variable y_2 which does not appear in $\chi(\bar{x}, y_1)$. Hence, $\chi(\bar{x}, y_1)$ is not a subformula of $\phi(\bar{x}) \wedge \phi_1(\bar{x}, y_1) \wedge \phi_2(\bar{x}, y_2)$, i.e., ψ is not a tautology. Consequently, ψ is a weakening of a Horn implication ψ' which has an SLD-derivation $\psi'_0, \dots, \psi'_s = \psi'$ from Φ' up to renaming variables. Recall that, since ψ is a weakening of ψ' , every atom from the premise of ψ' also appears in the premise of ψ .

Our first claim is that $\Phi'_2 \vdash \psi'$. We start by showing that ψ'_0 is a conjunct of Φ'_2 . Suppose, on the contrary, that ψ'_0 is a conjunct of Φ'_1 . Then the conclusion of ψ' is false. Note that there is no Horn implication in Φ' whose conclusion would contain an atom with a symbol occurring in the premise of a Horn implication from Φ'_1 . Consequently, we have $s = 0$. Also

note that every Horn implication from Φ'_1 is complete. Since there is no edge between y_1 and y_2 in the Gaifman graph of $\text{DB}(\exists \bar{x}, y_1, y_2(\phi \wedge \phi_1 \wedge \phi_2))$, and, for $i \in [2]$, each atom in ϕ_i contains the variable y_i , either ϕ_1 or ϕ_2 must be empty. Since the conclusion of ψ' contains no variables and $\phi(\bar{x}) \wedge \phi_i(\bar{x}, y_i) \leq_{\Phi'} \phi(\bar{x})$ for both $i \in [2]$, we get a contradiction to (6.44). Thus, ψ'_0 must be a conjunct of Φ'_2 . Since the conclusion of each Horn implication in Φ'_1 is false, no Horn implication from Φ'_1 can be used as a resolvent. We conclude that ψ' has an SLD-derivation from Φ'_2 modulo renaming variables, which confirms our first claim.

Our second claim is that ψ' contains no atoms with a symbol from $\{Q_1, Q_2, Q_3, Q\}$ and its conclusion equals false. Note that, by the construction of Φ'_2 , for every $i \in [s]$, if there exist variables z_1, z_2 in ψ'_{i-1} such that

$$\begin{aligned} &\text{every atom with a symbol from } \{Q_1, Q_2, Q_3, Q\} \text{ contains} \\ &z_1 \text{ in its first and } z_2 \text{ in its second argument, respectively.} \end{aligned} \quad (*)$$

then this is also the case for ψ'_i , for the same variables z_1, z_2 up to renaming. Since every possible choice of ψ'_0 from Φ'_2 initially satisfies (*), it follows via induction that (*) holds for $\psi' = \psi'_s$ for some variables z_1, z_2 . Also note that (6.41) is the only Horn implication in Φ'_2 that is not complete. However, the incompleteness is only due to one missing edge in the Gaifman graph between the two distinguished variables satisfying (*) for (6.41). We show that $\{z_1, z_2\} = \{y_1, y_2\}$ holds for the pair z_1, z_2 satisfying (*) for ψ' . Suppose, on the contrary, that both z_1 and z_2 are among \bar{x}, y_1 or \bar{x}, y_2 . For every $i \in [s]$, ψ'_i is a resolvent of ψ'_{i-1} and a Horn implication from Φ'_2 which is almost complete except possibly for one missing edge in the Gaifman graph between a pair of variables which must be substituted for the pair (z_1, z_2) satisfying (*) for ψ'_{i-1} . Since the variables y_1 and y_2 do not appear together in any atom in ψ' and $\{z_1, z_2\} \neq \{y_1, y_2\}$, they also do not appear together in any atom during the SLD-derivation. Then it follows from the fact that ϕ, ϕ_1 , and ϕ_2 satisfy the prerequisites of item 3 in Lemma 9 that we already have $\Phi'_2 \vdash \forall \bar{x}, y_1(\phi \wedge \phi_1 \Rightarrow \chi)$, a contradiction to (6.44). Since $\{z_1, z_2\} = \{y_1, y_2\}$ holds for the pair z_1, z_2 satisfying (*) for ψ' , ψ' cannot contain any atoms with a symbol from $\{Q_1, Q_2, Q_3, Q\}$ in the premise. Suppose that ψ'_0 is not of the form (6.42). Then the conclusion of ψ' is an atom with a symbol from $\{Q_1, Q_2, Q_3, Q\}$. But then, since z_1, z_2 with $\{z_1, z_2\} = \{y_1, y_2\}$ satisfy (*) for ψ' , the conclusion of ψ' would be an atom containing both variables y_1 and y_2 . This leads to a contradiction to (6.43) where we assume that the conclusion of ψ may only contain variables from \bar{x}, y_1 . Thus ψ'_0 is of the form (6.42), which means that χ equals false due to the minimality assumption. This concludes our second claim.

