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An Algebraic View on p-Admissible Concrete Domains for Lightweight Description Logics (Extended Version)

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An Algebraic View on p-Admissible Concrete Domains for Lightweight Description Logics (Extended Version)*

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Abstract. Concrete domains have been introduced in Description Logics (DLs) to enable reference to concrete objects (such as numbers) and predefined predicates on these objects (such as numerical comparisons) when defining concepts. To retain decidability when integrating a concrete domain into a decidable DL, the domain must satisfy quite strong restrictions. In previous work, we have analyzed the most prominent such condition, called ω-admissibility, from an algebraic point of view. This provided us with useful algebraic tools for proving ω-admissibility, which allowed us to find new examples for concrete domains whose integration leaves the prototypical expressive DL $\mathcal{ALC}$ decidable.
When integrating concrete domains into lightweight DLs of the $\mathcal{EL}$ family, achieving decidability is not enough. One wants reasoning in the resulting DL to be tractable. This can be achieved by using so-called p-admissible concrete domains and restricting the interaction between the DL and the concrete domain. In the present paper, we investigate p-admissibility from an algebraic point of view. Again, this yields strong algebraic tools for demonstrating p-admissibility. In particular, we obtain an expressive numerical p-admissible concrete domain based on the rational numbers. Although ω-admissibility and p-admissibility are orthogonal conditions that are almost exclusive, our algebraic characterizations of these two properties allow us to locate an infinite class of p-admissible concrete domains whose integration into $\mathcal{ALC}$ yields decidable DLs.

Keywords: Description logic · concrete domains · p-admissibility · convexity · ω-admissibility · finite boundedness · tractability · decidability · constraint satisfaction.

1 Introduction

Description Logics (DLs) [3,5] are a well-investigated family of logic-based knowledge representation languages, which are frequently used to formalize ontologies for application domains such as the Semantic Web [28] or biology and medicine [27]. A DL-based ontology consists of inclusion statements (so-called GCIs) between concepts defined using the DL at hand. For example, the GCI $\mathit{Human} \sqsubseteq \exists \mathit{parent}.\mathit{Human}$, which says that every human being has a human parent, uses concepts expressible in $\mathcal{EL}$. This GCI clearly implies the inclusion $\mathit{Human} \sqsubseteq \exists \mathit{parent}.\exists \mathit{parent}.\mathit{Human}$, i.e., *Human* is subsumed by $\exists \mathit{parent}.\exists \mathit{parent}.\mathit{Human}$ w.r.t. any ontology containing the above GCI. Keeping the subsumption problem decidable, and preferably of a low complexity, is an important design goal for DLs. While subsumption in the lightweight DL $\mathcal{EL}$ is tractable (i.e., decidable in polynomial time), it is ExpTime-complete in $\mathcal{ALC}$, which is obtain from $\mathcal{EL}$ by adding negation [5].

If information about the age of human beings is relevant in the application at hand, then one would like to associate humans with their ages and formulate constraints on these numbers. This becomes possible by integrating concrete domains into DLs [4]. Using the concrete domain $(\mathbb{Q}, >)$, we can express that parents cannot be younger than their children with the GCI $>(\mathit{age}, \mathit{parent}\,\mathit{age}) \sqsubseteq \bot$, where $\bot$ is the bottom concept (always interpreted as the empty set) and *age* is a concrete feature that maps from the abstract domain populating concepts into the concrete domain $(\mathbb{Q}, >)$. While integrating $(\mathbb{Q}, >)$ leaves $\mathcal{ALC}$ decidable [33], this is no

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longer the case if we integrate $(\mathbb{Q}, +_1)$, where $+_1$ is a binary predicate that is interpreted as incrementation [5,6]. In [34], ω-admissibility was introduced as a condition on concrete domains that ensures decidability. It was shown in that paper that Allen's interval logic [1] as well as the region connection calculus RCC8 [37] can be represented as ω-admissible concrete domains. Since ω-admissibility is a collection of rather complex technical conditions, it is not easy to show that a given concrete domain satisfies this property. In [6], we relate ω-admissibility to well-known notions from model theory, which allows us to prove ω-admissibility of certain concrete domains (among them Allen and RCC8) using known results from model theory. A different algebraic condition (called *EHD*) that ensures decidability was introduced in [19], and used in [32] to show decidability and complexity results for a concrete domains based on the integers.

When integrating a concrete domain into a lightweight DL like $\mathcal{EL}$, one wants to preserve tractability rather than just decidability. To achieve this, the notion of p-admissible concrete domains was introduced in [2] and paths of length > 1 were disallowed in concrete domain constraints. Regarding the latter restriction, note that, in the above example, we have used the path *parent age*, which has length 2. The restriction to paths of length 1 means (in our example) that we can no longer compare the ages of different humans, but we can still define concepts like teenager, using the GCI

$$\mathit{Teenager} \sqsubseteq \mathit{Human} \sqcap \geq_{10}(\mathit{age}) \sqcap \leq_{19}(\mathit{age}),$$

where $\geq_{10}$ and $\leq_{19}$ are unary predicates respectively interpreted as the rational numbers greater equal 10 and smaller equal 19. In a p-admissible concrete domain, satisfiability of conjunctions of atomic formulae and validity of implications between such conjunctions must be tractable. In addition, the concrete domain must be *convex*, which roughly speaking means that a conjunction cannot imply a true disjunction. For example, the concrete domain $(\mathbb{Q}, >, =, <)$ is ω-admissible [6], but it is not convex since $x < y \wedge x < z$ implies $y < z \vee y = z \vee y > z$, but none of the disjuncts. In [2], two p-admissible concrete domains were exhibited, where one of them is based on $\mathbb{Q}$ with unary predicates $=_p, >_p$ and binary predicates $+_p, =$. To the best of our knowledge, since then no other p-admissible concrete domains have been described in the literature.

One of the main contributions of the present paper is to devise algebraic characterizations of convexity in different settings. We start by noting that the definition of convexity given in [2] is ambiguous, and that what was really meant is what we call *guarded* convexity. However, in the presence of the equality predicate (which is available in the two p-admissible concrete domains introduced in [2]), the two notions of convexity coincide. Then we devise a general characterization of convexity based on the notion of *square embeddings*, which are embeddings of the product $\mathfrak{B}^2$ of a relational structure $\mathfrak{B}$ into $\mathfrak{B}$. We investigate the implications of this characterization further for so-called ω-categorical structures, finitely bounded structures, and numerical concrete domains. For *ω-categorical structures*, the square embedding criterion for convexity can be simplified, and we use this result to obtain new p-admissible concrete domains: countably infinite vector spaces over finite fields. *Finitely bounded structures* can be defined by specifying finitely many forbidden patterns, and are of great interest in the constraint satisfaction (CSP) community [16]. We show that, for such structures, convexity is a necessary *and sufficient* condition for p-admissibility. This result provides use with many examples of p-admissible concrete domains, but their usefulness in practice still needs to be investigated. Regarding *numerical concrete domains*, we exhibit a new and quite expressive p-admissible concrete domain based on the rational numbers, whose predicates are defined by linear equations over $\mathbb{Q}$.

Next, the paper investigates the connection between p-admissibility and ω-admissibility. We show that only trivial concrete domains can satisfy both properties. However, by combining the results on finitely bounded structures of the present paper with results in [6], we can show that convex finitely bounded homogeneous structures, which are p-admissible, can be integrated into $\mathcal{ALC}$ (even without the length 1 restriction on role paths) without losing decidability. Whereas these structures are not ω-admissible, they can be expressed in an ω-admissible concrete domain [6]. Finally, we show that, in general, the restriction to paths of length 1 is needed when integrating a p-admissible concrete domain into $\mathcal{EL}$, not only to stay tractable, but even to retain decidability.

2 Preliminaries

In this section, we introduce the algebraic and logical notions that will be used in the rest of the paper. The set $\{1,\dots,n\}$ is denoted by $[n]$. We use the bar notation for tuples; for a tuple $\bar{t}$ indexed by a set I, the value of $\bar{t}$ at the position $i \in I$ is denoted by $\bar{t}[i]$. For a function $f\colon A^k \to B$ and n-tuples $\bar{t}_1,\dots,\bar{t}_k \in A^n$, we use $f(\bar{t}_1,\dots,\bar{t}_k)$ as a shortcut for the tuple $\big(f(\bar{t}_1[1],\dots\bar{t}_k[1]),\dots,f(\bar{t}_1[n],\dots,\bar{t}_k[n])\big)$.

From a mathematical point of view, concrete domains are relational structures. A *relational signature* τ is a set of *relation symbols*, each with an associated natural number called *arity*. For a relational signature τ, a *relational τ-structure* $\mathfrak{A}$ (or simply τ-structure or structure) consists of a set A (the *domain*) together with the relations $R^{\mathfrak{A}} \subseteq A^k$ for each relation symbol $R \in \tau$ of arity k. Such a structure $\mathfrak{A}$ is *finite* if its domain A is finite. We often describe structures by listing their domain and relations, i.e., we write $(A, R_1^{\mathfrak{A}}, R_2^{\mathfrak{A}},\dots)$.

An *expansion* of a τ-structure $\mathfrak{A}$ is a σ-structure $\mathfrak{B}$ with $A = B$ such that $\tau \subseteq \sigma$ and $R^{\mathfrak{B}} = R^{\mathfrak{A}}$ for each relation symbol $R \in \tau$. Conversely, we call $\mathfrak{A}$ a *reduct* of $\mathfrak{B}$. The *product* of a family $(\mathfrak{A}_i)_{i\in I}$ of τ-structures is the τ-structure $\prod_{i\in I}\mathfrak{A}_i$ over $\prod_{i\in I} A_i$ such that, for each $R \in \tau$ of arity k, we have $(\bar{a}_1,\dots,\bar{a}_k) \in R^{\Pi_{i\in I}\mathfrak{A}_i}$ iff $(\bar{a}_1[i],\dots,\bar{a}_k[i]) \in R^{\mathfrak{A}_i}$ for every $i \in I$. We denote the binary product of a structure $\mathfrak{A}$ with itself as $\mathfrak{A}^2$.

A *homomorphism* $h\colon \mathfrak{A} \to \mathfrak{B}$ for τ-structures $\mathfrak{A},\mathfrak{B}$ is a mapping $h\colon A \to B$ that *preserves* each relation of $\mathfrak{A}$, i.e, if $\bar{t} \in R^{\mathfrak{A}}$ for some k-ary relation symbol $R \in \tau$, then $h(\bar{t}) \in R^{\mathfrak{B}}$. A homomorphism $h\colon \mathfrak{A} \to \mathfrak{B}$ is *strong* if it additionally satisfies the following condition: for every k-ary relation symbol $R \in \tau$ and $\bar{t} \in A^k$ we have $h(\bar{t}) \in R^{\mathfrak{B}}$ only if $\bar{t} \in R^{\mathfrak{A}}$. An *embedding* is an injective strong homomorphism. We write $\mathfrak{A} \hookrightarrow \mathfrak{B}$ if $\mathfrak{A}$ embeds into $\mathfrak{B}$. The class of all finite τ-structures that embed into $\mathfrak{B}$ is denoted by $\mathrm{Age}(\mathfrak{B})$. A *substructure* of $\mathfrak{B}$ is a structure $\mathfrak{A}$ over the domain $A \subseteq B$ such that the inclusion map $i\colon A \to B$ is an embedding. Conversely, we call $\mathfrak{B}$ an extension of $\mathfrak{A}$. An *isomorphism* is a surjective embedding. Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are *isomorphic* (written $\mathfrak{A} \cong \mathfrak{B}$) if there exists an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. An *automorphism* is an isomorphism from $\mathfrak{A}$ to $\mathfrak{A}$.

Given a relational signature τ, we can build first-order formulae using the relation symbols of τ in the usual way. Relational τ-structures are then just first-order interpretations. For a structure $\mathfrak{A}$ we denote its first-order theory, i.e., the first-order sentences that hold in $\mathfrak{A}$, with $\mathrm{Th}(\mathfrak{A})$. In the context of p-admissibility, we are interested in quite simple formulae. A τ-atom is of the form $R(x_1,\dots,x_n)$, where $R \in \tau$ is an n-ary relation symbol and $x_1,\dots,x_n$ are variables. For a fixed τ-structure $\mathfrak{A}$, the *constraint satisfaction problem (CSP)* for $\mathfrak{A}$ [10] asks whether a given conjunction of atoms is satisfiable in $\mathfrak{A}$. A conjunction of atoms with an existential quantifier prefix (quantifying over some of the variables occurring in the conjunction) is called a *conjunctive query.*

An *implication* is of the form $\forall\bar{x}.\,(\phi \Rightarrow \psi)$ where ϕ is a conjunction of atoms, ψ is a disjunction of atoms, and the tuple $\bar{x}$ consists of all the variables occurring in ϕ or ψ. Such an implication is a *Horn-implication* if ψ is the empty disjunction (corresponding to falsity $\bot$) or a single atom. The CSP for $\mathfrak{A}$ can be reduced in polynomial time to the validity problem for Horn-implications since ϕ is satisfiable in $\mathfrak{A}$ iff $\forall\bar{x}.\,(\phi \Rightarrow \bot)$ is not valid in $\mathfrak{A}$. Conversely, validity of Horn implications in a structure $\mathfrak{A}$ can be reduced in polynomial time to the CSP in the expansion $\mathfrak{A}^{\neg}$ of $\mathfrak{A}$ by the complements of all relations. In fact, the Horn implication $\forall\bar{x}.\,(\phi \Rightarrow \psi)$ is valid in $\mathfrak{A}$ iff $\phi \wedge \neg\psi$ is not satisfiable in $\mathfrak{A}^{\neg}$. Note that, in the signature of $\mathfrak{A}^{\neg}$, $\neg\psi$ can be expressed by an atom.

