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Camille Bourgaux Anni-Yasmin Turhan

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Postal Address:
Lehrstuhl für Automatentheorie
Institut für Theoretische Informatik
TU Dresden
01062 Dresden

http://lat.inf.tu-dresden.de

Visiting Address:
Nöthnitzer Str. 46
Dresden

Temporal Query Answering in DL-Lite over Inconsistent Data*

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Abstract

In ontology-based systems that process data stemming from different sources and that is received over time, as in context-aware systems, reasoning needs to cope with the temporal dimension and should be resilient against inconsistencies in the data. Motivated by such settings, this paper addresses the problem of handling inconsistent data in a temporal version of ontology-based query answering. We consider a recently proposed temporal query language that combines conjunctive queries with operators of propositional linear temporal logic and extend to this setting three inconsistency-tolerant semantics that have been introduced for querying inconsistent description logic knowledge bases. We investigate their complexity for DL-Lite$_{\mathcal{R}}$ temporal knowledge bases, and furthermore complete the picture for the consistent case.

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1 Introduction

Context-aware systems [17, 3] observe their environment over time and are able to detect situations while running in order to adapt their behaviour. They rely upon heterogeneous sources such as sensors (in a broad sense) or other applications that provide them with data. A context-aware system needs to integrate this data and should behave resilient towards erroneous or contradictory data. Since the collected data usually provides an incomplete description of the observed system, the closed world assumption employed by database systems, where facts not present are assumed to be false, is not appropriate. Moreover, it is convenient to use some knowledge about the system to reason with the data and get more complete answers to the queries than from the data alone. To address these requirements and facilitate data integration, *ontologies* have been used to implement situation recognition [17, 3, 13, 24].

Ontology-mediated query answering [14] performs database-style query answering over description logic (DL) knowledge bases that consist of an ontology (called a TBox) expressing conceptual knowledge about a domain and a dataset (or ABox) containing facts about particular individuals [5]. An important issue that may arise when querying data through ontology reasoning, especially in the context of situation recognition where the data comes from sensors and is changing frequently, is the inconsistency of the data w.r.t. the ontology. Indeed, under the classical semantics, every query is entailed from an inconsistent theory. Several inconsistency-tolerant semantics have thus been introduced in the context of DL knowledge bases (see [7] for a survey).

A situation is often defined not only w.r.t. the current state of the system but depends also on its history. For instance, a system that operates on a cluster of servers may need the list of servers which have been almost overloaded at least twice in the past ten time units. That is why research efforts have recently been devoted to temporalizing query answering [4, 11] by allowing to use operators of the *linear temporal logic* (LTL) [25] in the queries. In this setting, the query is answered over a *temporal knowledge base* consisting of a global TBox and a sequence of ABoxes that represents the data at different time points. The situation previously described can then be recognised by answering the query “$\Diamond^-(\mathsf{AlmostOverloaded}(x) \wedge \bigcirc^- \Diamond^- \mathsf{AlmostOverloaded}(x))$", where $\Diamond^-$ is the LTL operator “eventually in the past" and $\bigcirc^-$ the operator “previous", over the sequence of datasets that correspond to the last ten observations of the system, an ontology defining the concept $\mathsf{AlmostOverloaded}$. A lot of work has been dedicated to the temporalization of DL, combining different temporal logics and DL languages (see [22] for a survey). As efficiency is a primary concern, particular attention has been paid to temporalized DLs of the DL-Lite family [15] which underly the OWL 2 QL profile of the Semantic Web standard [23] and possess the notable property that query answering can be reduced to evaluation of standard database queries (see [2] for different temporal extensions of DL-Lite). The construction of temporal queries has attracted a lot of interest recently [18, 19, 1], and querying temporal databases has also been studied (see e.g., [16]). Here, we consider the setting proposed in [11] which does not allow for temporalized concepts or axioms in the TBox but focuses on querying sequences of ABoxes.

This work presents results on lifting inconsistency-tolerant reasoning to temporal query answering. To the best of our knowledge, this is the first investigation of temporal query answering under inconsistency-tolerant semantics. We consider three semantics that have been defined for DL knowledge bases and that we find particularly relevant. They are all based upon the notion of a *repair*, which is a maximal consistent subset of the data. The *AR semantics* [20, 21], inspired by consistent query answering in the database setting [6], considers the queries that hold in *every repair*. This semantics is arguably the most natural and is widely accepted to query inconsistent knowledge bases. However, AR query answering is intractable even for DL-Lite, which leads [20, 21] to propose a tractable approximation of AR, namely the *IAR semantics*, which queries the *intersection of the repairs*. Beside its better computational properties, this semantics is more cautious since it provides answers supported by facts that are not involved in any contradictions, so it may be interesting in our setting when the system should change its behaviour only if some situation has been recognised with a very high confidence. Finally, the *brave semantics* [9] returns every answer that holds in *some repair*, so is supported by some consistent set of facts. This less cautious semantics may be relevant for context recognition, when critical situations must imperatively be handled.

The contributions of this paper are as follows. In Section 3 we extend the AR, IAR and brave semantics to the setting of temporal query answering. We distinguish in our analysis three cases for *rigid predicates*, i.e., whose extensions stay unchanged across time points : no rigid predicates, rigid concepts only, or rigid concepts and roles. We show that when there is no rigid predicate, existing algorithms for temporal query answering and for IAR query answering can be combined to perform IAR temporal query answering. We also show that this method can sometimes be used for AR and provides in any case an approximation of the AR answers. In Section 4 we investigate the computational properties of the three semantics, considering both *data complexity* (in the size of the data only), and *combined complexity* (in the size of the whole problem), and distinguishing three different cases regarding the rigid symbols that are allowed. We show that in all cases except for brave semantics with rigid predicates, the data complexity is not higher than in the atemporal setting. In all cases, adding the temporal dimension does not increase the combined complexity. Our complexity analysis also leads us to close some open questions about temporal query answering under the classical semantics in the presence of rigid predicates. In particular, we show that it can often be reduced to the case without rigid predicates.

2 Preliminaries

We briefly recall the syntax and semantics of DLs, the three inconsistency-tolerant semantics we consider, and the setting of temporal query answering.

Syntax. A DL *knowledge base (KB)* $\mathcal{K}$ consists of an ABox $\mathcal{A}$ and a TBox $\mathcal{T}$, both constructed from three countably infinite sets: a set $\mathsf{N_C}$ of *concept names* (unary predicates), a set $\mathsf{N_R}$ of *role names* (binary predicates), and a set $\mathsf{N_I}$ of *individual names* (constants). The *ABox* (dataset) is a finite set of *concept assertions* $A(a)$ and *role assertions* $R(a,b)$, where $A \in \mathsf{N_C}$, $R \in \mathsf{N_R}$, $a, b \in \mathsf{N_I}$. The *TBox* (ontology) is a finite set of axioms whose form depends on the particular DL. In DL-Lite$_\mathcal{R}$, TBox axioms are either *concept inclusions* $B \sqsubseteq C$ or *role inclusions* $P \sqsubseteq S$ built according to the following syntax (where $A \in \mathsf{N_C}$ and $R \in \mathsf{N_R}$):

$$B := A \mid \exists P, \quad C := B \mid \neg B, \quad P := R \mid R^-, \quad S := P \mid \neg P$$

Inclusions of the form $B_1 \sqsubseteq B_2$ or $P_1 \sqsubseteq P_2$ are called *positive inclusions* (PI), those of the form $B_1 \sqsubseteq \neg B_2$ or $P_1 \sqsubseteq \neg P_2$ are called *negative inclusions* (NI).

Semantics. An *interpretation* has the form $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where $\Delta^\mathcal{I}$ is a non-empty set and $\cdot^\mathcal{I}$ maps each $a \in \mathsf{N_I}$ to $a^\mathcal{I} \in \Delta^\mathcal{I}$, each $A \in \mathsf{N_C}$ to $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, and each $R \in \mathsf{N_R}$ to $R^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$. We adopt the unique name assumption (i.e., for all $a, b \in \mathsf{N_I}$, $a^\mathcal{I} \neq b^\mathcal{I}$ if $a \neq b$). The function $\cdot^\mathcal{I}$ is straightforwardly extended to general concepts and roles, e.g., $(R^-)^\mathcal{I} = \{(d,e) \mid (e,d) \in R^\mathcal{I}\}$ and $(\exists P)^\mathcal{I} = \{d \mid \exists e : (d,e) \in P^\mathcal{I}\}$. An interpretation $\mathcal{I}$ satisfies an inclusion $G \sqsubseteq H$ if $G^\mathcal{I} \subseteq H^\mathcal{I}$; it satisfies $A(a)$ (resp. $R(a,b)$) if $a^\mathcal{I} \in A^\mathcal{I}$ (resp. $(a^\mathcal{I}, b^\mathcal{I}) \in R^\mathcal{I}$). We call $\mathcal{I}$ a *model* of $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ if $\mathcal{I}$ satisfies all axioms in $\mathcal{T}$ and all assertions in $\mathcal{A}$. A KB is *consistent* if it has a model, and we say that an ABox $\mathcal{A}$ is $\mathcal{T}$*-consistent* (or simply *consistent* for short), if the KB $\langle \mathcal{T}, \mathcal{A} \rangle$ is consistent.

Queries. A *conjunctive query* (CQ) takes the form $q = \exists \vec{y}\, \psi(\vec{x}, \vec{y})$, where ψ is a conjunction of atoms of the forms $A(t)$ or $R(t, t')$, with t, t' individuals or variables from $\vec{x} \cup \vec{y}$. A CQ is called

Boolean (BCQ) if it has no free variables (i.e. $\vec{x} = \emptyset$). A BCQ q is *entailed* from $\mathcal{K}$, written $\mathcal{K} \models q$, iff q holds in every model of $\mathcal{K}$. Given a CQ q with free variables $\vec{x} = (x_1, \ldots, x_k)$ and a tuple of individuals $\vec{a} = (a_1, \ldots, a_k)$, $\vec{a}$ is a *certain answer* to q over $\mathcal{K}$ just in the case that $\mathcal{K} \models q(\vec{a})$, where $q(\vec{a})$ is the BCQ resulting from replacing each x_j by a_j.

Inconsistency-tolerant semantics. A *repair* of $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ is an inclusion-maximal subset of $\mathcal{A}$ that is $\mathcal{T}$-consistent. We consider three semantics based on repairs. A tuple $\vec{a}$ is an answer to q over $\mathcal{K}$ under

- *AR semantics*, written $\mathcal{K} \models_{\text{AR}} q(\vec{a})$,
  iff $\langle \mathcal{T}, \mathcal{A}' \rangle \models q(\vec{a})$ for *every repair* $\mathcal{A}'$ of $\mathcal{K}$;
- *IAR semantics*, written $\mathcal{K} \models_{\text{IAR}} q(\vec{a})$,
  iff $\langle \mathcal{T}, \mathcal{A}^{\cap} \rangle \models q(\vec{a})$ where $\mathcal{A}^{\cap}$ is the *intersection of all repairs* of $\mathcal{K}$;
- *brave semantics*, written $\mathcal{K} \models_{\text{brave}} q(\vec{a})$,
  iff $\langle \mathcal{T}, \mathcal{A}' \rangle \models q(\vec{a})$ for *some repair* $\mathcal{A}'$ of $\mathcal{K}$.

In DL-Lite$_{\mathcal{R}}$, IAR or brave CQ answering is in P w.r.t. data complexity (in the size of the ABox) and NP-complete w.r.t. combined complexity (in the size of the whole KB and the query), and AR CQ answering is coNP-complete w.r.t. data complexity and Π_2^p-complete w.r.t. combined complexity [20, 9].

Temporal query answering. We consider the framework presented in [11].

Definition 1 (TKB)**.** A *temporal knowledge base* (TKB) $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leq i \leq n} \rangle$ consists of a TBox $\mathcal{T}$ and a finite sequence of ABoxes $(\mathcal{A}_i)_{0 \leq i \leq n}$. A sequence $\mathcal{J} = (\mathcal{I}_i)_{0 \leq i \leq n}$ of interpretations $\mathcal{I}_i = (\Delta, \cdot^{\mathcal{I}_i})$ over a fixed non-empty domain Δ is a *model* of $\mathcal{K}$ iff for all $0 \leq i \leq n$, $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i \rangle$, and for every $a \in \mathsf{N_I}$ and all $1 \leq i \leq j \leq n$, $a^{\mathcal{I}_i} = a^{\mathcal{I}_j}$. *Rigid predicates* are elements from the set of *rigid concepts* $\mathsf{N_{RC}} \subseteq \mathsf{N_C}$ or of *rigid roles* $\mathsf{N_{RR}} \subseteq \mathsf{N_R}$. A sequence of interpretations $\mathcal{J} = (\mathcal{I}_i)_{0 \leq i \leq n}$ *respects the rigid predicates* iff for every $X \in \mathsf{N_{RC}} \cup \mathsf{N_{RR}}$ and all $1 \leq i \leq j \leq n$, $X^{\mathcal{I}_i} = X^{\mathcal{I}_j}$. A TKB is *consistent* if it has a model that respects the rigid predicates. A sequence of ABoxes $(\mathcal{A}_i)_{0 \leq i \leq n}$ is $\mathcal{T}$*-consistent*, or simply consistent, if the TKB $\langle \mathcal{T}, (\mathcal{A}_i)_{0 \leq i \leq n} \rangle$ is consistent.

It is sometimes convenient to represent a sequence of ABoxes as a set of assertions associated with timestamps, which we call *timed-assertions*: $(\mathcal{A}_i)_{0 \leq i \leq n}$ becomes $\{(\alpha, i) \mid \alpha \in \mathcal{A}_i, 0 \leq i \leq n\}$. A *rigid assertion* is of the form $A(a)$ with $A \in \mathsf{N_{RC}}$ or $R(a, b)$ with $R \in \mathsf{N_{RR}}$. We distinguish three cases in our analysis : Case 1 with $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, Case 2 with $\mathsf{N_{RC}} \neq \emptyset$ and $\mathsf{N_{RR}} = \emptyset$, and Case 3 with $\mathsf{N_{RC}} \neq \emptyset$ and $\mathsf{N_{RR}} \neq \emptyset$. Note that since rigid roles can simulate rigid concepts, these three cases cover all possibilities. We denote by $\mathsf{N_C^{\mathcal{K}}}$, $\mathsf{N_R^{\mathcal{K}}}$, $\mathsf{N_{RC}^{\mathcal{K}}}$, $\mathsf{N_{RR}^{\mathcal{K}}}$, and $\mathsf{N_I^{\mathcal{K}}}$ respectively the sets of concepts, roles, rigid concepts, rigid roles, and individuals that occur in the TKB $\mathcal{K}$.

Definition 2 (TCQ)**.** *Temporal conjunctive queries* (TCQs) are built from CQs as follows: each CQ is a TCQ, and if ϕ_1 and ϕ_2 are TCQs, then so are $\phi_1 \wedge \phi_2$ (conjunction), $\phi_1 \vee \phi_2$ (disjunction), $\bigcirc \phi_1$ (strong next), $\bullet \phi_1$ (weak next), $\bigcirc^- \phi_1$ (strong previous), $\bullet^- \phi_1$ (weak previous), $\Box \phi_1$ (always), $\Box^- \phi_1$ (always in the past), $\Diamond \phi_1$ (eventually), $\Diamond^- \phi_1$ (some time in the past), $\phi_1 \mathsf{U} \phi_2$ (until), and $\phi_1 \mathsf{S} \phi_2$ (since). Given a TCQ ϕ with free variables $\vec{x} = (x_1, \ldots, x_k)$ and a tuple of individuals $\vec{a} = (a_1, \ldots, a_k)$, $\phi(\vec{a})$ denotes the Boolean TCQ (BTCQ) resulting from replacing each x_j by a_j. The tuple $\vec{a}$ is an answer to ϕ in a sequence of interpretations $\mathcal{J} = (\mathcal{I}_i)_{0 \leq i \leq n}$ at time point p $(0 \leq p \leq n)$ iff $\mathcal{J}, p \models \phi(\vec{a})$, where the entailment of a BTCQ ϕ is defined by induction on its structure as shown in Table 1. It is a *certain answer* to ϕ over $\mathcal{K}$ at time point p, written $\mathcal{K}, p \models \phi(\vec{a})$, iff $\mathcal{J}, p \models \phi(\vec{a})$ for every model $\mathcal{J}$ of $\mathcal{K}$ that respects the rigid predicates.

