TECHNISCHE UNIVERSITÄT DRESDEN

Faculty of Computer Science Institute of Theoretical Computer Science, Chair of Automata Theory

JOINING IMPLICATIONS IN FORMAL CONTEXTS AND INDUCTIVE LEARNING IN A HORN DESCRIPTION LOGIC (EXTENDED VERSION)

LTCS-REPORT 19-02

Francesco Kriegel

francesco.kriegel@tu-dresden.de

11th March 2019

Joining Implications in Formal Contexts and Inductive Learning in a Horn Description Logic

Francesco Kriegel[0000−0003−0219−0330]

Institute of Theoretical Computer Science,
Technische Universität Dresden, Dresden, Germany
francesco.kriegel@tu-dresden.de

Abstract. A joining implication is a restricted form of an implication where it is explicitly specified which attributes may occur in the premise and in the conclusion, respectively. A technique for sound and complete axiomatization of joining implications valid in a given formal context is provided. In particular, a canonical base for the joining implications valid in a given formal context is proposed, which enjoys the property of being of minimal cardinality among all such bases. Background knowledge in form of a set of valid joining implications can be incorporated. Furthermore, an application to inductive learning in a Horn description logic is proposed, that is, a procedure for sound and complete axiomatization of $\mathsf{Horn}\text{-}\mathcal{M}$ concept inclusions from a given interpretation is developed. A complexity analysis shows that this procedure runs in deterministic exponential time.

Keywords: Inductive learning · Data mining · Axiomatization · Formal concept analysis · Joining implication · Horn description logic · Concept inclusion

1 Introduction

Formal Concept Analysis (abbrv. FCA) [11] is subfield of lattice theory that allows to analyze data-sets that can be represented as formal contexts. Put simply, such a formal context binds a set of objects to a set of attributes by specifying which objects have which attributes. There are two major techniques that can be applied in various ways for purposes of data mining, machine learning, knowledge management, knowledge visualization, etc. On the one hand, it is possible to describe the hierarchical structure of such a data-set in form of a formal concept lattice [11]. On the other hand, the theory of implications (dependencies between attributes) valid in a given formal context can be axiomatized in a sound and complete manner by the so-called canonical base [12], which furthermore contains a minimal number of implications w.r.t. the properties of soundness and completeness. So far, some variations of the canonical base have been developed, e.g., incorporation of valid background knowledge [30], constraining premises and conclusions in implications by some closure operator [4], and incorporation of arbitrary background knowledge [23], among others. The canonical base in its default form as well as its variations can be be computed by the algorithm *NextClosures* [19,23] in a highly parallel way such the necessary computation time is almost inverse linear proportional to the number of available CPU cores.

Description Logic (abbrv. DL) [3] belongs to the field of knowledge representation and reasoning. DL researchers have developed a large family of logic-based languages, so-called *description logics* (abbrv. DLs). These logics allow their users to explicitly represent knowledge as *ontologies*, which are finite sets of (human- and machine-readable) axioms, and provide them with automated inference services to derive implicit knowledge. The landscape of decidability and computational complexity of common reasoning tasks for various description logics has been explored in large parts: there is always a trade-off between expressibility and reasoning costs. It is therefore not surprising that DLs are nowadays applied in a large variety of domains [3]: agriculture, astronomy, biology, defense, education, energy management, geography, geoscience, medicine, oceanography, and oil and gas. Furthermore, the most notable success of DLs is that these constitute the logical underpinning of the *Web Ontology Language* (abbrv. OWL) [14] in the *Semantic Web*.

Within this document, we propose the new notion of so-called *joining implications* in FCA. More specifically, we assume that there are two distinct sets of attributes: the first one containing the attributes that may occur in premises of implications, while conclusions must only contain attributes from the second set. A canonical base for the joining implications valid in a given formal context is developed and it is proven that it has minimal cardinality among all such bases. Then, an application to inductive learning in a *Horn description logic* [25] is provided. Roughly speaking, such a Horn DL is obtained from some DL by disallowing any disjunctions. Reasoning procedures can then work deterministically, i.e., *reasoning by case* is not required [15]. Hornness is not a new notion: Horn clauses in first-order logic are disjunctions of an arbitrary number of negated atomic formulae and at most one non-negated atomic formula. It is easy to see that such Horn clauses have an implicative character, since $\neg\phi_1 \vee \ldots \vee \neg\phi_n \vee \psi$ is equivalent to $\phi_1 \wedge \ldots \phi_n \to \psi$. A *logic program* is a set of Horn clauses, and a *Datalog programm* is a function-free logic program [8]. All commonly known Horn description logics can be translated into Datalog—more specifically, each $\mathsf{Horn}\text{-}\mathcal{DL}$ TBox $\mathcal{T}$ can be translated into some Datalog program $\mathcal{D}$ such that, for each simple ABox $\mathcal{A}$, the ontology $\mathcal{T} \cup \mathcal{A}$ is satisfiable if, and only if, the Datalog programm $\mathcal{D} \cup \mathcal{A}$ is satisfiable. For deeper insights please consider [13,16,25]. The most important advantage of Horn fragments is that these often have a significantly lower computational complexity. Using the canonical base of joining implications, we show how the $\mathsf{Horn}\text{-}\mathcal{M}$ concept inclusions valid in a given interpretation can be axiomatized. This continues a line of research that combines FCA and DL for the sake of inductive learning, cf. [5,6,10,20,21,22,28] just to name a few.

2 Joining Implications in Formal Contexts

Throughout this section, assume that $\mathbb{K} \coloneqq (G, M, I)$ is some *formal context*, that is, G is a set of *objects*, M is a set of *attributes*, and $I \subseteq G \times M$ is an *incidence relation*. If $(g, m) \in I$, then we say that g *has* m. It is well-known that the two mappings $\cdot^I \colon \wp(G) \to \wp(M)$ and $\cdot^I \colon \wp(M) \to \wp(G)$ defined below constitute a *Galois connection*, cf. [11].

$$A^I \coloneqq \{\, m \in M \mid (g, m) \in I \text{ for each } g \in A \,\} \quad \text{for any } A \subseteq G$$
$$B^I \coloneqq \{\, g \in G \mid (g, m) \in I \text{ for each } m \in M \,\} \quad \text{for any } B \subseteq M$$

In particular, this means that the following statements hold true for any sets $A, C \subseteq G$ and $B, D \subseteq M$.

1. $A \subseteq B^I$ if, and only if, $B \subseteq A^I$ if, and only if, $A \times B \subseteq I$
2. $A \subseteq A^{II}$
3. $A^I = A^{III}$
4. $A \subseteq C$ implies $C^I \subseteq A^I$
5. $B \subseteq B^{II}$
6. $B^I = B^{III}$
7. $B \subseteq D$ implies $D^I \subseteq B^I$

An *implication* over M is a term $X \to Y$ where $X, Y \subseteq M$. It is valid in $\mathbb{K}$ if $X^I \subseteq Y^I$ is satisfied, i.e., if each object that has all attributes in X also has all attributes in Y, and we shall then write $\mathbb{K} \models X \to Y$. A *model* of $X \to Y$ is a set $U \subseteq M$ such that $X \subseteq U$ implies $Y \subseteq U$, denoted as $U \models X \to Y$. An implication set $\mathcal{L}$ *entails* an implication $X \to Y$ if any model of $\mathcal{L}$, i.e., any set that is a model of all implications in $\mathcal{L}$, is also a model of $X \to Y$, and we denote this by $\mathcal{L} \models X \to Y$.

We are now interested in a restricted form of implications. In particular, we restrict the sets of attributes that may occur in the *premise* X and in the *conclusion* Y, respectively, of every implication $X \to Y$. Thus, let further M_{p} be a set of *premise attributes* and let M_{c} be a set of *conclusion attributes* such that $M_{\mathsf{p}} \cup M_{\mathsf{c}} \subseteq M$ holds true. For each $\mathsf{x} \in \{\mathsf{p}, \mathsf{c}\}$, we define the subcontext $\mathbb{K}_{\mathsf{x}} := (G, M_{\mathsf{x}}, I_{\mathsf{x}})$ where $I_{\mathsf{x}} := I \cap (G \times M_{\mathsf{x}})$. Furthermore, we may also write X^{x} instead of $X^{I_{\mathsf{x}}}$ for subsets $X \subseteq G$ or $X \subseteq M_{\mathsf{x}}$. Please note that then each pair $(\cdot^{\mathsf{x}}, \cdot^{\mathsf{x}})$ is a Galois connection between $(\wp(G), \subseteq)$ and $(\wp(M_{\mathsf{x}}), \subseteq)$, that is, similar statements like above are valid.

Definition 1. *A* joining implication *from* M_{p} *to* M_{c}*, or simply* pc-implication*, is an expression* $X \to Y$ *where* $X \subseteq M_{\mathsf{p}}$ *and* $Y \subseteq M_{\mathsf{c}}$*. It is valid in* $\mathbb{K}$*, written* $\mathbb{K} \models X \to Y$*, if* $X^{\mathsf{p}} \subseteq Y^{\mathsf{c}}$ *holds true.*

$\mathbb{K}_{\mathsf{illnesses}}$	Abrupt Onset	Fever	Aches	Chills	Fatigue	Sneezing	Cough	Stuffy Nose	Sore Throat	Headache	Cold	Flu
Bob	·	×	×	·	×	×	×	×	×	×	×	·
Alice	·	·	·	·	·	×	×	×	×	·	×	·
Tom	·	·	·	·	·	×	·	×	×	·	×	·
Julia	×	×	×	×	×	·	×	×	·	×	·	×
Keith	×	×	×	·	×	×	·	·	×	·	·	×
Wendy	×	×	×	×	×	×	×	·	·	×	·	×

M_{p} (Abrupt Onset – Headache) $\quad$ M_{c} (Cold, Flu)

Figure 1. The formal context $\mathbb{K}_{\mathsf{illnesses}}$.

Example. Consider the formal context $\mathbb{K}_{\mathsf{illnesses}}$ in Figure 1. It considers six persons as objects and their symptoms and illnesses as attributes. Furthermore, we regard the symptoms as premise attributes and the illnesses as conclusion attributes. Note that, in general, it is not required that the sets M_{p} and M_{c} form a partition of the attribute set. For other use cases both could overlap, one could be contained in the other, or their union could be a strict subset of the whole attribute set. The concept lattice is displayed in Figure 2.[1]

The expression $\{\mathsf{Cold},\ \mathsf{Cough}\} \to \{\mathsf{Chills}\}$ is no pc-implication, since the attribute Cold must not occur in a premise and, likewise, the attribute Chills must not occur in a conclusion. The expression $\{\mathsf{Sneezing},\ \mathsf{Cough},\ \mathsf{Stuffy\ Nose}\} \to \{\mathsf{Cold}\}$ is a well-formed joining implication and it is valid in $\mathbb{K}_{\mathsf{illnesses}}$, since $\{\mathsf{Sneezing},\ \mathsf{Cough},\ \mathsf{Stuffy\ Nose}\}^{\mathsf{p}} = \{\mathsf{Bob},\ \mathsf{Alice}\}$ is a subset of $\{\mathsf{Cold}\}^{\mathsf{c}} = \{\mathsf{Bob},\ \mathsf{Alice},\ \mathsf{Tom}\}$. Furthermore, the expression $\{\mathsf{Abrupt\ Onset}\} \to \{\mathsf{Cold}\}$ is a well-formed joining implication as well, but it is not valid in $\mathbb{K}_{\mathsf{illnesses}}$, as $\{\mathsf{Abrupt\ Onset}\}^{\mathsf{p}} = \{\mathsf{Julia},\ \mathsf{Keith},\ \mathsf{Wendy}\}$ is not a subset of $\{\mathsf{Cold}\}^{\mathsf{c}} = \{\mathsf{Bob},\ \mathsf{Alice},\ \mathsf{Tom}\}$.