3 Integrating p-Admissible Concrete Domains into $\mathcal{EL}$

Given countably infinite sets $\mathsf{N_C}$ and $\mathsf{N_R}$ of concept and role names, $\mathcal{EL}$ concepts are built using the concept constructors top concept ($\top$), conjunction ($C \sqcap D$), and existential restriction

($\exists r.C$). The semantics of the constructors is defined in the usual way (see, e.g., [3,5]). It assigns to every $\mathcal{EL}$ concept C a set $C^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, where $\Delta^{\mathcal{I}}$ is the interpretation domain of the given interpretation $\mathcal{I}$.

As mentioned before, a concrete domain is a τ-structure $\mathfrak{D}$ with a relational signature τ without constant symbols. To integrate such a structure into $\mathcal{EL}$, we complement concept and role names with a set of *feature names* $\mathsf{N_F}$, which provide the connection between the abstract domain $\Delta^{\mathcal{I}}$ and the concrete domain D. A *path* is of the form $r\,f$ or f where $r \in \mathsf{N_R}$ and $f \in \mathsf{N_F}$. In our example in the introduction, *age* is both a feature name and a path of length 1, and *parent age* is a path of length 2. The DL $\mathcal{EL}(\mathfrak{D})$ extends $\mathcal{EL}$ with the new concept constructor

$$R(p_1, \ldots, p_k) \quad \text{(concrete domain restriction)},$$

where $p_1, \ldots, p_k$ are paths, and $R \in \tau$ is a k-ary relation symbol. We use $\mathcal{EL}[\mathfrak{D}]$ to denote the sublanguage of $\mathcal{EL}(\mathfrak{D})$ where paths in concrete domain restrictions are required to have length 1. Note that $\mathcal{EL}(\mathfrak{D})$ is the restriction to $\mathcal{EL}$ of the way concrete domains were integrated into $\mathcal{ALC}$ in [34], whereas our definition of $\mathcal{EL}[\mathfrak{D}]$ describes how concrete domains were integrated into $\mathcal{EL}$ in [2].

To define the semantics of concrete domain restrictions, we assume that an interpretation $\mathcal{I}$ assigns functional binary relations $f^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times D$ to feature names $f \in \mathsf{N_F}$, where *functional* means that $(a, d) \in f^{\mathcal{I}}$ and $(a, d') \in f^{\mathcal{I}}$ imply $d = d'$. We extend the interpretation function to paths of the form $p = r\,f$ by setting

$$(r\,f)^{\mathcal{I}} = \{(a, d) \in \Delta^{\mathcal{I}} \times D \mid \text{there is } b \in \Delta^{\mathcal{I}} \text{ such that } (a, b) \in r^{\mathcal{I}} \text{ and } (b, d) \in f^{\mathcal{I}}\}.$$

The semantics of concrete domain restrictions is now defined as follows:

$$\begin{aligned} R(p_1, \ldots, p_k)^{\mathcal{I}} = \{a \in \Delta^{\mathcal{I}} \mid {} & \text{there are } d_1, \ldots, d_k \in D \text{ such that} \\ & (a, d_i) \in p_i^{\mathcal{I}} \text{ for all } i \in [k] \text{ and } (d_1, \ldots, d_k) \in R^{\mathfrak{D}}\}. \end{aligned}$$

As usual, an $\mathcal{EL}(\mathfrak{D})$ TBox is defined to be a finite set of GCIs $C \sqsubseteq D$, where C, D are $\mathcal{EL}(\mathfrak{D})$ concepts. The interpretation $\mathcal{I}$ is a *model* of such a TBox if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for all GCIs $C \sqsubseteq D$ occurring in it. Given $\mathcal{EL}(\mathfrak{D})$ concept descriptions C, D and an $\mathcal{EL}(\mathfrak{D})$ TBox $\mathcal{T}$, we say that C is *subsumed by* D w.r.t. $\mathcal{T}$ (written $C \sqsubseteq_{\mathcal{T}} D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for all models of $\mathcal{T}$. For the subsumption problem in $\mathcal{EL}[\mathfrak{D}]$, to which we restrict our attention for the moment, only $\mathcal{EL}[\mathfrak{D}]$ concepts may occur in $\mathcal{T}$, and C, D must also be $\mathcal{EL}[\mathfrak{D}]$ concepts.

Subsumption in $\mathcal{EL}$ is known to be decidable in polynomial time [17]. For $\mathcal{EL}[\mathfrak{D}]$, this is the case if the concrete domain is p-admissible [2]. According to [2], a concrete domain $\mathfrak{D}$ is *p-admissible* if it satisfies the following conditions: (i) satisfiability of conjunctions of atoms and validity of Horn implications in $\mathfrak{D}$ are tractable; and (ii) $\mathfrak{D}$ is convex. Unfortunately, the definition of convexity in [2] (below formulated using our notation) is ambiguous:

> ($*$) If a conjunction of atoms of the form $R(x_1, ..., x_k)$ implies a disjunction of such atoms, then it also implies one of its disjuncts.

The problem is that this definition does not say anything about which variables may occur in the left- and right-hand sides of such implications. To illustrate this, let us consider the structure $\mathfrak{N} = (\mathbb{N}, E, O)$ in which the unary predicates E and O are respectively interpreted as the even and odd natural numbers. If the right-hand side of an implication considered in the definition of convexity may contain variables not occurring on the left-hand side, then $\mathfrak{N}$ is not convex: $\forall x, y. (E(x) \Rightarrow E(y) \vee O(y))$ holds in $\mathfrak{N}$, but neither $\forall x, y. (E(x) \Rightarrow E(y))$ nor $\forall x, y. (E(x) \Rightarrow O(y))$ does. However, for *guarded implications*, where all variables occurring on the right-hand side must also occur on the left-hand side, the structure $\mathfrak{N}$ satisfies the convexity condition ($*$). We say that a structure is *convex* if ($*$) is satisfied without any restrictions on the occurrence of variables, and *guarded convex* if ($*$) is satisfied for guarded implications. Clearly, any convex structure is guarded convex, but the converse implication does not hold, as exemplified by $\mathfrak{N}$.

We claim that, what was actually meant in [2], was guarded convexity rather than convexity. In fact, it is argued in that paper that non-convexity of $\mathfrak{D}$ allows one to express disjunctions in $\mathcal{EL}[\mathfrak{D}]$, which makes subsumption in $\mathcal{EL}[\mathfrak{D}]$ ExpTime-hard. However, this argument works only if the counterexample to convexity is given by a guarded implication. Let us illustrate this again on our example $\mathfrak{N}$. Whereas $\forall x, y.\ (E(x) \Rightarrow E(y) \vee O(y))$ holds in $\mathfrak{N}$, the subsumption $E(f) \sqsubseteq_\emptyset E(g) \sqcup E(g)$ does not hold in the extension of $\mathcal{EL}[\mathfrak{D}]$ with disjunction since the feature g need not have a value. For this reason, we use guarded convexity rather than convexity in our definition of p-admissibility. For the same reason, one can also restrict the tractability requirement in this definition to validity of guarded Horn implications.

Definition 1. *A relational structure $\mathfrak{D}$ is* p-admissible *if it is guarded convex and validity of guarded Horn implications in $\mathfrak{D}$ is tractable*

Using this notion, the main results of [2] concerning concrete domains can now be summarized as follows.

Theorem 1 (Baader, Brandt, and Lutz [2]). *Let $\mathfrak{D}$ be a relational structure. Then subsumption in $\mathcal{EL}[\mathfrak{D}]$ is*

1. *decidable in polynomial time if $\mathfrak{D}$ is p-admissible;*
2. *ExpTime-hard if $\mathfrak{D}$ is not guarded convex.*

The two p-admissible concrete domains in [2] have equality as one of their relations. For such structures, convexity and guarded convexity obviously coincide since one can use $x = x$ as a trivially true guard. For example, the extension $\mathfrak{N}_=$ of $\mathfrak{N}$ with equality is no longer guarded convex since $\forall x. (x = x \Rightarrow E(x) \vee O(x))$ holds in $\mathfrak{N}_=$, but neither $\forall x. (x = x \Rightarrow E(x))$ nor $\forall x. (x = x \Rightarrow O(x))$.

In the next section, we will show algebraic characterizations of (guarded) convexity. Regarding the tractability condition in the definition of p-admissibility, we have seen that it is closely related to the constraint satisfaction problem for $\mathfrak{D}$ and $\mathfrak{D}^\neg$. Characterizing tractability of the CSP in a given structure is a very hard problem. Whereas the Feder-Vardi conjecture [23] has recently been confirmed after 25 years of intensive research in the field by giving an algebraic criterion that can distinguish between *finite* structures with tractable and with NP-complete CSP [38,18], finding comprehensive criteria that ensure tractability for the case of infinite structures is a wide open problem, though first results for special cases have been found (see, e.g., [14,15]).

4 Algebraic Characterizations of Convexity

Before we can formulate our characterization of (guarded) convexity, we need to introduce a semantic notion of guardedness. We say that the relational τ-structure $\mathfrak{A}$ is *guarded* if for every $a \in A$ there is a relation $R \in \tau$ such that a appears in a tuple in $R^{\mathfrak{A}}$.

Theorem 2. *For a relational τ structure $\mathfrak{B}$, the following are equivalent:*

1. *$\mathfrak{B}$ is* guarded *convex.*
2. *For every finite $\sigma \subseteq \tau$ and every $\mathfrak{A} \in \mathrm{Age}(\mathfrak{B}^2)$ whose σ-reduct is* guarded*, there exists a strong homomorphism from the σ-reduct of $\mathfrak{A}$ to the σ-reduct of $\mathfrak{B}$.*

Proof. "2 $\Rightarrow$ 1": Suppose to the contrary that the closed implication $\forall x_1, \ldots, x_n. (\phi \Rightarrow \psi)$ is valid in $\mathfrak{B}$, where ϕ is a conjunction of atoms such that each variable x_i is present in some atom of ϕ, and ψ is a disjunction of atoms $\psi_1, \ldots, \psi_k$, but we also have $\mathfrak{B} \not\models \forall x_1, \ldots, x_n. (\phi \Rightarrow \psi_i)$ for every $i \in [k]$. Without loss of generality, we assume that $\phi, \psi_1, \ldots, \psi_k$ all have the same free variables $x_1, \ldots, x_n$, some of which might not influence their truth value. For every $i \in [k]$, there exists a tuple $\bar{t}_i \in B^n$ such that

$$\mathfrak{B} \models \phi(\bar{t}_i) \wedge \neg\psi_i(\bar{t}_i). \qquad (*)$$

We show by induction on i that, for every $i \in [k]$, there exists a tuple $\bar{s}_i \in B^n$ that satisfies the *induction hypothesis*

$$\mathfrak{B} \models \phi(\bar{s}_i) \wedge \neg \bigvee_{\ell \in [i]} \psi_\ell(\bar{s}_i). \tag{†}$$

In the *base case* ($i = 1$), it follows from ($*$) that $\bar{s}_1 \coloneqq \bar{t}_1$ satisfies (†).

In the *induction step* ($i \to i+1$), let $\bar{s}_i \in B^n$ be any tuple that satisfies (†). Let $\sigma \subseteq \tau$ be the finite set of relation symbols occurring in the implication $\forall x_1, \dots, x_n.\, (\phi \Rightarrow \psi)$, and let $\mathfrak{A}_i$ be the substructure of $\mathfrak{B}^2$ on the set $\{(\bar{s}_i[1], \bar{t}_{i+1}[1]), \dots, (\bar{s}_i[n], \bar{t}_{i+1}[n])\}$. Since $\mathfrak{B} \models \phi(\bar{s}_i)$ by (†), $\mathfrak{B} \models \phi(\bar{t}_{i+1})$ by ($*$), and ϕ contains an atom for each variable x_i, we conclude that the σ-reduct of $\mathfrak{A}_i$ is guarded. By 2., there exists a strong homomorphism f_i from the σ-reduct of $\mathfrak{A}_i$ to the σ-reduct of $\mathfrak{B}$. Since ϕ is a conjunction of σ-atoms and f_i is a homomorphism, we have that $\mathfrak{B} \models \phi\big(f_i(\bar{s}_i, \bar{t}_{i+1})\big)$. Suppose that $\mathfrak{B} \models \psi_{i+1}\big(f_i(\bar{s}_i, \bar{t}_{i+1})\big)$. Since f_i is a strong homomorphism, we get $\mathfrak{B} \models \psi_{i+1}(\bar{t}_{i+1})$, a contradiction to ($*$). Now suppose that $\mathfrak{B} \models \psi_j\big(f_i(\bar{s}_i, \bar{t}_{i+1})\big)$ for some $j \leq i$. Since f_i is a strong homomorphism, we get $\mathfrak{B} \models \psi_j(\bar{s}_i)$, a contradiction to (†). We conclude that $\bar{s}_{i+1} \coloneqq f_i(\bar{s}_i, \bar{t}_{i+1})$ satisfies (†).

Since $\mathfrak{B} \models \forall x_1, \dots, x_n.\, (\phi \Rightarrow \psi)$, the existence of a tuple $\bar{s}_i \in B^n$ that satisfies (†) for $i = k$ leads to a contradiction. This completes the proof of of "$2 \Rightarrow 1$" of our theorem. Alternatively, we could have obtained this direction as an instance of McKinsey's lemma [25].

Before we proceed with the proof of "$1 \Rightarrow 2$", let us take a closer look at the contraposition of the guarded convexity condition. Suppose that we have a conjunction ϕ of τ-atoms and tuples $\bar{r}$ and $\bar{s}$ over B together with disjunctions $\psi_{\bar{r}}$ and $\psi_{\bar{s}}$ of τ-atoms such that $\mathfrak{B} \models (\phi \wedge \neg\psi_{\bar{r}})(\bar{r})$ and $\mathfrak{B} \models (\phi \wedge \neg\psi_{\bar{s}})(\bar{s})$, and the implications $\forall x_1, \dots, x_n.\, (\phi \Rightarrow \psi_{\bar{r}})$ and $\forall x_1, \dots, x_n.\, (\phi \Rightarrow \psi_{\bar{s}})$ are guarded. Then clearly there must exist a tuple $\bar{t}$ over B such that $\mathfrak{B} \models (\phi \wedge \neg\psi_{\bar{r}} \wedge \neg\psi_{\bar{s}})(\bar{t})$ as otherwise $\mathfrak{B} \models \forall x_1, \dots, x_n.\, (\phi \Rightarrow (\psi_{\bar{r}} \vee \psi_{\bar{s}}))$, but neither $\mathfrak{B} \models \forall x_1, \dots, x_n.\, (\phi \Rightarrow \psi_{\bar{r}})$ nor $\mathfrak{B} \models \forall x_1, \dots, x_n.\, (\phi \Rightarrow \psi_{\bar{s}})$ is true (which would lead to a contradiction to guarded convexity).