Remark 1. The additional LTL operators W (weak until), W^- (weak since), R (release), and R^- (past release) can be expressed w.r.t. our operator basis as follows: $\phi_1 \mathsf{W} \phi_2 \equiv (\phi_1 \mathsf{U} \phi_2) \vee$

ϕ	$\mathcal{J}, p \models \phi$ iff
$\exists \vec{y}\, \psi(\vec{y})$	$\mathcal{I}_p \models \exists \vec{y}\, \psi(\vec{y})$
$\phi_1 \wedge \phi_2$	$\mathcal{J}, p \models \phi_1$ and $\mathcal{J}, p \models \phi_2$
$\phi_1 \vee \phi_2$	$\mathcal{J}, p \models \phi_1$ or $\mathcal{J}, p \models \phi_2$
$\bigcirc \phi_1$	$p < n$ and $\mathcal{J}, p+1 \models \phi_1$
$\bullet \phi_1$	$p < n$ implies $\mathcal{J}, p+1 \models \phi_1$
$\bigcirc^- \phi_1$	$p > 0$ and $\mathcal{J}, p-1 \models \phi_1$
$\bullet^- \phi_1$	$p > 0$ implies $\mathcal{J}, p-1 \models \phi_1$
$\Box \phi_1$	$\forall k,\ p \leq k \leq n,\ \mathcal{J}, k \models \phi_1$
$\Box^- \phi_1$	$\forall k,\ 0 \leq k \leq p,\ \mathcal{J}, k \models \phi_1$
$\Diamond \phi_1$	$\exists k,\ p \leq k \leq n,\ \mathcal{J}, k \models \phi_1$
$\Diamond^- \phi_1$	$\exists k,\ 0 \leq k \leq p,\ \mathcal{J}, k \models \phi_1$
$\phi_1 \mathsf{U} \phi_2$	$\exists k,\ p \leq k \leq n,\ \mathcal{J}, k \models \phi_2$ and $\forall j,\ p \leq j < k, \mathcal{J}, j \models \phi_1$
$\phi_1 \mathsf{S} \phi_2$	$\exists k,\ 0 \leq k \leq p,\ \mathcal{J}, k \models \phi_2$ and $\forall j,\ k < j \leq p, \mathcal{J}, j \models \phi_1$

Table 1: Entailment of BTCQs.

$(\Box \phi_1)$, $\phi_1 \mathsf{W}^- \phi_2 \equiv (\phi_1 \mathsf{S} \phi_2) \vee (\Box^- \phi_1)$, $\phi_1 \mathsf{R} \phi_2 \equiv \phi_2 \mathsf{W}(\phi_2 \wedge \phi_1)$, and $\phi_1 \mathsf{R}^- \phi_2 \equiv \phi_2 \mathsf{W}^-(\phi_2 \wedge \phi_1)$. Since the top and bottom concepts $\top$ and $\bot$ are not allowed in every DL, $\Diamond$ and $\Box$ cannot be expressed w.r.t. the other operators as usual in LTL ($\Diamond \phi_1 \equiv \mathsf{true} \mathsf{U} \phi_1$, $\Box \phi_1 \equiv \phi_1 \mathsf{U}(\phi_1 \wedge \bullet \mathsf{false})$).

Note also that since disjunctions are allowed, TCQs could be defined with unions of conjunctive queries (UCQs) instead of CQs (in this case, in the first line of Table 1, the CQ $\exists \vec{y}\, \psi(\vec{y})$ would be replaced by a UCQ $\bigvee_{1 \leq j \leq m} \exists \vec{y_j}\, \psi_j(\vec{y_j})$). We use CQs for simplicity.

It follows from the definition of certain answers that TCQ answering is straightforwardly reduced to entailment of BTCQs and we can focus w.l.o.g. on the latter problem.

3 Temporal Query Answering over Inconsistent Data

We extend the three inconsistency-tolerant semantics to temporal query answering. The main difference to the atemporal case is that in the presence of rigid predicates, a TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leq i \leq n} \rangle$ may be inconsistent even if each KB $\langle \mathcal{T}, \mathcal{A}_i \rangle$ is consistent. In this case there need not exist a sequence of interpretations $\mathcal{J} = (\mathcal{I}_i)_{0 \leq i \leq n}$ such that each $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i \rangle$ and which respects rigid predicates. That is why we need to consider as repairs the $\mathcal{T}$-consistent sequences of subsets of the initial ABoxes that are component-wise maximal.

Definition 3 (Repair of a TKB)**.** A *repair* of a TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leq i \leq n} \rangle$ is a sequence of ABoxes $(\mathcal{A}'_i)_{0 \leq i \leq n}$ such that $\{(\alpha, i) \mid \alpha \in \mathcal{A}'_i, 0 \leq i \leq n\}$ is a maximal $\mathcal{T}$-consistent subset of $\{(\alpha, i) \mid \alpha \in \mathcal{A}_i, 0 \leq i \leq n\}$.We denote the set of repairs of $\mathcal{K}$ by $\mathit{Rep}(\mathcal{K})$.

The next example shows the influence of rigid predicates on the repairs.

Example 1. Consider the following TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{1 \leq i \leq 2} \rangle$. The TBox expresses that web servers and application servers are two distinct kinds of servers, and the ABoxes provide information about a server a that executes two processes.

$$\mathcal{T} = \{\mathsf{WebServer} \sqsubseteq \mathsf{Server},\ \mathsf{AppServer} \sqsubseteq \mathsf{Server},\ \mathsf{WebServer} \sqsubseteq \neg \mathsf{AppServer}\}$$
$$\mathcal{A}_1 = \{\mathsf{WebServer}(a),\ \mathsf{execute}(a, b)\}$$
$$\mathcal{A}_2 = \{\mathsf{AppServer}(a),\ \mathsf{WebServer}(a),\ \mathsf{execute}(a, c)\}$$

Assume that no predicate is rigid. The TKB $\mathcal{K}$ is inconsistent because the timed-assertions $(\mathsf{AppServer}(a), 2)$ and $(\mathsf{WebServer}(a), 2)$ violate the negative inclusion of $\mathcal{T}$, since $\mathsf{AppServer}(a)$ and $\mathsf{WebServer}(a)$ cannot both be true at time point 2. It follows that $\mathcal{K}$ has two repairs $(\mathcal{A}'_i)_{1\leq i\leq 2}$ and $(\mathcal{A}''_i)_{1\leq i\leq 2}$ with $\mathcal{A}'_1 = \mathcal{A}''_1 = \mathcal{A}_1$, and $\mathcal{A}'_2 = \{\mathsf{AppServer}(a), \mathsf{execute}(a, c)\}$ and $\mathcal{A}''_2 = \{\mathsf{WebServer}(a), \mathsf{execute}(a, c)\}$ which correspond to the two different ways of restoring consistency.

Assume now that $\mathsf{AppServer}$ is rigid. There is a new reason for $\mathcal{K}$ being inconsistent: the timed-assertions $(\mathsf{WebServer}(a), 1)$ and $(\mathsf{AppServer}(a), 2)$ violate the negative inclusion of $\mathcal{T}$ due to the rigidity of $\mathsf{AppServer}$ which implies that $\mathsf{AppServer}(a)$ and $\mathsf{WebServer}(a)$ should be both entailed at time point 1. Then $\mathcal{K}$ has two repairs $(\mathcal{A}'_i)_{1\leq i\leq 2}$ and $(\mathcal{A}''_i)_{1\leq i\leq 2}$ with $\mathcal{A}'_1 = \{\mathsf{execute}(a, b)\}$, $\mathcal{A}'_2 = \{\mathsf{AppServer}(a), \mathsf{execute}(a, c)\}$, and $\mathcal{A}''_1 = \mathcal{A}_1$, $\mathcal{A}''_2 = \{\mathsf{WebServer}(a), \mathsf{execute}(a, c)\}$. Note that even if $(\mathcal{A}'_i)_{1\leq i\leq 2}$ is maximal (since adding $\mathsf{WebServer}(a)$ to $\mathcal{A}'_1$ renders the TKB inconsistent), $\mathcal{A}'_1$ is not a repair of $\langle \mathcal{T}, \mathcal{A}_1\rangle$ since it is not maximal.

Next we extend the semantics AR, IAR, and brave to the temporal case in the natural way by regarding sequences of ABoxes.

Definition 4 (AR, IAR, brave semantics for TCQs)**.** A tuple $\vec{a}$ is an answer to a TCQ ϕ over a TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0\leq i\leq n}\rangle$ at time point p under

- AR semantics, written $\mathcal{K}, p \models_{\text{AR}} \phi(\vec{a})$,
  iff $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leq i\leq n}\rangle, p \models \phi(\vec{a})$ for every repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$;
- IAR semantics, written $\mathcal{K}, p \models_{\text{IAR}} \phi(\vec{a})$,
  iff $\langle \mathcal{T}, (\mathcal{A}^{IR}_i)_{0\leq i\leq n}\rangle, p \models \phi(\vec{a})$, with $\mathcal{A}^{IR}_i = \bigcap_{(\mathcal{A}'_j)_{0\leq j\leq n}\in Rep(\mathcal{K})} \mathcal{A}'_i$, $0 \leq i \leq n$;
- brave semantics, written $\mathcal{K}, p \models_{\text{brave}} \phi(\vec{a})$,
  iff $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leq i\leq n}\rangle, p \models \phi(\vec{a})$ for some repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$.

The following relationships between the semantics are implied by their definition:

$$\mathcal{K}, p \models_{\text{IAR}} \phi(\vec{a}) \quad \Rightarrow \quad \mathcal{K}, p \models_{\text{AR}} \phi(\vec{a}) \quad \Rightarrow \quad \mathcal{K}, p \models_{\text{brave}} \phi(\vec{a})$$

Next, we illustrate the effect of the different semantics in the temporal case.

Example 2 (Example 1 cont'd)**.** Consider the three temporal conjunctive queries:

$$\phi_1 = \Box(\exists y\ \mathsf{execute}(x, y)) \qquad \phi_2 = \Box(\exists y\ \mathsf{Server}(x) \wedge \mathsf{execute}(x, y))$$
$$\phi_3 = \Box(\exists y\ \mathsf{AppServer}(x) \wedge \mathsf{execute}(x, y))$$

In Case 1 with no rigid predicate, the intersection of the repairs is $(\mathcal{A}^{IR}_i)_{1\leq i\leq 2}$ with $\mathcal{A}^{IR}_1 = \mathcal{A}_1$, $\mathcal{A}^{IR}_2 = \{\mathsf{execute}(a, c)\}$. Then $\mathcal{K}, 1 \models_{\text{IAR}} \phi_1(a)$, since in every model of the intersection of the repairs a executes b at time point 1 and c at time point 2. For ϕ_2, $\mathcal{K}, 1 \models_{\text{AR}} \phi_2(a)$, since every model of every repair assigns a to $\mathsf{WebServer}$ at time point 1 and either to $\mathsf{AppServer}$ (in models of $(\mathcal{A}'_i)_{1\leq i\leq 2}$) or to $\mathsf{WebServer}$ (in models of $(\mathcal{A}''_i)_{1\leq i\leq 2}$) at time point 2, but $\mathcal{K}, 1 \not\models_{\text{IAR}} \phi_2(a)$. Finally, $\mathcal{K}, 1 \not\models_{\text{brave}} \phi_3(a)$ because no repair entails $\mathsf{AppServer}(a)$ at time point 1.

If $\mathsf{AppServer}$ is rigid, the intersection of the repairs is $(\mathcal{A}^{IR}_i)_{1\leq i\leq 2}$ with $\mathcal{A}^{IR}_1 = \{\mathsf{execute}(a, b)\}$, $\mathcal{A}^{IR}_2 = \{\mathsf{execute}(a, c)\}$. So still $\mathcal{K}, 1 \models_{\text{IAR}} \phi_1(a)$ holds. Since every model of every repair assigns a to Server at time points 1 and 2 (either because a is a web server or an application server), $\mathcal{K}, 1 \models_{\text{AR}} \phi_2(a)$, but $\mathcal{K}, 1 \not\models_{\text{IAR}} \phi_2(a)$. Finally, $\mathcal{K}, 1 \models_{\text{brave}} \phi_3(a)$ because every model of $\langle \mathcal{T}, (\mathcal{A}'_i)_{1\leq i\leq 2}\rangle$ assigns a to $\mathsf{AppServer}$ at any time point by rigidity of $\mathsf{AppServer}$, but $\mathcal{K}, 1 \not\models_{\text{AR}}$ $\phi_3(a)$.

ϕ	$\mathcal{K}, p \models_S \phi$ iff
$\exists \vec{y}\, \psi(\vec{y})$	$\langle \mathcal{T}, \mathcal{A}_p \rangle \models_S \exists \vec{y}\, \psi(\vec{y})$
$\phi_1 \wedge \phi_2$	$\mathcal{K}, p \models_S \phi_1$ and $\mathcal{K}, p \models_S \phi_2$
$\phi_1 \vee \phi_2$	$\mathcal{K}, p \models_S \phi_1$ or $\mathcal{K}, p \models_S \phi_2$
$\bigcirc\phi_1$	$p < n$ and $\mathcal{K}, p+1 \models_S \phi_1$
$\bullet\phi_1$	$p < n$ implies $\mathcal{K}, p+1 \models_S \phi_1$
$\bigcirc^-\phi_1$	$p > 0$ and $\mathcal{K}, p-1 \models_S \phi_1$
$\bullet^-\phi_1$	$p > 0$ implies $\mathcal{K}, p-1 \models_S \phi_1$
$\Box\phi_1$	$\forall k,\, p \leq k \leq n,\, \mathcal{K}, k \models_S \phi_1$
$\Box^-\phi_1$	$\forall k,\, 0 \leq k \leq p,\, \mathcal{K}, k \models_S \phi_1$
$\Diamond\phi_1$	$\exists k,\, p \leq k \leq n,\, \mathcal{K}, k \models_S \phi_1$
$\Diamond^-\phi_1$	$\exists k,\, 0 \leq k \leq p,\, \mathcal{K}, k \models_S \phi_1$
$\phi_1 \mathsf{U} \phi_2$	$\exists k,\, p \leq k \leq n,\, \mathcal{K}, k \models_S \phi_2$ and $\forall j, p \leq j < k, \mathcal{K}, j \models_S \phi_1$
$\phi_1 \mathsf{S} \phi_2$	$\exists k,\, 0 \leq k \leq p,\, \mathcal{K}, k \models_S \phi_2$ and $\forall j, k < j \leq p, \mathcal{K}, j \models_S \phi_1$

Table 2: Entailment under classical or IAR semantics without rigid predicates.

We point out some characteristics of Case 1. Since there is no rigid predicate, the interpretations $\mathcal{I}_i$ of a model $\mathcal{J} = (\mathcal{I}_i)_{0 \leq i \leq n}$ of $\mathcal{K}$ that respects the rigid predicates are independent, besides the interpretation of the constants.

Proposition 1. *If $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, then a TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leq i \leq n} \rangle$ is consistent iff every $\langle \mathcal{T}, \mathcal{A}_i \rangle$ is consistent. Moreover, if $\mathcal{K}$ is consistent, for every $0 \leq p \leq n$, $\mathcal{I}'_p$ is a model of $\langle \mathcal{T}, \mathcal{A}_p \rangle$ iff there exists a model $\mathcal{J} = (\mathcal{I}_i)_{0 \leq i \leq n}$ of $\mathcal{K}$ such that $\mathcal{I}_p = \mathcal{I}'_p$.*

Proof. If $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, a sequence of interpretations $\mathcal{J} = (\mathcal{I}_i)_{0 \leq i \leq n}$ is a model of $\mathcal{K}$ that respects the rigid predicates iff it is a model of $\mathcal{K}$, iff for every i, $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i \rangle$, and for every $a \in \mathsf{N_I}$ and all $1 \leq i \leq j \leq n$, $a^{\mathcal{I}_i} = a^{\mathcal{I}_j}$. It follows that $\mathcal{K}$ is consistent iff there exists $\mathcal{J} = (\mathcal{I}_i)_{0 \leq i \leq n}$ such that for every i, $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i \rangle$, and for every $a \in \mathsf{N_I}$ and all $1 \leq i \leq j \leq n$, $a^{\mathcal{I}_i} = a^{\mathcal{I}_j}$. We show that this is the case iff each $\langle \mathcal{T}, \mathcal{A}_i \rangle$ has a model. Let $\mathcal{I}'_0 = (\Delta^{\mathcal{I}'_0}, \cdot^{\mathcal{I}'_0}), \dots, \mathcal{I}'_n = (\Delta^{\mathcal{I}'_n}, \cdot^{\mathcal{I}'_n})$ be models of $\langle \mathcal{T}, \mathcal{A}_0 \rangle, \dots \langle \mathcal{T}, \mathcal{A}_n \rangle$ respectively, and $0 \leq p \leq n$. Let $\mathcal{J} = (\mathcal{I}_i)_{0 \leq i \leq n}$ with $\mathcal{I}_i = (\Delta, \cdot^{\mathcal{I}_i})$ where $\Delta = \Delta^{\mathcal{I}'_p}$ and for every $0 \leq i \leq n$, $\cdot^{\mathcal{I}_i}$ is defined as follows: $a^{\mathcal{I}_i} = a^{\mathcal{I}'_p}$ for every $a \in \mathsf{N_I}$, $A^{\mathcal{I}_i} = \{a^{\mathcal{I}'_p} \mid a^{\mathcal{I}'_i} \in A^{\mathcal{I}'_i}\}$ for every $A \in \mathsf{N_C}$, and $R^{\mathcal{I}_i} = \{(a^{\mathcal{I}'_p}, b^{\mathcal{I}'_p}) \mid (a^{\mathcal{I}'_i}, b^{\mathcal{I}'_i}) \in R^{\mathcal{I}'_i}\}$ for every $R \in \mathsf{N_R}$. Since we adopted the unique name assumption, each $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i \rangle$. It follows that $\mathcal{J} = (\mathcal{I}_i)_{0 \leq i \leq n}$ is such that for every i, $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i \rangle$, and for every $a \in \mathsf{N_I}$ and all $1 \leq i \leq j \leq n$, $a^{\mathcal{I}_i} = a^{\mathcal{I}_j}$. Moreover, $\mathcal{J}$ is such that $\mathcal{I}_p = \mathcal{I}'_p$. The other direction is trivial. □

Proposition 1 has several important consequences. First, the repairs of $\mathcal{K}$ are all possible sequences $(\mathcal{A}'_i)_{0 \leq i \leq n}$ where $\mathcal{A}'_i$ is a repair of $\langle \mathcal{T}, \mathcal{A}_i \rangle$, so the intersection of the repairs of $\mathcal{K}$ is $(\mathcal{A}_i^{\cap})_{0 \leq i \leq n}$ where $\mathcal{A}_i^{\cap}$ is the intersection of the repairs of $\langle \mathcal{T}, \mathcal{A}_i \rangle$. Second, we show that the entailment (resp. IAR entailment) of a BTCQ from a consistent (resp. possibly inconsistent) DL-Lite$_\mathcal{R}$ TKB can be equivalently defined w.r.t. the entailment (resp. IAR entailment) of the BCQs it contains as follows:

Proposition 2. *If $\mathcal{K}$ is a DL-Lite$_\mathcal{R}$ TKB and $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, then the entailments shown in Table 2 hold for S = classical when $\mathcal{K}$ is consistent, and for S = IAR.*

Proof. We start with the classical semantics when $\mathcal{K}$ is consistent.