In the following, we shall characterize the set of all joining implications valid in $\mathbb{K}$. Of course, the pc-implication set

$$\mathsf{Imp}_{\mathsf{pc}}(\mathbb{K}) \coloneqq \{\, X \to X^{\mathsf{pc}} \mid X \subseteq M_{\mathsf{p}} \,\}$$

contains only valid pc-implications and further entails any valid pc-implication, since $\mathbb{K} \models X \to Y$ is equivalent to $Y \subseteq X^{\mathsf{pc}}$ and so $\{X \to X^{\mathsf{pc}}\} \models X \to Y$.

Remark that a *closure operator* on M is some mapping $\phi\colon \wp(M) \to \wp(M)$ with the following properties for all subsets $X, Y \subseteq M$.

1. $X \subseteq \phi(X)$ *(extensive)*
2. $X \subseteq Y$ implies $\phi(X) \subseteq \phi(Y)$ *(monotonic)*
3. $\phi(\phi(X)) = \phi(X)$ *(idempotent)*

It is easy to verify that, for each Galois connection (f, g), the compositions $f \circ g$ and $g \circ f$ are closure operators. It is well-known that each implication set $\mathcal{L}$ induces a corresponding closure operator $\phi_{\mathcal{L}}$ such that the models of $\mathcal{L}$ are exactly the closures of $\phi_{\mathcal{L}}$, cf. [11,23]: for each $U \subseteq M$, the closure $\phi_{\mathcal{L}}(U)$ is the smallest superset of U such that $X \subseteq \phi_{\mathcal{L}}(U)$ implies $Y \subseteq \phi_{\mathcal{L}}(U)$ for any implication $X \to Y$ in $\mathcal{L}$. In particular, we can readily verify the following.

$$\begin{aligned} &\phi_{\mathcal{L}}(U) = U^{\mathcal{L}} \coloneqq \bigcup\{\, U^{\mathcal{L},n} \mid n \in \mathbb{N} \,\} \\ \text{where}\quad &V^{\mathcal{L},n+1} \coloneqq (V^{\mathcal{L},1})^{\mathcal{L},n} \\ \text{and}\quad &V^{\mathcal{L},1} \coloneqq V \cup \{\, Y \mid X \to Y \in \mathcal{L} \text{ and } X \subseteq V \,\} \quad \text{for each } V \subseteq M \end{aligned}$$

For the above joining implication set, we easily get that $\phi_{\mathbb{K}}^{\mathsf{pc}} \coloneqq \phi_{\mathsf{Imp}_{\mathsf{pc}}(\mathbb{K})}$ satisfies

$$\phi_{\mathbb{K}}^{\mathsf{pc}}(X) = X \cup (X \cap M_{\mathsf{p}})^{\mathsf{pc}}$$

for any $X \subseteq M$.

An implication $X \to Y$ is *valid* in a closure operator ϕ, written $\phi \models X \to Y$, if $Y \subseteq \phi(X)$ holds true, cf. [19]. Please note that this coincides with the notion of validity in a formal context $\mathbb{K}$ if we consider the closure operator $\phi_{\mathbb{K}}\colon X \mapsto X^{II}$ and, likewise, entailment by an implication set $\mathcal{L}$ is the same as validity in $\phi_{\mathcal{L}}$. Now consider an implication set $\mathcal{L}$. We say that $\mathcal{L}$ is *sound* for ϕ if $\phi \models \mathcal{L}$ holds true, that is, if $\phi \models X \to Y$ is satisfied for each implication $X \to Y \in \mathcal{L}$. Furthermore, $\mathcal{L}$ is *complete* for ϕ if, for any implication $X \to Y$, it holds true that $\phi \models X \to Y$ implies $\mathcal{L} \models X \to Y$.

[1] We have not introduced the notion of a concept lattice here, since it is not needed for our purposes; the interested reader is rather referred to [11].

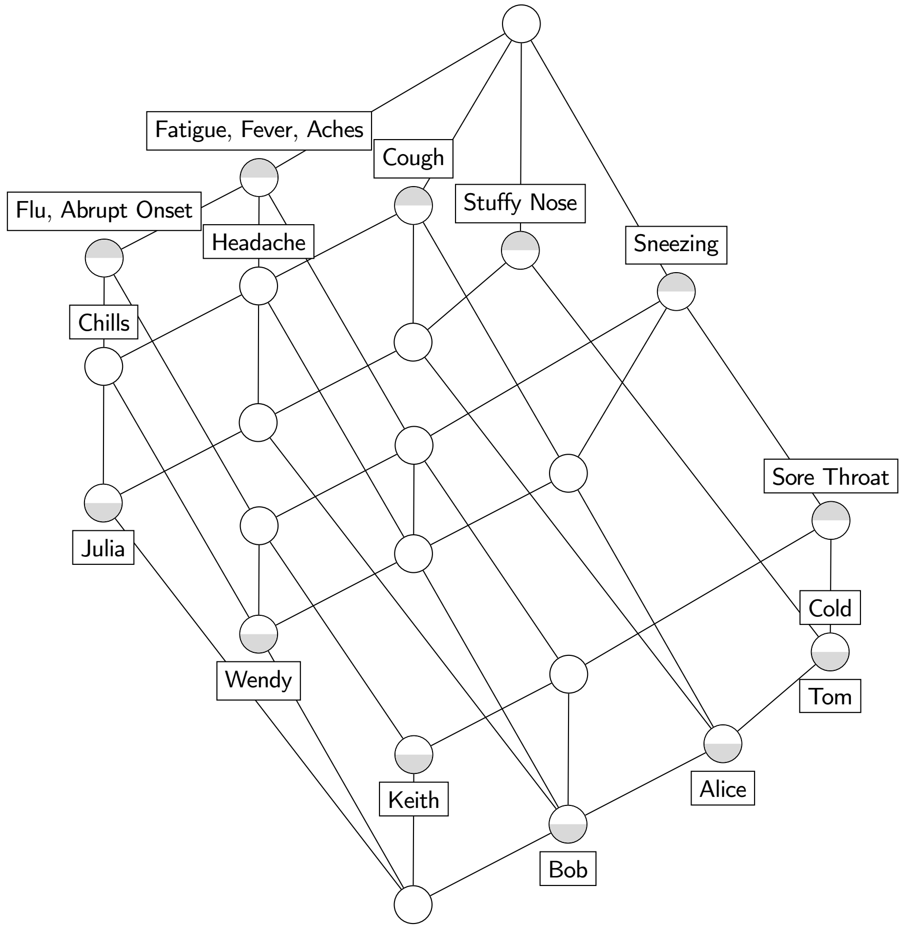

Figure 2. The concept lattice of $\mathbb{K}_{\mathsf{illnesses}}$.

Definition 2. *An implication set is* join-sound *or* pc-sound *if it is sound for* $\phi_{\mathbb{K}}^{\mathsf{pc}}$*, and it is* join-complete *or* pc-complete *if it is complete for* $\phi_{\mathbb{K}}^{\mathsf{pc}}$*. Fix some* pc*-sound implication set* $\mathcal{S}$*. A* pc*-implication set is called* joining implication base *or* pc-implication base *relative to* $\mathcal{S}$ *if it is* pc*-sound and its union with* $\mathcal{S}$ *is* pc*-complete.*

Obviously, the above $\mathsf{Imp}_{\mathsf{pc}}(\mathbb{K})$ is a joining implication base relative to $\emptyset$.

Further fix some implication set $\mathcal{S}$ as well as a closure operator ϕ such that $\phi \models \mathcal{S}$. Now remark that a *pseudo-closure* of ϕ relative to $\mathcal{S}$ is a set $P \subseteq M$ such that $P \neq \phi(P)$ and $P \models \mathcal{S}$ (i.e., $P = \phi_{\mathcal{S}}(P)$) hold true and $Q \subsetneq P$ implies $\phi(Q) \subseteq P$ for each pseudo-closure Q of ϕ relative to $\mathcal{S}$. We shall denote the set of all pseudo-closures of ϕ relative to $\mathcal{S}$ as $\mathsf{PsClo}(\phi, \mathcal{S})$. Then, the *canonical implication base* of ϕ relative to $\mathcal{S}$ is defined as $\mathsf{Can}(\phi, \mathcal{S}) := \{ P \to \phi(P) \mid P \in \mathsf{PsClo}(\phi, \mathcal{S}) \}$, and it is sound for ϕ and is further complete for ϕ relative to $\mathcal{S}$, i.e., $\phi \models X \to Y$ if, and only if, $\mathsf{Can}(\phi, \mathcal{S}) \cup \mathcal{S} \models X \to Y$

for each implication $X \to Y$, cf. [12,19,30]. It is easy to see that we can replace each implication $P \to \phi(P)$ by $P \to \phi(P) \setminus P$ to get an equivalent implication set.

Our aim for the sequel of this section is to find a *canonical* representation of the valid joining implications of some formal context, i.e., we shall provide a joining implication base that has *minimal* cardinality among all joining implication bases. For this purpose, we consider the canonical implication base of the above closure operator $\phi_{\mathbb{K}}^{\mathsf{pc}}$ and show how we can modify it to get a *canonical joining implication base*. We start with showing that we can rewrite any join-sound and join-complete implication set into a joining implication base in a certain normal form. For the remainder of this section, fix some arbitrary join-sound joining implication set $\mathcal{S}$ that is used as background knowledge.

Lemma 3. *Fix some join-sound implication set $\mathcal{L}$ over M. Further assume that $\mathcal{L} \cup \mathcal{S}$ is join-complete, and define the following set of joining implications.*

$$\mathcal{L}_{\mathsf{pc}} \coloneqq \{\, X \cap M_{\mathsf{p}} \to (X \cap M_{\mathsf{p}})^{\mathsf{pc}} \mid X \to Y \in \mathcal{L} \,\}$$

Then, $\mathcal{L}_{\mathsf{pc}}$ is a joining implication base relative to $\mathcal{S}$.

Proof. Since $\mathcal{L}_{\mathsf{pc}} \subseteq \mathsf{Imp}_{\mathsf{pc}}(\mathbb{K})$ obviously holds true, we know that $\mathcal{L}_{\mathsf{pc}}$ is join-sound. For join-completeness we show that $\mathcal{L}_{\mathsf{pc}} \cup \mathcal{S} \models \mathsf{Imp}_{\mathsf{pc}}(\mathbb{K})$. Thus, consider some $Z \subseteq M_{\mathsf{p}}$. As $\mathcal{L} \cup \mathcal{S}$ is join-complete, it must hold true that $\mathcal{L} \cup \mathcal{S} \models Z \to Z^{\mathsf{pc}}$, that is, there are implications $X_1 \to Y_1, \ldots, X_n \to Y_n$ in $\mathcal{L} \cup \mathcal{S}$ such that the following statements are satisfied.

$$\begin{aligned}
X_1 &\subseteq Z \\
X_2 &\subseteq Z \cup Y_1 \\
X_3 &\subseteq Z \cup Y_1 \cup Y_2 \\
&\vdots \\
X_n &\subseteq Z \cup Y_1 \cup Y_2 \cup \cdots \cup Y_{n-1} \\
Z^{\mathsf{pc}} &\subseteq Z \cup Y_1 \cup Y_2 \cup \cdots \cup Y_{n-1} \cup Y_n
\end{aligned}$$

Let $L \coloneqq \{\, k \mid k \in \{1, \ldots, n\} \text{ and } X_k \to Y_k \in \mathcal{L} \setminus \mathcal{S} \,\}$ and $S \coloneqq \{1, \ldots, n\} \setminus L$. Since $\mathcal{L}$ is join-sound, we have $Y_k \subseteq X_k \cup (X_k \cap M_{\mathsf{p}})^{\mathsf{pc}}$ for each index $k \in L$. Define $X_{n+1} \coloneqq Z^{\mathsf{pc}}$. An induction on $k \in \{1, \ldots, n+1\}$ shows the following.

$$\begin{aligned}
X_k \subseteq Z &\cup \bigcup \{\, Y_i \mid i \in \{1, \ldots, k-1\} \cap S \,\} \\
&\cup \bigcup \{\, (X_i \cap M_{\mathsf{p}})^{\mathsf{pc}} \mid i \in \{1, \ldots, k-1\} \cap L \,\}
\end{aligned}$$

Of course, $X_k \cap M_{\mathsf{p}} \subseteq X_k$ is satisfied for any index $k \in L$. We conclude that $\{\, X_k \to Y_k \mid k \in S \,\} \cup \{\, X_k \cap M_{\mathsf{p}} \to (X_k \cap M_{\mathsf{p}})^{\mathsf{pc}} \mid k \in L \,\}$ entails $Z \to Z^{\mathsf{pc}}$ and we are done. □

The transformation from Lemma 3 can now immediately be applied to the canonical implication base of the closure operator $\phi_{\mathbb{K}}^{\mathsf{pc}}$ to obtain a joining implication base, which we call *canonical*. This is due to fact that, by definition, $\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}})$ is both pc-sound and pc-complete.