Now we continue with the proof of "$1 \Rightarrow 2$". Let σ be an arbitrary finite subset of τ and let $\mathfrak{A} \in \mathrm{Age}\,(\mathfrak{B}^2)$ be an arbitrary finite substructure of $\mathfrak{B}^2$ whose σ-reduct is guarded. Let $\{(r_1, s_1), \dots, (r_n, s_n)\}$ be the domain of $\mathfrak{A}$. Consider the tuples $\bar{r} \coloneqq (r_1, \dots, r_n)$ and $\bar{s} \coloneqq (s_1, \dots, s_n)$. Let $\phi(x_1, \dots, x_n)$ be the conjunction of all σ-atoms such that

$$\mathfrak{A} \models \phi\big((r_1, s_1), \dots, (r_n, s_n)\big),$$

i.e., we consider all atoms built using a relation symbol from σ and containing variables from $\{x_1, \dots, x_n\}$, assign (r_i, s_i) to the variable x_i, and take those atoms for which the corresponding tuple of elements of $\mathfrak{A}$ belongs to the respective relation in $\mathfrak{A}$.

Clearly, the tuples $\bar{r}$ and $\bar{s}$ both satisfy ϕ in $\mathfrak{B}$ since the projection to a single coordinate is a homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$. Now let $\psi_{\bar{r}}$ be the disjunction of all σ-atoms which do not hold on the tuple $\bar{r}$ in $\mathfrak{B}$. Analogously, let $\psi_{\bar{s}}$ be the disjunction of all σ-atoms which do not hold on the tuple $\bar{s}$ in $\mathfrak{B}$. Without loss of generality, both disjunctions are non-empty since otherwise the projection onto one of the coordinates is a strong homomorphism and we are done. In addition, the implications $\forall x_1, \dots, x_n.\, (\phi \Rightarrow \psi_{\bar{r}})$ and $\forall x_1, \dots, x_n.\, (\phi \Rightarrow \psi_{\bar{s}})$ are guarded since the σ-reduct of $\mathfrak{A}$ is guarded.

We have that $\mathfrak{B} \models \phi \wedge \neg\psi_{\bar{r}}(\bar{r})$ and $\mathfrak{B} \models \phi \wedge \neg\psi_{\bar{s}}(\bar{s})$. Since $\mathfrak{B}$ is guarded convex, there must exist a tuple $\bar{t}$ such that $\mathfrak{B} \models \phi \wedge \neg\psi_{\bar{r}}(\bar{t}) \wedge \neg\psi_{\bar{s}}(\bar{t})$. Now consider the map f that sends, for every $i \in [n]$, the tuple (r_i, s_i) to $\bar{t}[i]$. Clearly f is a homomorphism from the σ-reduct of $\mathfrak{A}$ to the σ-reduct of $\mathfrak{B}$ because $\mathfrak{B} \models \phi(\bar{t})$. Moreover, f is a strong homomorphism because, if ψ is a formula consisting of a single σ-atom, then $\mathfrak{B} \models \psi(\bar{t})$ only if $\mathfrak{B} \models \psi(\bar{r})$ and $\mathfrak{B} \models \psi(\bar{s})$. ☐

As an easy consequence of Theorem 2, we also obtain a characterization of (unguarded) convexity. This is due to the fact that the structure $\mathfrak{B}$ is convex iff its expansion with the full unary predicate (interpreted as B) is guarded convex. In addition, in the presence of this predicate, any structure is guarded.

Corollary 1. *For a relational τ-structure $\mathfrak{B}$, the following are equivalent:*

1. *$\mathfrak{B}$ is convex.*
2. *For every finite $\sigma \subseteq \tau$ and every $\mathfrak{A} \in \mathrm{Age}(\mathfrak{B}^2)$, there exists a strong homomorphism from the σ-reduct of $\mathfrak{A}$ to the σ-reduct of $\mathfrak{B}$.*

As an example, the structure $\mathfrak{N} = (\mathbb{N}, E, O)$ introduced in the previous section is guarded convex, but not convex. According to the corollary, the latter should imply that there is a finite substructure $\mathfrak{A}$ of $\mathfrak{N}^2$ that has no strong homomorphism to $\mathfrak{N}$. In fact, if we take as $\mathfrak{A}$ the substructure of $\mathfrak{N}^2$ induced by the tuple $(1, 2)$, then this tuple belongs neither to E nor to O in the product. However, a strong homomorphism to $\mathfrak{N}$ would need to map this tuple either to an odd or an even number. But then the tuple would need to belong to either E or O since the homomorphism is strong. This example does not work for the case of guarded convexity, because the considered substructure is not guarded. In fact, a guarded substructure of $\mathfrak{N}^2$ can only contain tuples where both components are even or both components are odd. In the former case, the tuple can be mapped to an even number, and in the latter to an odd number.

In the presence of the equality predicate, strong homomorphisms are embeddings and guarded convexity is the same as convexity.

Corollary 2. *For a structure $\mathfrak{B}$ with a relational signature τ* with equality*, the following are equivalent:*

1. *$\mathfrak{B}$ is convex.*
2. *For every finite $\sigma \subseteq \tau$ and every $\mathfrak{A} \in \mathrm{Age}(\mathfrak{B}^2)$, the σ-reduct of $\mathfrak{A}$ embeds into the σ-reduct of $\mathfrak{B}$.*

The three results shown so far in this section follow the same general pattern, namely, they relate different versions of convexity to the existence of certain homomorphisms from all finite substructures of the square of the given structure into the structure. It would be nice if we could lift this property to a homomorphism from the whole square. We will see later (see Section 5.3) that this is not always possible for a given structure $\mathfrak{D}$ itself. However, we now show that it is always possible for some elementary extension of $\mathfrak{D}$. An extension $\mathfrak{M}$ of a τ-structure $\mathfrak{D}$ is *elementary* if, for every first-order τ-formula $\phi(x_1, \ldots, x_n)$ and every tuple $\bar{t} \in D^n$, we have that $\mathfrak{D} \models \phi(\bar{t})$ if and only if $\mathfrak{M} \models \phi(\bar{t})$. In particular, this means that the two structures have the same first-order theory, i.e., $\mathrm{Th}(\mathfrak{D}) = \mathrm{Th}(\mathfrak{M})$.

Theorem 3. *For a relational structure $\mathfrak{D}$* with equality*, the following are equivalent.*

1. *$\mathfrak{B}$ is convex.*
2. *There exists an elementary extension $\mathfrak{M}$ of $\mathfrak{D}$ such that $\mathfrak{M}^2 \hookrightarrow \mathfrak{M}$.*

The proof of this theorem requires some basic concepts in set theory such as transfinite induction. We allow the Axiom of Choice, i.e., every set can be well-ordered. A set S is an *ordinal* if it is *transitive*, i.e., if $S' \in S$, then $S' \subseteq S$; and if the membership relation is a well-order on S. Ordinals themselves are compared with each other using the membership relation—we write $S' < S$ for ordinals S, S' if $S' \in S$. An ordinal is a *cardinal* if it does not admit a bijection to an ordinal that is smaller w.r.t. to the membership order. Every set S admits a bijection to a unique cardinal, denoted by $|S|$.

A *theory* in a signature τ is a set of first-order τ-sentences. The *theory* of a τ-structure $\mathfrak{D}$ is the set $\mathrm{Th}(\mathfrak{D})$ of all first-order τ-sentences which are true in $\mathfrak{D}$. A 1-*type* of a theory T is a set S of first-order formulas with a single free variable such that $T \cup S$ is satisfiable. A 1-type S of $\mathrm{Th}(\mathfrak{D})$ is *realized* in $\mathfrak{D}$ if there exists $d \in D$ such that $\mathfrak{D} \models \phi(d)$ for each $\phi \in S$. For an infinite cardinal κ, a structure $\mathfrak{D}$ is κ-*saturated* if for every $\beta < \kappa$ and every expansion $\mathfrak{D}_\beta$ of $\mathfrak{D}$ by unary relation symbols $\{R_\alpha \mid \alpha < \beta\}$ which interpret in $\mathfrak{D}_\beta$ as singletons, every 1-type of $\mathrm{Th}(\mathfrak{D}_\beta)$ is realized in $\mathfrak{D}_\beta$. Given a family $\{R_\alpha \mid \alpha < \kappa\}$ of unary relation symbols which interpret in $\mathfrak{D}$ as singleton relations, we denote by $c_\alpha^{\mathfrak{D}}$ the unique element contained in $R_\alpha^{\mathfrak{D}}$.

Lemma 1 (c.f. Lemma 2.1 in [13]). *Let $\mathfrak{B}, \mathfrak{C}$ be τ-structures such that $\mathfrak{C}$ is $|B|$-saturated. Suppose that, for some cardinal $\kappa < |B|$, there are expansions $\mathfrak{B}_\kappa$ and $\mathfrak{C}_\kappa$ of $\mathfrak{B}$ and $\mathfrak{C}$ by unary symbols $\{R_\alpha \mid \alpha < \kappa\}$ for singleton relations such that every Boolean conjunctive query with atomic negation which holds in $\mathfrak{B}_\kappa$ also holds in $\mathfrak{C}_\kappa$. Then $\mathfrak{B}_\kappa$ admits a strong homomorphism to $\mathfrak{C}_\kappa$.*

Proof. Without loss of generality we assume that $\{c_\alpha^{\mathfrak{B}_\kappa} \mid \alpha < \kappa\}$ is a well-ordered set, in particular, it contains no repetitions. Let $\{R_\alpha \mid \alpha < |B|\}$ be a set of symbols with cardinality $|B|$ containing the original unary symbols, and let $\mathfrak{B}_{|B|}$ be an arbitrary $\tau \cup \{R_\alpha \mid \alpha < |B|\}$-expansion of $\mathfrak{B}_\kappa$ by singleton relations such that $\{c_\alpha^{\mathfrak{B}_{|B|}} \mid \alpha < |B|\}$ is a well-ordering of B. For every ordinal $\kappa < \lambda < |B|$, let $\mathfrak{B}_\lambda$ be the $\tau \cup \{R_\alpha \mid \alpha < \lambda\}$-reduct of $\mathfrak{B}_{|B|}$. We show by transfinite induction on λ up to $|B|$ that there exists a $\tau \cup \{R_\alpha \mid \alpha < |B|\}$-expansion $\mathfrak{C}_{|B|}$ of $\mathfrak{C}_\kappa$ by singleton relations such that every Boolean conjunctive query with atomic negation that holds in $\mathfrak{B}_{|B|}$ also holds in $\mathfrak{C}_{|B|}$. Then $f(c_\alpha^{\mathfrak{B}_{|B|}}) := c_\alpha^{\mathfrak{C}_{|B|}}$ is the desired strong homomorphisms.

The *base case* $\lambda = \kappa$ follows from the assumptions in Lemma 1.

For the *inductive step*, we first consider limit ordinals λ. There the inductive hypothesis holds as each Boolean conjunctive query can only contain finitely many symbols R_α whose indices are less than some $\gamma < \lambda$. Now suppose that $\lambda = \gamma + 1$ is a successor ordinal. Let Σ_γ be the set of all conjunctive queries with atomic negation $\phi(x)$ in the signature $\tau \cup \{R_\alpha \mid \alpha < \gamma\}$ such that $\mathfrak{B}_\gamma \models \phi(c_\lambda^{\mathfrak{B}_{|B|}})$. By the induction hypothesis, we have $\mathfrak{C}_\gamma \models \exists x.\, \phi(x)$ for every formula $\phi(x)$ from Σ_γ. Suppose that $\Sigma_\gamma \cup \mathrm{Th}(\mathfrak{C}_\gamma)$ is not satisfiable. Then, by compactness of first-order logic, there is a finite subset $\{\phi_1(x), \dots, \phi_k(x)\}$ of Σ_γ such that $\{\phi_1(x), \dots, \phi_k(x)\} \cup \mathrm{Th}(\mathfrak{C}_\gamma)$ is not satisfiable. This means that $\{\phi_1(x) \wedge \dots \wedge \phi_k(x)\} \cup \mathrm{Th}(\mathfrak{C}_\gamma)$ is not satisfiable. But since $\phi_1(x) \wedge \dots \wedge \phi_k(x) \in \Sigma_\gamma$, this leads to a contradiction to the definition of Σ_γ. Thus $\Sigma_\gamma \cup \mathrm{Th}(\mathfrak{C}_\gamma)$ is satisfiable, which means that Σ_γ is a 1-type of $\mathfrak{C}_\gamma$. By $|B|$-saturation of $\mathfrak{C}$, it is realized by some element $c \in C$. We define $\mathfrak{C}_\lambda$ by setting $c_\lambda^{\mathfrak{C}_\lambda} := c$. □

Proof of Theorem 3: Let τ be the signature of $\mathfrak{D}$.

"$2 \Rightarrow 1$": Suppose that $\mathfrak{D}$ has such an elementary extension $\mathfrak{M}$. Then $\mathfrak{M}$ is convex by Item 1 of Corollary 2. Note that convexity of $\mathfrak{D}$ can be axiomatized by a set of first-order sentences of the form $(\forall x_1, \dots, x_n.\, (\phi \Rightarrow \bigvee_{i=1}^k \psi_i)) \Rightarrow \bigvee_{i=1}^k (\forall x_1, \dots, x_n.\, (\phi \Rightarrow \psi_i))$. Since $\mathrm{Th}(\mathfrak{M}) = \mathrm{Th}(\mathfrak{D})$, we conclude that $\mathfrak{D}$ is convex as well.