It remains to show that the existence of such ψ' implies the existence of a tiling $f : [2^n] \times [m] \rightarrow [k]$ satisfying the given input parameters. We show that this follows from our second claim and the construction of Φ' . By the previous paragraph, ψ'_0 is of the form (6.42). Since ψ' does not contain any atoms with a symbol from $\{Q_1, Q_2, Q_3, Q\}$, the last such atom must have been eliminated from ψ'_{s-1} by taking a resolvent with (6.41). By the construction of Φ'_2 , clearly all Horn implications introduced between (6.41) and (6.42) must have been used to obtain ψ'

6. Towards user-definable concrete domains

through an SLD-derivation $\psi'_0, \dots, \psi'_s = \psi'$ from Φ'_2 . Recall that we have replaced each variable b_i in Φ_2 with either ℓ or r while rewriting Φ_2 as an equality-free Horn sentence. For every Horn implication in Φ'_2 , the conclusion is either `false` or an atom with a symbol from $\{Q_1, Q_2, Q_3, Q\}$ containing all variables from the premise, with the following two exceptions: (i) verifier nodes are not carried over in any atoms because their only contribution is the encoding of a unique number, (ii) after a pair of successive rows has been checked by deriving a Q_2 -atom containing a 2^n -th verifier node, the variable representing the lower row is not carried over in any atom because it is no longer needed. Thus, we may assume that the SLD-derivation from above is linear (and not tree-like). Since ϕ, ϕ_1 , and ϕ_2 satisfy the prerequisites of item 3 in Lemma 9, no ill-behaved variable identifications might have occurred during the SLD-derivation above as otherwise, we would have $\Phi'_1 \models \forall \bar{x}, y_1 (\phi \wedge \phi_1 \Rightarrow \chi)$, a contradiction to (6.44). Consequently, the SLD-derivation must have the full intended length $(2^n + 1) \cdot m$ for some $m \geq 1$, because every intermediate stage starts and ends with verifier nodes encoding the numbers 2^n and 1, respectively, and one can only progress in steps which decrement the encoded number by one. This finishes the proof. \square

Conclusion

The notions of ω -admissibility and p -admissibility were respectively introduced in [70] and [3] to obtain decidable and tractable extensions of DLs by concrete domains. In each of these papers, two examples of concrete domains satisfying the respective restrictions were given. To the best of our knowledge, no other ω -admissible or p -admissible concrete domains had been exhibited in the literature before our investigations in [10] and [12]. This appears to be mainly due to the fact that it is not easy to show the conditions required by ω -admissibility or p -admissibility “by hand”.

7.1

Contributions and future outlook

The main contribution of this thesis is that it provides us with useful algebraic tools for proving ω -admissibility and p -admissibility.

We have shown that ω -admissibility is closely related to well-known notions from model theory such as homogeneity and finite boundedness. Given the fact that a large number of homogeneous structures are known from the literature [71] and that homogeneous and finitely bounded structures play an important role in the CSP community, we believe that our work will turn out to be useful for locating new ω -admissible concrete domains.