Proposition 4. *The following is a joining implication base relative to $\mathcal{S}$ and is called* canonical joining implication base *or* canonical pc-implication base *of* $\mathbb{K}$ *relative to* $\mathcal{S}$.

$$\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S}) \coloneqq \{\, P \cap M_{\mathsf{p}} \to (P \cap M_{\mathsf{p}})^{\mathsf{pc}} \mid P \in \mathsf{PsClo}(\phi_{\mathbb{K}}^{\mathsf{pc}}, \mathcal{S}) \,\}$$

Proof. Remark that $\phi_{\mathbb{K}}^{\mathsf{pc}}(P) = P \cup (P \cap M_{\mathsf{p}})^{\mathsf{pc}}$ holds true and, consequently, the canonical implication base for $\phi_{\mathbb{K}}^{\mathsf{pc}}$ relative to $\mathcal{S}$ evaluates to

$$\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}}, \mathcal{S}) = \{\, P \to (P \cap M_{\mathsf{p}})^{\mathsf{pc}} \mid P \in \mathsf{PsClo}(\phi_{\mathbb{K}}^{\mathsf{pc}}, \mathcal{S}) \,\}.$$

We already know that $\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}}, \mathcal{S})$ is join-sound and its union with $\mathcal{S}$ is join-complete. Since $(\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}}, \mathcal{S}))_{\mathsf{pc}} = \mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})$ holds true, an application of Lemma 3 shows that $\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})$ is indeed a joining implication base relative to $\mathcal{S}$. □

Example. We continue with investigating our exemplary formal context $\mathbb{K}_{\mathsf{illnesses}}$. In order to compute the canonical joining implication base of it (relative to $\emptyset$), we first need to construct the canonical base of the closure operator $\phi_{\mathbb{K}_{\mathsf{illnesses}}}^{\mathsf{pc}}$.[2]

$$\mathsf{Can}(\phi_{\mathbb{K}_{\mathsf{illnesses}}}^{\mathsf{pc}}, \emptyset) = \left\{ \begin{array}{r} \{\mathsf{Headache},\ \mathsf{Sore\ Throat}\} \to \{\mathsf{Cold}\} \\ \{\mathsf{Abrupt\ Onset}\} \to \{\mathsf{Flu}\} \\ \{\mathsf{Sore\ Throat},\ \mathsf{Stuffy\ Nose}\} \to \{\mathsf{Cold}\} \\ \{\mathsf{Flu},\ \mathsf{Sore\ Throat},\ \mathsf{Chills}\} \to \{\mathsf{Cold}\} \\ \{\mathsf{Stuffy\ Nose},\ \mathsf{Sneezing}\} \to \{\mathsf{Cold}\} \\ \{\mathsf{Chills}\} \to \{\mathsf{Flu}\} \\ \{\mathsf{Sore\ Throat},\ \mathsf{Cough}\} \to \{\mathsf{Cold}\} \end{array} \right\}$$

Now applying the transformation from Lemma 3 yields the following set of joining implications, which is the canonical joining implication base. In particular, only the forth implication is altered.

$$\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}_{\mathsf{illnesses}}, \emptyset) = \left\{ \begin{array}{l} \qquad \{\mathsf{Headache},\ \mathsf{Sore\ Throat}\} \to \{\mathsf{Cold}\} \\ \qquad\qquad \{\mathsf{Abrupt\ Onset}\} \to \{\mathsf{Flu}\} \\ \{\mathsf{Sore\ Throat},\ \mathsf{Stuffy\ Nose}\} \to \{\mathsf{Cold}\} \\ \qquad \{\mathsf{Sore\ Throat},\ \mathsf{Chills}\} \to \{\mathsf{Flu},\ \mathsf{Cold}\} \\ \quad \{\mathsf{Stuffy\ Nose},\ \mathsf{Sneezing}\} \to \{\mathsf{Cold}\} \\ \qquad\qquad\qquad \{\mathsf{Chills}\} \to \{\mathsf{Flu}\} \\ \qquad \{\mathsf{Sore\ Throat},\ \mathsf{Cough}\} \to \{\mathsf{Cold}\} \end{array} \right\}$$

The canonical base of $\mathbb{K}_{\mathsf{illnesses}}$, which coincides with the canonical base of the induced closure operator $\phi_{\mathbb{K}_{\mathsf{illnesses}}}$, is as follows. Note that it is sound and complete for *all* implications valid in $\mathbb{K}_{\mathsf{illnesses}}$, i.e., no constraints on premises and conclusions are imposed.

$\mathsf{Can}(\mathbb{K}_{\mathsf{illnesses}}, \emptyset) =$

[2] The result has not been obtained by hand, but instead the implementation of the algorithm *NextClosures* [19] in *ConceptExplorer FX* [18] has been utilized. Thus, no intermediate computation steps are provided.

$$\left\{\begin{aligned}
\{\mathsf{Fever}\} &\to \{\mathsf{Fatigue},\ \mathsf{Aches}\}\\
\{\mathsf{Sore\ Throat}\} &\to \{\mathsf{Sneezing}\}\\
\{\mathsf{Chills}\} &\to \left\{\begin{matrix}\mathsf{Headache},\ \mathsf{Flu},\ \mathsf{Fatigue},\ \mathsf{Cough},\\ \mathsf{Fever},\ \mathsf{Aches},\ \mathsf{Abrupt\ Onset}\end{matrix}\right\}\\
\{\mathsf{Cold}\} &\to \{\mathsf{Sore\ Throat},\ \mathsf{Stuffy\ Nose},\ \mathsf{Sneezing}\}\\
\{\mathsf{Headache}\} &\to \{\mathsf{Fatigue},\ \mathsf{Cough},\ \mathsf{Fever},\ \mathsf{Aches}\}\\
\left\{\begin{matrix}\mathsf{Headache},\ \mathsf{Flu},\ \mathsf{Fatigue},\ \mathsf{Cough},\\ \mathsf{Fever},\ \mathsf{Aches},\ \mathsf{Abrupt\ Onset}\end{matrix}\right\} &\to \{\mathsf{Chills}\}\\
\{\mathsf{Aches}\} &\to \{\mathsf{Fatigue},\ \mathsf{Fever}\}\\
\{\mathsf{Stuffy\ Nose},\ \mathsf{Sneezing}\} &\to \{\mathsf{Sore\ Throat},\ \mathsf{Cold}\}\\
\{\mathsf{Fatigue}\} &\to \{\mathsf{Fever},\ \mathsf{Aches}\}\\
\{\mathsf{Sore\ Throat},\ \mathsf{Sneezing},\ \mathsf{Cough}\} &\to \{\mathsf{Stuffy\ Nose},\ \mathsf{Cold}\}\\
\{\mathsf{Fatigue},\ \mathsf{Stuffy\ Nose},\ \mathsf{Fever},\ \mathsf{Aches}\} &\to \{\mathsf{Headache},\ \mathsf{Cough}\}\\
\{\mathsf{Fatigue},\ \mathsf{Cough},\ \mathsf{Fever},\ \mathsf{Aches}\} &\to \{\mathsf{Headache}\}\\
\{\mathsf{Abrupt\ Onset}\} &\to \{\mathsf{Flu},\ \mathsf{Fatigue},\ \mathsf{Fever},\ \mathsf{Aches}\}\\
\{\mathsf{Flu}\} &\to \{\mathsf{Fatigue},\ \mathsf{Fever},\ \mathsf{Aches},\ \mathsf{Abrupt\ Onset}\}
\end{aligned}\right\}$$

If we apply the transformation from Lemma 3 to $\mathsf{Can}(\mathbb{K}_{\mathsf{illnesses}}, \emptyset)$, then we obtain the following set of joining implications. Obviously, it is not complete, since it does not entail the valid joining implication $\{\mathsf{Headache},\ \mathsf{Sore\ Throat}\} \to \{\mathsf{Cold}\}$.

$$\left\{\begin{aligned}
\{\mathsf{Chills}\} &\to \{\mathsf{Flu}\}\\
\{\mathsf{Stuffy\ Nose},\ \mathsf{Sneezing}\} &\to \{\mathsf{Cold}\}\\
\{\mathsf{Sore\ Throat},\ \mathsf{Sneezing},\ \mathsf{Cough}\} &\to \{\mathsf{Cold}\}\\
\{\mathsf{Abrupt\ Onset}\} &\to \{\mathsf{Flu}\}
\end{aligned}\right\}$$

We close this section with two further important properties of the canonical joining implication base. On the one hand, we shall show that it has minimal cardinality among all joining implication bases or, more generally, even among all join-sound, join-complete implication bases. On the other hand, we investigate the computational complexity of actually computing the canonical joining implication base.

Proposition 5. *The canonical joining implication base* $\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})$ *has minimal cardinality among all implication sets that are join-sound and have a union with* $\mathcal{S}$ *that is join-complete.*

Proof. Consider some implication set $\mathcal{L}$ such that $\mathcal{L} \cup \mathcal{S}$ is join-sound and join-complete. According to Lemma 3, we can assume that—without loss of generality—$\mathcal{L} \subseteq \mathsf{Imp}_{\mathsf{pc}}(\mathbb{K})$ holds true. In particular, note that $|\mathcal{L}_{\mathsf{pc}}| \leq |\mathcal{L}|$ is always true.

Join-soundness and join-completeness of $\mathcal{L} \cup \mathcal{S}$ yield that $\mathcal{L} \cup \mathcal{S}$ and $\mathsf{Can}(\phi^{\mathsf{pc}}_{\mathbb{K}}, \mathcal{S}) \cup \mathcal{S}$ are equivalent. It is well-known [12,30] that $\mathsf{Can}(\phi^{\mathsf{pc}}_{\mathbb{K}}, \mathcal{S})$ has minimal cardinality among all implication bases for $\phi^{\mathsf{pc}}_{\mathbb{K}}$ relative to $\mathcal{S}$, and so it follows that $|\mathcal{L}| \geq |\mathsf{Can}(\phi^{\mathsf{pc}}_{\mathbb{K}}, \mathcal{S})|$.