"$1 \Rightarrow 2$": Now suppose that $\mathfrak{D}$ is convex. If D is finite, then it follows directly from Corollary 2 that $(D; =)^2 \hookrightarrow (D; =)$, which is impossible. Thus D is infinite. We build an elementary extension $\mathfrak{M}$ of $\mathfrak{D}$ inductively. Let $\mathfrak{M}_0$ be the empty τ-structure, and let $\mathfrak{M}_1 := \mathfrak{D}$. Suppose that $\mathfrak{M}_i$ is already defined for some $i > 1$. Then we take as $\mathfrak{M}_{i+1}$ an arbitrary elementary extension of $\mathfrak{M}_i$ that is $\max(|\tau|, |M_i^2|)$-saturated. Such an extension always exists by Corollary 8.2.2 in [26]. Finally, we set $\mathfrak{M} := \bigcup_{i \in \mathbb{N}} \mathfrak{M}_i$. We claim that $\mathfrak{M}$ is an elementary extension of $\mathfrak{D}$. Let $\bar{t} \in D^k$ be arbitrary, and let ϕ be an arbitrary k-ary τ-formula. Let $x_1, \dots, x_\ell$ be the quantified variables of ϕ. Suppose that $\mathfrak{D} \models \phi(\bar{t})$. Note that, if $\bar{s} \in M^\ell$ is a tuple of elements that are to be substituted for the variables $x_1, \dots, x_\ell$ in order to check whether $\mathfrak{M} \models \phi(\bar{t})$, then $\bar{s} \in M_i$ for some $i \geq 1$. Since each $\mathfrak{M}_i$ is an elementary extension of $\mathfrak{D}$ and thus $\mathfrak{M}_i \models \phi(\bar{t})$, we conclude that $\mathfrak{M} \models \phi(\bar{t})$. The other direction where we start with $\mathfrak{M} \models \phi(\bar{t})$ is analogous.

Next, we construct an embedding from $\mathfrak{M}^2$ to $\mathfrak{M}$. We show by induction on i that every embedding $f_i \colon \mathfrak{M}_i^2 \hookrightarrow \mathfrak{M}_{i+1}$ can be extended to an embedding $f_{i+1} \colon \mathfrak{M}_{i+1}^2 \hookrightarrow \mathfrak{M}_{i+2}$. Since $\mathfrak{M}_0$ is the empty structure which trivially embeds into $\mathfrak{M}_1$, this gives us a chain of embeddings $f_0, f_1, \dots$ such that $f_{i+1}|_{M_i} = f_i$ for every $i \geq 0$.

In the *base case* $i = 0$, there exists only one strong homomorphism from $\mathfrak{M}_0^2$ to $\mathfrak{M}_1$, namely the empty map. Thus we only need to show that there exists an embedding from $\mathfrak{M}_1^2$ to $\mathfrak{M}_2$. Let ϕ be a Boolean conjunctive query with atomic negation that holds in $\mathfrak{M}_1^2$. Then ϕ is of the form $\exists x_1, \dots, x_n.\, \psi(x_1, \dots, x_n)$ for some conjunction ψ of atoms and negated atoms. There exist $\bar{t}_1, \dots, \bar{t}_n \in M_1^2$ such that $\mathfrak{M}_1^2 \models \psi(\bar{t}_1, \dots, \bar{t}_n)$. Let σ be the finite set of all symbols from τ which appear in ψ, and let $\mathfrak{A}$ be the substructure of $\mathfrak{M}_1^2$ on $\{\bar{t}_1, \dots, \bar{t}_n\}$. Since $\mathfrak{D}$ is convex, by Corollary 2, there exists an embedding e from the σ-reduct of $\mathfrak{A}$ to the σ-reduct of $\mathfrak{M}_1$.

Thus $\mathfrak{M}_1 \models \psi(e(\bar{t}_1), \ldots, e(\bar{t}_n))$ and hence $\mathfrak{M}_1 \models \phi$. Since $\mathrm{Th}(\mathfrak{M}_1) = \mathrm{Th}(\mathfrak{M}_2)$, we have that $\mathfrak{M}_2 \models \phi$. Since $\mathfrak{M}_2$ is $|M_1^2|$-saturated, it follows from Lemma 1 that there exists an embedding $f_1 \colon \mathfrak{M}_1^2 \hookrightarrow \mathfrak{M}_2$.

In the *induction step*, we assume the existence of an embedding $f_i \colon \mathfrak{M}_i^2 \hookrightarrow \mathfrak{M}_{i+1}$ for some $i > 0$. Let $\kappa \coloneqq |M_i^2|$, $\mathfrak{B} \coloneqq \mathfrak{M}_{i+1}^2$, and $\mathfrak{C} \coloneqq \mathfrak{M}_{i+1}$. Moreover, let $\mathfrak{B}_\kappa$ and $\mathfrak{C}_\kappa$ be $\tau \cup \{R_\alpha \mid \alpha < \kappa\}$-expansions of $\mathfrak{B}$ and $\mathfrak{C}$, respectively, such that $\{c_\alpha^{\mathfrak{B}_\kappa} \mid \alpha < \kappa\}$ is a well-ordering on M_i^2, and $c_\alpha^{\mathfrak{C}_\kappa} \coloneqq f_i(c_\alpha^{\mathfrak{B}_\kappa})$. Let ϕ be a Boolean conjunctive query with atomic negation that holds in $\mathfrak{B}_\kappa$. Then ϕ is of the form $\exists x_1, \ldots, x_n.\, \psi(x_1, \ldots, x_n)$ for some conjunction ψ of atoms and negated atoms. There exist $\bar{t}_1, \ldots, \bar{t}_n \in M_{i+1}^2$ such that $\mathfrak{B}_\kappa \models \psi(\bar{t}_1, \ldots, \bar{t}_n)$. Let $R_{\alpha_1}, \ldots, R_{\alpha_m}$ be the finitely many new unary symbols which appear in ψ. Let $\psi_+(x_1, \ldots, x_n)$ be the conjunction of all non-negated atoms in ψ. Then we clearly have $(\mathfrak{M}_{i+1}; \{c_{\alpha_1}^{\mathfrak{B}_\kappa}[j]\}, \ldots, \{c_{\alpha_m}^{\mathfrak{B}_\kappa}[j]\}) \models \psi_+(\bar{t}_1[j], \ldots, \bar{t}_n[j])$ for both $j = 1$ and $j = 2$. However, for formulas $\psi_-(x_1, \ldots, x_n)$ consisting of a single negated atom from ψ, we only have $(\mathfrak{M}_{i+1}; \{c_{\alpha_1}^{\mathfrak{B}_\kappa}[j]\}, \ldots, \{c_{\alpha_m}^{\mathfrak{B}_\kappa}[j]\}) \models \psi_-(\bar{t}_1[j], \ldots, \bar{t}_n[j])$ for $j = 1$ or $j = 2$. For $j \in \{1, 2\}$, let $\psi_{j,-}(x_1, \ldots, x_n)$ be the conjunction of all negated atoms ψ_- from ψ such that $(\mathfrak{M}_{i+1}; \{c_{\alpha_1}^{\mathfrak{B}_\kappa}[j]\}, \ldots, \{c_{\alpha_m}^{\mathfrak{B}_\kappa}[j]\}) \models \psi_-(\bar{t}_1[j], \ldots, \bar{t}_n[j])$. Then $(\mathfrak{M}_{i+1}; \{c_{\alpha_1}^{\mathfrak{B}_\kappa}[j]\}, \ldots, \{c_{\alpha_m}^{\mathfrak{B}_\kappa}[j]\}) \models \psi_+(\bar{t}_1[j], \ldots, \bar{t}_n[j]) \wedge \psi_{-,j}(\bar{t}_1[j], \ldots, \bar{t}_n[j])$ for both $j = 1$ and $j = 2$. Since $\mathfrak{M}_{i+1}$ is an elementary extension of $\mathfrak{M}_i$, there exist $\bar{s}_1, \ldots, \bar{s}_n \in M_i^2$ such that $(\mathfrak{M}_i; \{c_{\alpha_1}^{\mathfrak{B}_\kappa}[j]\}, \ldots, \{c_{\alpha_m}^{\mathfrak{B}_\kappa}[j]\}) \models \psi_+(\bar{s}_1[j], \ldots, \bar{s}_n[j]) \wedge \psi_{-,j}(\bar{s}_1[j], \ldots, \bar{s}_n[j])$ for both $j = 1$ and $j = 2$, i.e., $(\mathfrak{M}_i^2; \{c_{\alpha_1}^{\mathfrak{B}_\kappa}\}, \ldots, \{c_{\alpha_m}^{\mathfrak{B}_\kappa}\}) \models \psi(\bar{s}_1, \ldots, \bar{s}_n)$. Since f_i is an embedding, $(\mathfrak{M}_{i+1}; \{f_i(c_{\alpha_1}^{\mathfrak{B}_\kappa})\}, \ldots, \{f_i(c_{\alpha_m}^{\mathfrak{B}_\kappa})\}) \models \psi(f_i(\bar{s}_1), \ldots, f_i(\bar{s}_n))$. By the definition of $\mathfrak{C}_\kappa$, it follows that $\mathfrak{C}_\kappa \models \phi$. Since ϕ was chosen arbitrarily and $\mathfrak{C}_\kappa$ is $|B|$-saturated, it follows from Lemma 1 that $\mathfrak{B}_\kappa$ embeds to $\mathfrak{C}_\kappa$. By the definition of $\mathfrak{B}_\kappa$ and $\mathfrak{C}_\kappa$, this means that there exists an embedding $f_{i+1} \colon \mathfrak{M}_{i+1}^2 \hookrightarrow \mathfrak{M}_{i+2}$ which extends f_i.

Now it is easy to see that f defined by $f \coloneqq \bigcup_{i \in \mathbb{N}} f_i$ is an embedding from $\mathfrak{M}^2$ to $\mathfrak{M}$. $\square$

5 Examples of Convex and p-Admissible Structures

We consider three different kinds of structures (ω-categorical, finitely bounded, numerical) and show under which conditions such structures are convex. This provides us with new examples for p-admissible concrete domains.

5.1 Convex ω-Categorical Structures

A structure is called *ω-categorical* if its first-order theory has a unique countable model up to isomorphism. A well-known example of such a structure is $(\mathbb{Q}, <)$, whose first-order theory is the theory of linear orders without first and last element. Such structures have drawn considerable attention in the CSP community since their CSPs can, to some extent, be investigated using the algebraic tools originally developed for finite structures. Countably infinite ω-categorical structures can be characterized using automorphisms and orbits. For every structure $\mathfrak{A}$, the set of all automorphisms of $\mathfrak{A}$, denoted by $\mathrm{Aut}(\mathfrak{A})$, forms a permutation group with composition as group operation [26]. The *orbit* of a tuple $\bar{t} \in A^k$ under $\mathrm{Aut}(\mathfrak{A})$ is the set $\{(g(\bar{t}[1]), ..., g(\bar{t}[k])) \mid g \in \mathrm{Aut}(\mathfrak{A})\}$. The following result is due to Engeler, Ryll-Nardzewski, and Svenonius (see Theorem 6.3.1 in [26]).

Theorem 4. *For a countably infinite structure $\mathfrak{D}$ with a countable signature, the following are equivalent:*

1. *$\mathfrak{D}$ is ω-categorical.*
2. *Every relation preserved by $\mathrm{Aut}(\mathfrak{D})$ has a first-order definition in $\mathfrak{D}$.*
3. *For every $k \geq 1$, there are only finitely many orbits of k-tuples under $\mathrm{Aut}(\mathfrak{D})$.*

For countably infinite ω-categorical structures the characterization of convexity of Corollary 2 can be improved to the following simpler statement, which is similar to the characterization in Theorem 3, but using the structure itself rather than an elementary extension.

Theorem 5. *For a countably infinite ω-categorical relational structure $\mathfrak{B}$ with a countable signature τ with equality, the following are equivalent:*

1. *$\mathfrak{B}$ is convex.*
2. *$\mathfrak{B}^2$ embeds into $\mathfrak{B}$.*

The proof of this theorem combines the proof of Corollary 2 with the following two facts, which are implied by ω-categoricity of $\mathfrak{B}$. First, there exists a strong homomorphism from $\mathfrak{B}^2$ to $\mathfrak{B}$ iff there exists a strong homomorphism from $\mathfrak{A}$ to $\mathfrak{B}$ for every $\mathfrak{A} \in \mathrm{Age}(\mathfrak{B}^2)$ (see, e.g., Lemma 3.1.5 in [10]). Second, to deal with the fact that τ may be infinite (which is problematic for the proof of "1 ⇒ 2"), we can use Theorem 4, which ensures that, for every $k \geq 1$, there are only finitely many inequivalent k-ary formulae over $\mathfrak{B}$ consisting of a single τ-atom. This ensures that the formulae $\phi, \psi_{\bar{r}}, \psi_{\bar{s}}$ constructed in the proof of "1 ⇒ 2" of Corollary 2 can be assumed to be finite.

In the CSP literature, one can find two examples of countably infinite ω-categorical structure that satisfy the square embedding condition of the above theorem: atomless Boolean algebras and countably infinite vector spaces over finite fields. Since the CSP for atomless Boolean algebras is NP-complete [8], this example does not provide us with a p-admissible concrete domain. Things are more rosy for the vector space example.

As shown in [12], the relational representation $\mathfrak{V}_q = (V_q, R^+, R^{s_0}, \ldots, R^{s_{q-1}})$ of the countably infinite vector space over a finite field $\mathrm{GF}(q)$ is ω-categorical, satisfies $\mathfrak{V}_q^2 \cong \mathfrak{V}_q$, and its CSP is decidable in polynomial time, even if the complements of all predicates are added. Here R^+ is a ternary predicate corresponding to addition of vectors, and the R^{s_i} are binary predicates corresponding to scalar multiplication of a vector with the element s_i of $\mathrm{GF}(q)$. We can show that these properties are preserved if we add finitely many unary predicates R^{e_i} that correspond to unit vectors $e_1, \ldots, e_k$.