This is not the first model-theoretic description of a sufficient condition for decidability of reasoning in DLs with concrete domains in the presence of TBoxes. The *existence of homomorphism is definable (EHD)* property was used in [36] to obtain decidability results for DLs with concrete domains. However, the way the concrete domain is integrated into the DL in [36] is different from the classical one employed by us and used in all other works on DLs with concrete domains. In [36], constraints are always placed along a linear path stemming from a single individual, which is rather similar to the use of constraints in temporal logics [35, 42]. In contrast, in the classical setting of DLs with concrete domains, one can compare feature values of siblings of an individual. Compared to homogeneity and finite boundedness, the EHD property is not as well-investigated. To the best of our knowledge, the only article besides [36] where concrete domains satisfying the EHD property are studied in the context of \mathcal{ALC} with GCIs is [65].¹ There, the authors consider specific concrete domains based on integers equipped with a linear order and provide an exponential upper bound for reasoning using an

¹Though EHD is not used in the proofs in [65].

7. Conclusion

automata-theoretic algorithm. Interestingly, their upper bound holds not only for constraints along paths, but also for the traditional integration of concrete domain into DLs. The results in [36, 65] demonstrate that ω -admissibility is not necessary for decidable reasoning. However, all known non- ω -admissible concrete domains with the EHD property are based on “discrete” versions of ω -admissible concrete domains, which are patchworks but lack homomorphism ω -compactness, e.g., $(\mathbb{Z}; <, =, >)$ or Allen’s interval algebra over integers. Motivated by this observation, we identify homomorphism ω -compactness in its current form as the most obvious “flaw” of the ω -admissibility condition, in the sense that it may be too strong. In fact, the correctness of the tableau algorithm from [70] only requires very specific infinite structures to have a homomorphism to the concrete domain, e.g., their treewidth is always bounded by a computable function in the size of the input concept and TBox. But even if we restrict the inputs to homomorphism ω -compactness appropriately, the tableau algorithm from [70] is not correct for “discrete” versions of ω -admissible concrete domains, as illustrated by Example 1. We conclude that, although a modified version of ω -admissibility could in theory also be necessary for decidable reasoning in \mathcal{ALC} with concrete domains in the presence of GCIs, showing this might require a non-trivial combination of the methods in [36, 65, 70].

For p-admissibility, we have developed a very useful algebraic tool for showing convexity: the square embedding property. We have shown that this tool can indeed be used to exhibit new p-admissible concrete domains, such as countably infinite vector spaces over finite fields, the countable homogeneous partial order, and numerical concrete domains over \mathbb{R} and \mathbb{Q} whose relations are defined by linear equations. The usefulness of these numerical concrete domains for defining concepts should be evident. For the other two we have indicated their potential usefulness by small examples. We have shown that, when embedding p-admissible concrete domains into \mathcal{EL} , the restriction to paths of length one in concrete domain restrictions (indicated by the square brackets) is needed since there is a p-admissible concrete domain \mathcal{D} such that subsumption in $\mathcal{EL}(\mathcal{D})$ is undecidable. We have also shown that, for finitely bounded structures, convexity is equivalent to p-admissibility, and that this corresponds to the finite substructures being definable by a universal Horn sentence. Interestingly, this provides us with infinitely many examples of countable p-admissible concrete domains, which all yield a different extension of \mathcal{EL} : the Henson digraphs. From a theoretical point of view, this is quite a feat, given that before only two p-admissible concrete domains were known. It is less clear whether these concrete domains are useful for defining concepts. Finitely bounded structures also provide us with examples of structures \mathcal{D} that can be used both in the context of \mathcal{EL} and \mathcal{ALC} , in the sense that subsumption is tractable in $\mathcal{EL}[\mathcal{D}]$ and decidable in $\mathcal{ALC}(\mathcal{D})$.