Clearly, the choice $\mathcal{L} \coloneqq \mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})$ implies $|\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})| \geq |\mathsf{Can}(\phi^{\mathsf{pc}}_{\mathbb{K}}, \mathcal{S})|$. It is further apparent that $|\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})| \leq |\mathsf{Can}(\phi^{\mathsf{pc}}_{\mathbb{K}}, \mathcal{S})|$ holds true and we infer that, in particular, $\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})$ and $\mathsf{Can}(\phi^{\mathsf{pc}}_{\mathbb{K}}, \mathcal{S})$ must contain the same number of implications. □

The next proposition shows that computing the canonical joining implication base is not more expensive than computing the canonical implication base where no constraints on the premises and conclusions must be satisfied. It uses the fact that canonical implication bases of closure operators can be computed using the algorithm *NextClosures* [19].

Proposition 6. *The canonical joining implication base can be computed in exponential time, and there exist formal contexts for which the canonical joining implication base cannot be encoded in polynomial space.*

Proof. The canonical implication base of the closure operator $\phi_{\mathbb{K}}^{\mathsf{pc}}$ relative to some background implication set $\mathcal{S}$ can be computed in exponential time by means of the algorithm *NextClosures*, cf. [6,19], which is easy to verify. The transformation of $\mathsf{Can}(\phi_{\mathbb{K}}^{\mathsf{pc}}, \mathcal{S})$ into $\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})$ as described in Lemma 3 can be done in polynomial time.

Kuznetsov and Obiedkov have shown in [27, Theorem 4.1] that the number of implications in the canonical implication base $\mathsf{Can}(\mathbb{K})$ of a formal context $\mathbb{K} \coloneqq (G, M, I)$ can be exponential in $|G| \cdot |M|$. Clearly, if we let $\mathcal{S} \coloneqq \emptyset$ and set both M_{p} and M_{c} to M, then $\mathsf{Can}(\mathbb{K})$ and $\mathsf{Can}_{\mathsf{pc}}(\mathbb{K}, \mathcal{S})$ coincide. □

We have seen in the running example that the canonical pc-implication base can be used to characterize implications between symptoms and diagnoses/illnesses. A further applications is, for instance, formal contexts encoding observations between attributes satisfied *yesterday* and *today*, i.e., we could construct the canonical base of pc-implications and then use it as a forecast stating which combinations of attributes being satisfied *today* would imply which combinations of attributes being satisfied *tomorrow*. In general, we could think of the premise attributes as *observable attributes* and the conclusion attributes as *goal/decision attributes*. By constructing the canonical pc-implication base from some formal context in which the goal/decision attributes have been manually assessed, we would obtain a set of rules with which we could analyze new data sets for which only the observable attributes are specified.

3 The Description Logic Horn-$\mathcal{M}$

A *Horn description logic* [13,16,25] is some description logic that, basically, does not allow for any usage of disjunction. While Hornness decreases expressiveness, it often also significantly lowers the computational complexity of some common reasoning tasks, e.g., *instance checking* or *query answering*. These are, thus, of importance in practical applications where computation times and costs must not be too high.

In the sequel of this section, we introduce the description logic Horn-$\mathcal{M}$, which is the Horn variant of $\mathcal{M} \coloneqq \mathcal{ALQ}^{\geq}\mathcal{N}^{\leq}(\mathsf{Self})$ [20]. Restrictions are imposed on concept inclusions only and, generally speaking, premises must always be $\mathcal{EL}^{*} \coloneqq \mathcal{EL}^{\bot}(\mathsf{Self})$ concept descriptions while conclusions may be arbitrary $\mathcal{M}^{\leq 1} \coloneqq \mathcal{ALQ}^{\geq}\mathcal{N}^{\leq 1}(\mathsf{Self})$ concept descriptions, that is, $\mathcal{M}$ concept descriptions except that in unqualified smaller-than restrictions $\exists \leq n.\, r$ only the case $n = 1$ is allowed. More specifically, a Horn-$\mathcal{M}$ *concept inclusion* is an expression $C \sqsubseteq D$ where the *concept descriptions* C and D are built by means of the following grammar. Beforehand, fix some *signature* Σ, which is a disjoint union of a set Σ_{I} of *individual names*, a set Σ_{C} of *concept names*, and a set Σ_{R} of *role names*. In the below grammar, A can be replaced by an arbitrary concept name from Σ_{C} and, likewise, r can be replaced by an arbitrary role name from Σ_{R}.

$$C \coloneqq \bot \mid \top \mid A \mid C \sqcap C \mid \exists\, r.\, C \mid \exists\, r.\, \mathsf{Self}$$

$$D \coloneqq \bot \mid \top \mid A \mid \neg A \mid D \sqcap D \mid \exists {\geq} n.\, r.\, D \mid \exists {\leq} 1.\, r \mid \forall r.\, D \mid \exists r.\, \mathsf{Self}$$

As usual, we denote by $\mathcal{DL}(\Sigma)$ the set of all $\mathcal{DL}$ concept descriptions over Σ for each description logic $\mathcal{DL}$. The *role depth* $\mathsf{rd}(E)$ of a concept description E is the maximal number of nestings of restrictions within E.[3] We then further denote by $\mathcal{DL}_d(\Sigma)$ the set of all $\mathcal{DL}$ concept descriptions over Σ with a role depth not greater than d. Note that the above syntactic characterization follows easily from the results in [13,16,25]. A finite set of concept inclusions is called *terminological box* (abbrv. TBox).

As it has already been pointed out in [16], the following properties can be expressed in a sufficiently strong Horn DL, e.g., in $\mathsf{Horn}\text{-}\mathcal{M}$.

Inclusion of Simple Concepts. $A \sqsubseteq B$ states that each individual being A is also B.
Concept Disjointness. $A \sqcap B \sqsubseteq \bot$ states that there are no individuals that are both A and B.
Domain Restrictions. $\exists r.\, \top \sqsubseteq A$ states that each individual having an r-successor must be an A.
Range Restrictions. $\top \sqsubseteq \forall r.\, A$ states that each individual being an r-successor must be an A.
Functionality Restrictions. $\top \sqsubseteq \exists {\leq} 1.\, r$ states that each individual has at most one r-successor.
Participation Constraints. $A \sqsubseteq \exists r.\, B$ states that each individual that is an A has an r-successor that is a B.

A *concept assertion* is an expression $a \sqsubseteq E$ where a is an individual name from Σ_I and E is some concept description, and further a *role assertion* is an expression $(a, b) \sqsubseteq r$ where $a, b \in \Sigma_\mathsf{I}$ and r is some role name from Σ_R. A finite set of concept and role assertions is called *assertional box* (abbrv. ABox). The union of a terminological and an assertional box yields an *ontology*. We often call the assertional part of an ontology the *data* and the terminological part of an ontology the *schema*. If a question of the form $\mathcal{O} \models \alpha$? is to be decided, then we also call the axiom α the *query*.

An *interpretation* $\mathcal{I}$ is a pair $(\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consisting of a non-empty set $\Delta^\mathcal{I}$ of *objects*, called *domain*, and an *extension mapping* $\cdot^\mathcal{I}$ such that $a^\mathcal{I} \in \Delta^\mathcal{I}$ for $a \in \Sigma_\mathsf{I}$, $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$ for each $A \in \Sigma_\mathsf{C}$, and $r^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$ for each $r \in \Sigma_\mathsf{R}$. The extension mapping is then extended to all concept descriptions in the following recursive manner; the names of these concept descriptions are shown in the right column.

$$\bot^\mathcal{I} \coloneqq \emptyset \qquad \textit{(bottom concept description)}$$
$$\top^\mathcal{I} \coloneqq \Delta^\mathcal{I} \qquad \textit{(top concept description)}$$
$$(\neg A)^\mathcal{I} \coloneqq \Delta^\mathcal{I} \setminus A^\mathcal{I} \qquad \textit{(negated concept name)}$$
$$(E \sqcap F)^\mathcal{I} \coloneqq E^\mathcal{I} \cap F^\mathcal{I} \qquad \textit{(conjunction)}$$
$$(\exists r.\, E)^\mathcal{I} \coloneqq \{\, \delta \mid (\delta, \epsilon) \in r^\mathcal{I} \text{ and } \epsilon \in E^\mathcal{I} \text{ for some } \epsilon \,\} \qquad \textit{(existential restriction)}$$
$$(\exists {\geq} n.\, r.\, E)^\mathcal{I} \coloneqq \{\, \delta \mid |\{\, \epsilon \mid (\delta, \epsilon) \in r^\mathcal{I} \text{ and } \epsilon \in E^\mathcal{I} \,\}| \geq n \,\} \qquad \textit{(qualified at-least restr.)}$$
$$(\exists {\leq} 1.\, r)^\mathcal{I} \coloneqq \{\, \delta \mid |\{\, \epsilon \mid (\delta, \epsilon) \in r^\mathcal{I} \,\}| \leq 1 \,\} \qquad \textit{(local functionality restriction)}$$

[3] Formally, the *role depth* is recursively defined as follows: $\mathsf{rd}(\bot) \coloneqq \mathsf{rd}(\top) \coloneqq \mathsf{rd}(A) \coloneqq \mathsf{rd}(\neg A) \coloneqq 0$, and $\mathsf{rd}(E \sqcap F) \coloneqq \mathsf{rd}(E) \vee \mathsf{rd}(F)$, and $\mathsf{rd}(\exists r.\, E) \coloneqq \mathsf{rd}(\exists {\geq} n.\, r.\, E) \coloneqq \mathsf{rd}(\forall r.\, E) \coloneqq 1 + \mathsf{rd}(E)$, and $\mathsf{rd}(\exists {\leq} 1.\, r) \coloneqq \mathsf{rd}(\exists r.\, \mathsf{Self}) \coloneqq 1$.

$$(\exists\, r.\,\mathsf{Self})^{\mathcal{I}} := \{\, \delta \mid (\delta, \delta) \in r^{\mathcal{I}} \,\} \qquad \textit{(existential self-restriction)}$$
$$(\forall\, r.\, E)^{\mathcal{I}} := \{\, \delta \mid (\delta, \epsilon) \in r^{\mathcal{I}} \text{ implies } \epsilon \in E^{\mathcal{I}} \text{ for each } \epsilon \,\} \qquad \textit{(value restriction)}$$

Now a concept inclusion $C \sqsubseteq D$ is *valid* in $\mathcal{I}$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds true, written $\mathcal{I} \models C \sqsubseteq D$. A concept assertion $a \mathrel{\varepsilon} E$ is *valid* in $\mathcal{I}$ if $a^{\mathcal{I}} \in E^{\mathcal{I}}$ is satisfied, and we shall denote this as $\mathcal{I} \models a \mathrel{\varepsilon} E$. Likewise, a role assertion $(a, b) \mathrel{\varepsilon} r$ is *valid* in $\mathcal{I}$ if $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$ holds true, and we symbolize this as $\mathcal{I} \models (a, b) \mathrel{\varepsilon} r$. If $\mathcal{O}$ is an ontology, then $\mathcal{I}$ is a *model* of $\mathcal{O}$ if $\mathcal{I} \models \alpha$ holds true for each axiom $\alpha \in \mathcal{O}$, and we shall denote this as $\mathcal{I} \models \mathcal{O}$. Furthermore, an ontology $\mathcal{O}_1$ *entails* another ontology $\mathcal{O}_2$, written $\mathcal{O}_1 \models \mathcal{O}_2$, if each model of $\mathcal{O}_1$ is a model of $\mathcal{O}_2$ too. In case $\mathcal{O} \models \{\alpha\}$ for some single axiom α, we shall omit set parenthesis and simply write $\mathcal{O} \models \alpha$. Note that, if $x * y$ is an axiom and $\mathcal{Z}$ is either an interpretation or an ontology, then we sometimes write $x *_{\mathcal{Z}} y$ instead of $\mathcal{Z} \models x * y$.

There are several standard reasoning tasks as follows.