Corollary 3. *The structure $\mathfrak{V}_q$ expanded with predicates $R^{e_1}, \ldots, R^{e_k}$ for unit vectors $e_1, \ldots, e_k$ is p-admissible.*

Proof. We have $\mathfrak{V}_q^2 \cong \mathfrak{V}_q$, i.e., both structures are vector spaces over $\mathrm{GF}(q)$ of countably infinite dimension. Now if we fix finitely many unit vectors $e_1, \ldots, e_k \in V_q$ by expanding $\mathfrak{V}_q$ with the unary predicates $R^{e_1}, \ldots, R^{e_k}$, we can still extend the map which sends (e_i, e_i) to e_i for each $i \in [k]$ to a bijection between bases of both vector spaces. This bijection then naturally extends to an isomorphism from $(\mathfrak{V}_q, R^{e_1}, \ldots, R^{e_k})^2$ to $(\mathfrak{V}_q, R^{e_1}, \ldots, R^{e_k})$. Thus the convexity of $(\mathfrak{V}_q, R^{e_1}, \ldots, R^{e_k})$ follows from Corollary 2. The CSP in its expansion $(\mathfrak{V}_q, R^{e_1}, \ldots, R^{e_k})^{\neg}$ by complements of all relations can be solved, similarly as in the Gaussian elimination algorithm, by iterated elimination of variables from equations and subsequent search for unsatisfiable equalities and/or inequalities between unit vectors (e.g., $e_1 \neq e_1$) [12]. This implies that testing validity of Horn implications in $(\mathfrak{V}_q, R^{e_1}, \ldots, R^{e_k})$ is tractable as well. We conclude that $(\mathfrak{V}_q, R^{e_1}, \ldots, R^{e_k})$ is p-admissible. □

For the case $q = 2$, the vectors in V_q are one-sided infinite tuples of zeros and ones containing only finitely many ones, which can be viewed as representing finite subsets of $\mathbb{N}$. For example, $(0, 1, 1, 0, 1, 0, 0, \ldots)$ represents the set $\{1, 2, 4\}$. Thus, if we use $\mathfrak{V}_2$ as concrete domain, the features assign finite sets of natural numbers to individuals. For example, assume that the feature *dages* assigns the set of ages of female children to a person, and *sages* the set of ages of male children. Then $R^+(\mathit{dages}, \mathit{sages}, \mathit{zero})$ describes persons that, for every age, have either both a son and a daughter of this age, or no child at all of this age. The feature *zero* is supposed to point to the zero vector, which can, e.g., be enforced using the GCI $\top \sqsubseteq R^+(\mathit{zero}, \mathit{zero}, \mathit{zero})$. If e_1 is the unit vector $(0, 1, 0, 0, \ldots)$ and e_4 is the unit vetor $(0, 0, 0, 0, 1, 0, 0, \ldots)$, then the concept $\mathit{Human} \sqcap R^+(\mathit{one}, \mathit{four}, \mathit{dages})$ describes humans that have daughters of age one and four, and of no other age, if the TBox contains the GCI $\top \sqsubseteq R^{e_1}(\mathit{one}) \sqcap R^{e_4}(\mathit{four})$.

5.2 Convex Structures with Forbidden Patterns

For a class $\mathcal{F}$ of τ-structures, $\mathrm{Forb}_e(\mathcal{F})$ stands for the class of all finite τ-structures that do not embed any member of $\mathcal{F}$. A structure $\mathfrak{B}$ is *finitely bounded* if its signature is finite and $\mathrm{Age}(\mathfrak{B}) = \mathrm{Forb}_e(\mathcal{F})$ for some finite set $\mathcal{F}$ of *bounds*. Alternatively, one can say that $\mathfrak{B}$ is finitely bounded if its signature is finite and there is a universal first-order sentence Φ with equality[1] such that $\mathrm{Age}(\mathfrak{B})$ consists precisely of the finite models of Φ [7]. A well-known example of a finitely bounded structure is $(\mathbb{Q}, >, =)$, for which the self loop, the 2-cycle, the 3-cycle, and two isolated vertices can be used as bounds (see Fig. 1 in [6]). As universal sentence defining $\mathrm{Age}(\mathbb{Q}, >, =)$ we can take the conjunction of the usual axioms defining linear orders. For finitely bounded structures, p-admissibility turns out to be equivalent to convexity.

Theorem 6. *Let $\mathfrak{B}$ be a finitely bounded τ-structure with equality. The following are equivalent:*

1. *$\mathfrak{B}$ is convex,*
2. *$\mathrm{Age}(\mathfrak{B})$ is defined by a conjunction Φ of Horn implications,*
3. *$\mathfrak{B}$ is p-admissible.*

Proof. "1 $\Rightarrow$ 2": Using the logical reformulation of finite boundedness mentioned above (see, e.g., [7]), we know that $\mathfrak{B}$ is finitely bounded if its signature is finite and there is a universal first-order sentence Φ such that $\mathrm{Age}(\mathfrak{B})$ consists precisely of the finite models of Φ. We bring Φ into prenex normal form, and then transform its quantifier-free part in conjunctive normal form. This shows that we can assume that Φ is a conjunction of implications (in the sense defined in Section 2). Note that a universal sentence holds in a relational structure iff it holds in each of its finite substructures. In particular, we have $\mathfrak{B} \models \Phi$. For every implication in Φ where the conclusion consists of at least two atoms, we apply the definition of convexity and reduce Φ to a conjunction of Horn implications Φ' such that $\mathfrak{B} \models \Phi'$. This implies that Φ' holds in all elements of $\mathrm{Age}(\mathfrak{B})$. In addition, by the construction of Φ', the original formula Φ is a logical consequence of Φ'. Thus, if a finite τ-structure satisfies Φ', it also satisfies Φ, and thus belongs to $\mathrm{Age}(\mathfrak{B})$. This shows that Φ' defines $\mathrm{Age}(\mathfrak{B})$.

"2 $\Rightarrow$ 3": We first show that $\mathfrak{B}$ is convex using Corollary 2. We set $\sigma \coloneqq \tau$ and select an arbitrary finite substructure $\mathfrak{A}$ of $\mathfrak{B}^2$. Let $\forall \bar{x}. (\phi_i \Rightarrow \psi_i)$ be one of the Horn implications whose conjunction Φ over $i \in [\ell]$ defines $\mathrm{Age}(\mathfrak{B})$. Let $\bar{t}$ be a tuple over A such that $\mathfrak{A} \models \phi_i(\bar{t})$ for some $i \in [\ell]$ and let k be its arity. By the definition of $\mathfrak{A}$, $\bar{t}$ is of the form $((x_1, y_1), \ldots, (x_k, y_k))$ such that $\mathfrak{B} \models \phi_i(x_1, \ldots, x_k)$ and $\mathfrak{B} \models \phi_i(y_1, \ldots, y_k)$. Since the substructure of $\mathfrak{B}$ on $\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ satisfies $\forall \bar{x}. (\phi_i \Rightarrow \psi_i)$, we have $\mathfrak{B} \models \psi_i(x_1, \ldots, x_k) \wedge \psi_i(y_1, \ldots, y_k)$, and thus $\mathfrak{A} \models \psi_i(\bar{t})$. Since the tuple $\bar{t}$ and the index $i \in [\ell]$ were chosen arbitrarily, we know that that $\mathfrak{A} \models \forall \bar{x}. (\phi_i \Rightarrow \psi_i)$ for all $i \in [\ell]$. Thus, we have $\mathfrak{A} \models \Phi$, which implies $\mathfrak{A} \in \mathrm{Age}(\mathfrak{B})$, i.e., $\mathfrak{A}$ embeds into $\mathfrak{B}$.

Regarding tractability, note that the structure $\mathfrak{B}$ satisfies a given Horn implication $\forall \bar{x}. (\phi \Rightarrow \psi)(\bar{x})$ iff this implication is satisfied by all elements of $\mathrm{Age}(\mathfrak{B})$. This is the case iff the conjunction of Horn implications Φ that defines $\mathrm{Age}(\mathfrak{B})$ implies the Horn implication $\forall \bar{x}. (\phi \Rightarrow \psi)(\bar{x})$. It is well-known that the entailment problem is decidable in polynomial time for Horn implications [22].

"3 $\Rightarrow$ 1": This direction is trivial. $\square$

The structure $(\mathbb{Q}, >, =)$ is not convex. In fact, since it is also ω-categorical, convexity would imply that its square $(\mathbb{Q}, >, =) \times (\mathbb{Q}, >, =)$ embeds into $(\mathbb{Q}, >, =)$, by Theorem 5. This cannot be the case since the product contains incomparable elements, whereas $(\mathbb{Q}, >, =)$ does not. In the universal sentence defining $\mathrm{Age}(\mathbb{Q}, >, =)$, the totality axiom $\forall x, y. (x < y \vee x = y \vee x > y)$ is the culprit since it is not Horn. If we remove this axiom, we obtain the theory of strict partial orders.

Example 1. It is well-known that there exists a unique countable homogeneous[2] strict partial order $\mathfrak{O}$ [36], whose age is defined by the universal sentence $\forall x, y, z. (x < y \wedge y < z \Rightarrow x <$

[1] Here "with equality" means that the sentence may use equality even if the signature τ does not contain it.

[2] A structure is *homogeneous* if every isomorphism between its finite substructures extends to an automorphism of the whole structure.

$z) \wedge \forall x. (x < x \Rightarrow \bot)$, which is a Horn implication. Thus, $\mathfrak{O}$ extended with equality is finitely bounded and convex. Using $\mathfrak{O}$ as a concrete domain means that the feature values satisfy the theory of strict partial orders, but not more. One can, for instance, use this concrete domain to model preferences of people; e.g., the concept $\mathit{Italian} \sqcap >(\mathit{pizzapref}, \mathit{pastapref})$ describes Italians that like pizza more than pasta. Using $\mathfrak{O}$ here means that preferences may be incomparable. As we have seen above, adding totality would break convexity and thus p-admissibility.

Beside finitely bounded structures, the literature also considers structures whose age can be described by a finite set of forbidden homomorphic images [21,29]. For a class $\mathcal{F}$ of τ-structures, $\mathrm{Forb}_h(\mathcal{F})$ stands for the class of all finite τ-structures that do not contain a homomorphic image of any member of $\mathcal{F}$. A structure is *connected* if its so-called Gaifman graph is connected. The *Gaifman graph* of a structure $\mathfrak{A}$ is the undirected graph (A, E) such that there is an edge in E between two elements $a, a' \in A$ iff they occur together in a tuple from a relation of $\mathfrak{A}$.

Theorem 7 (Cherlin, Shelah, and Shi [21]). *Let $\mathcal{F}$ be a finite family of connected relational structures with a finite signature τ. Then there exists an ω-categorical τ-structure $\mathrm{CSS}(\mathcal{F})$ that is a reduct of a finitely bounded homogeneous structure and such that $\mathrm{Age}\,(\mathrm{CSS}(\mathcal{F})) = \mathrm{Forb}_h(\mathcal{F})$.*

We can show that the structures of the form $\mathrm{CSS}(\mathcal{F})$ provided by this theorem are always p-admissible.

Proposition 1. *Let $\mathcal{F}$ be a finite family of connected relational structures with a finite signature τ. Then the expansion $\mathrm{CSS}^{=}(\mathcal{F})$ of $\mathrm{CSS}(\mathcal{F})$ by the equality predicate is p-admissible.*

Proof. Let $\mathfrak{B} := \mathrm{CSS}(\mathcal{F})$. By Theorem 7, we have $\mathfrak{A} \in \mathrm{Age}\,(\mathfrak{B})$ iff $\mathfrak{A}$ does not contain a homomorphic image of any $\mathfrak{F} \in \mathcal{F}$ as a substructure. If we can show $\mathrm{Age}\,(\mathfrak{B}^2) \subseteq \mathrm{Age}\,(\mathfrak{B})$, then we trivially also get $\mathrm{Age}\,\big((\mathfrak{B}, =)^2\big) \subseteq \mathrm{Age}\,\big((\mathfrak{B}, =)\big)$, and it follows from Corollary 2 that $\mathrm{CSS}^{=}(\mathcal{F})$ is convex. Suppose that there exists $\mathfrak{A} \in \mathrm{Age}\,(\mathfrak{B}^2)$ such that $\mathfrak{A} \notin \mathrm{Age}\,(\mathfrak{B})$. Then there exists $\mathfrak{F} \in \mathcal{F}$ such that $\mathfrak{F} \to \mathfrak{A}$. Since the projection to a single component is a homomorphism, this shows that there is a homomorphism $\mathfrak{F} \to \mathfrak{B}$. But then the image of $\mathfrak{F}$ under this homomorphism is a finite substructure of $\mathfrak{B}$ that does not belong to $\mathrm{Forb}_h(\mathcal{F})$, which contradicts the fact that $\mathrm{Age}\,(\mathfrak{B}) = \mathrm{Forb}_h(\mathcal{F})$. Thus indeed $\mathrm{Age}\,(\mathfrak{B}^2) \subseteq \mathrm{Age}\,(\mathfrak{B})$ and $\mathrm{CSS}^{=}(\mathcal{F})$ is convex.

Since there are, up to isomorphisms, only finitely many homomorphic images of each $\mathfrak{F} \in \mathcal{F}$ in $\mathfrak{B}$, there exists a finite set $\mathcal{F}'$ of finite structures such that $\mathrm{Age}\,(\mathfrak{B}) = \mathrm{Forb}_e(\mathcal{F}')$, which means that $\mathfrak{B}$ is finitely bounded. Note that $\mathrm{CSS}^{=}(\mathcal{F})$ is also finitely bounded: we can simply expand the universal sentence ϕ defining $\mathrm{Age}\,(\mathfrak{B})$ by an additional conjunct that ensures the the binary relation symbol $R_{=}$ in the signature of $\mathrm{CSS}^{=}(\mathcal{F})$, which should be interpreted as the equality predicate, indeed is interpreted in this way; e.g., we can append $\forall x, y. (R(x, y) \Leftrightarrow x = y)$ to ϕ.