For CQs we apply Proposition 1:

- $\mathcal{K}, p \models \exists \vec{y}\, \psi(\vec{y})$
  iff for every model $\mathcal{J} = (\mathcal{I}_i)_{0 \leq i \leq n}$ of $\mathcal{K}$ that respects the rigid predicates, $\mathcal{I}_p \models \exists \vec{y}\, \psi(\vec{y})$
  iff for every model $\mathcal{I}_p$ of $\langle \mathcal{T}, \mathcal{A}_p \rangle$, $\mathcal{I}_p \models \exists \vec{y}\, \psi(\vec{y})$ by Proposition 1
  iff $\langle \mathcal{T}, \mathcal{A}_p \rangle \models \exists \vec{y}\, \psi(\vec{y})$.

For the other cases where ϕ is built from TCQs ϕ_1, ϕ_2, we make use of the *canonical model* of $\mathcal{K}$. Indeed, it has been shown in [10] that if $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, for any DL-Lite$_\mathcal{R}$ TKB $\mathcal{K}$, there exists a canonical model $\mathcal{J}_\mathcal{K}$ of $\mathcal{K}$ such that for every BTCQ ϕ, and time point p, $\mathcal{K}, p \models \phi$ iff $\mathcal{J}_\mathcal{K}, p \models \phi$. Applying the definitions of Table 1 with $\mathcal{J}_\mathcal{K}$ gives the relations of Table 2.

For IAR semantics, let $(\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n}$ denote the intersection of the repairs of $\mathcal{K}$ and $\mathcal{A}_i^{\Cap}$ denote the intersection of the repairs of $\langle \mathcal{T}, \mathcal{A}_i \rangle$:

- $\mathcal{K}, p \models_{\text{IAR}} \exists \vec{y}\, \psi(\vec{y})$
  iff $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p \models \exists \vec{y}\, \psi(\vec{y})$
  iff $\langle \mathcal{T}, \mathcal{A}_p^{\mathit{IR}} \rangle \models \exists \vec{y}\, \psi(\vec{y})$ since $(\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n}$ is consistent
  iff $\langle \mathcal{T}, \mathcal{A}_p^{\Cap} \rangle \models \exists \vec{y}\, \psi(\vec{y})$ since the repairs of $\mathcal{K}$ are the sequences of the repairs of the $\langle \mathcal{T}, \mathcal{A}_i \rangle$
  iff $\langle \mathcal{T}, \mathcal{A}_p \rangle \models_{\text{IAR}} \exists \vec{y}\, \psi(\vec{y})$

- $\mathcal{K}, p \models_{\text{IAR}} \phi_1 \wedge \phi_2$
  iff $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p \models \phi_1 \wedge \phi_2$
  iff $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p \models \phi_1$ and $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p \models \phi_2$
  iff $\mathcal{K}, p \models_{\text{IAR}} \phi_1$ and $\mathcal{K}, p \models_{\text{IAR}} \phi_2$

- $\mathcal{K}, p \models_{\text{IAR}} \phi_1 \vee \phi_2$
  iff $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p \models \phi_1 \vee \phi_2$
  iff $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p \models \phi_1$ or $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p \models \phi_2$
  iff $\mathcal{K}, p \models_{\text{IAR}} \phi_1$ or $\mathcal{K}, p \models_{\text{IAR}} \phi_2$

- $\mathcal{K}, p \models_{\text{IAR}} \bigcirc \phi_1$
  iff $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p \models \bigcirc \phi_1$
  iff $p < n$ and $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p + 1 \models \phi_1$
  iff $p < n$ and $\mathcal{K}, p + 1 \models_{\text{IAR}} \phi_1$

- $\mathcal{K}, p \models_{\text{IAR}} \bullet \phi_1$
  iff $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p \models \bullet \phi_1$
  iff $p < n$ implies $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p + 1 \models \phi_1$
  iff $p < n$ implies $\mathcal{K}, p + 1 \models_{\text{IAR}} \phi_1$

- $\mathcal{K}, p \models_{\text{IAR}} \Box \phi_1$
  iff $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p \models \Box \phi_1$
  iff for every k, $p \leq k \leq n$, $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, k \models \phi_1$
  iff for every k, $p \leq k \leq n$, $\mathcal{K}, k \models_{\text{IAR}} \phi_1$

- $\mathcal{K}, p \models_{\text{IAR}} \Diamond \phi_1$
  iff $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p \models \Diamond \phi_1$
  iff there exists k, $p \leq k \leq n$, $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, k \models \phi_1$
  iff there exists k, $p \leq k \leq n$, $\mathcal{K}, k \models_{\text{IAR}} \phi_1$

- $\mathcal{K}, p \models_{\text{IAR}} \phi_1 \mathsf{U} \phi_2$
  iff $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, p \models \phi_1 \mathsf{U} \phi_2$
  iff there exists k, $p \leq k \leq n$, $\langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, k \models \phi_2$ and for every j, $p \leq j < k, \langle \mathcal{T}, (\mathcal{A}_i^{\mathit{IR}})_{0 \leq i \leq n} \rangle, j \models \phi_1$
  iff there exists k, $p \leq k \leq n$, $\mathcal{K}, k \models_{\text{IAR}} \phi_2$ and for every j, $p \leq j < k, \mathcal{K}, j \models_{\text{IAR}} \phi_1 \mathsf{U} \phi_2$

- $\mathcal{K}, p \models_{\text{IAR}} \bigcirc^- \phi_1$, $\mathcal{K}, p \models_{\text{IAR}} \bullet^- \phi_1$, $\mathcal{K}, p \models_{\text{IAR}} \Box^- \phi_1$, $\mathcal{K}, p \models_{\text{IAR}} \Diamond^- \phi_1$, $\mathcal{K}, p \models_{\text{IAR}} \phi_1 \mathsf{S} \phi_2$:
  similar to the corresponding future operators

$\square$

This is a remarkable result, since it follows from it that answering temporal CQs under IAR semantics can be done with the algorithms developed for the consistent case [10, 11] by replacing classical CQ answering by IAR CQ answering (see [21, 8, 26] for algorithms). The following example shows that this is unfortunately not true for brave or AR semantics.

Example 3. Consider the following TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{1\leq i\leq n}\rangle$ and TCQ ϕ.

$$\mathcal{T} = \{T \sqsubseteq \neg F\} \qquad \mathcal{A}_i = \{T(a), F(a)\} \text{ for } 1 \leq i \leq n \qquad \phi = \Box^-(T(a) \wedge \newmoon^- F(a))$$

Now, $\mathcal{K}, k \models_{\text{brave}} T(a) \wedge \newmoon^- F(a)$ for every $0 \leq k \leq n$, but $\mathcal{K}, n \not\models_{\text{brave}} \phi$. This is because the same repair cannot entail $T(a) \wedge \newmoon^- F(a)$ both at time point k and $k+1$, since it would contain both $(T(a), k)$ and $(F(a), k)$ which is not possible. For AR semantics, consider $\phi = T(a) \vee F(a)$ over the TKB $\mathcal{K}$: while ϕ holds under AR semantics at each time point, neither $T(a)$ nor $F(a)$ does.

However, if the operators allowed in the TCQ are restricted to $\wedge, \bigcirc, \newmoon, \bigcirc^-, \newmoon^-, \Box$, and $\Box^-$, then AR TCQ answering can be done with the algorithms developed for the consistent case by simply replacing classical CQ answering by AR CQ answering (see [8] for algorithms). Indeed, for these operators, the relations of Proposition 2 hold for S = AR:

- $\mathcal{K}, p \models_{\text{AR}} \exists \vec{y}\, \psi(\vec{y})$
  iff for every repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$, $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leq i\leq n}\rangle, p \models \exists \vec{y}\, \psi(\vec{y})$
  iff for every repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$, $\langle \mathcal{T}, \mathcal{A}'_p\rangle \models \exists \vec{y}\, \psi(\vec{y})$
  iff for every repair $\mathcal{A}'_p$ of $\langle \mathcal{T}, \mathcal{A}_p\rangle$, $\langle \mathcal{T}, \mathcal{A}'_p\rangle \models \exists \vec{y}\, \psi(\vec{y})$ since the repairs of $\mathcal{K}$ are the sequences of the repairs of the $\langle \mathcal{T}, \mathcal{A}_i\rangle$
  iff $\langle \mathcal{T}, \mathcal{A}_p\rangle \models_{\text{AR}} \exists \vec{y}\, \psi(\vec{y})$

- $\mathcal{K}, p \models_{\text{AR}} \phi_1 \wedge \phi_2$
  iff for every repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$, $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leq i\leq n}\rangle, p \models \phi_1 \wedge \phi_2$
  iff for every repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$, $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leq i\leq n}\rangle, p \models \phi_1$ and $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leq i\leq n}\rangle, p \models \phi_2$
  iff $\mathcal{K}, p \models_{\text{AR}} \phi_1$ and $\mathcal{K}, p \models_{\text{AR}} \phi_2$

- $\mathcal{K}, p \models_{\text{AR}} \bigcirc\phi_1$
  iff for every repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$, $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leq i\leq n}\rangle, p \models \bigcirc\phi_1$
  iff for every repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$, $p < n$ and $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leq i\leq n}\rangle, p+1 \models \phi_1$
  iff $p < n$ and $\mathcal{K}, p+1 \models_{\text{AR}} \phi_1$

- $\mathcal{K}, p \models_{\text{AR}} \newmoon\phi_1$
  iff for every repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$, $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leq i\leq n}\rangle, p \models \newmoon\phi_1$
  iff for every repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$, $p < n$ implies $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leq i\leq n}\rangle, p+1 \models \phi_1$
  iff $p < n$ implies $\mathcal{K}, p+1 \models_{\text{AR}} \phi_1$

- $\mathcal{K}, p \models_{\text{AR}} \Box\phi_1$
  iff for every repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$, $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leq i\leq n}\rangle, p \models \Box\phi_1$
  iff for every repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$, for every k, $p \leq k \leq n$, $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leq i\leq n}\rangle, k \models \phi_1$
  iff for every k, $p \leq k \leq n$, $\mathcal{K}, k \models_{\text{AR}} \phi_1$

- $\mathcal{K}, p \models_{\text{AR}} \bigcirc^-\phi_1$, $\mathcal{K}, p \models_{\text{AR}} \newmoon^-\phi_1$, $\mathcal{K}, p \models_{\text{AR}} \Box^-\phi_1$: similar to the corresponding future operators

The following counter examples show that this is not the case for the other operators $(\vee, \Diamond, \Diamond^-, \mathsf{U}, \mathsf{S})$:

- $\mathcal{K}, 1 \models_{\text{AR}} \phi_1 \vee \phi_2$ but $\mathcal{K}, 1 \not\models_{\text{AR}} \phi_1$ and $\mathcal{K}, 1 \not\models_{\text{AR}} \phi_2$:

$$\mathcal{T} = \{A \sqsubseteq \neg B\} \qquad \phi_1 = A(a) \qquad \phi_2 = B(a) \qquad \mathcal{A}_1 = \{A(a), B(a)\}$$

- $\mathcal{K}, 1 \models_{\text{AR}} \Diamond\phi_1$ but for every k, $1 \leq k \leq 3$, $\mathcal{K}, k \not\models_{\text{AR}} \phi_1$:

$$\begin{array}{lll} \mathcal{T} = \{A \sqsubseteq \neg B\} & \phi_1 = A(a) \wedge \bigcirc B(a) & \\ \mathcal{A}_1 = \{A(a)\} & \mathcal{A}_2 = \{A(a), B(a)\} & \mathcal{A}_3 = \{B(a)\} \end{array}$$

- $\mathcal{K}, 1 \models_{\text{AR}} \phi_1 \mathsf{U} \phi_2$ but for every k, $1 \leq k \leq 3$, either $\mathcal{K}, k \not\models_{\text{AR}} \phi_2$ or there exists j, $1 \leq j < k$, $\mathcal{K}, j \not\models_{\text{AR}} \phi_1$:

$$\begin{array}{lll} \mathcal{T} = \{A \sqsubseteq \neg B\} & \phi_1 = A(a) & \phi_2 = B(a) \\ \mathcal{A}_1 = \{A(a)\} & \mathcal{A}_2 = \{A(a), B(a)\} & \mathcal{A}_3 = \{B(a)\} \end{array}$$

- Similar counter example to $\Diamond$ for $\Diamond^-$ and to U for S.

Interestingly, contrary to the brave semantics, even for general TCQs the "if" direction of Proposition 2 is true:

- if $\mathcal{K}, p \models_{\text{AR}} \phi_1$ or $\mathcal{K}, p \models_{\text{AR}} \phi_2$, then $\mathcal{K}, p \models_{\text{AR}} \phi_1 \vee \phi_2$
- if there exists k, $p \leq k \leq n$, $\mathcal{K}, k \models_{\text{AR}} \phi_1$, then $\mathcal{K}, p \models_{\text{AR}} \Diamond\phi_1$
- if there exists k, $0 \leq k \leq p$, $\mathcal{K}, k \models_{\text{AR}} \phi_1$, then $\mathcal{K}, p \models_{\text{AR}} \Diamond^-\phi_1$
- if there exists k, $p \leq k \leq n$, $\mathcal{K}, k \models_{\text{AR}} \phi_2$ and for every j, $p \leq j < k$, $\mathcal{K}, j \models_{\text{AR}} \phi_1$, then $\mathcal{K}, p \models_{\text{AR}} \phi_1 \mathsf{U} \phi_2$
- if there exists k, $0 \leq k \leq p$, $\mathcal{K}, k \models_{\text{AR}} \phi_2$ and for every j, $k < j \leq p$, $\mathcal{K}, j \models_{\text{AR}} \phi_1$, then $\mathcal{K}, p \models_{\text{AR}} \phi_1 \mathsf{S} \phi_2$

It follows that even for unrestricted TCQs, combining algorithms for TCQ answering with algorithms for AR query answering will provide *a sound approximation* of AR answers.

4 Complexity Analysis for DL-Lite$_{\mathcal{R}}$

In this section, $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leq i \leq n} \rangle$ is a DL-Lite$_{\mathcal{R}}$ TKB and ϕ is a BTCQ. The set of constants of ϕ is denoted by $\mathsf{N}_\mathsf{I}^\phi$. We make use of the following notations: for a role P and two constants or variables x and y, $P^- := S$ if $P = S^-$ and $P(x, y)$ denotes $S(x, y)$ if $P = S$ and $S(y, x)$ if $P = S^-$. We assume w.l.o.g. that no $x \in \mathsf{N}_\mathsf{I}^\mathcal{K}$ is of the form x_w^e where w, e are words built over $\mathsf{N}_\mathsf{I}^\mathcal{K} \cup \mathsf{N}_\mathsf{C}^\mathcal{K} \cup \mathsf{N}_\mathsf{R}^\mathcal{K}$ and $\mathbb{N}$ respectively.