Knowledge Base Consistency. Given an ontology $\mathcal{O}$, is there a model of $\mathcal{O}$?

Concept Satisfiability. Given a concept description E and an ontology $\mathcal{O}$, is there a model of $\mathcal{O}$ in which E has a non-empty extension?

Concept Subsumption. Given two concept descriptions C and D and an ontology $\mathcal{O}$, does $\mathcal{O}$ entail $C \sqsubseteq D$?

Instance Checking. Given an individual a, a concept description E, and an ontology $\mathcal{O}$, does $\mathcal{O}$ entail $a \mathrel{\varepsilon} E$?

There are two approaches to determining the computational complexity of the above tasks.

Combined Complexity. This is the default. Necessary time and space for solving the reasoning problem is measured as a function in the size of the whole input. For instance, if $a \mathrel{\varepsilon}_{\mathcal{O}} E$ is to be decided, then time and space requirements are measured as a function of $||a \mathrel{\varepsilon} E|| + ||\mathcal{O}||$.

Data Complexity. Determining data complexity is more meaningful for practical purposes, as in most cases the size of the stored data easily outgrows the size of the schema and query. In particular, time and space needed for solving the reasoning problem is measured as a function in the size of the ABox only. If, e.g., $a \mathrel{\varepsilon}_{\mathcal{O}} E$ is to be decided where $\mathcal{O}$ is the union of an ABox $\mathcal{A}$ and some TBox $\mathcal{T}$, then necessary time and space is only measured as a function of $||\mathcal{A}||$.

So far, the computational complexity of reasoning in $\mathcal{M}$ and its sibling Horn-$\mathcal{M}$ has not been determined and, thus, we shall catch up on this here. Since for a large variety of description logics complexity results have been obtained, we can immediately find the following results for $\mathcal{M}^- := \mathcal{ALQ}^{\geq}\mathcal{N}^{\leq}$, the sublogic of $\mathcal{M}$ in which we disallow existential self-restrictions $\exists\, r.\,\mathsf{Self}$. Note that we always consider the case of a *general* TBox, i.e., where no restrictions are imposed on the concept inclusions (except those possibly implied by Hornness).

Concept subsumption in $\mathcal{M}^-$ is **EXP***-complete (combined complexity).* Since $\mathcal{M}^-$ is a sublogic of $\mathcal{SHIQ}$ and concept subsumption in $\mathcal{SHIQ}$ is in **EXP** [29,31], it follows that concept subsumption in $\mathcal{M}^-$ is in **EXP** as well. Furthermore, $\mathcal{FL}_0$ is a sublogic of $\mathcal{M}^-$ in which concept subsumption is **EXP**-hard [2]. We conclude that concept subsumption in $\mathcal{M}^-$ must be **EXP**-hard too.

Concept subsumption in Horn-$\mathcal{M}^-$ *is* **EXP***-complete (combined complexity).* Horn-$\mathcal{M}^-$ is a sublogic of Horn-$\mathcal{SHIQ}$ and for the latter concept subsumption is known to be in **EXP** [25]. Thus, concept subsumption in Horn-$\mathcal{M}^-$ is also in **EXP**. Since $\mathcal{ELF}$ is a sublogic of Horn-$\mathcal{M}^-$ in which concept subsumption is **EXP**-hard [2], we infer that the same problem in Horn-$\mathcal{M}^-$ must be **EXP**-hard too.

Instance checking in $\mathcal{M}^-$ *is* **co NP***-complete (data complexity).* Instance checking in $\mathcal{SHIQ}$ is in **co NP** (data complexity) [15] and since $\mathcal{M}^-$ is a sublogic of $\mathcal{SHIQ}$, it follows that instance checking in $\mathcal{M}^-$ is also in **co NP** (data complexity). Furthermore, $\mathcal{EL}^{kf}$ is a sublogic of $\mathcal{M}^-$ in which instance checking is **co NP**-hard (data complexity) [24], and this result immediately transfers to $\mathcal{M}^-$.

Instance checking in Horn-$\mathcal{M}^-$ *is* **P***-complete (data complexity).* As instance checking in Horn-$\mathcal{SHIQ}$ is in **P** (data complexity) [15] and Horn-$\mathcal{M}^-$ is a sublogic of Horn-$\mathcal{SHIQ}$, we conclude that the similar problem in Horn-$\mathcal{M}^-$ is in **P** (data complexity) as well. Furthermore, $\mathcal{EL}$ is a sublogic of Horn-$\mathcal{M}^-$ and instance checking in $\mathcal{EL}$ is **P**-hard (data complexity) [7]. Consequently, instance checking in Horn-$\mathcal{M}^-$ is **P**-hard as well.

We see that terminological reasoning in Horn-$\mathcal{M}^-$ is not cheaper than in $\mathcal{M}^-$, but that assertional reasoning with knowledge bases containing both a schema (TBox) and data (ABox) is considerably cheaper in Horn-$\mathcal{M}^-$ than in $\mathcal{M}^-$ if we only take into account the size of the ABox (data complexity), unless **P** = **NP**. It is obvious that the hardness results transfer from $\mathcal{M}^-$ to $\mathcal{M}$ and accordingly for the Horn variants. Furthermore, since $\mathcal{M}$ and Horn-$\mathcal{M}$ can each be seen as a sublogic of $\mu\mathcal{ALCQ}$ in which concept subsumption is **EXP**-complete [9,26], we can infer that concept subsumption in $\mathcal{M}$ as well as in Horn-$\mathcal{M}$ is **EXP**-complete (combined complexity) as well. Unfortunately, the author cannot provide sharp upper bounds for the data complexity of instance checking in $\mathcal{M}$ and Horn-$\mathcal{M}$. If one takes a closer look on the proofs in [16], one could get the impression that it might suffice to include the case $\pi_y(\exists\, R.\,\mathsf{Self}, X) \coloneqq R(X, X)$ for the translation of concept descriptions into first-order logic. While the author conjectures that this extended translation allows for obtaining the same complexity results, it is necessary to check whether all later steps in the proof indeed work as before.

Henceforth, it makes sense to use a Horn-$\mathcal{M}$ TBox as the schema for *ontology-based data access* (abbrv. OBDA) applications. This motivates the development of a procedure that can learn Horn-$\mathcal{M}$ concept inclusions from observations in form of an interpretation.

The next section makes use of the notion of a *model-based most specific concept description*, which we shall define now. Fix some description logic $\mathcal{DL}$, an interpretation $\mathcal{I}$, a subset $X \subseteq \Delta^{\mathcal{I}}$, as well as some role-depth bound $d \in \mathbb{N}$. The *model-based most specific concept description* (abbrv. MMSC) of X in $\mathcal{I}$ is then some $\mathcal{DL}$ concept description E that satisfies the following conditions.

1. $\mathsf{rd}(E) \leq d$
2. $X \subseteq E^{\mathcal{I}}$
3. $X \subseteq F^{\mathcal{I}}$ implies $E \sqsubseteq_{\emptyset} F$ for each $\mathcal{DL}$ concept description F satisfying $\mathsf{rd}(F) \leq d$.

Since MMSCs are unique up to equivalence, we shall denote these as $X^{\mathcal{I}_d^{\mathcal{DL}}}$. In [20] the author has shown how MMSCs can be computed in the description logic $\mathcal{M}$. For any sublogic of $\mathcal{M}$, the computation method can suitably be adapted by simply ignoring unsupported concept constructors.

It is easy to see that this MMSC mapping $\cdot^{\mathcal{I}_d^{\mathcal{DL}}} \colon \wp(\Delta^{\mathcal{I}}) \to \mathcal{DL}_d(\Sigma)$ is the *adjoint* of the extension mapping $\cdot^{\mathcal{I}} \colon \mathcal{DL}_d(\Sigma) \to \wp(\Delta^{\mathcal{I}})$, that is, the pair of both constitutes a galois connection just like it is the case for the pair of derivation operators $\cdot^{I}$ induced by a formal context. This implies that the following statements hold true, where X and Y are arbitrary subsets of the domain $\Delta^{\mathcal{I}}$, and E and F are any $\mathcal{DL}$ concept descriptions with a role depth of at most d.

1. $X \subseteq E^{\mathcal{I}}$ if, and only if, $X^{\mathcal{I}_d^{\mathcal{DL}}} \sqsubseteq_{\emptyset} E$
2. $X \subseteq X^{\mathcal{I}_d^{\mathcal{DL}}\mathcal{I}}$
3. $X^{\mathcal{I}_d^{\mathcal{DL}}} = X^{\mathcal{I}_d^{\mathcal{DL}}\mathcal{I}\mathcal{I}_d^{\mathcal{DL}}}$
4. $X \subseteq Y$ implies $X^{\mathcal{I}_d^{\mathcal{DL}}} \sqsubseteq_{\emptyset} Y^{\mathcal{I}_d^{\mathcal{DL}}}$
5. $E^{\mathcal{I}\mathcal{I}_d^{\mathcal{DL}}} \sqsubseteq_{\emptyset} E$
6. $E^{\mathcal{I}} \equiv_{\emptyset} E^{\mathcal{I}\mathcal{I}_d^{\mathcal{DL}}\mathcal{I}}$
7. $E \sqsubseteq_{\emptyset} F$ implies $E^{\mathcal{I}} \subseteq F^{\mathcal{I}}$

Compared to the FCA setting, we have replaced intent descriptions using sets of attributes by intent descriptions using $\mathcal{DL}$ concept descriptions.

4 Inductive Learning in Horn-$\mathcal{M}$

Now fix some finitely representable interpretation $\mathcal{I}$ over a signature Σ, and further let $d \in \mathbb{N}$ be a role-depth bound. Similarly to [6,10,20], we define the *induced formal context* $\mathbb{K}_{\mathcal{I},d} \coloneqq (\Delta^{\mathcal{I}}, M, I)$ where $M \coloneqq M_{\mathsf{p}} \cup M_{\mathsf{c}}$, the premise attribute set is defined by

$$M_{\mathsf{p}} \coloneqq \{\bot\} \cup \Sigma_{\mathsf{C}} \cup \{\, \exists\, r.\, \mathsf{Self} \mid r \in \Sigma_{\mathsf{R}} \,\} \cup \{\, \exists\, r.\, X^{\mathcal{I}_{d-1}^{\mathcal{EL}^*}} \mid r \in \Sigma_{\mathsf{R}} \text{ and } \emptyset \neq X \subseteq \Delta^{\mathcal{I}} \,\}$$

while the conclusion attribute set is given as

$$\begin{aligned} M_{\mathsf{c}} \coloneqq{}& \{\bot\} \cup \{\, A, \neg A \mid A \in \Sigma_{\mathsf{C}} \,\} \cup \{\, \exists\, r.\, \mathsf{Self},\ \exists \leq 1.\, r,\ \forall\, r.\, \bot \mid r \in \Sigma_{\mathsf{R}} \,\} \\ &\cup \left\{ \circlearrowleft r.\, X^{\mathcal{I}_{d-1}^{\mathcal{M}^{\leq 1}}} \;\middle|\; \begin{array}{l} \circlearrowleft \in \{\, \exists \geq n \mid 1 \leq n \leq |\Delta^{\mathcal{I}}| \,\} \cup \{\forall\}, \\ r \in \Sigma_{\mathsf{R}},\ \text{and}\ \emptyset \neq X \subseteq \Delta^{\mathcal{I}} \end{array} \right\}, \end{aligned}$$

and $(\delta, C) \in I$ if $\delta \in C^{\mathcal{I}}$. Our interest is to axiomatize the Horn-$\mathcal{M}$ concept inclusions valid in $\mathcal{I}$. Of course, it holds true that $\bigsqcap \mathbf{X} \sqsubseteq \bigsqcap \mathbf{Y}$ is a Horn-$\mathcal{M}$ concept inclusion for each subset $\mathbf{X} \subseteq M_{\mathsf{p}}$ and each subset $\mathbf{Y} \subseteq M_{\mathsf{c}}$. As in [6,10,20], such a concept inclusion is valid in $\mathcal{I}$ if, and only if, the joining implication $\mathbf{X} \to \mathbf{Y}$ is valid in the induced formal context $\mathbb{K}_{\mathcal{I},d}$. As we are only interested in axiomatizing these concept inclusions that are valid in $\mathcal{I}$ and are no tautologies, we define the following joining implication set that we shall use as background knowledge on the FCA side.