Finally, we have addressed the question of user-definability of ω - and p-admissible concrete domains. To this question we have associated a natural decision problem, namely whether a given finite description of a class of finite structures defines the finite substructures of some ω - or p-admissible structure. We have considered the two settings where the input is either a set of forbidden substructures or a universal sentence. Both input specifications are well motivated by the rest of the thesis, and one can be converted to the other with an at most exponential blow-up in size. In the case of p-admissibility, we have shown that this meta-

problem is undecidable already for universal Horn sentences containing only binary symbols. In the case of ω -admissibility, we have shown decidability for binary signatures by establishing a close connection to the problem of testing the amalgamation property, which was previously studied in the literature. Additionally we have provided an upper bound at the second level of the polynomial hierarchy for the case where the input is a set of forbidden substructures. For signatures involving symbols of higher arities, the decidability is open. In fact, it has now been open for almost 40 years [66]. Our contributions here are two new lower bounds; PSPACE-hardness if the input is an equality-free universal Horn sentence with at most ternary symbols, and EXPSpace-hardness if the input is an arbitrary universal sentence and the arities of the occurring symbols are unrestricted. Obtaining an upper bound or even any lower bound stronger than EXPSpace-hardness is a challenging task that might require development of new proof techniques. We believe that this problem deserves more attention from the research community than it currently has.

Chapter
A

Concrete Domains without Equality

In this section, we briefly discuss the setting where the default equality predicate is not allowed as an atomic formula in concrete domain restrictions. We will see that the resulting theory is neither more elegant nor does it produce additional useful examples.

Recall that the notion of p-admissibility was introduced in [3] to capture precisely those concrete domains \mathcal{D} for which subsumption in $\mathcal{EL}[\mathcal{D}]$ is decidable in PTIME. It is argued in that paper that non-convexity of \mathcal{D} allows one to express disjunctions in $\mathcal{EL}[\mathcal{D}]$, which makes subsumption EXPTIME-hard. However, if equality cannot be used in concrete domain restrictions, then this argument works only if the counterexample to convexity is given by a guarded implication. For this reason, we must use guarded convexity rather than convexity in our definition of p-admissibility. For the same reason, one must also restrict the tractability requirement in this definition to validity of guarded Horn implications. In sum, disallowing the default equality predicate in concrete domain restrictions enables new structures that satisfy some implications which are not equivalent to Horn implications, but only because such implications cannot be encoded into a TBox and thus have no impact on reasoning.

Similarly to convexity, guarded convexity can also be characterized using an algebraic condition, essentially by restricting the SEP to guarded structures. We say that the relational τ -structure \mathfrak{A} is *guarded* if for every $a \in A$ there is a relation $R \in \tau$ such that a appears in a tuple in $R^{\mathfrak{A}}$.

Theorem 22. *For a relational τ structure \mathfrak{B} , the following are equivalent:*

1. \mathfrak{B} is guarded convex.
2. *For every finite $\sigma \subseteq \tau$ and every $\mathfrak{A} \in \text{Age}(\mathfrak{B}^2)$ whose σ -reduct is guarded, there exists a strong homomorphism from the σ -reduct of \mathfrak{A} to the σ -reduct of \mathfrak{B} .*

The proof of this theorem is similar to the proof of Theorem 9, but we include it anyway for the sake of completeness. When we speak of guarded convexity, we implicitly assume that the default equality predicate is not allowed as an atomic formula. In the presence of the equality predicate, strong homomorphisms are embeddings and guarded convexity is the same as convexity. Thus, Theorem 9 is technically a corollary to Theorem 22.

Proof. “2 \Rightarrow 1”: Suppose to the contrary that the closed implication $\forall x_1, \dots, x_n (\phi \Rightarrow \psi)$ is valid in \mathfrak{B} , where ϕ is a conjunction of atoms such that each variable x_i is present in some atom of ϕ , and ψ is a disjunction of atoms ψ_1, \dots, ψ_s , but we also have $\mathfrak{B} \not\models \forall x_1, \dots, x_n (\phi \Rightarrow \psi_i)$