We recall the definitions of the complexity classes that appear in this section:

- P: problems which are solvable in polynomial time.
- NP: problems which are solvable in non-deterministic polynomial time.
- coNP: problems whose complement is in NP.
- Σ_2^p: problems which are solvable in non-deterministic polynomial time with an NP oracle.
- Π_2^p: problems whose complement is in Σ_2^p.

- ALogTime: class of languages decidable in logarithmic time by a random access alternating Turing machine. In this work, we only use that ALogTime $\subseteq$ P.
- PSpace: problems which are solvable in polynomial space.

We conclude this introductory paragraph with the notions of conflicts and causes that will be used in some proofs. A *conflict* for a KB $\mathcal{K} = \langle \mathcal{T}, \mathcal{A} \rangle$ is a minimal $\mathcal{T}$-inconsistent subset of $\mathcal{A}$. A *cause* for a BCQ q w.r.t. $\mathcal{K}$ is a minimal $\mathcal{T}$-consistent subset $\mathcal{C} \subseteq \mathcal{A}$ such that $\langle \mathcal{T}, \mathcal{C} \rangle \models q$. The following definitions extend these notions to the temporal setting.

Definition 5 (Conflicts of a TKB)**.** A *conflict* of a TKB $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leq i \leq n} \rangle$ is a sequence of ABoxes $(\mathcal{A}'_i)_{0 \leq i \leq n}$ such that $\{(\alpha, i) \mid \alpha \in \mathcal{A}'_i, 0 \leq i \leq n\}$ is a minimal $\mathcal{T}$-inconsistent subset of $\{(\alpha, i) \mid \alpha \in \mathcal{A}_i, 0 \leq i \leq n\}$.

Because of DL-Lite$_{\mathcal{R}}$ syntax, the conflicts of a DL-Lite$_{\mathcal{R}}$ TKB are at most binary, i.e., contain at most two timed-assertions.

Definition 6 (Causes for a BTCQ in a TKB)**.** A *cause* for a BTCQ ϕ at time point p in $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leq i \leq n} \rangle$ is a sequence of ABoxes $(\mathcal{C}_i)_{0 \leq i \leq n}$ such that $\{(\alpha, i) \mid \alpha \in \mathcal{C}_i, 0 \leq i \leq n\}$ is a minimal $\mathcal{T}$-consistent subset of $\{(\alpha, i) \mid \alpha \in \mathcal{A}_i, 0 \leq i \leq n\}$ such that $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle, p \models \phi$.

Note that a KB (resp. TKB) is consistent iff it has no conflict, and that a BCQ (resp. BTCQ) is entailed from a KB (resp. a TKB) $\mathcal{K}$ under brave semantics iff it has some cause in $\mathcal{K}$, since such a cause can be extended to a repair that entails the query.

4.1 Complexity of TCQ answering for the classical semantics

The complexity of TCQ answering under the classical semantics in DL-Lite$_{\mathcal{R}}$ with *negations* in the query has been shown ALogTime-complete w.r.t. data complexity and PSpace-complete w.r.t. combined complexity, rigid concepts and roles being present or not [12]. In our case, i.e., without negations, CQ evaluation over databases provides a NP lower bound for combined complexity and it has been shown in [10, 11] that TCQs in DL-Lite$_{\mathcal{R}}$ are rewritable so that they can be answered over a temporal database—albeit for a restricted setting without rigid roles and with rigid concepts only for TCQs that are rooted. The NP membership of TCQ answering in Case 1 for combined complexity is implied by this latter work as follows: it is possible to guess for each time point i and CQ q from the TCQ either a rewriting q' of q that holds in $\mathcal{A}_i$ together with the rewriting steps that produce q' and the variables assignment that maps q' in $\mathcal{A}_i$, or to guess "false". Checking that q' is indeed a rewriting of q and holds in $\mathcal{A}_i$ can be done in polynomial time and there are polynomially many such pairs of a time point and a CQ to test. Moreover, verifying that the propositional LTL formula obtained by replacing the CQs by propositional variables is satisfied by the sequence of truth assignments that assigns the propositional abstraction of q to false at time point i if "false" has been guessed and to true otherwise is in P since the formula does not contain negation. It follows that TCQ answering is NP-complete w.r.t. combined complexity. To alleviate the limitations imposed in [10, 11], we first show that TCQ answering without negations is NP-complete w.r.t. combined complexity even in the presence of rigid concepts and roles, with the restriction that a rigid role can only have rigid sub-roles. Indeed, we show that under this restriction, TCQ answering in Case 3 can be reduced to TCQ answering in Case 1 by adding to every ABox a set of assertions that models rigid consequences of the TKB and is computable in polynomial time.

As a first step, we assume that $\mathcal{K}$ is consistent and construct a model $\mathcal{J}_{\mathcal{K}}$ of $\mathcal{K}$ such that for any BTCQ ϕ such that $\mathsf{N}_{\mathsf{I}}^{\phi} \subseteq \mathsf{N}_{\mathsf{I}}^{\mathcal{K}}$, $\mathcal{K}, p \models \phi$ iff $\mathcal{J}_{\mathcal{K}}, p \models \phi$. This model will be used latter to prove that in the case where $\mathcal{K}$ is consistent, TCQ answering gives the same answers over $\mathcal{K}$ and over the TKB we will construct by adding a set of assertions to each ABox and $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$. We build a sequence of (possibly infinite) ABoxes $(\mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_i))_{0\leq i\leq n}$ similar to the chase presented in [14] for KBs. Let $\mathcal{S}$ be a set of DL-Lite$_\mathcal{R}$ assertions. A PI α is applicable in $\mathcal{S}$ to an assertion $\beta \in \mathcal{S}$ if

- $\alpha = A_1 \sqsubseteq A_2$, $\beta = A_1(a)$, and $A_2(a) \notin \mathcal{S}$
- $\alpha = A \sqsubseteq \exists P$, $\beta = A(a)$, and there is no b such that $P(a, b) \in \mathcal{S}$
- $\alpha = \exists P \sqsubseteq A$, $\beta = P(a, b)$, and $A(a) \notin \mathcal{S}$
- $\alpha = \exists P_1 \sqsubseteq \exists P_2$, $\beta = P_1(a_1, a_2)$, and there is no b such that $P_2(a_1, b) \in \mathcal{S}$
- $\alpha = P_1 \sqsubseteq P_2$, $\beta = P_1(a_1, a_2)$, and $P_2(a_1, a_2) \notin \mathcal{S}$.

Applying a PI α to an assertion β means adding a new suitable assertion β_{new} to $\mathcal{S}$ such that α is not applicable to β in $\mathcal{S} \cup \{\beta_{\mathrm{new}}\}$.

Definition 7 (Rigid chase of a TKB)**.** Let $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0\leq i\leq n}\rangle$ be a DL-Lite$_\mathcal{R}$ TKB. Let $(\mathcal{A}'_i)_{0\leq i\leq n} = (\mathcal{A}_i \cup \{\beta \mid \exists k, \beta \in \mathcal{A}_k \text{ and } \beta \text{ is rigid}\})_{0\leq i\leq n}$, let $\mathcal{T}_p$ be the set of positive inclusions in $\mathcal{T}$, and let N_i be the number of assertions in $\mathcal{A}'_i$. Assume that the assertions of each $\mathcal{A}'_i$ are numbered from $N_1 + \cdots + N_{i-1} + 1$ to $N_1 + \cdots + N_i$ following their lexicographic order. Consider the sequences of sets of assertions $\mathcal{S}^j = (\mathcal{S}^j_i)_{0\leq i\leq n}$ defined as follows:

$$\mathcal{S}^0 = (\mathcal{A}'_i)_{0\leq i\leq n} \quad \text{and} \quad \mathcal{S}^{j+1} = \mathcal{S}^j \cup \mathcal{S}^{\mathrm{new}} = (\mathcal{S}^j_i \cup \mathcal{S}^{\mathrm{new}}_i)_{0\leq i\leq n},$$

where $\mathcal{S}^{\mathrm{new}}$ is defined in terms of the assertion β_{new} obtained as follows: let $\beta \in \mathcal{S}^j_{i_\beta}$ be the first assertion in $\mathcal{S}^j$ such that there exists a PI in $\mathcal{T}_p$ applicable in $\mathcal{S}^j_{i_\beta}$ to β and let α be the lexicographically first PI applicable in $\mathcal{S}^j_{i_\beta}$ to β. In case α, β are of the form

- $\alpha = A_1 \sqsubseteq A_2$ and $\beta = A_1(a)$ then $\beta_{\mathrm{new}} = A_2(a)$
- $\alpha = A \sqsubseteq \exists P$ and $\beta = A(a)$ then $\beta_{\mathrm{new}} = P(a, a_{\mathrm{new}})$
- $\alpha = \exists P \sqsubseteq A$ and $\beta = P(a, b)$ then $\beta_{\mathrm{new}} = A(a)$
- $\alpha = \exists P_1 \sqsubseteq \exists P$ and $\beta = P_1(a, b)$ then $\beta_{\mathrm{new}} = P(a, a_{\mathrm{new}})$
- $\alpha = P_1 \sqsubseteq P_2$ and $\beta = P_1(a_1, a_2)$ then $\beta_{\mathrm{new}} = P_2(a_1, a_2)$

where a_{new} is constructed from α and β as follows:

- if $a \in \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$ then $a_{\mathrm{new}} = x^{i_\beta}_{aP}$
- otherwise $a \notin \mathsf{N}^{\mathcal{K}}_{\mathsf{I}}$, then let $a = x^{i_1\dots i_l}_{a'P_1\dots P_l}$ and define $a_{\mathrm{new}} = x^{i_1\dots i_l i_\beta}_{a'P_1\dots P_l P}$.

If β_{new} is rigid, then $\mathcal{S}^{\mathrm{new}} = (\{\beta_{\mathrm{new}}\})_{0\leq i\leq n}$, otherwise, $\mathcal{S}^{\mathrm{new}} = (\mathcal{S}^{\mathrm{new}}_i)_{0\leq i\leq n}$ with $\mathcal{S}^{\mathrm{new}}_{i_\beta} = \{\beta_{\mathrm{new}}\}$ and $\mathcal{S}^{\mathrm{new}}_i = \emptyset$ for $i \neq i_\beta$.

Let N be the total number of assertions in $\mathcal{S}^j$. The assertion(s) added are numbered as follows: if β_{new} is not rigid, β_{new} is numbered by $N + 1$, otherwise for every $0 \leq i \leq n$, the assertion $\beta_{\mathrm{new}} \in \mathcal{S}^{\mathrm{new}}_i$ added to $\mathcal{S}^j_i$ is numbered by $N + 1 + i$.

We call the rigid chase of $\mathcal{K}$, denoted by $\mathit{chase}_{\mathrm{rig}}(\mathcal{K}) = (\mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_i))_{0\leq i\leq n}$, the sequence of sets of assertions obtained as the infinite union of all $\mathcal{S}^j$, i.e.,

$$\mathit{chase}_{\mathrm{rig}}(\mathcal{K}) = (\mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_i))_{0\leq i\leq n} = \bigcup_{j\in\mathbb{N}} \mathcal{S}^j = (\bigcup_{j\in\mathbb{N}} \mathcal{S}^j_i)_{0\leq i\leq n}.$$

Let Γ_N be the set of individuals that appear in $\mathit{chase}_{\mathrm{rig}}(\mathcal{K})$ but not in $\mathcal{K}$. The following properties of $\mathit{chase}_{\mathrm{rig}}(\mathcal{K})$ will be useful:

Proposition 3. *$\mathit{chase}_{rig}(\mathcal{K})$ is such that:*

(P1) $x^{i_1}_{aP_1} \in \Gamma_N \implies P_1(a, x^{i_1}_{aP_1}) \in \mathit{chase}^{\mathcal{K}}_{rig}(\mathcal{A}_{i_1})$

(P2) $x^{i_1 \ldots i_l}_{aP_1 \ldots P_l} \in \Gamma_N, l > 1 \implies P_l(x^{i_1 \ldots i_{l-1}}_{aP_1 \ldots P_{l-1}}, x^{i_1 \ldots i_l}_{aP_1 \ldots P_l}) \in \mathit{chase}^{\mathcal{K}}_{rig}(\mathcal{A}_{i_l})$

(P3) $\mathit{chase}^{\mathcal{K}}_{rig}(\mathcal{A}_i) \models B(x^{i_1 \ldots i_l}_{aP_1 \ldots P_l}) \implies \mathcal{T} \models \exists P_l^- \sqsubseteq B$

(P4) $x^{i_1 \ldots i_l}_{aP_1 \ldots P_l} \in \Gamma_N, l > 1 \implies \mathcal{T} \models \exists P_{l-1}^- \sqsubseteq \exists P_l$

(P5) $\mathit{chase}^{\mathcal{K}}_{rig}(\mathcal{A}_i) \models B(a), a \in \mathsf{N}_{\mathsf{I}}^{\mathcal{K}} \implies \langle \mathcal{T}, \mathcal{A}_i \rangle \models B(a)$ *or there exists* $B' := A | \exists R | \exists R^-$ *with* $A \in \mathsf{N}_{\mathsf{RC}}, R \in \mathsf{N}_{\mathsf{RR}}$ *such that* $\mathcal{T} \models B' \sqsubseteq B$ *and there exists* j *such that* $\langle \mathcal{T}, \mathcal{A}_j \rangle \models B'(a)$

(P6) $\mathit{chase}^{\mathcal{K}}_{rig}(\mathcal{A}_i) \models B(x^{i_1 \ldots i_l}_{aP_1 \ldots P_l}) \implies i = i_l$ *or there exists* $B' := A | \exists R | \exists R^-$ *with* $A \in \mathsf{N}_{\mathsf{RC}}, R \in \mathsf{N}_{\mathsf{RR}}$ *such that* $\mathcal{T} \models B' \sqsubseteq B$ *and* $\mathit{chase}^{\mathcal{K}}_{rig}(\mathcal{A}_{i_l}) \models B'(x^{i_1 \ldots i_l}_{aP_1 \ldots P_l})$

(P7) $P(a, b) \in \mathit{chase}^{\mathcal{K}}_{rig}(\mathcal{A}_i), a, b \in \mathsf{N}_{\mathsf{I}}^{\mathcal{K}} \implies \langle \mathcal{T}, \mathcal{A}_i \rangle \models P(a, b)$ *or there exists* $P' := R | R^-$ *with* $R \in \mathsf{N}_{\mathsf{RR}}$ *such that* $\mathcal{T} \models P' \sqsubseteq P$ *and there exists* j *such that* $\langle \mathcal{T}, \mathcal{A}_j \rangle \models P'(a, b)$

(P8) $P(a, x^{i_1}_{aP_1}) \in \mathit{chase}^{\mathcal{K}}_{rig}(\mathcal{A}_i), a \in \mathsf{N}_{\mathsf{I}}^{\mathcal{K}}, i_1 = i \implies \mathcal{T} \models P_1 \sqsubseteq P$ *and* $\langle \mathcal{T}, \mathcal{A}_i \rangle \models \exists x P_1(a, x)$ *or there exists* $B := A | \exists R | \exists R^-$ *with* $A \in \mathsf{N}_{\mathsf{RC}}, R \in \mathsf{N}_{\mathsf{RR}}$ *such that* $\mathcal{T} \models B \sqsubseteq \exists P_1$ *and there exists* j *such that* $\langle \mathcal{T}, \mathcal{A}_j \rangle \models B(a)$

(P9) $P(a, x^{i_1}_{aP_1}) \in \mathit{chase}^{\mathcal{K}}_{rig}(\mathcal{A}_i), a \in \mathsf{N}_{\mathsf{I}}^{\mathcal{K}}, i_1 \neq i \implies$ *there exists* $P' := R | R^-$ *with* $R \in \mathsf{N}_{\mathsf{RR}}$ *such that* $\mathcal{T} \models P_1 \sqsubseteq P' \sqsubseteq P$

(P10) $P(x, y) \in \mathit{chase}^{\mathcal{K}}_{rig}(\mathcal{A}_i), x, y \in \Gamma_N \implies x = x^{i_1 \ldots i_l}_{aP_1 \ldots P_l}, y = x^{i_1 \ldots i_l i_{l+1}}_{aP_1 \ldots P_l P_{l+1}}$ *and* $\mathcal{T} \models P_{l+1} \sqsubseteq P$ *or* $x = x^{i_1 \ldots i_l i_{l+1}}_{aP_1 \ldots P_l P_{l+1}}, y = x^{i_1 \ldots i_l}_{aP_1 \ldots P_l}$ *and* $\mathcal{T} \models P_{l+1} \sqsubseteq P^-$

(P11) $P(x^{i_1 \ldots i_l}_{aP_1 \ldots P_l}, x^{i_1 \ldots i_l i_{l+1}}_{aP_1 \ldots P_l P_{l+1}}) \in \mathit{chase}^{\mathcal{K}}_{rig}(\mathcal{A}_i), i_{l+1} \neq i \implies$ *there exists* $P' := R | R^-$ *with* $R \in \mathsf{N}_{\mathsf{RR}}$ *such that* $\mathcal{T} \models P_{l+1} \sqsubseteq P' \sqsubseteq P$ *and* $P'(x^{i_1 \ldots i_l}_{aP_1 \ldots P_l}, x^{i_1 \ldots i_l i_{l+1}}_{aP_1 \ldots P_l P_{l+1}}) \in \mathit{chase}^{\mathcal{K}}_{rig}(\mathcal{A}_{i_{l+1}})$

(P12) $P_l(x^{i_1 \ldots i_{l-1}}_{aP_1 \ldots P_{l-1}}, x^{i_1 \ldots i_l}_{aP_1 \ldots P_l}) \in \mathit{chase}^{\mathcal{K}}_{rig}(\mathcal{A}_{i_l}) \implies \exists j, \langle \mathcal{T}, \mathcal{A}_j \rangle \models \exists xy P_{l-1}(x, y)$

Proof. (P1) If $x^{i_1}_{aP_1} \in \Gamma_N$, $x^{i_1}_{aP_1}$ has been introduced to construct $P_1(a, x^{i_1}_{aP_1})$ at some step j of the construction of the chase by applying a PI to an assertion $\beta \in \mathcal{S}^j_{i_1}$, so $P_1(a, x^{i_1}_{aP_1}) \in \mathcal{S}^{j+1}_{i_1}$, so $P_1(a, x^{i_1}_{aP_1}) \in \mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_{i_1})$.