$$\begin{aligned} \mathcal{S} \coloneqq{}& \{\, \{C\} \to \{D\} \mid C \in M_{\mathsf{p}},\ D \in M_{\mathsf{c}},\ \text{and}\ C \sqsubseteq_{\emptyset} D \,\} \\ &\cup \{\, \{C, \exists\, r.\, \mathsf{Self}\} \to \{D\} \mid C \in M_{\mathsf{p}},\ r \in \Sigma_{\mathsf{R}},\ D \in M_{\mathsf{c}},\ \text{and}\ C \sqcap \exists\, r.\, \mathsf{Self} \sqsubseteq_{\emptyset} D \,\} \end{aligned}$$

We will see at the end of this section that the model-based most specific concept descriptions $X^{\mathcal{I}_d}$ can have an exponential size w.r.t. $|\Delta^{\mathcal{I}}|$ and d in $\mathcal{M}^{\leq 1}$. Since the problem of deciding subsumption in Horn-$\mathcal{M}$ is **EXP**-complete, we infer that a naïve approach of computing $\mathcal{S}$ needs double exponential time. However, a more sophisticated analysis yields that most concept inclusions cannot be valid. In particular, a concept description from M_{p} only contains concept names and existential (self-)restrictions and, thus, these can never be subsumed (w.r.t. $\emptyset$) by a concept description from M_{c} containing a negated concept name, a local functionality restriction, a qualified at-least restriction where $n > 1$, or a value restriction. Thus, we conclude from the characterization in [20, Section 8] that $\mathcal{S}$ does not contain any implication $\{C\} \to \{D\}$ or $\{C, \exists\, r.\,\mathsf{Self}\} \to \{D\}$ except for the trivial cases where $C = \bot$, $C = D$, or $D = \exists\, r.\,\mathsf{Self}$ (only for the second form), and it can hence be computed in single exponential time. Even in the case where the tautological TBox $\mathcal{S}$ is not that simple, e.g., for another description logic where subsumption is also **EXP**-complete and model-based most specific concept descriptions can have exponential sizes, we could also dispense with the expensive computation of $\mathcal{S}$, since the canonical base can then still be computed in single exponential time with the only drawback that it could contain tautologies.

In the remainder of this section, we show how the techniques from Section 2 can be applied to axiomatize Horn-$\mathcal{M}$ concept inclusions valid in $\mathcal{I}$. Note that the proofs are suitable adaptations of those for the $\mathcal{EL}^{\bot}$ case [6] or of those for the $\mathcal{M}$ case [20]. We first show that the TBox consisting of the concept inclusions $C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$ for all $\mathcal{EL}^*$ concept descriptions C with a role depth not exceeding d is sound and complete.

Lemma 7. *The following* Horn-$\mathcal{M}$ *TBox is sound and complete for the* Horn-$\mathcal{M}$ *concept inclusions valid in $\mathcal{I}$ and having role depths at most d.*

$$\mathcal{B}_0 \coloneqq \{\, C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \mid C \in \mathcal{EL}^*(\Sigma) \text{ and } \mathsf{rd}(C) \leq d \,\}$$

Proof. Fix some Horn-$\mathcal{M}$ concept inclusion $C \sqsubseteq D$ such that both C and D have role depths not exceeding d. Then, [20, Lemma 7.3] yields that the Horn-$\mathcal{M}$ concept inclusion $C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$ is valid in $\mathcal{I}$. If further $C \sqsubseteq D$ is valid in $\mathcal{I}$, then another immediate consequence of [20, Lemma 7.3] is that $\{C \sqsubseteq C^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}\}$ entails $C \sqsubseteq D$. ◻

As a further step, we prove by means of structural induction that also the TBox containing the concept inclusions $\bigsqcap \mathbf{C} \sqsubseteq (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$ where $\mathbf{C}$ is a subset of M_{p} is sound and complete.

Lemma 8. *The following* Horn-$\mathcal{M}$ *TBox is sound and complete for the* Horn-$\mathcal{M}$ *concept inclusions valid in $\mathcal{I}$ and having role depths at most d.*

$$\mathcal{B}_1 \coloneqq \{\, \bigsqcap \mathbf{C} \sqsubseteq (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \mid \mathbf{C} \subseteq M_{\mathsf{p}} \,\}$$

Proof. Soundness is obvious, and for completeness we show that $\mathcal{B}_1 \models \mathcal{B}_0$. We do this by structural induction on $C \in \mathcal{EL}^*(\Sigma)$. The base cases where $C \in \{\bot\} \cup \Sigma_{\mathsf{C}} \cup \{\, \exists\, r.\,\mathsf{Self} \mid r \in \Sigma_{\mathsf{R}} \,\}$ are obvious.

Case $C = D \sqcap E$. Two applications of the induction hypothesis yield that $\mathcal{B}_1$ entails $D \sqsubseteq D^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$ and $E \sqsubseteq E^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$. We infer that

$$D \sqcap E \sqsubseteq_{\mathcal{B}_1} D^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqcap E^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}.$$

Since $\mathcal{EL}^*$ is less expressive than $\mathcal{M}^{\leq 1}$, we can show by induction on the role-depth bound d and using the recursive characterizations in [20, Section 8] that $X^{\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqsubseteq_{\emptyset} X^{\mathcal{I}_d^{\mathcal{EL}^*}}$ holds true for each subset $X \subseteq \Delta^{\mathcal{I}}$. Consequently, it follows that

$$D^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqcap E^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqsubseteq_{\emptyset} D^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}} \sqcap E^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}}.$$

Now the latter concept description is expressible in terms of M_{p}, and so we infer that $\mathcal{B}_1$ must entail the concept inclusion

$$D^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}} \sqcap E^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}} \sqsubseteq (D^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}} \sqcap E^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}.$$

As $D^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}}$ is subsumed by D and $E^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}}$ is subsumed by E with respect to $\emptyset$, we conclude that

$$(D^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}} \sqcap E^{\mathcal{I}\mathcal{I}_d^{\mathcal{EL}^*}})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqsubseteq_\emptyset (D \sqcap E)^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$$

and we are done.

Case $C = \exists\, r.\, D$. Using the induction hypothesis, we get that $\mathcal{B}_1$ entails $\exists\, r.\, D \sqsubseteq \exists\, r.\, D^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$. It is further readily verified that

$$\exists\, r.\, D^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqsubseteq_\emptyset \exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{M}^{\leq 1}}} \sqsubseteq_\emptyset \exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}}$$

holds true. Since $\exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}}$ is an element of M_{p} (choose $X \coloneqq D^{\mathcal{I}}$), it must be the case that $\mathcal{B}_1$ entails $\exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}} \sqsubseteq (\exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$. Eventually, we know that $D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}} \sqsubseteq D$ is a tautology, and we conclude that $\exists\, r.\, D^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}} \sqsubseteq (\exists\, r.\, D)^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$ follows from $\mathcal{B}_1$. □

Furthermore, when computing the MMSC of a conjunction $\bigsqcap \mathbf{C}$ where $\mathbf{C} \subseteq M_{\mathsf{p}}$ we do not have to do this on the DL side, which is expensive, but it suffices to compute the result $\mathbf{C}^{\mathsf{pc}}$ on the FCA side by applying the derivation operators $\cdot^{\mathsf{p}}$ and $\cdot^{\mathsf{c}}$. The conjunction $\bigsqcap \mathbf{C}^{\mathsf{pc}}$ is then (equivalent to) the MMSC in the DL $\mathcal{M}^{\leq 1}$.

Lemma 9. *For any subset* $\mathbf{C} \subseteq M_{\mathsf{p}}$*, the following equivalence holds true.*

$$\bigsqcap \mathbf{C}^{\mathsf{pc}} \equiv_\emptyset (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$$

Proof. The equivalence follows by a suitable variation of [20, Statement 1 of Lemma 10.9]. However, we shall present the full proof in the following. In particular, we show the following subclaims.

1. $\mathbf{C}^{\mathsf{pc}} = \pi_{\mathsf{c}}((\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}})$ where the projection mapping $\pi_{\mathsf{c}} \colon \mathcal{M}^{\leq 1}(\Sigma) \to \wp(M_{\mathsf{c}})$ is defined by $\pi_{\mathsf{c}}(D) \coloneqq \{\, E \in M_{\mathsf{c}} \mid D \sqsubseteq_\emptyset E \,\}$.
2. The concept description $(\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}$ is expressible in terms of M_{c}.
3. If C is expressible in terms of M_{c}, then $C \equiv_\emptyset \bigsqcap \pi_{\mathsf{c}}(C)$ holds true.

Our main claim then follows easily:

$$\bigsqcap \mathbf{C}^{\mathsf{pc}} \stackrel{1}{\equiv}_\emptyset \bigsqcap \pi_{\mathsf{c}}((\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}) \stackrel{2,3}{\equiv}_\emptyset (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}.$$

Statement 3 has already been shown in [20, Statement 5 of Lemma 10.3], and Statement 2 has been shown in [20, Lemma 10.8]. Eventually, the proof of Statement 1 is similar to the one of [20, Statement 4 of Lemma 10.3], and is as follows.

$$\pi_{\mathsf{c}}((\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}) = \{\, D \in M_{\mathsf{c}} \mid (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}^{\leq 1}}} \sqsubseteq_\emptyset D \,\}$$

$$
\begin{aligned}
&= \{\, D \in M_{\mathsf{c}} \mid (\textstyle\bigsqcap \mathbf{C})^{\mathcal{I}} \subseteq D^{\mathcal{I}} \,\} \\
&= \{\, D \in M_{\mathsf{c}} \mid \mathbf{C}^{\mathsf{p}} \subseteq \{D\}^{\mathsf{c}} \,\} \\
&= \{\, D \in M_{\mathsf{c}} \mid D \in \mathbf{C}^{\mathsf{pc}} \,\} \\
&= \mathbf{C}^{\mathsf{pc}}
\end{aligned}
$$
□

The main result for inductive learning of Horn-$\mathcal{M}$ concept inclusions is as follows. It states that (the premises of) each pc-implication base of the induced context $\mathbb{K}_{\mathcal{I},d}$ give rise to a base of Horn-$\mathcal{M}$ concept inclusions for $\mathcal{I}$.

Proposition 10. *If $\mathcal{L}$ is a joining implication base for $\mathbb{K}_{\mathcal{I},d}$ relative to $\mathcal{S}$, then the following TBox is sound and complete for the* Horn-$\mathcal{M}$ *concept inclusions that are valid in $\mathcal{I}$ and have role depths not exceeding d.*

$$\mathcal{B}_2 := \{\, \textstyle\bigsqcap \mathbf{C} \sqsubseteq (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}\leq 1}} \mid \mathbf{C} \to \mathbf{D} \in \mathcal{L} \,\}$$

Proof. It suffices to prove that $\mathcal{B}_2$ entails $\mathcal{B}_1$, i.e., we show that $\mathcal{B}_2$ entails $\bigsqcap \mathbf{C} \sqsubseteq (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}\leq 1}}$ for each subset $\mathbf{C} \subseteq M_{\mathsf{p}}$. Thus, fix some model $\mathcal{J}$ of $\mathcal{B}_2$ and consider some subset $\mathbf{C} \subseteq M_{\mathsf{p}}$. We define the formal context $\mathbb{K}_{\mathcal{J}} := (\Delta^{\mathcal{J}}, M, J)$ where $(\delta, C) \in J$ if $\delta \in C^{\mathcal{J}}$.