A. Concrete Domains without Equality

for every $i \in [k]$. Without loss of generality, we assume that $\phi, \psi_1, \dots, \psi_k$ all have the same free variables x_1, \dots, x_n , some of which might not influence their truth value. For every $i \in [k]$, there exists a tuple $\bar{t}_i \in B^n$ such that

$$\mathfrak{B} \models \phi(\bar{t}_i) \wedge \neg \psi_i(\bar{t}_i). \quad (\text{A.1})$$

We show by induction on i that, for every $i \in [k]$, there exists a tuple $\bar{s}_i \in B^n$ that satisfies the *induction hypothesis*

$$\mathfrak{B} \models \phi(\bar{s}_i) \wedge \neg \left(\bigvee_{\ell \in [i]} \psi_\ell(\bar{s}_i) \right). \quad (\text{A.2})$$

In the *base case* ($i = 1$), it follows from (5.1) that $\bar{s}_1 := \bar{t}_1$ satisfies (A.2).

In the *induction step* ($i \rightarrow i + 1$), let $\bar{s}_i \in B^n$ be any tuple that satisfies (A.2). Let $\sigma \subseteq \tau$ be the finite set of relation symbols occurring in the implication $\forall x_1, \dots, x_n (\phi \Rightarrow \psi)$, and let \mathfrak{A}_i be the substructure of \mathfrak{B}^2 on the set $\{(\bar{s}_i[1], \bar{t}_{i+1}[1]), \dots, (\bar{s}_i[n], \bar{t}_{i+1}[n])\}$. Since $\mathfrak{B} \models \phi(\bar{s}_i)$ by (A.2), $\mathfrak{B} \models \phi(\bar{t}_{i+1})$ by (A.1), and ϕ contains an atom for each variable x_i , we conclude that the σ -reduct of \mathfrak{A}_i is guarded. By 2., there exists a strong homomorphism f_i from the σ -reduct of \mathfrak{A}_i to the σ -reduct of \mathfrak{B} . Since ϕ is a conjunction of σ -atoms and f_i is a homomorphism, we have that $\mathfrak{B} \models \phi(f_i(\bar{s}_i, \bar{t}_{i+1}))$. Suppose that $\mathfrak{B} \models \psi_{i+1}(f_i(\bar{s}_i, \bar{t}_{i+1}))$. Since f_i is a strong homomorphism, we get $\mathfrak{B} \models \psi_{i+1}(\bar{t}_{i+1})$, a contradiction to (A.1). Now suppose that $\mathfrak{B} \models \psi_j(f_i(\bar{s}_i, \bar{t}_{i+1}))$ for some $j \leq i$. Since f_i is a strong homomorphism, we get $\mathfrak{B} \models \psi_j(\bar{s}_i)$, a contradiction to (A.2). We conclude that $\bar{s}_{i+1} := f_i(\bar{s}_i, \bar{t}_{i+1})$ satisfies (A.2).

Since $\mathfrak{B} \models \forall x_1, \dots, x_n (\phi \Rightarrow \psi)$, the existence of a tuple $\bar{s}_i \in B^n$ that satisfies (A.2) for $i = k$ leads to a contradiction. This completes the proof of “ $2 \Rightarrow 1$ ” of our theorem.

Before we proceed with the proof of “ $1 \Rightarrow 2$ ”, let us take a closer look at the contraposition of the guarded convexity condition. Suppose that we have a conjunction ϕ of τ -atoms and tuples \bar{r} and \bar{s} over B together with disjunctions $\psi_{\bar{r}}$ and $\psi_{\bar{s}}$ of τ -atoms such that $\mathfrak{B} \models (\phi \wedge \neg \psi_{\bar{r}})(\bar{r})$ and $\mathfrak{B} \models (\phi \wedge \neg \psi_{\bar{s}})(\bar{s})$, and the implications $\forall x_1, \dots, x_n (\phi \Rightarrow \psi_{\bar{r}})$ and $\forall x_1, \dots, x_n (\phi \Rightarrow \psi_{\bar{s}})$ are guarded. Then clearly there must exist a tuple \bar{t} over B such that $\mathfrak{B} \models (\phi \wedge \neg \psi_{\bar{r}} \wedge \neg \psi_{\bar{s}})(\bar{t})$ as otherwise $\mathfrak{B} \models \forall x_1, \dots, x_n (\phi \Rightarrow \psi_{\bar{r}} \vee \psi_{\bar{s}})$, but neither $\mathfrak{B} \models \forall x_1, \dots, x_n (\phi \Rightarrow \psi_{\bar{r}})$ nor $\mathfrak{B} \models \forall x_1, \dots, x_n (\phi \Rightarrow \psi_{\bar{s}})$ is true (which would lead to a contradiction to guarded convexity).