(P2) If $x^{i_1 \ldots i_l}_{aP_1 \ldots P_l} \in \Gamma_N$, $x^{i_1 \ldots i_l}_{aP_1 \ldots P_l}$ has been introduced to construct $P_l(x^{i_1 \ldots i_{l-1}}_{aP_1 \ldots P_{l-1}}, x^{i_1 \ldots i_l}_{aP_1 \ldots P_l})$ at some step j of the construction of the chase by applying a PI to an assertion $\beta \in \mathcal{S}^j_{i_l}$, so $P_l(x^{i_1 \ldots i_{l-1}}_{aP_1 \ldots P_{l-1}}, x^{i_1 \ldots i_l}_{aP_1 \ldots P_l}) \in \mathcal{S}^{j+1}_{i_l}$, so $P_l(x^{i_1 \ldots i_{l-1}}_{aP_1 \ldots P_{l-1}}, x^{i_1 \ldots i_l}_{aP_1 \ldots P_l}) \in \mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_{i_l})$.

(P3) We show that if there is some i and step j such that $\mathcal{S}_i^j \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$ then $\mathcal{T} \models \exists P_l^- \sqsubseteq B$ by induction on $p = j - s$ where s is the step where $x^{i_1\dots i_l}_{aP_1\dots P_l}$ has been introduced to produce $P_l(x^{i_1\dots i_{l-1}}_{aP_1\dots P_{l-1}}, x^{i_1\dots i_l}_{aP_1\dots P_l})$. If $p = 0$, since $P_l(x^{i_1\dots i_{l-1}}_{aP_1\dots P_{l-1}}, x^{i_1\dots i_l}_{aP_1\dots P_l})$ is the only assertion of $\mathcal{S}^j$ that contains the individual $x^{i_1\dots i_l}_{aP_1\dots P_l}$, if $\mathcal{S}_i^j \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$, it follows that $P_l(x^{i_1\dots i_{l-1}}_{aP_1\dots P_{l-1}}, x^{i_1\dots i_l}_{aP_1\dots P_l}) \in \mathcal{S}_i^j$ and that $P_l(x^{i_1\dots i_{l-1}}_{aP_1\dots P_{l-1}}, x^{i_1\dots i_l}_{aP_1\dots P_l}) \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$, so $B = \exists P_l^-$ and $\mathcal{T} \models \exists P_l^- \sqsubseteq B$. For $p > 0$, assume that $\mathcal{S}_i^j \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$. If there exists i' such that $\mathcal{S}_{i'}^{j-1} \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$, $\mathcal{T} \models \exists P_l^- \sqsubseteq B$ by induction hypothesis. Otherwise, let $\beta \in \mathcal{S}_i^j$ be such that $\beta \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$. Since $\beta \notin \mathcal{S}_i^{j-1}$, it has been created at step j by applying a PI $\alpha \in \mathcal{T}$ to an assertion $\beta' \in \mathcal{S}_{i'}^{j-1}$. Either $\beta' = A(x^{i_1\dots i_l}_{aP_1\dots P_l})$ for some concept A, so by induction hypothesis $\mathcal{T} \models \exists P_l^- \sqsubseteq A$, and since $\alpha, \beta' \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$, $\mathcal{T} \models A \sqsubseteq B$ and $\mathcal{T} \models \exists P_l^- \sqsubseteq B$, or $\beta' = P(x^{i_1\dots i_l}_{aP_1\dots P_l}, x)$ for some role P, so by induction hypothesis $\mathcal{T} \models \exists P_l^- \sqsubseteq \exists P$, and since $\alpha, \beta' \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$, $\mathcal{T} \models \exists P \sqsubseteq B$ and $\mathcal{T} \models \exists P_l^- \sqsubseteq B$.

(P4) By (P2) $\mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_{i_l}) \models \exists P_l(x^{i_1\dots i_{l-1}}_{aP_1\dots P_{l-1}})$ so by (P3), it follows that $\mathcal{T} \models \exists P_{l-1}^- \sqsubseteq \exists P_l$.

(P5) We show that if $\mathcal{S}_i^j \models B(a)$ then $\langle \mathcal{T}, \mathcal{A}_i \rangle \models B(a)$ or there exist $B' := A | \exists R | \exists R^-$ with $A \in \mathsf{N_{RC}}, R \in \mathsf{N_{RR}}$ such that $\mathcal{T} \models B' \sqsubseteq B$ and i' such that $\langle \mathcal{T}, \mathcal{A}_{i'} \rangle \models B'(a)$ by induction on j. If $j = 0$, since $\mathcal{S}_i^0 = \mathcal{A}_i \cup \{\beta \mid \exists k, \beta \in \mathcal{A}_k \text{ and } \beta \text{ is rigid}\}$, then either $\mathcal{A}_i \models B(a)$ or there exist k and a rigid assertion $\beta \in \mathcal{A}_k$ such that $\beta \models B(a)$, so there exists $B' := A | \exists R | \exists R^-$ with $A \in \mathsf{N_{RC}}, R \in \mathsf{N_{RR}}$ such that $\mathcal{T} \models B' \sqsubseteq B$ and $\mathcal{A}_k \models B'(a)$. For $j > 0$, assume that $\mathcal{S}_i^j \models B(a)$. If $\mathcal{S}_i^{j-1} \models B(a)$, we apply the induction hypothesis. Otherwise, let $\beta \in \mathcal{S}_i^j$ be such that $\beta \models B(a)$. Since $\beta \notin \mathcal{S}_i^{j-1}$, it has been created at step j by applying a PI $\alpha \in \mathcal{T}$ to an assertion $\beta' \in \mathcal{S}_{i'}^{j-1}$. Either $\beta' = A(a)$ for some concept A, and since $\alpha, \beta' \models B(a)$, $\mathcal{T} \models A \sqsubseteq B$, or $\beta' = P(a, x)$ for some role P, and since $\alpha, \beta' \models B(a)$, $\mathcal{T} \models \exists P \sqsubseteq B$. Let $C = A$ in the first case, $C = \exists P$ in the second case. $\mathcal{S}_{i'}^{j-1} \models C(a)$ so by induction hypothesis $\langle \mathcal{T}, \mathcal{A}_{i'} \rangle \models C(a) \models B(a)$ or there exist a rigid concept C' such that $\mathcal{T} \models C' \sqsubseteq C \sqsubseteq B$ and i'' such that $\langle \mathcal{T}, \mathcal{A}_{i''} \rangle \models C'(a)$. In the first case, either $i' = i$ and $\langle \mathcal{T}, \mathcal{A}_i \rangle \models B(a)$, or $i' \neq i$, so since $\beta \in \mathcal{S}_i^j$, β is rigid and B is rigid.

(P6) We show that if $\mathcal{S}_i^j \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$ then $i = i_l$ or there exist $B' := A | \exists R | \exists R^-$ with $A \in \mathsf{N_{RC}}, R \in \mathsf{N_{RR}}$ such that $\mathcal{T} \models B' \sqsubseteq B$ and i' such that $\mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_{i'}) \models B'(x^{i_1\dots i_l}_{aP_1\dots P_l})$ by induction on $p = j - s$ where s is the step where $x^{i_1\dots i_l}_{aP_1\dots P_l}$ has been introduced to produce $P_l(x^{i_1\dots i_{l-1}}_{aP_1\dots P_{l-1}}, x^{i_1\dots i_l}_{aP_1\dots P_l})$. If $p = 0$, since $P_l(x^{i_1\dots i_{l-1}}_{aP_1\dots P_{l-1}}, x^{i_1\dots i_l}_{aP_1\dots P_l})$ is the only assertion of $\mathcal{S}^j$ that contains $x^{i_1\dots i_l}_{aP_1\dots P_l}$, if $\mathcal{S}_i^j \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$, $B = \exists P_l^-$ and $P_l(x^{i_1\dots i_{l-1}}_{aP_1\dots P_{l-1}}, x^{i_1\dots i_l}_{aP_1\dots P_l}) \in \mathcal{S}_i^j$ so $i = i_l$ or P_l is rigid. For $p > 0$, assume that $\mathcal{S}_i^j \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$. If $\mathcal{S}_i^{j-1} \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$, we apply the induction hypothesis. Otherwise, let $\beta \in \mathcal{S}_i^j$ be such that $\beta \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$. Since $\beta \notin \mathcal{S}_i^{j-1}$, it has been created at step j by applying a PI $\alpha \in \mathcal{T}$ to an assertion $\beta' \in \mathcal{S}_{i'}^{j-1}$. Either $\beta' = A(x^{i_1\dots i_l}_{aP_1\dots P_l})$ for some concept A, and since $\alpha, \beta' \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$, $\mathcal{T} \models A \sqsubseteq B$, or $\beta' = P(x^{i_1\dots i_l}_{aP_1\dots P_l}, x)$ for some role P, and since $\alpha, \beta' \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$, $\mathcal{T} \models \exists P \sqsubseteq B$. Let $C = A$ in the first case, $C = \exists P$ in the second case. $\mathcal{S}_{i'}^{j-1} \models C(x^{i_1\dots i_l}_{aP_1\dots P_l})$ so by induction hypothesis $i' = i_l$ or there exist a rigid concept C' such that $\mathcal{T} \models C' \sqsubseteq C \sqsubseteq B$ and i'' such that $\mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_{i''}) \models C'(x^{i_1\dots i_l}_{aP_1\dots P_l})$.

(P7) We show that if $P(a, b) \in \mathcal{S}_i^j$ then $\langle \mathcal{T}, \mathcal{A}_i \rangle \models P(a, b)$ or there exist a rigid role P' such that $\mathcal{T} \models P' \sqsubseteq P$ and i' such that $\langle \mathcal{T}, \mathcal{A}_{i'} \rangle \models P'(a, b)$ by induction on j. If $j = 0$, since $\mathcal{S}_i^0 = \mathcal{A}_i \cup \{\beta \mid \exists k, \beta \in \mathcal{A}_k \text{ and } \beta \text{ is rigid}\}$, then either $P(a, b) \in \mathcal{A}_i$ or P is rigid and there exist k such that $P(a, b) \in \mathcal{A}_k$. For $j > 0$, assume that $P(a, b) \in \mathcal{S}_i^j$. If $P(a, b) \in \mathcal{S}_i^{j-1}$, we apply the induction hypothesis. Otherwise, since $P(a, b) \notin \mathcal{S}_i^{j-1}$, it has been created at step j by applying a PI $P' \sqsubseteq P \in \mathcal{T}$ to an assertion $P'(a, b) \in \mathcal{S}_{i'}^{j-1}$, so by induction hypothesis $\langle \mathcal{T}, \mathcal{A}_{i'} \rangle \models P'(a, b) \models P(a, b)$ or there exist a rigid role P'' such that $\mathcal{T} \models P'' \sqsubseteq P' \sqsubseteq P$ and i''

such that $\langle \mathcal{T}, \mathcal{A}_{i''} \rangle \models P''(a, b)$. In the first case, either $i' = i$ and $\langle \mathcal{T}, \mathcal{A}_i \rangle \models P(a, b)$, or $i' \neq i$, so since $P(a, b) \in \mathcal{S}_i^j$, P is rigid.

(P8) First, since $P(a, x^{i_1}_{aP_1}) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{i_1})$, $\mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{i_1}) \models \exists P_1(a)$, so by (P5), either $\langle \mathcal{T}, \mathcal{A}_{i_1} \rangle \models \exists x P_1(a, x)$ or there exist a rigid concept B such that $\mathcal{T} \models B \sqsubseteq \exists P_1$ and i such that $\langle \mathcal{T}, \mathcal{A}_i \rangle \models B(a)$. We then show that if $P(a, x^{i_1}_{aP_1}) \in \mathcal{S}_i^j$ for some i, then $\mathcal{T} \models P_1 \sqsubseteq P$ by induction on $p = j - s$ where s is the step where $x^{i_1}_{aP_1}$ has been introduced to produce $P_1(a, x^{i_1}_{aP_1})$. If $p = 0$, since $P_1(a, x^{i_1}_{aP_1})$ is the only assertion of $\mathcal{S}^j$ that contains $x^{i_1}_{aP_1}$, $P = P_1$. For $p > 0$, assume that $P(a, x^{i_1}_{aP_1}) \in \mathcal{S}_i^j$. If $P(a, x^{i_1}_{aP_1}) \in \mathcal{S}_i^{j-1}$, we apply the induction hypothesis. Otherwise, since $P(a, x^{i_1}_{aP_1}) \notin \mathcal{S}_i^{j-1}$, it has been created at step j by applying a PI $P' \sqsubseteq P \in \mathcal{T}$ to an assertion $P'(a, x^{i_1}_{aP_1}) \in \mathcal{S}_{i'}^{j-1}$, so by induction hypothesis $\mathcal{T} \models P_1 \sqsubseteq P' \sqsubseteq P$.

(P9) We show that if $P(a, x^{i_1}_{aP_1}) \in \mathcal{S}_i^j$ for some $i \neq i_1$, then there exists a rigid P' such that $\mathcal{T} \models P_1 \sqsubseteq P' \sqsubseteq P$ by induction on $p = j - s$ where s is the step where $x^{i_1}_{aP_1}$ has been introduced to produce $P_1(a, x^{i_1}_{aP_1})$. If $p = 0$, since $P_1(a, x^{i_1}_{aP_1})$ is the only assertion of $\mathcal{S}^j$ that contains $x^{i_1}_{aP_1}$, $P = P_1$ and is rigid since $i \neq i_1$. For $p > 0$, assume that $P(a, x^{i_1}_{aP_1}) \in \mathcal{S}_i^j$. If $P(a, x^{i_1}_{aP_1}) \in \mathcal{S}_i^{j-1}$, we apply the induction hypothesis. Otherwise, since $P(a, x^{i_1}_{aP_1}) \notin \mathcal{S}_i^{j-1}$, it has been created at step j by applying a PI $P' \sqsubseteq P \in \mathcal{T}$ to an assertion $P'(a, x^{i_1}_{aP_1}) \in \mathcal{S}_{i'}^{j-1}$, so either $i' \neq i_1$ and by induction hypothesis there exists a rigid P'' such that $\mathcal{T} \models P_1 \sqsubseteq P'' \sqsubseteq P' \sqsubseteq P$, or $i' = i_1$ and since $i \neq i_1$, P is rigid, and by (P8), $\mathcal{T} \models P_1 \sqsubseteq P' \sqsubseteq P$.

(P10) We show that if $P(x, y) \in \mathcal{S}_i^j$ for some i, then $x = x^{i_1 \dots i_l}_{aP_1 \dots P_l}, y = x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}$ and $\mathcal{T} \models P_{l+1} \sqsubseteq P$, or $x = x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}, y = x^{i_1 \dots i_l}_{aP_1 \dots P_l}$ and $\mathcal{T} \models P_{l+1} \sqsubseteq P^-$ by induction on $p = j - s$ where s is the maximum of the steps where x or y has been introduced. If $p = 0$, either (i) $P(x, y)$ has been created by applying a PI of the form $B \sqsubseteq \exists P$ to an assertion $B(x)$, and if $x = x^{i_1 \dots i_l}_{aP_1 \dots P_l}$, then $y = x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}$ with $P_{l+1} = P$, or (ii) $P(x, y)$ has been created by applying a PI of the form $B \sqsubseteq \exists P^-$ to an assertion $B(y)$, and if $y = x^{i_1 \dots i_l}_{aP_1 \dots P_l}$, then $x = x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}$ with $P_{l+1} = P^-$. For $p > 0$, $P(x, y)$ has been created by applying a PI of the form $P' \sqsubseteq P$ to an assertion $P'(x, y) \in \mathcal{S}_{i'}^{j-1}$, so by induction $x = x^{i_1 \dots i_l}_{aP_1 \dots P_l}, y = x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}$ and $\mathcal{T} \models P_{l+1} \sqsubseteq P' \sqsubseteq P$ or $x = x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}, y = x^{i_1 \dots i_l}_{aP_1 \dots P_l}$ and $\mathcal{T} \models P_{l+1} \sqsubseteq P'^- \sqsubseteq P^-$.