Since $\mathrm{CSS}^{=}(\mathcal{F})$ is convex and finitely bounded, its p-admissibility follows by Theorem 6. $\square$

This proposition actually provides us with infinitely many examples of countable p-admissible concrete domains, which all yield a different extension of $\mathcal{EL}$: the so-called Henson digraphs [24] (see Example 2 below). The usefulness of these concrete domains for defining interesting concepts is, however, unclear.

Example 2. A directed graph is a *tournament* if every two distinct vertices in it are connected by exactly one directed edge. A *Henson digraph* is a homogeneous directed graph whose age equals $\mathrm{Forb}_e(\mathcal{N})$ for some set $\mathcal{N}$ consisting of finite tournaments plus the loop and the 2-cycle such that no member of $\mathcal{N}$ is embeddable into any other member of $\mathcal{N}$.

We claim that $\mathrm{Forb}_e(\mathcal{N}) = \mathrm{Forb}_h(\mathcal{N})$ holds for any such set $\mathcal{N}$. The inclusion $\mathrm{Forb}_h(\mathcal{N}) \subseteq \mathrm{Forb}_e(\mathcal{N})$ is true simply because every embedding is a homomorphism. To show the other inclusion, suppose that $\mathfrak{A} \in \mathrm{Forb}_e(\mathcal{N})$. The loop clearly does not homomorphically map to $\mathfrak{A}$ because every homomorphism from the loop to $\mathfrak{A}$ is an embedding. Since the loop does not homomorphically map to $\mathfrak{A}$, every homomorphism from the 2-cycle to $\mathfrak{A}$ is an embedding. Thus, the 2-cycle does not homomorphically map to $\mathfrak{A}$. Since the loop and the 2-cycle do not homomorphically map to $\mathfrak{A}$, every homomorphism from a tournament to $\mathfrak{A}$ is an embedding. Thus, $\mathfrak{A}$ does not admit any homomorphic image of a structure from $\mathcal{N}$. We conclude that $\mathrm{Forb}_e(\mathcal{N}) \subseteq \mathrm{Forb}_h(\mathcal{N})$.

For every selection $\mathcal{N}$ of finitely many tournaments that do not embed into each other, the set $\mathcal{N}$ consists of connected structures since tournaments as well as the loop and the 2-cycle are connected. Moreover, if $\mathcal{N}_1, \mathcal{N}_2$ are two distinct such sets, then $\mathrm{Forb}_h(\mathcal{N}_1) \neq \mathrm{Forb}_h(\mathcal{N}_2)$ [35]. Since there are infinitely many such families $\mathcal{N}$, Theorem 7 yields infinitely many non-isomorphic p-admissible and finitely bounded concrete domains that have different ages. Consequently, the ages of these structures are defined by conjunctions of Horn implications that are not equivalent. This implies that, in the extension of $\mathcal{EL}$ with these concrete domains, different subsumptions hold.

To make this more precise, assume that $\forall \bar{x}. (\phi \Rightarrow \psi)$ is a Horn implication that is satisfied by all elements of $\mathrm{Forb}_h(\mathcal{N}_1)$, but for which there is an element $\mathfrak{G}$ of $\mathrm{Forb}_h(\mathcal{N}_1)$ that does not satisfy it. We can easily turn the conjunction of atoms ϕ and the atom ψ into concepts C_ϕ and C_ψ of the DLs $\mathcal{EL}[\mathrm{CSS}^{=}(\mathcal{N}_1)]$ and $\mathcal{EL}[\mathrm{CSS}^{=}(\mathcal{N}_2)]$ by viewing the variables in $\bar{x}$ as features and replacing the conjunct operators $\wedge$ in ϕ by DL conjunction $\sqcap$. If we additionally ensure that all these features are defined (using GCIs $\top \sqsubseteq =(x,x)$ for all x occurring in $\bar{x}$), then C_ϕ is subsumed by C_ψ w.r.t. these GCIs in $\mathcal{EL}[\mathrm{CSS}^{=}(\mathcal{N}_1)]$, but not in $\mathcal{EL}[\mathrm{CSS}^{=}(\mathcal{N}_2)]$.

A more general class of p-admissible structures can be obtained from connected MMSNP sentences. A *connected MMSNP* sentence Φ in a finite relational signature τ is of the form $\exists P_1, \ldots, P_n. \forall. \bar{x} \bigwedge_i \neg(\alpha_i \wedge \beta_i)$ where

- $P_1, \ldots, P_n$ are unary relation symbols not in τ,
- each α_i is a conjunction of τ-atoms,
- each β_i is a conjunction of atoms and/or negated atoms involving relation symbols from $\{P_1, \ldots, P_n\}$,
- the *canonical database* $\mathrm{DB}(\exists \bar{x}_i. \alpha_i)$ is connected for every i where $\bar{x}_i$ represents all free variables in α_i; this is the τ-structure whose domain consists of the quantified variables $\bar{x}_i$ and whose relations are specified by the quantifier-free part α_i.

Note that, for every family $\mathfrak{F}$ as in Theorem 7, the class $\mathrm{Age}\,(\mathrm{CSS}(\mathcal{F}))$ consists of all finite models of a particular MMSNP sentence of the form $\forall \bar{x}. \bigwedge_i \neg\alpha_i$ where each α_i encodes a single structure $\mathfrak{F} \in \mathcal{F}$ up to homomorphic equivalence using a conjunctive query. The following proposition can be viewed as a generalization of Theorem 7 to more complicated forbidden patterns involving existentially quantified unary predicates.

Theorem 8 (Theorem 7 in [11]). *For every connected MMSNP sentence Φ in a finite signature τ, there exists an ω-categorical τ-structure $\mathfrak{B}_\Phi$ that is a reduct of a finitely bounded homogeneous structure and such that* $\mathrm{Age}\,(\mathfrak{B}_\Phi)$ *consists of all finite models of Φ.*

Similarly as in the case of Theorem 7, this theorem can be used to produce p-admissible concrete domains. However, in contrast to Theorem 7, connected MMSNP is known to exhibit a complexity dichotomy between P and NP-complete [15]. It follows from Proposition 2 below that already within the class of reducts of finitely bounded homogeneous structures, p-admissibility does not only depend on the convexity requirement, in contrast to what one might expect when coming from Theorem 6.

Proposition 2. *Let Φ be a connected MMSNP sentence in a finite signature τ. Then the expansion of $\mathfrak{B}_\Phi$ by the equality predicate is convex, and it is p-admissible if, and only if, satisfiability of Φ in finite τ-structures can be tested in polynomial time.*

Proof. We start with convexity. Let $\mathfrak{A}$ be an arbitrary finite substructure of $\mathfrak{B}_\Phi$. Then $\mathfrak{A} \models \Phi$ and this is witnessed by some $P_1, \ldots, P_n \subseteq A$. Suppose that $\mathfrak{A}^2 \not\models \Phi$. For every $i \in [n]$, we set $P'_i \coloneqq P_i \times A$. Since $\mathfrak{A}^2 \not\models \Phi$, there exists a tuple $\bar{s}$ over A^2 such that $(\mathfrak{A}^2, P'_1, \ldots, P'_n) \models (\alpha_i \wedge \beta_i)(\bar{s})$ for some i. Let $\bar{r}$ be the tuple over A obtained from $\bar{s}$ by taking the projection of each entry in

$\bar{s}$ to the first coordinate. By the definition of the product of structures and by the definition of P_i', we have $(\mathfrak{A}, P_1, \ldots, P_n) \models (\alpha_i \wedge \beta_i)(\bar{r})$ which contradicts $\mathfrak{A} \models \Phi$. Thus $\mathfrak{A}^2 \models \Phi$, which implies $\mathrm{Age}(\mathfrak{B}_\Phi^2) \subseteq \mathrm{Age}(\mathfrak{B}_\Phi)$ and we trivially also get $\mathrm{Age}\big((\mathfrak{B}_\Phi, =)^2\big) \subseteq \mathrm{Age}\big((\mathfrak{B}_\Phi, =)\big)$. By Corollary 2, the expansion of $\mathfrak{B}_\Phi$ by the equality predicate is convex.

It remains to determine in which cases we can test validity of Horn implications in the expansion of $\mathfrak{B}_\Phi$ by the equality predicate in polynomial time. Let $\forall \bar{x}. (\phi \Rightarrow \psi)$ be an arbitrary Horn implication. Since $\mathrm{CSP}(\mathfrak{B}_\Phi) = \mathrm{Age}(\mathfrak{B}_\Phi)$ by the proof of Theorem 7 in [11], we have $\mathfrak{B}_\Phi \models \exists \bar{x}. (\phi \wedge \neg\psi)$ if and only if $\mathfrak{B}_\Phi \models \exists \bar{x}. \phi$ and ϕ does not contain ψ as a conjunct.[3] We can assume that ϕ contains no occurrence of the equality predicate; otherwise we remove them by repeated identification of variables. By a standard result in database theory, $\mathfrak{B}_\Phi \models \exists \bar{x}. \phi$ iff the canonical database $\mathrm{DB}(\exists \bar{x}. \phi)$ homomorphically maps to $\mathfrak{B}_\Phi$ [20]. We conclude that $\mathfrak{B}_\Phi \models \forall \bar{x} (\phi \Rightarrow \psi)$ iff ϕ contains ψ as a conjunct or $\mathrm{DB}(\exists \bar{x}. \phi(\bar{x})) \not\models \Phi$. This can be tested in polynomial time iff testing satisfiability of Φ in finite structures can be done in polynomial time. □

5.3 Convex Numerical Structures

We exhibit two new p-admissible concrete domain that are respectively based on the real and the rational numbers, and whose predicates are defined by linear equations. Let $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ be the relational structure over $\mathbb{R}$ that has, for every linear equation system $A\bar{x} = \bar{b}$ over $\mathbb{Q}$, a relation consisting of all its solutions in $\mathbb{R}$. We define $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ as the substructure of $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ on $\mathbb{Q}$. For example, using the matrix $A = (2\ 1\ {-1})$ and the vector $\bar{b} = (0)$ one obtains the ternary relation $\{(p, q, r) \in \mathbb{Q}^3 \mid 2p + q = r\}$ in $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$.

Theorem 9. *The relational structures $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ and $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ are p-admissible.*

Our proof of Theorem 9 uses the following simple observation about first-order definable relations. Let $\mathfrak{D}$ be a relational τ-structure. A relation $R \subseteq D^k$ is *defined* by a τ-formula ϕ in $\mathfrak{D}$ if it is of the form $R = \{\bar{t} \in D^k \mid \mathfrak{D} \models \phi(\bar{t})\}$. We say that R is *first-order definable* in $\mathfrak{D}$ if there exists a first-order formula that defines R in $\mathfrak{D}$.

Lemma 2. *Let $\mathfrak{D}$ be a structure for which there exists an isomorphism f from $\mathfrak{D}^2$ to $\mathfrak{D}$.*

1. *If R is a relation definable in $\mathfrak{D}$ using a conjunctive query, then f is an isomorphism from $(\mathfrak{D}, R)^2$ to $(\mathfrak{D}, R)$.*
2. *If R is the complement of a relation of $\mathfrak{D}$, then f is a homomorphism from $(\mathfrak{D}, R)^2$ to $(\mathfrak{D}, R)$.*

Proof. (1) By a standard result in model theory, f is also a homomorphism from $(\mathfrak{D}, R)^2$ to $(\mathfrak{D}, R)$ (see, e.g., Proposition 5.2.2 in [10]). Since f is bijective, it only remains to show that f is even a strong homomorphism from $(\mathfrak{D}, R)^2$ to $(\mathfrak{D}, R)$. Let $\phi(x_1, \ldots, x_k) \coloneqq \exists x_{k+1}, \ldots, x_\ell. \psi(x_1, \ldots, x_\ell)$ be the conjunctive query that defines R in $\mathfrak{D}$, where ψ is the quantifier-free part of ϕ. Let $\bar{r} \in R$ be an arbitrary tuple of the form $\bar{r} = f(\bar{r}_1, \bar{r}_2)$ for some $\bar{r}_1, \bar{r}_2 \in D^k$. Then there exists $\bar{s} \in D^{\ell-k}$ such that $\mathfrak{D} \models \psi(\bar{r}[1], \ldots, \bar{r}[k], \bar{s}[1], \ldots, \bar{s}[\ell - k])$. Since f is surjective, there exist $\bar{s}_1, \bar{s}_2 \in D^{\ell-k}$ such that $\bar{s} = f(\bar{s}_1, \bar{s}_2)$. Since f is a strong homomorphism from $\mathfrak{D}^2$ to $\mathfrak{D}$, we have $\mathfrak{D} \models \psi(\bar{r}_i[1], \ldots, \bar{r}_i[k], \bar{s}_i[1], \ldots, \bar{s}_i[\ell - k])$ for both $i \in \{1, 2\}$. This means that $\bar{r}_1, \bar{r}_2 \in R$, which confirms our claim.

(2) This is an immediate consequence of the fact that an isomorphism is a strong homomorphism, and thus does not only preserve the relations from the signature, but also the complements of these relations. □

Proof of Theorem 9: To prove this theorem for $\mathbb{R}$, we start with the well-known fact that $(\mathbb{R}, +, 0)^2$ and $(\mathbb{R}, +, 0)$ are isomorphic [31], and show that this property can be extended to

[3] Recall that, for a given structure $\mathfrak{B}$, $\mathrm{CSP}(\mathfrak{B})$ consists of the finite structures that can homomorphically be mapped to $\mathfrak{B}$.

$\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$. The first isomorphism exists because $(\mathbb{R},+,0)^2$ and $(\mathbb{R},+,0)$ are both vector spaces over $\mathbb{Q}$ whose dimensions are uncountably infinite and of the same cardinality. Thus every bijective map from a basis of $(\mathbb{R},+,0)^2$ to a basis of $(\mathbb{R},+,0)$ extends to an isomorphism. Now we simply choose any two bases of $(\mathbb{R},+,0)^2$ and $(\mathbb{R},+,0)$, respectively, such that the first basis contains $(1,1)$ and the second basis contains 1. Then we choose an arbitrary bijection from the first basis to the second basis that sends $(1,1)$ to 1. This bijection extends to an isomorphism $f\colon (\mathbb{R},+,0,1)^2 \to (\mathbb{R},+,0,1)$. It is easy to see that every relation of $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ can be defined in $(\mathbb{R},+,0,1)$ using a conjunctive query. By Item 1 of Lemma 2, f is an isomorphism from $\mathfrak{D}^2_{\mathbb{R},\mathrm{lin}}$ to $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$. By Corollary 2, this yields convexity.