Now we continue with the proof of “ $1 \Rightarrow 2$ ”. Let σ be an arbitrary finite subset of τ and let $\mathfrak{A} \in \text{Age}(\mathfrak{B}^2)$ be an arbitrary finite substructure of \mathfrak{B}^2 whose σ -reduct is guarded. Let $\{(r_1, s_1), \dots, (r_n, s_n)\}$ be the domain of \mathfrak{A} . Consider the tuples $\bar{r} := (r_1, \dots, r_n)$ and $\bar{s} := (s_1, \dots, s_n)$. Let $\phi(x_1, \dots, x_n)$ be the conjunction of all σ -atoms such that

$$\mathfrak{A} \models \phi((r_1, s_1), \dots, (r_n, s_n)),$$

i.e., we consider all atoms built using a relation symbol from σ and containing variables from $\{x_1, \dots, x_n\}$, assign (r_i, s_i) to the variable x_i , and take those atoms for which the corresponding tuple of elements of \mathfrak{A} belongs to the respective relation in \mathfrak{A} .

Clearly, the tuples \bar{r} and \bar{s} both satisfy ϕ in \mathfrak{B} since the projection to a single coordinate is a

homomorphism from \mathfrak{A} to \mathfrak{B} . Now let $\psi_{\bar{r}}$ be the disjunction of all σ -atoms which do not hold on the tuple \bar{r} in \mathfrak{B} . Analogously, let $\psi_{\bar{s}}$ be the disjunction of all σ -atoms which do not hold on the tuple \bar{s} in \mathfrak{B} . Without loss of generality, both disjunctions are non-empty since otherwise the projection onto one of the coordinates is a strong homomorphism and we are done. In addition, the implications $\forall x_1, \dots, x_n (\phi \Rightarrow \psi_{\bar{r}})$ and $\forall x_1, \dots, x_n (\phi \Rightarrow \psi_{\bar{s}})$ are guarded since the σ -reduct of \mathfrak{A} is guarded.

We have that $\mathfrak{B} \models \phi \wedge \neg\psi_{\bar{r}}(\bar{r})$ and $\mathfrak{B} \models \phi \wedge \neg\psi_{\bar{s}}(\bar{s})$. Since \mathfrak{B} is guarded convex, there must exist a tuple \bar{t} such that $\mathfrak{B} \models \phi \wedge \neg\psi_{\bar{r}}(\bar{t}) \wedge \neg\psi_{\bar{s}}(\bar{t})$. Now consider the map f that sends, for every $i \in [n]$, the tuple (r_i, s_i) to $\bar{t}[i]$. Clearly f is a homomorphism from the σ -reduct of \mathfrak{A} to the σ -reduct of \mathfrak{B} because $\mathfrak{B} \models \phi(\bar{t})$. Moreover, f is a strong homomorphism because, if ψ is a formula consisting of a single σ -atom, then $\mathfrak{B} \models \psi(\bar{t})$ only if $\mathfrak{B} \models \psi(\bar{r})$ and $\mathfrak{B} \models \psi(\bar{s})$. \square

As an example, the structure $\mathfrak{N} = (\mathbb{N}; E, O)$ introduced above Definition 4 is guarded convex, but not convex. According to Theorem 9, the latter should imply that there is a finite substructure \mathfrak{A} of \mathfrak{N}^2 that does not embed to \mathfrak{N} . In fact, if we take as \mathfrak{A} the substructure of \mathfrak{N}^2 induced by the tuple $(1, 2)$, then this tuple belongs neither to E nor to O in the product. However, a strong homomorphism to \mathfrak{N} would need to map this tuple either to an odd or an even number. But then the tuple would need to belong to either E or O since the homomorphism is strong. This example does not work for the case of guarded convexity, because the considered substructure is not guarded. In fact, a guarded substructure of \mathfrak{N}^2 can only contain tuples where both components are even or both components are odd. In the former case, the tuple can be mapped to an even number, and in the latter to an odd number.