(P11) We show that if $P(x^{i_1 \dots i_l}_{aP_1 \dots P_l}, x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}) \in \mathcal{S}_i^j$ for some $i \neq i_{l+1}$, then there exists a rigid role P' such that $\mathcal{T} \models P_{l+1} \sqsubseteq P' \sqsubseteq P$ and the assertion $P'(x^{i_1 \dots i_l}_{aP_1 \dots P_l}, x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{i_{l+1}})$ by induction on $p = j - s$ where s is the step where the individual $x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}$ has been introduced to produce $P_{l+1}(x^{i_1 \dots i_l}_{aP_1 \dots P_l}, x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}})$. If $p = 0$, since $P_{l+1}(x^{i_1 \dots i_l}_{aP_1 \dots P_l}, x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}})$ is the only assertion of $\mathcal{S}^j$ that contains $x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}$, $P = P_{l+1}$ and is rigid since $i \neq i_{l+1}$, and $P_{l+1}(x^{i_1 \dots i_l}_{aP_1 \dots P_l}, x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{i_{l+1}})$. For $p > 0$, assume that $P(x^{i_1 \dots i_l}_{aP_1 \dots P_l}, x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}) \in \mathcal{S}_i^j$. If $P(x^{i_1 \dots i_l}_{aP_1 \dots P_l}, x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}) \in \mathcal{S}_i^{j-1}$, we apply the induction hypothesis. Otherwise, since $P(x^{i_1 \dots i_l}_{aP_1 \dots P_l}, x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}) \notin \mathcal{S}_i^{j-1}$, it has been created at step j by applying a PI $P' \sqsubseteq P \in \mathcal{T}$ to an assertion $P'(x^{i_1 \dots i_l}_{aP_1 \dots P_l}, x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}) \in \mathcal{S}_{i'}^{j-1}$, so either $i' \neq i_{l+1}$ and by induction hypothesis there exists a rigid P'' such that $\mathcal{T} \models P_{l+1} \sqsubseteq P'' \sqsubseteq P' \sqsubseteq P$ and $P''(x^{i_1 \dots i_l}_{aP_1 \dots P_l}, x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{i_{l+1}})$, or $i' = i_{l+1}$ and since $i \neq i_{l+1}$, P is rigid so $P(x^{i_1 \dots i_l}_{aP_1 \dots P_l}, x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{i_{l+1}})$.

(P12) We show that if $P_l(x^{i_1 \dots i_{l-1}}_{aP_1 \dots P_{l-1}}, x^{i_1 \dots i_l}_{aP_1 \dots P_l}) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{i_l})$ then there is some j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models \exists xy P_{l-1}(x, y)$ by induction on l. If $l = 2$, by (P1), $P_{l-1}(a, x^{i_1 \dots i_{l-1}}_{aP_1 \dots P_{l-1}}) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{i_{l-1}})$ so $\mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{i_{l-1}}) \models \exists P_{l-1}(a)$ so by (P5), there is some j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models$

$\exists P_{l-1}(a)$, so $\langle \mathcal{T}, \mathcal{A}_j \rangle \models \exists xy P_{l-1}(x,y)$. For $l > 2$, by (P2), $P_{l-1}(x^{i_1 \dots i_{l-2}}_{aP_1 \dots P_{l-2}}, x^{i_1 \dots i_{l-1}}_{aP_1 \dots P_{l-1}}) \in$ *chase*$^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_{i_{l-1}})$ so by induction there is some j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models \exists xy P_{l-2}(x,y)$, and since by (P4) $\mathcal{T} \models \exists P^-_{l-2} \sqsubseteq \exists P_{l-1}$, $\langle \mathcal{T}, \mathcal{A}_j \rangle \models \exists xy P_{l-1}(x,y)$. □

Based on the rigid chase of $\mathcal{K}$, we construct the sequence of interpretations $\mathfrak{J}_\mathcal{K} = (\mathcal{I}_i)_{0 \leq i \leq n}$ where $\mathcal{I}_i = (\Delta, \cdot^{\mathcal{I}_i})$ is defined as follows: $\Delta = \mathsf{N}_\mathsf{I}^\mathcal{K} \cup \Gamma_N$, $a^{\mathcal{I}_i} = a$ for every $a \in \Delta$, $A^{\mathcal{I}_i} = \{a \mid A(a) \in \mathit{chase}^\mathcal{K}_{\text{rig}}(\mathcal{A}_i)\}$ for every $A \in \mathsf{N}_\mathsf{C}$, and $R^{\mathcal{I}_i} = \{(a,b) \mid R(a,b) \in \mathit{chase}^\mathcal{K}_{\text{rig}}(\mathcal{A}_i)\}$ for every $R \in \mathsf{N}_\mathsf{R}$. We show that $\mathfrak{J}_\mathcal{K}$ is a model of $\mathcal{K}$ that respects the rigid predicates and such that for any BTCQ ϕ such that $\mathsf{N}_\mathsf{I}^\phi \subseteq \mathsf{N}_\mathsf{I}^\mathcal{K}$, $\mathcal{K}, p \models \phi$ iff $\mathfrak{J}_\mathcal{K}, p \models \phi$.

Lemma 1. *If $\mathcal{K}$ is consistent, then $\mathfrak{J}_\mathcal{K}$ is a model of $\mathcal{K}$ that respects the rigid predicates.*

Proof. We first show that $\mathfrak{J}_\mathcal{K}$ is a model of $\mathcal{K}$, i.e., that for every $1 \leq i \leq n$, $\mathcal{I}_i \models \mathcal{A}_i$ and $\mathcal{I}_i \models \mathcal{T}$. It is easy to see that $\mathcal{I}_i \models \mathcal{A}_i$ because $\mathcal{A}_i \subseteq \mathit{chase}^\mathcal{K}_{\text{rig}}(\mathcal{A}_i)$. We can show that $\mathcal{I}_i$ satisfies every positive inclusion of $\mathcal{T}$ with similar arguments as those used in [14]. Indeed, if a PI $\alpha \in \mathcal{T}_p$ is not satisfied, there is an assertion $\beta \in \mathit{chase}^\mathcal{K}_{\text{rig}}(\mathcal{A}_i)$ such that α is applicable to β in $\mathit{chase}^\mathcal{K}_{\text{rig}}(\mathcal{A}_i)$. This is impossible given that every PI applicable to β in $\mathcal{S}_i^j$ at step j of the construction of the rigid chase becomes not applicable to β in $\mathcal{S}_i^k$ for some $k \geq j$, since there are not infinitely many assertions before β nor infinitely many PIs applied to some assertion that precedes β because a PI can be applied only once to a given assertion. Finally, $\mathcal{I}_i$ satisfies every negative inclusion of $\mathcal{T}$ because $\mathcal{K}$ is consistent. Indeed, if a negative inclusion is not satisfied, this implies that there is a conflict $\mathcal{B}$ in $\mathit{chase}^\mathcal{K}_{\text{rig}}(\mathcal{A}_i)$. If $\mathcal{B} = \{\alpha\}$, the timed-assertion $(\alpha', j) \in (\mathcal{A}_i)_{0 \leq i \leq n}$ from which α has been derived by applying PIs from $\mathcal{T}_p$ is clearly inconsistent. Otherwise $\mathcal{B} = \{\alpha, \beta\}$ with α derived from (α', j), β derived from (β', k). If $j = k$, $\{(\alpha', j), (\beta', k)\}$ is clearly inconsistent. If $j \neq k$, since α and β belong to $\mathit{chase}^\mathcal{K}_{\text{rig}}(\mathcal{A}_i)$, if $j \neq i$ (resp. $k \neq i$) there exists $\alpha'' \in \mathit{chase}^\mathcal{K}_{\text{rig}}(\mathcal{A}_i)$ rigid such that α derives from α'' which derives from α' (resp. $\beta'' \in \mathit{chase}^\mathcal{K}_{\text{rig}}(\mathcal{A}_i)$ rigid such that β derives from β'' which derives from β'), so $\{(\alpha', j), (\beta', k)\}$ is inconsistent because no sequence of interpretations that respects rigid predicates can be a model of $\mathcal{K}$.

Moreover, the model $\mathfrak{J}_\mathcal{K}$ respects the rigid predicates because if an assertion β of $\mathit{chase}^\mathcal{K}_{\text{rig}}(\mathcal{A}_i)$ is rigid, either $\beta \in \mathcal{A}_i$ and by construction $\beta \in \mathcal{S}_k^0 = \mathcal{A}'_k$ for every k, or β has been derived at some step j by applying some PI to an assertion of $\mathcal{S}^j$ and $\beta \in \mathcal{S}_k^{j+1}$ for every k, so in both cases $\beta \in \mathit{chase}^\mathcal{K}_{\text{rig}}(\mathcal{A}_k)$ for every k. □

Lemma 2. *If $\mathcal{K}$ is consistent, then for any BTCQ ϕ such that $\mathsf{N}_\mathsf{I}^\phi \subseteq \mathsf{N}_\mathsf{I}^\mathcal{K}$, $\mathcal{K}, p \models \phi$ iff $\mathfrak{J}_\mathcal{K}, p \models \phi$.*

Proof. Since $\mathfrak{J}_\mathcal{K} = (\mathcal{I}_i)_{0 \leq i \leq n}$ with $\mathcal{I}_i = (\Delta, \cdot^{\mathcal{I}_i})$ is a model of $\mathcal{K}$ that respects the rigid predicates, the first direction is clear and we only need to show that if $\mathfrak{J}_\mathcal{K}, p \models \phi$ then $\mathcal{K}, p \models \phi$. Let $\mathfrak{J} = (\mathcal{I}'_i)_{0 \leq i \leq n}$ with $\mathcal{I}'_i = (\Delta', \cdot^{\mathcal{I}'_i})$ be a model of $\mathcal{K}$ that respects rigid predicates. We show by structural induction on ϕ that if $\mathfrak{J}_\mathcal{K}, p \models \phi$ then $\mathfrak{J}, p \models \phi$.

If ϕ is a CQ $\exists \vec{y}\, \psi(\vec{y})$, we show that if there exists a homomorphism π of $\exists \vec{y}\, \psi(\vec{y})$ into $\mathcal{I}_p$, then $\mathcal{I}'_p \models \exists \vec{y}\, \psi(\vec{y})$. We define a mapping h from Δ into Δ' (we assume w.l.o.g. that Δ and Δ' are disjoint) as follows:

- for every $a \in \mathsf{N}_\mathsf{I}^\mathcal{K}$, $h(a^{\mathcal{I}_p}) = a^{\mathcal{I}'_p}$
- for every $x^{i_1}_{aP_1} \in \Gamma_N$, $h(x^{i_1 \mathcal{I}_p}_{aP_1}) = y$ where $(a^{\mathcal{I}'_p}, y) \in P_1^{\mathcal{I}'_{i_1}}$ (if there are several such y, choose one of them randomly)
- for every $x^{i_1 \dots i_l}_{aP_1 \dots P_l} \in \Gamma_N$ with $l > 1$, $h(x^{i_1 \dots i_l \mathcal{I}_p}_{aP_1 \dots P_l}) = y$ where $(h(x^{i_1 \dots i_{l-1} \mathcal{I}_p}_{aP_1 \dots P_{l-1}}), y) \in P_l^{\mathcal{I}'_{i_l}}$ (if there are several such y, choose one of them randomly).

We first show that h is well defined, i.e., that in the two latter cases there always exists a y as required by induction on l. In the case of $l = 1$, since $x^{i_1}_{aP_1} \in \Gamma_N$, by (P1) $P_1(a, x^{i_1}_{aP_1}) \in \mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_{i_1})$ so by (P8) either (i) $\langle \mathcal{T}, \mathcal{A}_{i_1} \rangle \models \exists x P_1(a, x)$ and since $\mathcal{I}'_{i_1}$ is a model of $\langle \mathcal{T}, \mathcal{A}_{i_1} \rangle$, there is some $(a^{\mathcal{I}'_p}, y) \in P_1^{\mathcal{I}'_{i_1}}$, or (ii) there exists $B := A|\exists R|\exists R^-$ with $A \in \mathsf{N_{RC}}$, $R \in \mathsf{N_{RR}}$, such that $\mathcal{T} \models B \sqsubseteq \exists P_1$ and there exists j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models B(a)$. In the latter case, since $\mathcal{J}$ is a model of $\mathcal{K}$ that respects the rigid predicates, $\mathcal{I}'_{i_1} \models B(a)$, so since $\mathcal{I}'_{i_1}$ is a model of $\mathcal{T}$, there is some $(a^{\mathcal{I}'_p}, y) \in P_1^{\mathcal{I}'_{i_1}}$. Then, for $l > 1$, since $x^{i_1 \dots i_l}_{aP_1 \dots P_l} \in \Gamma_N$, by (P4), $\mathcal{T} \models \exists P^-_{l-1} \sqsubseteq \exists P_l$. It follows that since by induction there is an $(x, h(x^{i_1 \dots i_{l-1} \mathcal{I}_p}_{aP_1 \dots P_{l-1}})) \in P_{l-1}^{\mathcal{I}'_{i_l}}$, then there is some $(h(x^{i_1 \dots i_{l-1} \mathcal{I}_p}_{aP_1 \dots P_{l-1}}), y) \in P_l^{\mathcal{I}'_{i_l}}$.

We then show that h is a homomorphism of $\mathcal{I}_p$ into $\mathcal{I}'_p$, which implies that $h \circ \pi$ is a homomorphism of $\exists \vec{y}\, \psi(\vec{y})$ into $\mathcal{I}'_p$:

For every $a \in \mathsf{N}_{\mathsf{I}}^{\mathcal{K}}$ and concept A, if $a^{\mathcal{I}_p} \in A^{\mathcal{I}_p}$, i.e., $A(a) \in \mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_p)$, then by (P5), either (i) $\langle \mathcal{T}, \mathcal{A}_p \rangle \models A(a)$, and since $\mathcal{I}'_p$ is a model of $\langle \mathcal{T}, \mathcal{A}_p \rangle$, then $h(a^{\mathcal{I}_p}) = a^{\mathcal{I}'_p} \in A^{\mathcal{I}'_p}$, or (ii) there exists $B := C|\exists R|\exists R^-$ with $C \in \mathsf{N_{RC}}$, $R \in \mathsf{N_{RR}}$, such that $\mathcal{T} \models B \sqsubseteq A$ and there exists j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models B(a)$. In the latter case, since $\mathcal{J}$ is a model of $\mathcal{K}$ that respects the rigid predicates, $\mathcal{I}'_p \models B(a) \models A(a)$ so $h(a^{\mathcal{I}_p}) = a^{\mathcal{I}'_p} \in A^{\mathcal{I}'_p}$. For every pair $a, b \in \mathsf{N}_{\mathsf{I}}^{\mathcal{K}}$ and role P, if $(a^{\mathcal{I}_p}, b^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$, by (P7), similar arguments can be used to prove that $(h(a^{\mathcal{I}_p}), h(b^{\mathcal{I}_p})) = (a^{\mathcal{I}'_p}, b^{\mathcal{I}'_p}) \in P^{\mathcal{I}'_p}$.

For every $x^{i_1 \dots i_l}_{aP_1 \dots P_l} \in \Gamma_N$, such that $x^{i_1 \dots i_l \mathcal{I}_p}_{aP_1 \dots P_l} \in A^{\mathcal{I}_p}$, i.e., $A(x^{i_1 \dots i_l}_{aP_1 \dots P_l}) \in \mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_p)$, by (P6) we are in one of the following cases:

(i) $i_l = p$. By (P3), $\mathcal{T} \models \exists P_l^- \sqsubseteq A$ and by construction of h, $h(x^{i_1 \dots p \mathcal{I}_p}_{aP_1 \dots P_l}) = y$ with $(h(x^{i_1 \dots i_{l-1} \mathcal{I}_p}_{aP_1 \dots P_{l-1}}), y) \in P_l^{\mathcal{I}'_p}$ (note that if $l = 1$, $x^{i_1 \dots i_{l-1}}_{aP_1 \dots P_{l-1}} = a$). It follows that since $\mathcal{I}'_p$ is a model of $\mathcal{T}$, then $y \in A^{\mathcal{I}'_p}$.