1. We show that $\mathbb{K}_{\mathcal{J}} \models \mathcal{L}$. Fix some joining implication $\mathbf{C} \to \mathbf{D}$ contained in $\mathcal{L}$. We know that $\mathbf{C}^{J_{\mathsf{p}}} = \mathbf{C}^{J} = (\bigsqcap \mathbf{C})^{\mathcal{J}}$ holds true and, analogously, that $\mathbf{D}^{J_{\mathsf{c}}} = \mathbf{D}^{J} = (\bigsqcap \mathbf{D})^{\mathcal{J}}$ is satisfied. Since
$$\textstyle\bigsqcap \mathbf{C} \sqsubseteq_{\mathcal{B}_2} (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}\leq 1}} \equiv_{\emptyset} \bigsqcap \mathbf{C}^{I_{\mathsf{p}} I_{\mathsf{c}}} \sqsubseteq_{\emptyset} \bigsqcap \mathbf{D},$$
we infer that $(\bigsqcap \mathbf{C})^{\mathcal{J}} \subseteq (\bigsqcap \mathbf{D})^{\mathcal{J}}$. Consequently, we have that $\mathbf{C}^{J_{\mathsf{p}}} \subseteq \mathbf{D}^{J_{\mathsf{c}}}$, that is, the considered joining implication $\mathbf{C} \to \mathbf{D}$ is valid in $\mathbb{K}_{\mathcal{J}}$.
2. We show that $\mathbb{K}_{\mathcal{J}} \models \mathcal{S}$. Let $\{C_1, C_2\} \to \{D\} \in \mathcal{S}$, i.e., $C_1 \sqcap C_2 \sqsubseteq D$ is a tautology. In particular, we infer that $C_1^{\mathcal{J}} \cap C_2^{\mathcal{J}} \subseteq D^{\mathcal{J}}$, which implies $\{C_1\}^{J_{\mathsf{p}}} \cap \{C_2\}^{J_{\mathsf{p}}} \subseteq \{D\}^{J_{\mathsf{c}}}$, i.e., the implication $\{C_1, C_2\} \to \{D\}$ is valid in $\mathbb{K}_{\mathcal{J}}$ as well.
3. We finish the proof by demonstrating that $\bigsqcap \mathbf{C} \sqsubseteq (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}\leq 1}}$ is valid in $\mathcal{J}$. As $\mathcal{L} \cup \mathcal{S}$ is sound and complete for the joining implications valid in $\mathbb{K}_{\mathcal{I},d}$, we have that $\mathcal{L} \cup \mathcal{S} \models \mathbf{C} \to \mathbf{C}^{\mathsf{pc}}$, and so $\mathbf{C} \to \mathbf{C}^{\mathsf{pc}}$ must be valid in $\mathbb{K}_{\mathcal{J}}$ too. Clearly, this shows that the concept inclusion $\bigsqcap \mathbf{C} \sqsubseteq \bigsqcap \mathbf{C}^{\mathsf{pc}}$ is valid in $\mathcal{J}$. Using the equivalence $\bigsqcap \mathbf{C}^{\mathsf{pc}} \equiv_{\emptyset} (\bigsqcap \mathbf{C})^{\mathcal{I}\mathcal{I}_d^{\mathcal{M}\leq 1}}$ yields the claim. □

Instantiating the previous proposition with the canonical pc-implication base now yields the following corollary.

Corollary 11. *The following* Horn-$\mathcal{M}$ *TBox, called* canonical Horn-$\mathcal{M}$ concept inclusion base *for $\mathcal{I}$ and d, is sound and complete for the* Horn-$\mathcal{M}$ *concept inclusions that are valid in $\mathcal{I}$ and have role depths at most d.*

$$\mathsf{Can}_{\mathsf{Horn}\text{-}\mathcal{M}}(\mathcal{I}, d) := \{\, \textstyle\bigsqcap (P \cap M_{\mathsf{p}}) \sqsubseteq \bigsqcap (P \cap M_{\mathsf{p}})^{\mathsf{pc}} \mid P \in \mathsf{PsClo}(\phi_{\mathcal{I},d}^{\mathsf{Horn}\text{-}\mathcal{M}}, \mathcal{S}) \,\}$$

The closure operator $\phi_{\mathcal{I},d}^{\mathsf{Horn}\text{-}\mathcal{M}} \colon \wp(M) \to \wp(M)$ is defined by $X \mapsto X \cup (X \cap M_{\mathsf{p}})^{\mathsf{pc}}$.

In the sequel of this section, we investigate the computational complexity of computing the canonical Horn-$\mathcal{M}$ concept inclusion base. As it turns out, the complexity is the same as for computing the canonical pc-implication base—both can be obtained in exponential time. Afterwards, we investigate whether we can show that the canonical Horn-$\mathcal{M}$ concept inclusion base has minimal cardinality.

Proposition 12. *The canonical* Horn-$\mathcal{M}$ *concept inclusion base for a finitely representable interpretation $\mathcal{I}$ and role depth bound $d \geq 1$ can be computed in exponential time with respect to d and the cardinality of the domain $\Delta^{\mathcal{I}}$, and further there exist finitely representable interpretations $\mathcal{I}$ for which the canonical* Horn-$\mathcal{M}$ *concept inclusion base cannot be encoded in polynomial space.*

Proof. The proof is very similar to [21,22, Proof of Proposition 2] and heavily depends on an argument from Albano [1] too. However, we need to elaborate on the size of model-based most specific concept descriptions as well as on the complexity of computing these. The remaining argumentation is then the same.

We make use of the recursive formula in [20, Section 8]. Fix some finitely representable interpretation $\mathcal{I}$ such that $\Delta^{\mathcal{I}}$ contains n objects. We start with the $\mathcal{M}^{\leq 1}$ case. In particular, we inductively construct upper estimates[4] u_d such that $||X^{\mathcal{I}_d}|| \preceq u_d$ is satisfied for any subset $X \subseteq \Delta^{\mathcal{I}}$. For $d = 0$, $X^{\mathcal{I}_d}$ can only contain concept names and negations of concept names, i.e., we set $u_0 \coloneqq 2 \cdot |\Sigma_{\mathsf{C}}| \leq 2 \cdot |\Sigma|$. Furthermore, for computing $X^{\mathcal{I}_d}$ we only need to traverse through Σ_{C} and check, for each concept name A, whether $X \subseteq A^{\mathcal{I}}$ or $X \cap A^{\mathcal{I}} = \emptyset$ is satisfied; clearly, this needs only polynomial time w.r.t. n.

For a depth $d > 0$, $X^{\mathcal{I}_d}$ can contain concept names and negations thereof, one unqualified at-most restriction $\exists \leq n. r$ for each role name $r \in \Sigma$ as well as one existential self-restriction $\exists r. \mathsf{Self}$ for each role name $r \in \Sigma_{\mathsf{R}}$, and $X^{\mathcal{I}_d}$ can further contain qualified at-least restrictions $\exists \geq k. r. Y^{\mathcal{I}_{d-1}}$ for each $k \in \{1, \ldots, n\}$, $r \in \Sigma_{\mathsf{R}}$, and $Y \subseteq \Delta^{\mathcal{I}}$ as well as universal restrictions $\forall r. Y^{\mathcal{I}_{d-1}}$ for each role name $r \in \Sigma_{\mathsf{R}}$ and each subset $Y \subseteq \Delta^{\mathcal{I}}$, which yields an upper estimate of $u_d \coloneqq 2 \cdot |\Sigma_{\mathsf{C}}| + 2 \cdot |\Sigma_{\mathsf{R}}| + (n+1) \cdot |\Sigma_{\mathsf{R}}| \cdot 2^n \cdot u_{d-1}$. Consequently, we obtain the following estimate.

$$||X^{\mathcal{I}_d^{\mathcal{M}^{\leq 1}}}|| \preceq \sum_{k=0}^{d} ((n+1) \cdot |\Sigma_{\mathsf{R}}| \cdot 2^n)^k \cdot 2 \cdot |\Sigma|$$

Furthermore, finding all top-level conjuncts of the form A or $\neg A$ can be done in polynomial time w.r.t. n just like for $d = 0$. Determining all top-level conjuncts of the form $\exists \leq n. r$ or $\exists r. \mathsf{Self}$ can be achieved in polynomial time w.r.t. n as well. For the qualified at-least restrictions and the universal restrictions occurring in the top-level conjunction, the crucial task is to determine the minimal successor sets $\mathsf{Min}(\mathsf{Suc}(X, \circlearrowleft r))$: while for $\circlearrowleft = \forall$ this is polynomial, it can take exponential time w.r.t. n for $\circlearrowleft = \exists \geq k$. An induction on d shows that $X^{\mathcal{I}_d}$ can be computed in exponential time w.r.t. n and d.

In order to show that the above exponential bound is tight, we consider the following interpretation $\mathcal{I}_n$ over the signature Σ_n where $(\Sigma_n)_{\mathsf{C}} \coloneqq \{A_1, \ldots, A_n\}$ and $(\Sigma_n)_{\mathsf{R}} \coloneqq \{r\}$.

$$\Delta^{\mathcal{I}_n} \coloneqq \{\delta_1, \ldots, \delta_n\}$$

[4] Following Knuth [17], we write $f \preceq g$ for $f \in \mathcal{O}(g)$, that is, if $\exists c \in \mathbb{R}_+ \, \exists n_0 \in \mathbb{N} \, \forall n \geq n_0 \colon f(n) \leq c \cdot g(n)$, and then say that f is *asymptotically bounded above* by g.

$$\mathcal{I}_n : \begin{cases} A_i \mapsto \{\, \delta_j \mid j \in \{1,\ldots,n\} \text{ and } i \neq j \,\} & \text{for each } i \in \{1,\ldots,n\} \\ r \mapsto \{\, (\delta_i,\delta_j) \mid i,j \in \{1,\ldots,n\} \,\} \end{cases}$$

It is then easy to verify that, in $\mathcal{I}_n$, the model-based most specific concept description of $\{\delta_1,\ldots,\delta_n\}$ for the role-depth bound 1 in $\mathcal{M}^{\leq 1}$ contains the mutually $\sqsubseteq_\emptyset$-incomparable top-level conjuncts $\exists \geq k.\, r.\, \bigsqcap \mathbf{A}$ for each $k \in \{2,\ldots,n\}$ and each $\mathbf{A} \in \binom{(\Sigma_n)_\mathsf{C}}{n-k}$.

Similarly for the easier $\mathcal{EL}^*$ case, we can prove by induction on d that we can compute $X^{\mathcal{I}_d}$ in exponential time w.r.t. n and d, and that we get the following estimate.

$$||X^{\mathcal{I}_d^{\mathcal{EL}^*}}|| \preceq \sum_{k=0}^{d} (|\Sigma_\mathsf{R}| \cdot 2^n)^k \cdot |\Sigma|$$

We conclude that model-based most specific concept descriptions always have a size that is at most exponential in n and d and can be computed in exponential time w.r.t. n and d in both description logics $\mathcal{EL}^*$ and $\mathcal{M}^{\leq 1}$. □

Note that in order to save space for representing the model-based most specific concept descriptions, we could also represent them in the form $X^{\mathcal{I}}\restriction_d$ where $X^{\mathcal{I}}$ is the model-based most specific concept description without any bound on the role depth and $E\restriction_d$ denotes the *unraveling* of some concept description E (formulated in a DL with greatest fixed-point semantics) up to role depth d. In general, these unbounded MMSCs $X^{\mathcal{I}}$ only exist in extensions of the considered DL with greatest fixed-point semantics. The advantage is that then the size of $X^{\mathcal{I}}\restriction_d$ is exponential only in $|\Delta^{\mathcal{I}}|$ but not in d.