Regarding tractability, recall that validity of Horn implications in $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ can be tested in polynomial time if the CSP for $\mathfrak{D}^{\neg}_{\mathbb{R},\mathrm{lin}}$ can be tested in polynomial time. By Item 2 of Lemma 2, f is a homomorphism from $(\mathfrak{D}^{\neg}_{\mathbb{R},\mathrm{lin}})^2$ to $\mathfrak{D}^{\neg}_{\mathbb{R},\mathrm{lin}}$. It follows from Corollary 5.10 in [13] that the CSP for $\mathfrak{D}^{\neg}_{\mathbb{R},\mathrm{lin}}$ is decidable in polynomial time. We conclude that $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ is p-admissible.

For $\mathbb{Q}$, we cannot employ the same argument since $(\mathbb{Q},+,0)^2$ does not even admit a strong homomorphism to $(\mathbb{Q},+,0)$. Instead, we use the well-known fact that the structures $(\mathbb{Q},+,0)$ and $(\mathbb{R},+,0)$ satisfy the same first-order-sentences [31] to show that convexity of $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ implies convexity of $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$. We claim that a stronger statement is true, namely, that $\mathrm{Th}(\mathbb{Q},+,0,1) = \mathrm{Th}(\mathbb{R},+,0,1)$. Let ϕ be an arbitrary first-order sentence in the signature of $(\mathbb{R},+,0,1)$. We obtain the formula $\psi(x)$ in the signature of $(\mathbb{R},+,0)$ by replacing the constant 1 in ϕ by a fresh free variable x, i.e., $(\mathbb{R},+,0,1) \models \phi$ iff $(\mathbb{R},+,0) \models \psi(1)$. For every $c \in \mathbb{R} \setminus \{0\}$, the map $x \mapsto cx$ is an automorphism of $(\mathbb{R},+,0)$ that sends 1 to c. Since $\{x \in \mathbb{R} \mid (\mathbb{R},+,0) \models \psi(x)\}$ has a first-order definition in $(\mathbb{R},+,0)$, it is preserved by all automorphisms of $(\mathbb{R},+,0)$ [26]. Now we distinguish the following two cases. If $(\mathbb{R},+,0) \models \psi(0)$, then $(\mathbb{R},+,0,1) \models \phi$ iff $(\mathbb{R},+,0) \models \exists x.\, \psi(x)$. Otherwise $(\mathbb{R},+,0,1) \models \phi$ iff $(\mathbb{R},+,0) \models \exists x.\, \big(\neg(x=0) \wedge \psi(x)\big)$. Using an analogous argument we have either $(\mathbb{Q},+,0,1) \models \phi$ iff $(\mathbb{Q},+,0) \models \exists x.\, \psi(x)$ in the case where $(\mathbb{Q},+,0) \models \psi(0)$, or $(\mathbb{Q},+,0,1) \models \phi$ iff $(\mathbb{Q},+,0) \models \exists x.\, \big(\neg(x=0) \wedge \psi(x)\big)$. Since ϕ was chosen arbitrarily and $\mathrm{Th}(\mathbb{Q},+,0) = \mathrm{Th}(\mathbb{R},+,0)$, we conclude that indeed $\mathrm{Th}(\mathbb{Q},+,0,1) = \mathrm{Th}(\mathbb{R},+,0,1)$.

Since the relations of $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ are definable in $(\mathbb{Q},+,0,1)$ using the same conjunctive queries as their counterparts in $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ and $\mathrm{Th}(\mathbb{Q},+,0,1) = \mathrm{Th}(\mathbb{R},+,0,1)$, we conclude that p-admissibility of $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ implies p-admissibility of $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$. In fact, a counterexample to convexity in $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ would then yield a counterexample to convexity in $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ since it depends on the validity status of certain first-order sentences in $(\mathbb{Q},+,0,1)$. Similarly, the CSPs in $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ and $\mathfrak{D}_{\mathbb{R},\mathrm{lin}}$ are determined by the validity status of certain first-order sentences in $(\mathbb{Q},+,0,1)$ and $(\mathbb{R},+,0,1)$, repectively □

It is tempting to claim that $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ is considerably more expressive than the p-admissible concrete domain $\mathfrak{D}_{\mathbb{Q},\mathrm{dist}}$ with domain $\mathbb{Q}$, unary predicates $=_p, >_p$, and binary predicates $+_p, =$ exhibited in [2]. However, formally speaking, this is not true since the relations $>_p$ cannot be expressed in $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$. In fact, adding such a relation to $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ would destroy convexity: $x+y>0$ implies $x>0 \vee y>0$, but neither $x>0$ nor $y>0$. This example also works the other way round, i.e., it shows that adding the ternary relation $R_+ = \{(x,y,z) \in \mathbb{Q}^3 \mid x+y=z\}$ of $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ to $\mathfrak{D}_{\mathbb{Q},\mathrm{dist}}$ destroys convexity. In particular, it shows that the concrete domain $(\mathbb{Q}, R_+, >_0)$ is not convex.

Theorem 10. *TBoxes of the DLs $\mathcal{EL}^{++}[\mathfrak{D}_{\mathbb{Q},\mathrm{dist}}]$ and $\mathcal{EL}^{++}[\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}]$ have incomparable expressive power, i.e., there is an $\mathcal{EL}^{++}[\mathfrak{D}_{\mathbb{Q},\mathrm{dist}}]$ TBox $\mathcal{T}_1$ that cannot be expressed by an $\mathcal{EL}^{++}[\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}]$ TBox and there is an $\mathcal{EL}^{++}[\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}]$ TBox $\mathcal{T}_2$ that cannot be expressed by an $\mathcal{EL}^{++}[\mathfrak{D}_{\mathbb{Q},\mathrm{dist}}]$ TBox.*

Proof. We define

$$\mathcal{T}_1 := \{A \sqsubseteq >_0(f), >_0(f) \sqsubseteq A\} \text{ and } \mathcal{T}_2 := \{A \sqsubseteq R_+(f_1,f_2,f_3), R_+(f_1,f_2,f_3) \sqsubseteq A\}.$$

Assume that there is an $\mathcal{EL}^{++}[\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}]$ TBox $\mathcal{T}_1'$ that is a conservative extension of $\mathcal{T}_1$, i.e., restricted to A, f, its models coincide with the ones of $\mathcal{T}_1$. Using $\mathcal{T}_1'$ (possibly in renamed variants) and the expressiveness of $\mathcal{EL}^{++}[\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}]$, we can the express TBoxes of $\mathcal{EL}[(\mathbb{Q}, R_+, >_0)]$ by polynomially large TBoxes of $\mathcal{EL}^{++}[\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}]$. However, by Theorem 1, reasoning in $\mathcal{EL}^{++}[\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}]$ is polynomial, whereas it is ExpTime-hard in $\mathcal{EL}[(\mathbb{Q}, R_+, >_0)]$, which yields a contradiction.

The assumption that there is an $\mathcal{EL}^{++}[\mathfrak{D}_{\mathbb{Q},\mathrm{dist}}]$ TBox $\mathcal{T}_2'$ that is a conservative extension of $\mathcal{T}_2$ can similarly be shown to lead to a contradiction. □

We expect, however, that $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ will turn out to be more useful than $\mathfrak{D}_{\mathbb{Q},\mathrm{dist}}$ in practice.

6 ω-Admissibility versus p-Admissibility

The notion of ω-admissibility was introduced in [34] as a condition on concrete domains $\mathfrak{D}$ that ensures that the subsumption problem in $\mathcal{ALC}(\mathfrak{D})$ w.r.t. TBoxes remains decidable. This is a rather complicated condition, but for our purposes it is sufficient to know that, according to [34], an ω-admissible concrete domain $\mathfrak{D}$ has finitely many binary relations, which are jointly exhaustive (i.e., their union yields $D \times D$) and pairwise disjoint (i.e., for two different relation symbols R_i, R_j we have $R_i^{\mathfrak{D}} \cap R_j^{\mathfrak{D}} = \emptyset$).

Only very simple structures can be at the same time jointly exhaustive, pairwise disjoint, and guarded convex.

Proposition 3. *Let $\mathfrak{D}$ be a structure with a finite binary relational signature. Then the following are equivalent:*

1. *$\mathfrak{D}$ is guarded convex and its relations are jointly exhaustive and pairwise disjoint.*
2. *There exists a partition $V_1, \dots, V_m$ of D such that the non-empty relations of $\mathfrak{D}$ are exactly the ones of the form $V_j \times V_k$ for every $(j,k) \in [m]^2$.*

Proof. "$1 \Rightarrow 2$": Let $R_1, \dots, R_\ell$ be an enumeration of those symbols from τ that are interpreted in $\mathfrak{D}$ as a non-empty relation. For every $i \in [\ell]$, we have $\mathfrak{D} \models \forall x, y.\, \big(R_i(x,y) \Rightarrow (\bigvee_{j\in[\ell]} R_j(x,x)) \wedge (\bigvee_{k\in[\ell]} R_k(y,y))\big)$ because the relations of $\mathfrak{D}$ are jointly exhaustive. Using the distributive law and the guarded convexity of $\mathfrak{D}$ we conclude that, for every $i \in [\ell]$, there exists a pair $(j,k) \in [\ell]^2$ such that $\mathfrak{D} \models \forall x, y.\, \big(R_i(x,y) \Rightarrow (R_j(x,x) \wedge R_k(y,y))\big)$. Since the relations of $\mathfrak{D}$ are pairwise disjoint and each R_i is non-empty, there can only be one such pair (j,k) for every $i \in [\ell]$. Also, for every such pair (j,k) corresponding to a fixed $i \in [\ell]$, we have $\mathfrak{D} \models \forall x, y.\, \big((R_j(x,x) \wedge R_k(y,y)) \Rightarrow (R_1(x,y) \vee \dots \vee R_\ell(x,y))\big)$ because the relations of $\mathfrak{D}$ are jointly exhaustive. Using the guarded convexity of $\mathfrak{D}$ we conclude that there exists an $i' \in [\ell]$ such that $\mathfrak{D} \models \forall x, y.\, \big((R_j(x,x) \wedge R_k(y,y)) \Rightarrow R_{i'}(x,y)\big)$. Since the relations of $\mathfrak{D}$ are pairwise disjoint, the index i' must be the original i we started with. In sum, we have schown that

$$\begin{array}{l}\text{for every } i \in [\ell], \text{ there exists precisely one pair } (j,k) \in [\ell]^2 \text{ such that} \\ \mathfrak{D} \models \forall x, y.\, \big(R_i(x,y) \Leftrightarrow (R_j(x,x) \wedge R_k(y,y))\big).\end{array} \quad (*)$$

For a given $i \in [\ell]$, we distinguish the following two cases:

1. If there exists $x \in D$ such that $(x,x) \in R_i$, then $(*)$ implies $i = j = k$ and $R_i = V_i^2$ where $V_i := \{x \in D \mid (x,x) \in R_i\}$.
2. If there exists no $x \in D$ such that $(x,x) \in R_i$, then $(*)$ implies the existence of $j, k \in [\ell]$ such that i, j, k are all pairwise distinct and $R_i = V_j \times V_k$.

Without loss of generality, $R_1, \dots, R_m$ are the relations of the first kind, and $R_{m+1}, \dots, R_\ell$ are the relations of the second kind. Since the relations of $\mathfrak{D}$ are jointly exhaustive, $V_1, \dots, V_m$ form a partition of D.

"$2 \Rightarrow 1$": Clearly, the relations of $\mathfrak{D}$ are jointly exhaustive and pairwise disjoint. We use Theorem 2 to show that $\mathfrak{D}$ is guarded convex. Thus, let $\mathfrak{A}$ be an arbitrary guarded structure in $\mathrm{Age}(\mathfrak{D}^2)$. Observe that no pair $(x,y) \in D^2$ such that $x \in V_j$ and $y \in V_k$ for $j \neq k$ can be an component of a tuple from a relation of $\mathfrak{D}^2$. This is a direct consequence of our assumptions about the relations of $\mathfrak{D}$ and the definition of the product of structures. Thus, since $\mathfrak{A}$ is guarded and embeds into $\mathfrak{D}^2$, for every $(x, y) \in A$, we have $x, y \in V_i$ for some $i \in [m]$. It is easy to see that, in this particular case, the projection map $(x, y) \mapsto x$ is a strong homomorphism from $\mathfrak{A}$ to $\mathfrak{D}$. Since $\mathfrak{A}$ was chosen arbitrarily, it follows from Theorem 2 that $\mathfrak{D}$ is guarded convex. $\square$

In the presence of equality, only trivial structures can be jointly exhaustive, pairwise disjoint, and guarded convex (which in the presence of equality is the same as convex).

Proposition 4. *Let $\mathfrak{D}$ be a structure with a finite binary relational signature τ that includes equality. If $\mathfrak{D}$ is convex, jointly exhaustive, and pairwise disjoint, then its domain D satisfies $|D| \leq 1$.*

Proof. If $|D| = 0$, then we are done. Thus, assume that $|D| \geq 1$ and let $\tau = \{R_1, \dots, R_\ell\}$. Then $\mathfrak{D} \models \forall x, y. (x = x \wedge y = y \Rightarrow R_1(x, y) \vee \dots \vee R_\ell(x, y))$ since $\mathfrak{D}$ is jointly exhaustive. Convexity implies that there is an $i \in [\ell]$ such that

$$\mathfrak{D} \models \forall x, y. R_i(x, y). \tag{1}$$

Since the equality predicate is non-empty, pairwise disjointness implies that R_i must be equality. But then (1) yields $|D| = 1$. $\square$

This proposition shows that there are no non-trivial concrete domains with equality that are at the same time p-admissible and ω-admissible. Nevertheless, by combining the results of Section 5.2 with Corollary 2 in [6], we obtain non-trivial p-admissible concrete domains with equality for which subsumption in $\mathcal{ALC}(\mathfrak{D})$ is decidable.