When it comes to ω -admissibility, we already mentioned that the original notion was formulated for binary signatures only, and also without the JD requirement because the DL did not have access to the default equality predicate. And indeed the tableau algorithm from [70] works just fine in the setting where elements of the concrete domain cannot be distinguished using atomic formulas. However, it seems odd to require each pair of elements to be present in a unique binary relation but not caring whether they can be told apart. Moreover, the decomposition of relations into orbits that we use to obtain ω -admissible structures yields JD for free. One good reason for dropping the JD axiom would be if it allowed us to locate new interesting concrete domains in the intersection of ω - and p-admissibility. We show that this is not the case; only very simple structures can be at the same time jointly exhaustive, pairwise disjoint, and guarded convex. For the sake of brevity, the statement here is formulated for binary signatures only.

Proposition 24. *Let \mathfrak{D} be a structure with a finite binary relational signature. Then the following are equivalent:*

1. \mathfrak{D} is guarded convex and JEPD.
2. There exists a partition V_1, \dots, V_m of D such that the non-empty relations of \mathfrak{D} are exactly the ones of the form $V_j \times V_k$ for every $(j, k) \in [m]^2$.

Proof. “1 \Rightarrow 2”: Let R_1, \dots, R_ℓ be an enumeration of those symbols from τ that are interpreted

in \mathcal{D} as a non-empty relation. We first prove the following.

Observation 6. *For every $i \in [\ell]$, there exists precisely one pair $(j, k) \in [\ell]^2$ such that*

$$\mathcal{D} \models \forall x, y (R_i(x, y) \Leftrightarrow R_j(x, x) \wedge R_k(y, y)).$$

Proof of Observation 6. For every $i \in [\ell]$, we have $\mathcal{D} \models \forall x, y (R_i(x, y) \Rightarrow \bigvee_{j \in [\ell]} R_j(x, x))$ and $\mathcal{D} \models \forall x, y (R_i(x, y) \Rightarrow \bigvee_{k \in [\ell]} R_k(y, y))$ because the relations of \mathcal{D} are jointly exhaustive. Using the guarded convexity of \mathcal{D} we conclude that, for every $i \in [\ell]$, there exists a pair $(j, k) \in [\ell]^2$ such that $\mathcal{D} \models \forall x, y (R_i(x, y) \Rightarrow R_j(x, x) \wedge R_k(y, y))$. Since the relations of \mathcal{D} are pairwise disjoint and each R_i is non-empty, there can only be one such pair (j, k) for every $i \in [\ell]$. Also, for every such pair (j, k) corresponding to a fixed $i \in [\ell]$, we have $\mathcal{D} \models \forall x, y (R_j(x, x) \wedge R_k(y, y) \Rightarrow R_1(x, y) \vee \dots \vee R_\ell(x, y))$ because the relations of \mathcal{D} are jointly exhaustive. Using the guarded convexity of \mathcal{D} we conclude that there exists an $i' \in [\ell]$ such that $\mathcal{D} \models \forall x, y (R_j(x, x) \wedge R_k(y, y) \Rightarrow R_{i'}(x, y))$. Since the relations of \mathcal{D} are pairwise disjoint, the index i' must be the original i we started with. \square

For a given $i \in [\ell]$, we distinguish the following two cases:

1. If there exists $x \in D$ such that $(x, x) \in R_i$, then Observation 6 implies $i = j = k$ and $R_i = V_i^2$ where $V_i := \{x \in D \mid (x, x) \in R_i\}$.
2. If there exists no $x \in D$ such that $(x, x) \in R_i$, then Observation 6 implies the existence of $j, k \in [\ell]$ such that i, j, k are all pairwise distinct and $R_i = V_j \times V_k$.