(ii) there exists $B := C|\exists R|\exists R^-$ with $C \in \mathsf{N_{RC}}, R \in \mathsf{N_{RR}}$ such that $\mathcal{T} \models B \sqsubseteq A$ and $\mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_{i_l}) \models B(x^{i_1 \dots i_l}_{aP_1 \dots P_l})$. As in case (i), by (P3) and definition of h we have that $h(x^{i_1 \dots i_l \mathcal{I}_p}_{aP_1 \dots P_l}) = y \in B^{\mathcal{I}'_{i_l}}$. Since B is rigid, $y \in B^{\mathcal{I}'_p}$. It follows that since $\mathcal{I}'_p$ is a model of $\mathcal{T}$, then $y \in A^{\mathcal{I}'_p}$.

For every pair $x, y \in \Gamma_N$ and role P, such that $(x^{\mathcal{I}_p}, y^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$, by (P10) $x = x^{i_1 \dots i_l}_{aP_1 \dots P_l}, y = x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}$ and $\mathcal{T} \models P_{l+1} \sqsubseteq P$ or $x = x^{i_1 \dots i_l i_{l+1}}_{aP_1 \dots P_l P_{l+1}}, y = x^{i_1 \dots i_l}_{aP_1 \dots P_l}$ and $\mathcal{T} \models P_{l+1} \sqsubseteq P^-$. We can assume w.l.o.g. that we are in the first case (otherwise we consider $(y^{\mathcal{I}_p}, x^{\mathcal{I}_p}) \in P^{-\mathcal{I}_p}$). If $i_{l+1} = p$, by definition of h, $(h(x^{\mathcal{I}_p}), h(y^{\mathcal{I}_p})) \in P_{l+1}^{\mathcal{I}'_p}$, so since $\mathcal{I}'_p$ is a model of $\mathcal{T}$, $(h(x^{\mathcal{I}_p}), h(y^{\mathcal{I}_p})) \in P^{\mathcal{I}'_p}$. Otherwise, by (P11), there exists $P' := R|R^-$ with $R \in \mathsf{N_{RR}}$ such that $\mathcal{T} \models P_{l+1} \sqsubseteq P' \sqsubseteq P$ and $P'(x, y) \in \mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_{i_{l+1}})$. With the same arguments as in the first case we show that $(h(x^{\mathcal{I}_p}), h(y^{\mathcal{I}_p})) \in P'^{\mathcal{I}'_{i_{l+1}}}$, and since P' is rigid $(h(x^{\mathcal{I}_p}), h(y^{\mathcal{I}_p})) \in P'^{\mathcal{I}'_p}$. It follows that since $\mathcal{I}'_p$ is a model of $\mathcal{T}$, then $(h(x^{\mathcal{I}_p}), h(y^{\mathcal{I}_p})) \in P^{\mathcal{I}'_p}$.

Finally, if $a \in \mathsf{N}_{\mathsf{I}}^{\mathcal{K}}$ and $x \in \Gamma_N$, $(a^{\mathcal{I}_p}, x^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$ only if $x = x^{i_1}_{aP_1}$. If $i_1 = p$, by definition of h, $(h(a^{\mathcal{I}_p}), h(x^{\mathcal{I}_p})) \in P_1^{\mathcal{I}'_p}$. Since by (P8) $\mathcal{T} \models P_1 \sqsubseteq P$ and $\mathcal{I}'_p$ is a model of $\mathcal{T}$, it follows that $(h(a^{\mathcal{I}_p}), h(x^{\mathcal{I}_p})) \in P^{\mathcal{I}'_p}$. If $i_1 \neq p$, by (P9), there exists P' rigid such that $\mathcal{T} \models P_1 \sqsubseteq P' \sqsubseteq P$ and since by definition of h, $(h(a^{\mathcal{I}_p}), h(x^{\mathcal{I}_p})) \in P_1^{\mathcal{I}'_{i_1}}$, then $(h(a^{\mathcal{I}_p}), h(x^{\mathcal{I}_p})) \in P'^{\mathcal{I}'_{i_1}}$. Since $\mathcal{J}$ respects rigid predicates, it follows that $(h(a^{\mathcal{I}_p}), h(x^{\mathcal{I}_p})) \in P'^{\mathcal{I}'_p}$ and $(h(a^{\mathcal{I}_p}), h(x^{\mathcal{I}_p})) \in P^{\mathcal{I}'_p}$.

We have thus shown that if $\mathcal{J}_\mathcal{K}, p \models \exists \vec{y}\, \psi(\vec{y})$ then $\mathcal{J}, p \models \exists \vec{y}\, \psi(\vec{y})$.

Assume that for two BTCQs ϕ_1, ϕ_2 such that $\mathsf{N}_\mathsf{I}^{\phi_1} \subseteq \mathsf{N}_\mathsf{I}^{\mathcal{K}}$ and $\mathsf{N}_\mathsf{I}^{\phi_2} \subseteq \mathsf{N}_\mathsf{I}^{\mathcal{K}}$, if $\mathcal{J}_\mathcal{K}, p \models \phi_i$ then $\mathcal{J}, p \models \phi_i$ ($i \in \{1, 2\}$). Then:

- If $\mathcal{J}_\mathcal{K}, p \models \phi_1 \wedge \phi_2$ then $\mathcal{J}_\mathcal{K}, p \models \phi_1$ and $\mathcal{J}_\mathcal{K}, p \models \phi_2$
  so by assumption $\mathcal{J}, p \models \phi_1$ and $\mathcal{J}, p \models \phi_2$
  then $\mathcal{J}, p \models \phi_1 \wedge \phi_2$
- If $\mathcal{J}_\mathcal{K}, p \models \phi_1 \vee \phi_2$ then $\mathcal{J}_\mathcal{K}, p \models \phi_1$ or $\mathcal{J}_\mathcal{K}, p \models \phi_2$
  so by assumption $\mathcal{J}, p \models \phi_1$ or $\mathcal{J}, p \models \phi_2$
  then $\mathcal{J}, p \models \phi_1 \vee \phi_2$
- If $\mathcal{J}_\mathcal{K}, p \models \bigcirc\phi_1$ then $p < n$ and $\mathcal{J}_\mathcal{K}, p+1 \models \phi_1$
  so by assumption $p < n$ and $\mathcal{J}, p+1 \models \phi_1$
  then $\mathcal{J}, p \models \bigcirc\phi_1$
- If $\mathcal{J}_\mathcal{K}, p \models \bullet\phi_1$ then $p = n$ or $\mathcal{J}_\mathcal{K}, p+1 \models \phi_1$
  so by assumption $p = n$ or $\mathcal{J}, p+1 \models \phi_1$
  then $\mathcal{J}, p \models \bullet\phi_1$
- If $\mathcal{J}_\mathcal{K}, p \models \Box\phi_1$ then for every k, $p \leq k \leq n$, $\mathcal{J}_\mathcal{K}, k \models \phi_1$
  so by assumption for every k, $p \leq k \leq n$, $\mathcal{J}, k \models \phi_1$
  then $\mathcal{J}, p \models \Box\phi_1$
- If $\mathcal{J}_\mathcal{K}, p \models \Diamond\phi_1$ then there exists k, $p \leq k \leq n$, $\mathcal{J}_\mathcal{K}, k \models \phi_1$
  so by assumption $\mathcal{J}, k \models \phi_1$
  then $\mathcal{J}, p \models \Diamond\phi_1$
- If $\mathcal{J}_\mathcal{K}, p \models \phi_1 \mathsf{U} \phi_2$ then there exists k, $p \leq k \leq n$, $\mathcal{J}_\mathcal{K}, k \models \phi_2$ and for every j, $p \leq j < k$, $\mathcal{J}_\mathcal{K}, j \models \phi_1$
  so by assumption $\mathcal{J}, k \models \phi_2$ and for every j, $p \leq j < k$, $\mathcal{J}, j \models \phi_1$
  then $\mathcal{J}, p \models \phi_1 \mathsf{U} \phi_2$
- $\bigcirc^-\phi_1$, $\bullet^-\phi_1$, $\Box^-\phi_1$, $\Diamond^-\phi_1$, $\phi_1 \mathsf{S} \phi_2$: similar to the corresponding future operators

We conclude by induction that for every BTCQ ϕ such that $\mathsf{N}_\mathsf{I}^{\phi} \subseteq \mathsf{N}_\mathsf{I}^{\mathcal{K}}$, if $\mathcal{J}_\mathcal{K}, p \models \phi$ then $\mathcal{J}, p \models \phi$. It follows that if $\mathcal{J}_\mathcal{K}, p \models \phi$ then $\mathcal{K}, p \models \phi$.

We have thus shown that for every BTCQ ϕ such that $\mathsf{N}_\mathsf{I}^{\phi} \subseteq \mathsf{N}_\mathsf{I}^{\mathcal{K}}$, $\mathcal{K}, p \models \phi$ iff $\mathcal{J}_\mathcal{K}, p \models \phi$. □

To show that TCQ answering in Case 3 reduces to TCQ answering in Case 1, we want to construct in polynomial time a set of assertions $\mathcal{R}$ that captures all relevant information about rigid concepts and roles for consistency checking and TCQ answering, i.e., such that TCQ answering over $\mathcal{K}$ with $\mathsf{N}_{\mathsf{RC}} \neq \emptyset$, $\mathsf{N}_{\mathsf{RR}} \neq \emptyset$ can be done by TCQ answering over $\langle \mathcal{T}, (\mathcal{A}_i \cup \mathcal{R})_{0 \leq i \leq n} \rangle$ with $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$. Without any restriction on the TBox, $\mathcal{R}$ may be infinite, as illustrated in the following example.

Example 4. Consider $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0 \leq i \leq n} \rangle$ with $\mathcal{T} = \{A \sqsubseteq \exists P, \exists P^- \sqsubseteq \exists R, \exists R^- \sqsubseteq \exists R, R \sqsubseteq S\}$ with S rigid, and $(\mathcal{A}_i)_{0 \leq i \leq n}$ with $\mathcal{A}_0 = \{A(a)\}$, and $\mathcal{A}_i = \emptyset$ for $1 \leq i \leq n$. A model of $\mathcal{K}$ that respects rigid predicates is such that $\phi = \exists x_1 ... x_{k+1} S(x_1, x_2) \wedge ... \wedge S(x_k, x_{k+1})$ holds for any $k > 0$ and at any time point. Since with $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$, $\mathcal{K}$ entails such a query only at time point 0, $\mathcal{R}$ should be such that $\langle \mathcal{T}, \mathcal{R} \rangle$ entails such a query, so that $\langle \mathcal{T}, (\mathcal{A}_i \cup \mathcal{R})_{0 \leq i \leq n} \rangle$ entails it at any time point. Moreover, a model of $\mathcal{K}$ that respects rigid predicates can be such that neither $\exists x_1 ... x_k S(x_1, x_2) \wedge ... \wedge S(x_k, x_1)$, nor $\exists x A(x)$, $\exists xy P(x, y)$ or $\exists xy R(x, y)$ holds at some time point $i > 0$, so $\mathcal{R}$ should not contain any cycle of S, or any A, P or R assertions. It follows that $\mathcal{R}$ has to contain an infinite chain of S.

Therefore we assume the restriction that rigid roles only have rigid sub-roles, i.e., $\mathcal{T}$ does not entail any role inclusion of the form $P_1 \sqsubseteq P_2$ with $P_1 := R_1|R_1^-$, $R_1 \in \mathsf{N_R}\backslash\mathsf{N_{RR}}$ and $P_2 := R_2|R_2^-$, $R_2 \in \mathsf{N_{RR}}$. This condition avoids that there may be chains of rigid roles in the anonymous part of $\mathcal{J}_{\mathcal{K}}$ that cannot be entailed by a single rigid assertion. In the example above, if rigid roles only have rigid sub-roles, R has to be rigid, so adding the single assertion $R(x, y)$ to every $\mathcal{A}_i$ is sufficient for $\phi = \exists x_1 ... x_{k+1} R(x_1, x_2) \wedge ... \wedge R(x_k, x_{k+1})$ being entailed at every time point for any $k > 0$, thus sufficient for $\phi = \exists x_1 ... x_{k+1} S(x_1, x_2) \wedge ... \wedge S(x_k, x_{k+1})$ being entailed at every time point for any $k > 0$ since $R \sqsubseteq S$.

Proposition 4. *Let $\mathcal{R}$ be as follows:*

$$\begin{aligned}
\mathcal{R} =&\{A(a) \mid A \in \mathsf{N}_{\mathsf{RC}}^{\mathcal{K}}, a \in \mathsf{N}_{\mathsf{I}}^{\mathcal{K}}, \exists i, \langle \mathcal{T}, \mathcal{A}_i \rangle \models_{brave} A(a)\} \cup \\
&\{R(a,b) \mid R \in \mathsf{N}_{\mathsf{RR}}^{\mathcal{K}}, a, b \in \mathsf{N}_{\mathsf{I}}^{\mathcal{K}}, \exists i, \langle \mathcal{T}, \mathcal{A}_i \rangle \models_{brave} R(a,b)\} \cup \\
&\{P(a, x_{aP}) \mid R \in \mathsf{N}_{\mathsf{RR}}^{\mathcal{K}}, P := R|R^-, a \in \mathsf{N}_{\mathsf{I}}^{\mathcal{K}}, \exists i, \langle \mathcal{T}, \mathcal{A}_i \rangle \models_{brave} \exists x P(a,x)\} \cup \\
&\{A(x_{P_1}) \mid S \in \mathsf{N}_{\mathsf{R}}^{\mathcal{K}} \backslash \mathsf{N}_{\mathsf{RR}}^{\mathcal{K}}, P_1 := S|S^-, A \in \mathsf{N}_{\mathsf{RC}}^{\mathcal{K}}, \\
&\quad \exists i, \langle \mathcal{T}, \mathcal{A}_i \rangle \models_{brave} \exists xy P_1(x,y) \text{ and } \mathcal{T} \models \exists P_1^- \sqsubseteq A\} \cup \\
&\{P_2(x_{P_1}, x_{P_1P_2}) \mid S \in \mathsf{N}_{\mathsf{R}}^{\mathcal{K}} \backslash \mathsf{N}_{\mathsf{RR}}^{\mathcal{K}}, P_1 := S|S^-, R \in \mathsf{N}_{\mathsf{RR}}^{\mathcal{K}}, P_2 := R|R^-, \\
&\quad \exists i, \langle \mathcal{T}, \mathcal{A}_i \rangle \models_{brave} \exists xy P_1(x,y) \text{ and } \mathcal{T} \models \exists P_1^- \sqsubseteq \exists P_2\}
\end{aligned}$$

The set $\mathcal{R}$ is computable in polynomial time and such that

1. *$\mathcal{K}$ is consistent iff $\mathcal{K}_{\mathcal{R}} = \langle \mathcal{T}, (\mathcal{A}_i \cup \mathcal{R})_{0 \leq i \leq n} \rangle$ is consistent with $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, and*
2. *for any BTCQ ϕ such that $\mathsf{N}_{\mathsf{I}}^{\phi} \subseteq \mathsf{N}_{\mathsf{I}}^{\mathcal{K}}$, $\mathcal{K}, p \models \phi$ iff $\mathcal{K}_{\mathcal{R}}, p \models \phi$ with $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$.*

The size of $\mathcal{R}$ is polynomial in the size of $\mathsf{N}_{\mathsf{C}}^{\mathcal{K}}, \mathsf{N}_{\mathsf{R}}^{\mathcal{K}}$, and $\mathsf{N}_{\mathsf{I}}^{\mathcal{K}}$, and since atomic query answering under brave semantics as well as subsumption checking can be done in polynomial time, $\mathcal{R}$ can be computed in P. The first three parts of $\mathcal{R}$ retain information about the participation of individuals of $\mathsf{N}_{\mathsf{I}}^{\mathcal{K}}$ in rigid predicates. The last two witness the participation in rigid predicates of the role-successors w.r.t. non-rigid roles, thus take into account also anonymous individuals that are created in $chase_{\text{rig}}(\mathcal{K})$ when applying PIs whose right-hand side is an existential restriction with a non-rigid role. Note that the individuals created in $chase_{\text{rig}}(\mathcal{K})$ when applying such a PI with a rigid role are witnessed by the x_{aP} or $x_{P_1P_2}$ if they do not follow from a rigid role assertion, and do not need to be witnessed otherwise, since the assertion $P_2(x_{P_1}, x_{P_1P_2})$ is sufficient to trigger all the anonymous part implied by the fact that $x_{P_1P_2}$ is in the range of P_2. We use the brave semantics to define $\mathcal{R}$ because there is no guarantee that every $\langle \mathcal{T}, \mathcal{A}_i \rangle$ is consistent, and everything would be entailed under classical semantics if it is inconsistent. The brave semantics allows us to derive any relevant fact because if some fact is entailed from some $\langle \mathcal{T}, \mathcal{A}_i \rangle$ under the classical semantics but not under brave semantics, this means that $\langle \mathcal{T}, \mathcal{A}_i \rangle$ is inconsistent, so $\mathcal{K}$ is already inconsistent with $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, and $\mathcal{K}_{\mathcal{R}}$ is also inconsistent since $\langle \mathcal{T}, \mathcal{A}_i \cup \mathcal{R} \rangle$ is inconsistent (and in this case any BTCQ ϕ is entailed from both $\mathcal{K}$ and $\mathcal{K}_{\mathcal{R}}$ at any time point).