Corollary 4. *Let $\mathfrak{D}$ be a finitely bounded convex structure with equality that is a reduct of a finitely bounded homogeneous structure. Then subsumption w.r.t. TBoxes is tractable in $\mathcal{EL}[\mathfrak{D}]$ and decidable in $\mathcal{ALC}(\mathfrak{D})$.*

The Henson digraphs already mentioned in Section 5.2 provide us with infinitely many examples of structures that satisfy the conditions of this corollary. In general, however, p-admissibility of $\mathfrak{D}$ does *not* guarantee decidability of subsumption in $\mathcal{ALC}(\mathfrak{D})$. For example, subsumption w.r.t. TBoxes is undecidable in $\mathcal{ALC}(\mathfrak{D}_{\mathbb{Q},\mathrm{dist}})$ and $\mathcal{ALC}(\mathfrak{D}_{\mathbb{Q},\mathrm{lin}})$ since this is already true for their common reduct $(\mathbb{Q}, +_1)$ [6].

Even for $\mathcal{EL}$, integrating a p-admissible concrete domain may cause undecidability if we allow for role paths of length 2. To show this, we consider the relational structure $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$ over $\mathbb{Q}^2$, which has, for every affine transformation $\mathbb{Q}^2 \to \mathbb{Q}^2 : \bar{x} \mapsto A\bar{x} + \bar{b}$, the binary relation $R_{A,\bar{b}} \coloneqq \{(\bar{x}, \bar{y}) \in (\mathbb{Q}^2)^2 \mid \bar{y} = A\bar{x} + \bar{b}\}$.

Theorem 11. *The relational structure $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$ is p-admissible, which implies that subsumption w.r.t. TBoxes is tractable in $\mathcal{EL}[\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}]$. However, subsumption w.r.t. TBoxes is undecidable in $\mathcal{EL}(\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}})$.*

We show p-admissibility of $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$ using the fact that $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ is p-admissible. Tractability of subsumption in $\mathcal{EL}[\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}]$ is then an immediate consequence of Theorem 1. Undecidability of subsumption w.r.t. TBoxes in $\mathcal{EL}(\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}})$ can be shown by a reduction from *2-Dimensional Affine Reachability*, which is undecidable by Corollary 4 in [9]. For this problem, one is given vectors $\bar{v}, \bar{w} \in \mathbb{Q}^2$ and a finite set S of affine transformations from $\mathbb{Q}^2$ to $\mathbb{Q}^2$. The question is then whether $\bar{w}$ can be obtained from $\bar{v}$ by repeated application of transformations from S. It is not hard to show that 2-Dimensional Affine Reachability can effectively be reduced to subsumption w.r.t. TBoxes in $\mathcal{EL}(\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}})$.

Lemma 3. *The relational structure $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$ is p-admissible.*

Proof. First, note that validity of Horn implications in $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$ can be reduced in linear time to satisfiability of conjunctive queries and Horn implications in $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$.

It thus remains to show that $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$ is convex. Let σ be a finite subset of the signature of $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$ and $\mathfrak{A}$ a finite substructure of $\mathfrak{D}^2_{\mathbb{Q}^2,\mathrm{aff}}$. We claim that the σ-reduct of $\mathfrak{A}$ embeds into the σ-reduct of $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$. For every binary relation $R_{M,\bar{v}}$ of $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$ we consider the 4-ary relation $\{(\bar{x}[1],\bar{x}[2],\bar{y}[1],\bar{y}[2]) \in \mathbb{Q}^4 \mid \bar{y} = M\bar{x}+\bar{v}\}$ of $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$, which we denote by $R'_{M,\bar{v}}$. Consider the substructure $\mathfrak{A}'$ of $\mathfrak{D}^2_{\mathbb{Q},\mathrm{lin}}$ on the set of all pairs $(x_1,x_2) \in \mathbb{Q}^2$ for which there exists $(\bar{x}_1,\bar{x}_2) \in A$ such that $(x_1,x_2) = (\bar{x}_1[1],\bar{x}_2[1])$ or $(x_1,x_2) = (\bar{x}_1[2],\bar{x}_2[2])$. Let σ' be a finite subset of the signature of $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ which contains a symbol for every relation $R'_{M,\bar{v}}$ for which there exists a symbol in σ which interprets as $R_{M,\bar{v}}$ in $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$. Since $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$ is convex, by Corollary 2 there exists an embedding f' from the σ'-reduct of $\mathfrak{A}'$ to the σ'-reduct of $\mathfrak{D}_{\mathbb{Q},\mathrm{lin}}$. Let f be the map from A to $\mathbb{Q}^2$ defined by $f(\bar{x}_1,\bar{x}_2) \coloneqq \big(f'(\bar{x}_1[1],\bar{x}_2[1]), f'(\bar{x}_1[2],\bar{x}_2[2])\big)$. It is well-defined by the definition of $\mathfrak{A}'$. Let $(\bar{x}_1,\bar{x}_2),(\bar{y}_1,\bar{y}_2) \in A$ be arbitrary. Then $(\bar{x}_1[1],\bar{x}_2[1]),(\bar{x}_1[2],\bar{x}_2[2]),(\bar{y}_1[1],\bar{y}_2[1]),(\bar{y}_1[2],\bar{y}_2[2]) \in A'$ and, for every affine transformation $\bar{x} \mapsto M\bar{x}+\bar{v}$, we have the following chain of equivalent statements.

$$\underbrace{\begin{pmatrix}\bar{x}_1\\ \bar{y}_1\end{pmatrix}\begin{pmatrix}\bar{x}_2\\ \bar{y}_2\end{pmatrix}}_{\in R_{M,\bar{v}}} \overset{\text{def.} R'_{M,\bar{v}}}{\iff} \underbrace{\begin{pmatrix}\bar{x}_1[1]\\ \bar{x}_1[2]\\ \bar{y}_1[1]\\ \bar{y}_1[2]\end{pmatrix}\begin{pmatrix}\bar{x}_2[1]\\ \bar{x}_2[2]\\ \bar{y}_2[1]\\ \bar{y}_2[2]\end{pmatrix}}_{\in R'_{M,\bar{v}}} \overset{f'\text{emb.}}{\iff} \underbrace{\begin{pmatrix}f'(\bar{x}_1[1],\bar{x}_2[1])\\ f'(\bar{x}_1[2],\bar{x}_2[2])\\ f'(\bar{y}_1[1],\bar{y}_2[1])\\ f'(\bar{y}_1[2],\bar{y}_2[2])\end{pmatrix}}_{\in R'_{M,\bar{v}}} \overset{\text{def.} f}{\iff} \underbrace{\begin{pmatrix}f(\bar{x}_1,\bar{x}_2)\\ f(\bar{y}_1,\bar{y}_2)\end{pmatrix}}_{\in R_{M,\bar{v}}}$$

This chain also holds for the equality predicate which can be written as $R_{E,\bar{z}}$ for E the 2×2 identity matrix and $\bar{z} = (0,0)$. It follows that f is an embedding from $\mathfrak{A}$ to $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$. By Corollary 2, $\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}}$ is convex. □

Lemma 4. *Subsumption w.r.t. TBoxes is undecidable in $\mathcal{EL}(\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}})$.*

Proof. We give a computable reduction of *2-dimensional Affine Reachability* to subsumption w.r.t. general TBoxes in $\mathcal{EL}(\mathfrak{D}_{\mathbb{Q}^2,\mathrm{aff}})$. For given vectors $\bar{v},\bar{w} \in \mathbb{Q}^2$ and affine transformations $S = \{\bar{x} \mapsto M_1\bar{x}+\bar{v}_1,\dots,\bar{x} \mapsto M_k\bar{x}+\bar{v}_k\}$, we define the TBox $\mathcal{T}$ as follows:

- for every $i \in [k]$, $\mathcal{T}$ contains the GCI $\top \sqsubseteq \exists f, gf.\, R_{M_i,\bar{v}_i}$,
- $\mathcal{T}$ contains the GCI $\exists g.\, L \sqsubseteq L$, and
- $\mathcal{T}$ contains the GCI $\exists f,f.\, R_{Z,\bar{w}} \sqsubseteq L$, where Z is the 2×2 matrix consisting of zeros only.

Note that $(\bar{x},\bar{x}) \in R_{Z,\bar{w}}$ if and only if $\bar{x} = \bar{w}$.

We claim that $\exists f,f.\, R_{Z,\bar{v}}$ is subsumed by L w.r.t. $\mathcal{T}$ iff $\bar{w}$ can be obtained from $\bar{v}$ through repeated application of affine transformations from S.

"$\Leftarrow$": Suppose that there is a sequence of applications of affine transformations from S to $\bar{v}$ that yields $\bar{w}$. Let $\mathcal{I}$ be a model of $\mathcal{T}$ and let a be an arbitrary element of $\Delta^{\mathcal{I}}$ with $a \in \exists f,f.\, R_{Z,\bar{v}}$, i.e., $f^{\mathcal{I}}(a) = \bar{v}$. Since $\mathcal{T}$ contains $\top \sqsubseteq \exists f, gf.\, R_{M_i,\bar{v}_i}$ for every $i \in [k]$ and $\bar{w}$ is reachable from $\bar{v}$ through repeated application of affine transformations from S, there exists a role path $a \to_{g^{\mathcal{I}}} \cdots \to_{g^{\mathcal{I}}} b$ to some individual b with $f^{\mathcal{I}}(b) = \bar{w}$. Since $\mathcal{T}$ contains the GCI $\exists f,f.\, R_{Z,\bar{w}} \sqsubseteq L$, we have that b is contained in $L^{\mathcal{I}}$. Since $\mathcal{T}$ contains the GCI $\exists g.\, L \sqsubseteq L$, we have that a is contained in $L^{\mathcal{I}}$. This finishes the proof of this direction since a was chosen arbitrarily.

"$\Rightarrow$": Suppose that $\exists f,f.\, R_{Z,\bar{v}}$ is subsumed by L w.r.t. $\mathcal{T}$. We construct the following interpretation $\mathcal{I}$:

- the domain of $\mathcal{I}$ is $\mathbb{Q}^2$,
- we define $f^{\mathcal{I}}$ as the identity map on $\mathbb{Q}^2$,
- we set $g^{\mathcal{I}} \coloneqq \{(\bar{x},\bar{y}) \in (\mathbb{Q}^2)^2 \mid \exists i \in [k] \text{ such that } \bar{y} = M_i\bar{x}+\bar{v}_i\}$,
- we set $L^{\mathcal{I}} \coloneqq \{\bar{w}\} \cup \{\bar{x} \in \mathbb{Q}^2 \mid \text{there exists a role path } \bar{x} \to_{g^{\mathcal{I}}} \cdots \to_{g^{\mathcal{I}}} \bar{w}\}$.

It is easy to check that $\mathcal{I}$ is a model of $\mathcal{T}$. Since $\bar{v} \in (\exists f,f.\, R_{Z,\bar{v}})^{\mathcal{I}}$ and this concept is subsumed by L w.r.t. $\mathcal{T}$, we have $\bar{v} \in L^{\mathcal{I}}$. It follows from the definition of $L^{\mathcal{I}}$ that $\bar{v} = \bar{w}$ or that $\bar{w}$ is must be reachable from $\bar{v}$ through a role path for the role g. In both cases, $\bar{w}$ is reachable from $\bar{v}$ by a sequence of applications of affine transformations from S. □

It is not clear whether undecidability also holds for $\mathcal{EL}(\mathfrak{D}_{\mathbb{Q},\mathrm{aff}})$, where $\mathfrak{D}_{\mathbb{Q},\mathrm{aff}}$ has domain $\mathbb{Q}$ and the relations are defined by 1-dimensional affine transformations. However, Proposition 2 in [30] at least yields an NP lower bound for the subsumption problem in $\mathcal{EL}(\mathfrak{D}_{\mathbb{Q},\mathrm{aff}})$.

7 Conclusion

The notion of p-admissible concrete domains was introduced in [2], where it was shown that integrating such concrete domains into the lightweight DL $\mathcal{EL}$ (and even the more expressive DL $\mathcal{EL}^{++}$) leaves the subsumption problem tractable. The paper [2] contains two examples of p-admissible concrete domains, and since then no new examples have been exhibited in the literature. This appears to be mainly due to the fact that it is not easy to show the convexity condition required by p-admissibility "by hand". The main contribution of the present paper is that it provides us with a useful algebraic tool for showing convexity: the square embedding condition. We have shown that this tool can indeed be used to exhibit new p-admissible concrete domains, such as countably infinite vector spaces over finite field, the countable homogeneous partial order, and numerical concrete domains over $\mathbb{R}$ and $\mathbb{Q}$ whose relations are defined by linear equations. The usefulness of these numerical concrete domains for defining concepts should be evident. For the other two we have indicated their potential usefulness by small examples.

We have also shown that, for finitely bounded structures, convexity is equivalent to p-admissibility, and that this corresponds to the finite substructures being definable by a conjunction of Horn implications. Interestingly, this provides us with infinitely many examples of countable p-admissible concrete domains, which all yield a different extension of $\mathcal{EL}$: the Henson digraphs. From a theoretical point of view, this is quite a feat, given that before only two p-admissible concrete domains were known. It is less clear whether these concrete domains will turn out to be useful for defining concepts in practice.

Finitely bounded structures also provide us with examples of structures $\mathfrak{D}$ that can be used both in the context of $\mathcal{EL}$ and $\mathcal{ALC}$, in the sense that subsumption is tractable in $\mathcal{EL}[\mathfrak{D}]$ and decidable in $\mathcal{ALC}(\mathfrak{D})$. Finally, we have shown that, when embedding p-admissible concrete domains into $\mathcal{EL}$, the restriction to paths of length 1 in concrete domain restrictions (indicated by the square brackets) is needed since there is a p-admissible concrete domains $\mathfrak{D}$ such that subsumption in $\mathcal{EL}(\mathfrak{D})$ is undecidable.

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