Without loss of generality, R_1, \dots, R_m are the relations of the first kind, and R_{m+1}, \dots, R_ℓ are the relations of the second kind. Since the relations of \mathcal{D} are jointly exhaustive, V_1, \dots, V_m form a partition of D .

“ $2 \Rightarrow 1$ ”: Clearly, the relations of \mathcal{D} are jointly exhaustive and pairwise disjoint. We use Theorem 22 to show that \mathcal{D} is guarded convex. Thus, let \mathfrak{A} be an arbitrary guarded structure in $\text{Age}(\mathcal{D}^2)$. Observe that no pair $(x, y) \in D^2$ such that $x \in V_j$ and $y \in V_k$ for $j \neq k$ can be a component of a tuple from a relation of \mathcal{D}^2 . This is a direct consequence of our assumptions about the relations of \mathcal{D} and the definition of the product of structures. Thus, since \mathfrak{A} is guarded and embeds into \mathcal{D}^2 , for every $(x, y) \in A$, we have $x, y \in V_i$ for some $i \in [m]$. It is easy to see that, in this particular case, the projection map $(x, y) \mapsto x$ is a strong homomorphism from \mathfrak{A} to \mathcal{D} . Since \mathfrak{A} was chosen arbitrarily, it follows from Theorem 22 that \mathcal{D} is guarded convex. \square

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List of Figures

4.1. The basic relations of Allen and RCC8.	34
4.2. The basic relations of the Cardinal Direction Calculus.	45
6.1. An illustration of the situation in Lemma 8 for $n = 6$	73
6.2. An illustration of the situation in Lemma 11 for $n = 5$	84
6.3. An illustration of the encoding.	92

Alphabetical Index

- ω -admissible, 17
- ω -categorical, 23
- 2RM, *see* two-register machine
- age, 6
- amalgamation
 - class, 25
 - property, 25
 - strong, 80
- arity, 5
- automorphism, 6
- bound, 29
- compactness
 - ω -embedding, 37
 - ω -homomorphism, 16
- complete
 - equality-free Horn implication, 79
- concept, 9
- conclusion, 6
- connected
 - equality-free Horn implication, 69
 - structure, 56
- constraint satisfaction problem, 6
 - solution, 7
- convex, 18, 66
- core, 34
- countable model property, 37
- decomposition, 28
- definable, 6
- domain, 5
- embedding, 6
- equality, 5
- expansion, 5
- extension, 6
 - property, 38
- feature, 9
- finitely bounded, 29
- formula
 - k -ary, 5
 - equality-free, 6
 - existential positive, 6
 - primitive positive, 6
 - quantifier-free, 6
- Fraïssé limit, 25
- Gaifman graph, 56
- GCI, *see* general concept inclusion
- general concept inclusion, 9
- guarded
 - convex, 19
 - implication, 19
 - structure, 103
- Henson digraph, 57
- homomorphism, 6
 - strong, 6
- Horn
 - implication, 6
 - sentence, 6
- implication, 6
- isomorphism, 6
- JD, 17
- JE, 17
- joint embedding property, 49
- MMSNP, 57

ALPHABETICAL INDEX

- p-admissible, 19
- patchwork, 17
- PD, 17
- premise, 6
- preserves, 6, 23
- product
 - algebraic, 41
 - direct, 5
- quantifier elimination, 25
- random graph, 38
- reduct, 5
- relational
 - signature, 5
 - structure, 5
 - symbol, 5
- restriction
 - concrete domain, 10
 - existential, 9
 - universal, 9
- role, 9
- SLD, 66
- square embedding property, 47
- substructure, 6
- subsumption, 9
- tautology, 6
- TBox, 9
- theory, 5
- tournament, 56
- two-register machine, 11
- union, 40
 - disjoint, 40

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Jakub Rydval
Dresden, March 29, 2022