The author conjectures that, for each finitely representable interpretation $\mathcal{I}$, the canonical Horn-$\mathcal{M}$ concept inclusion base $\mathsf{Can}_{\mathsf{Horn}\text{-}\mathcal{M}}(\mathcal{I}, d)$ has *minimal cardinality* among all Horn-$\mathcal{M}$ concept inclusion bases for $\mathcal{I}$ and d. However, it is not immediately possible to suitably adapt the minimality proof for the $\mathcal{EL}$ case described in [6,10], since not all notions from $\mathcal{EL}$ are available in more expressive description logics. The crucial point is that we need the validity of the following claim, which resembles [10, Lemma 5.16] or [6, Lemma A.9], respectively, for our case of Horn-$\mathcal{M}$.

Claim. Fix some Horn-$\mathcal{M}$ TBox $\mathcal{T} \cup \{C \sqsubseteq D\}$ in which all occurring concept descriptions have role depths not exceeding d. Further assume that $\mathcal{I}$ is a finitely representable model of $\mathcal{T}$ such that, for each subconcept $\exists\, r.\, X$ of C, the filler X is (equivalent to) some model-based most specific concept description of $\mathcal{I}$ in the description logic $\mathcal{EL}^*$; more specifically, we assume that $Y \equiv Y^{\mathcal{I}\mathcal{I}_{d-1}^{\mathcal{EL}^*}}$ is satisfied for each $\exists\, r.\, Y \in \mathsf{Conj}(C)$. If $C \not\sqsubseteq_\emptyset D$ and $C \sqsubseteq_\mathcal{T} D$, then $C \sqsubseteq_\emptyset E$ and $C \not\sqsubseteq_\emptyset F$ holds true for some concept inclusion $E \sqsubseteq F$ contained in $\mathcal{T}$.

However, the author has just developed a computation procedure for so-called most specific consequences, cf. [21,22, Definition 3], in a description logic that is more expressive than $\mathcal{EL}^\bot$. A proof of the above claim can then be obtained as a by-product. This will be subject of a future publication.

5 Conclusion

In Formal Concept Analysis, a restricted form of implications has been introduced: so-called *joining implications*. From the underlying attribute set M two subsets M_p

and M_{c} are declared, and then only those implications are considered in which the premises only contain attributes from M_{p} and in which the conclusions only contain attributes from M_{c}. A canonical base for the joining implications valid in some given formal context has been devised, and it has been proven that it has minimal cardinality and can be computed in deterministic exponential time.

The former results have then been applied to the problem of inductive learning in the Horn description logic Horn-$\mathcal{M}$. More specifically, we have proposed a canonical base for the Horn-$\mathcal{M}$ concept inclusions valid in a given interpretation. While the author conjectures that it has minimal cardinality, it has been demonstrated that it can be computed in deterministic exponential time.

Future research could deliver the proof for the claimed minimality of the canonical Horn-$\mathcal{M}$ concept inclusion base, or could investigate means that allow for the integration of existing knowledge to make incremental learning possible. The author believes that both tasks can be tackled as soon as computation procedures for most specific consequences in Horn-$\mathcal{M}$ are available. This will be subject of future publications.

Eventually, the author wants to point out that incremental learning from a sequence of interpretations [21,22, Section 8.4 for the $\mathcal{EL}^{\bot}$ case] is probably more practical than *model exploration* or *ABox exploration* [10], since new observations that could show invalidity of concept inclusions are not requested from the expert at a certain time point, but are rather processed upon availability (*"push instead of pull"*). However, completeness of the eventual result for the considered domain of interest is only achieved if *all* typical individuals occur in the sequence at some time point.

Acknowledgments The author gratefully thanks Sebastian Rudolph for the very idea of learning in Horn description logics as well as for a helpful discussion on basics of Horn description logics. The author further thanks the reviewers for their constructive remarks.

References

1. Albano, A.L.J.H.: Polynomial Growth of Concept Lattices, Canonical Bases and Generators: Extremal Set Theory in Formal Concept Analysis. Doctoral thesis, Technische Universität Dresden, Dresden, Germany (2017)
2. Baader, F., Brandt, S., Lutz, C.: Pushing the $\mathcal{EL}$ Envelope. In: Kaelbling, L.P., Saffiotti, A. (eds.) IJCAI-05, Proceedings of the Nineteenth International Joint Conference on Artificial Intelligence, Edinburgh, Scotland, UK, July 30 - August 5, 2005. pp. 364–369. Professional Book Center (2005)
3. Baader, F., Horrocks, I., Lutz, C., Sattler, U.: An Introduction to Description Logic. Cambridge University Press (2017)
4. Belohlávek, R., Vychodil, V.: Closure-based Constraints in Formal Concept Analysis. Discrete Applied Mathematics **161**(13-14), 1894–1911 (2013)
5. Borchmann, D.: Learning Terminological Knowledge with High Confidence from Erroneous Data. Doctoral thesis, Technische Universität Dresden, Dresden, Germany (2014)
6. Borchmann, D., Distel, F., Kriegel, F.: Axiomatisation of General Concept Inclusions from Finite Interpretations. Journal of Applied Non-Classical Logics **26**(1), 1–46 (2016)
7. Calvanese, D., De Giacomo, G., Lembo, D., Lenzerini, M., Rosati, R.: Data Complexity of Query Answering in Description Logics. In: Doherty, P., Mylopoulos, J., Welty, C.A. (eds.) Proceedings, Tenth International Conference on Principles of Knowledge Representation and Reasoning, Lake District of the United Kingdom, June 2-5, 2006. pp. 260–270. AAAI Press (2006)

8. Dantsin, E., Eiter, T., Gottlob, G., Voronkov, A.: Complexity and Expressive Power of Logic Programming. ACM Computing Surveys **33**(3), 374–425 (2001)
9. De Giacomo, G., Lenzerini, M.: A Uniform Framework for Concept Definitions in Description Logics. Journal of Artificial Intelligence Research **6**, 87–110 (1997)
10. Distel, F.: Learning Description Logic Knowledge Bases from Data using Methods from Formal Concept Analysis. Doctoral thesis, Technische Universität Dresden, Dresden, Germany (2011)
11. Ganter, B., Wille, R.: Formal Concept Analysis: Mathematical Foundations. Springer (1999)
12. Guigues, J.L., Duquenne, V.: Famille minimale d'implications informatives résultant d'un tableau de données binaires. Mathématiques et Sciences Humaines **95**, 5–18 (1986)
13. Hernich, A., Lutz, C., Papacchini, F., Wolter, F.: Horn-Rewritability vs. PTime Query Evaluation in Ontology-Mediated Querying. In: Lang, J. (ed.) Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence, IJCAI 2018, July 13-19, 2018, Stockholm, Sweden. pp. 1861–1867. ijcai.org (2018)
14. Hitzler, P., Krötzsch, M., Rudolph, S.: Foundations of Semantic Web Technologies. Chapman and Hall/CRC Press (2010)
15. Hustadt, U., Motik, B., Sattler, U.: Data Complexity of Reasoning in Very Expressive Description Logics. In: Kaelbling, L.P., Saffiotti, A. (eds.) IJCAI-05, Proceedings of the Nineteenth International Joint Conference on Artificial Intelligence, Edinburgh, Scotland, UK, July 30 - August 5, 2005. pp. 466–471. Professional Book Center (2005)
16. Hustadt, U., Motik, B., Sattler, U.: Reasoning in Description Logics by a Reduction to Disjunctive Datalog. Journal of Automated Reasoning **39**(3), 351–384 (2007)
17. Knuth, D.E.: Big Omicron and Big Omega and Big Theta. SIGACT News **8**(2), 18–24 (Apr 1976)
18. Kriegel, F.: Concept Explorer FX (2010–2019), `https://github.com/francesco-kriegel/conexp-fx`, Software for Formal Concept Analysis with Description Logic Extensions
19. Kriegel, F.: NextClosures with Constraints. In: Huchard, M., Kuznetsov, S. (eds.) Proceedings of the Thirteenth International Conference on Concept Lattices and Their Applications, Moscow, Russia, July 18-22, 2016. CEUR Workshop Proceedings, vol. 1624, pp. 231–243. CEUR-WS.org (2016)
20. Kriegel, F.: Acquisition of Terminological Knowledge from Social Networks in Description Logic. In: Missaoui, R., Kuznetsov, S.O., Obiedkov, S.A. (eds.) Formal Concept Analysis of Social Networks, pp. 97–142. Lecture Notes in Social Networks, Springer (2017)
21. Kriegel, F.: Most Specific Consequences in the Description Logic $\mathcal{EL}$. LTCS-Report 18-11, Chair of Automata Theory, Institute of Theoretical Computer Science, Technische Universität Dresden, Dresden, Germany (2018), `https://tu-dresden.de/inf/lat/reports#Kr-LTCS-18-11`, accepted for publication in Discrete Applied Mathematics
22. Kriegel, F.: Most Specific Consequences in the Description Logic $\mathcal{EL}$. Discrete Applied Mathematics (2019), to appear.
23. Kriegel, F., Borchmann, D.: NextClosures: Parallel Computation of the Canonical Base with Background Knowledge. International Journal of General Systems **46**(5), 490–510 (2017)
24. Krisnadhi, A., Lutz, C.: Data Complexity in the $\mathcal{EL}$ Family of Description Logics. In: Dershowitz, N., Voronkov, A. (eds.) Logic for Programming, Artificial Intelligence, and Reasoning, 14th International Conference, LPAR 2007, Yerevan, Armenia, October 15-19, 2007, Proceedings. Lecture Notes in Computer Science, vol. 4790, pp. 333–347. Springer (2007)
25. Krötzsch, M., Rudolph, S., Hitzler, P.: Complexities of Horn Description Logics. ACM Transactions on Computational Logic **14**(1), 2:1–2:36 (2013)
26. Kupferman, O., Sattler, U., Vardi, M.Y.: The Complexity of the Graded μ-Calculus. In: Voronkov, A. (ed.) Automated Deduction - CADE-18, 18th International Conference on Automated Deduction, Copenhagen, Denmark, July 27-30, 2002, Proceedings. Lecture Notes in Computer Science, vol. 2392, pp. 423–437. Springer (2002)

27. Kuznetsov, S.O., Obiedkov, S.A.: Some Decision and Counting Problems of the Duquenne-Guigues Basis of Implications. Discrete Applied Mathematics **156**(11), 1994–2003 (2008)
28. Rudolph, S.: Relational Exploration: Combining Description Logics and Formal Concept Analysis for Knowledge Specification. Doctoral thesis, Technische Universität Dresden, Dresden, Germany (2006)
29. Schild, K.: A Correspondence Theory for Terminological Logics: Preliminary Report. In: Mylopoulos, J., Reiter, R. (eds.) Proceedings of the 12th International Joint Conference on Artificial Intelligence. Sydney, Australia, August 24-30, 1991. pp. 466–471. Morgan Kaufmann (1991)
30. Stumme, G.: Attribute Exploration with Background Implications and Exceptions, pp. 457–469. Studies in Classification, Data Analysis, and Knowledge Organization, Springer, Berlin, Heidelberg (1996)
31. Tobies, S.: Complexity Results and Practical Algorithms for Logics in Knowledge Representation. Doctoral thesis, Rheinisch-Westfälische Technische Hochschule Aachen, Aachen, Germany (2001)