We break the proof of Proposition 4 in several lemmas.

Lemma 3. *$\mathcal{K}$ is consistent iff $\mathcal{K}_{\mathcal{R}}$ is consistent with* $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$.

Proof. $\mathcal{K}_{\mathcal{R}}$ is consistent with $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$ iff each $\langle \mathcal{T}, \mathcal{A}_i \cup \mathcal{R} \rangle$ is consistent by Proposition 1. We show that $\mathcal{K}$ is consistent iff each $\langle \mathcal{T}, \mathcal{A}_i \cup \mathcal{R} \rangle$ is consistent.

If $\mathcal{K}$ is not consistent, let $\mathcal{B}$ be a conflict of $\mathcal{K}$. Then $\mathcal{B}$ is either internal to some $\mathcal{A}_i$, and $\langle \mathcal{T}, \mathcal{A}_i \cup \mathcal{R} \rangle$ is inconsistent, or is of the form $\mathcal{B} = \{(\alpha, i), (\beta, j)\}$ with $i \neq j$. In the latter case, $\{\alpha, \beta\}$ violates some negative inclusion of the closure of the TBox that involves at least a rigid concept A or a rigid role R by assigning an individual a (or two individuals a, b) to two disjoint concepts (or roles). We can then assume w.l.o.g. that $\langle\mathcal{T}, \alpha\rangle \models A(a)$ (resp. $\langle\mathcal{T}, \alpha\rangle \models \exists x R(a, x)$, resp. $\langle\mathcal{T}, \alpha\rangle \models \exists x R(x, a)$, resp. $\langle\mathcal{T}, \alpha\rangle \models R(a, b)$). It follows that $\langle\mathcal{T}, \mathcal{A}_i\rangle \models_{\text{brave}} A(a)$ (resp. $\langle\mathcal{T}, \mathcal{A}_i\rangle \models_{\text{brave}} \exists x R(a, x)$, resp. $\langle\mathcal{T}, \mathcal{A}_i\rangle \models_{\text{brave}} \exists x R(x, a)$, resp. $\langle\mathcal{T}, \mathcal{A}_i\rangle \models_{\text{brave}} R(a, b)$) since α is consistent (otherwise $\{(\alpha, i), (\beta, j)\}$ is not a conflict). By construction of $\mathcal{R}$, $A(a) \in \mathcal{R}$ (resp. $R(a, x_{aR}) \in \mathcal{R}$, resp. $R(x_{aR^-}, a) \in \mathcal{R}$, resp. $R(a, b) \in \mathcal{R}$), so $\langle\mathcal{T}, \mathcal{A}_j \cup \mathcal{R}\rangle$ is inconsistent.

In the other direction, if there exists i, $0 \leq i \leq n$, such that $\langle\mathcal{T}, \mathcal{A}_i \cup \mathcal{R}\rangle$ is inconsistent, let $\mathcal{B}$ be a conflict of $\langle\mathcal{T}, \mathcal{A}_i \cup \mathcal{R}\rangle$. If $\mathcal{B}$ is internal to $\mathcal{A}_i$, $\mathcal{K}$ is clearly inconsistent. Otherwise $\mathcal{B}$ is of the form $\{\alpha, \beta\}$ and involves at least an assertion of $\mathcal{R}$. The assertions α and β assign an individual x to two disjoint concepts (that may be existential restrictions of roles) C_1, C_2 or two individuals x, y to two disjoint roles R_1, R_2. Suppose for a contradiction that x appears only in $\mathcal{R}$. If $x = x_{aP}$ (resp. $x = x_{P_1P_2}$), since $P(a, x_{aP})$ (resp. $P_2(x_{P_1}, x_{P_1P_2})$) is the only assertion of $\mathcal{R}$ that contains x, it implies that $\exists P^-$ (resp. $\exists P_2^-$) is unsatisfiable. This contradicts the fact that there exists j such that $\langle\mathcal{T}, \mathcal{A}_j\rangle \models_{\text{brave}} \exists x P(a, x)$ (resp. $\langle\mathcal{T}, \mathcal{A}_j\rangle \models_{\text{brave}} \exists xy P_1(x, y)$ and $\mathcal{T} \models \exists P_1^- \sqsubseteq \exists P_2$). If $x = x_{P_1}$, since x_{P_1} appears only in concepts that subsume $\exists P_1^-$, it implies that $\exists P_1^-$ is unsatisfiable, which contradicts the fact that there exists j such that $\langle\mathcal{T}, \mathcal{A}_j\rangle \models_{\text{brave}} \exists xy P_1(x, y)$. It follows that $x \in \mathsf{N}_\mathsf{I}^\mathcal{K}$. Since α or β is in $\mathcal{R}$, at least one of C_1, C_2 (or R_1, R_2) is rigid. Let c_α be a cause for the brave entailment that triggered the addition of α to $\mathcal{R}$ if $\alpha \notin \mathcal{A}_i$ (in this case c_α belongs to some $\mathcal{A}_{j_\alpha}$), and otherwise $(c_\alpha, j_\alpha) = (\alpha, i)$, and c_β be a cause for the brave entailment that triggered the addition of β to $\mathcal{R}$ if $\beta \notin \mathcal{A}_i$ (in this case c_β belongs to some $\mathcal{A}_{j_\beta}$), and otherwise $(c_\beta, j_\beta) = (\beta, i)$. Then $\{(c_\alpha, j_\alpha), (c_\beta, j_\beta)\}$ is a conflict of $\mathcal{K}$ because c_α and c_β have for consequence that a (or a, b) is assigned to two disjoint concepts (or disjoint roles) such that at least one of them is rigid. $\square$

We now assume that $\mathcal{K}$ and $\mathcal{K}_\mathcal{R}$ are consistent. Note that if it is not the case, they both trivially entail any BTCQ. The brave entailments in the construction of $\mathcal{R}$ correspond thus to classical entailments. The two following lemmas show that if a Boolean conjunctive query $q = \exists \vec{y}\, \psi(\vec{y})$ is such that $\mathsf{N}_\mathsf{I}^q \subseteq \mathsf{N}_\mathsf{I}^\mathcal{K}$, then $\mathcal{K}_\mathcal{R}, p \models q$ iff $\mathcal{K}, p \models q$ iff $\mathcal{I}_p \models q$.

Lemma 4. *If* $q = \exists \vec{y}\, \psi(\vec{y})$ *is such that* $\mathsf{N}_\mathsf{I}^q \subseteq \mathsf{N}_\mathsf{I}^\mathcal{K}$*, if* $\mathcal{K}_\mathcal{R}, p \models q$ *then* $\mathcal{I}_p \models q$*.*

Proof. Assume that $\mathcal{K}_\mathcal{R}, p \models \exists \vec{y}\, \psi(\vec{y})$, i.e., $\langle\mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models \exists \vec{y}\, \psi(\vec{y})$ (since $\mathsf{N}_\mathsf{RC} = \mathsf{N}_\mathsf{RR} = \emptyset$). Let $\mathcal{I}_p^\mathcal{R} = (\Delta^{\mathcal{I}_p^\mathcal{R}}, \cdot^{\mathcal{I}_p^\mathcal{R}})$ be the canonical model of $\langle\mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$. There exists a homomorphism π of $\exists \vec{y}\, \psi(\vec{y})$ into $\mathcal{I}_p^\mathcal{R}$. We first define a mapping σ from $\{x^{\mathcal{I}_p^\mathcal{R}} \mid x \in \mathsf{N}_\mathsf{I}^\mathcal{K}$ or occurs in $\mathcal{R}\}$ into $\{x^{\mathcal{I}_p} \mid x \in \mathsf{N}_\mathsf{I}^\mathcal{K} \cup \Gamma_N, x$ occurs in $\mathit{chase}_{\text{rig}}^\mathcal{K}(\mathcal{A}_p)\}$ (we assume that Δ and $\Delta^{\mathcal{I}_p^\mathcal{R}}$ are disjoint) as follows:

- $\sigma(a^{\mathcal{I}_p^\mathcal{R}}) = a^{\mathcal{I}_p}$ for $a \in \mathsf{N}_\mathsf{I}^\mathcal{K}$
- $\sigma(x_{aP}^{\mathcal{I}_p^\mathcal{R}}) = x^{\mathcal{I}_p}$ such that $P(a, x) \in \mathit{chase}_{\text{rig}}^\mathcal{K}(\mathcal{A}_p)$
- $\sigma(x_P^{\mathcal{I}_p^\mathcal{R}}) = x^{\mathcal{I}_p}$ such that there exists $P(y, x) \in \bigcup_{i=0}^n \mathit{chase}_{\text{rig}}^\mathcal{K}(\mathcal{A}_i)$
- $\sigma(x_{PP'}^{\mathcal{I}_p^\mathcal{R}}) = x^{\mathcal{I}_p}$ such that $P'(y, x) \in \mathit{chase}_{\text{rig}}^\mathcal{K}(\mathcal{A}_p)$ with $\sigma(x_P^{\mathcal{I}_p^\mathcal{R}}) = y^{\mathcal{I}_p}$

Claim 1. σ is well defined:

If x_{aP} occurs in $\mathcal{R}$, there exists i such that $\langle\mathcal{T}, \mathcal{A}_i\rangle \models \exists x P(a, x)$, and since $\mathcal{I}_i$ is a model of $\langle\mathcal{T}, \mathcal{A}_i\rangle$, there is some $P(a, x) \in \mathit{chase}_{\text{rig}}^\mathcal{K}(\mathcal{A}_i)$. Moreover, since P is rigid, $P(a, x) \in \mathit{chase}_{\text{rig}}^\mathcal{K}(\mathcal{A}_p)$.

If x_P occurs in $\mathcal{R}$, there exists i such that $\langle\mathcal{T}, \mathcal{A}_i\rangle \models \exists xy P(x, y)$, so since $\mathcal{I}_i$ is a model of $\langle\mathcal{T}, \mathcal{A}_i\rangle$, there exist $x, y \in \mathsf{N}_\mathsf{I}^\mathcal{K} \cup \Gamma_N$ such that $P(y, x) \in \mathit{chase}_{\text{rig}}^\mathcal{K}(\mathcal{A}_i)$. Moreover, x occurs in $\mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_p)$ because there exists $B := A|\exists R|\exists R^-$ with $A \in \mathsf{N_{RC}}$ and $R \in \mathsf{N_{RR}}$ such that $\mathcal{T} \models \exists P^- \sqsubseteq B$, so there is a rigid assertion $\beta \models B(x)$ such that $\beta \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_p)$.

If $x_{PP'}$ occurs in $\mathcal{R}$, x_P occurs in $\mathcal{R}$, so there exist i and $y \in \mathsf{N}_\mathsf{I}^{\mathcal{K}} \cup \Gamma_N$ such that $P(y, \sigma(x_P^{\mathcal{I}_p^{\mathcal{R}}})) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$, and since by construction of $\mathcal{R}$ P' is rigid and such that $\mathcal{T} \models \exists P^- \sqsubseteq \exists P'$ and $\mathcal{I}_i$ is a model of $\mathcal{T}$, there exists $x \in \mathsf{N}_\mathsf{I}^{\mathcal{K}} \cup \Gamma_N$ such that $P'(\sigma(x_P^{\mathcal{I}_p^{\mathcal{R}}}), x) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$, and $P'(\sigma(x_P^{\mathcal{I}_p^{\mathcal{R}}}), x) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_p)$.

Claim 2. σ is a partial homomorphism of $\mathcal{I}_p^{\mathcal{R}}$ into $\mathcal{I}_p$:

For every $a \in \mathsf{N}_\mathsf{I}^{\mathcal{K}}$ and concept A, if $a^{\mathcal{I}_p^{\mathcal{R}}} \in A^{\mathcal{I}_p^{\mathcal{R}}}$, since $\mathcal{I}_p^{\mathcal{R}}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models A(a)$. Let $\{\alpha\}$ be a cause for $A(a)$. If $\alpha \in \mathcal{A}_p$, $\alpha \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_p)$, so since $\mathcal{I}_p$ is a model of $\mathcal{T}$ and $\langle \mathcal{T}, \alpha\rangle \models A(a)$, $\sigma(a^{\mathcal{I}_p^{\mathcal{R}}}) = a^{\mathcal{I}_p} \in A^{\mathcal{I}_p}$. Otherwise $\alpha \in \mathcal{R}$ and is either of the form $A'(a)$ with $A' \in \mathsf{N_{RC}}$, $P(a, b)$, or $P(a, x_{aP})$ with P rigid. In the two first cases, there exists i such that $\langle \mathcal{T}, \mathcal{A}_i\rangle \models \alpha$ so since $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i\rangle$, $\alpha \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$. Since α is rigid, $\alpha \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_p)$ so since $\mathcal{I}_p$ is a model of $\mathcal{T}$ and $\langle \mathcal{T}, \alpha\rangle \models A(a)$, $\sigma(a^{\mathcal{I}_p^{\mathcal{R}}}) = a^{\mathcal{I}_p} \in A^{\mathcal{I}_p}$. If $\alpha = P(a, x_{aP})$, there exists i such that $\langle \mathcal{T}, \mathcal{A}_i\rangle \models \exists x P(a, x)$. Since $\mathcal{I}_i$ is a model of $\langle \mathcal{T}, \mathcal{A}_i\rangle$, there is some $P(a, x) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$. Since P is rigid, $P(a, x) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_p)$ so since $\mathcal{I}_p$ is a model of $\mathcal{T}$ and $\langle \mathcal{T}, P(a, x)\rangle \models A(a)$, $\sigma(a^{\mathcal{I}_p^{\mathcal{R}}}) = a^{\mathcal{I}_p} \in A^{\mathcal{I}_p}$.

For every pair $a, b \in \mathsf{N}_\mathsf{I}^{\mathcal{K}}$ and role P, if $(a^{\mathcal{I}_p^{\mathcal{R}}}, b^{\mathcal{I}_p^{\mathcal{R}}}) \in P^{\mathcal{I}_p^{\mathcal{R}}}$, we can use similar arguments to show that $(\sigma(a^{\mathcal{I}_p^{\mathcal{R}}}), \sigma(b^{\mathcal{I}_p^{\mathcal{R}}})) = (a^{\mathcal{I}_p}, b^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$.

For every x_{aP} that occurs in $\mathcal{R}$ and $A \in \mathsf{N_C}$, if $x_{aP}^{\mathcal{I}_p^{\mathcal{R}}} \in A^{\mathcal{I}_p^{\mathcal{R}}}$, since $\mathcal{I}_p^{\mathcal{R}}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models A(x_{aP})$. Let $\{\alpha\}$ be a cause for $A(x_{aP})$. By construction, the only assertion of $\mathcal{A}_p \cup \mathcal{R}$ that involves x_{aP} is $P(a, x_{aP})$ so $\alpha = P(a, x_{aP})$ and $\langle \mathcal{T}, P(a, x_{aP})\rangle \models A(x_{aP})$. Since $\sigma(x_{aP}^{\mathcal{I}_p^{\mathcal{R}}}) = x^{\mathcal{I}_p}$ is such that $P(a, x) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_p)$ and $\mathcal{I}_p$ is a model of $\mathcal{T}$, then $\sigma(x_{aP}^{\mathcal{I}_p^{\mathcal{R}}}) \in A^{\mathcal{I}_p}$.

For every $a \in \mathsf{N}_\mathsf{I}^{\mathcal{K}}$, $x \notin \mathsf{N}_\mathsf{I}^{\mathcal{K}}$ that occurs in $\mathcal{R}$, and role P, if $(a^{\mathcal{I}_p^{\mathcal{R}}}, x^{\mathcal{I}_p^{\mathcal{R}}}) \in P^{\mathcal{I}_p^{\mathcal{R}}}$, since $\mathcal{I}_p^{\mathcal{R}}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models P(a, x)$. Let $\{\alpha\}$ be a cause for $P(a, x)$. By construction of $\mathcal{R}$, $x = x_{aP_1}$, and $\alpha = P_1(a, x_{aP_1})$ so by definition of σ, $(\sigma(a^{\mathcal{I}_p^{\mathcal{R}}}), \sigma(x_{aP_1P}^{\mathcal{I}_p^{\mathcal{R}}})) \in P_1^{\mathcal{I}_p}$. Since $\langle \mathcal{T}, P_1(a, x)\rangle \models P(a, x)$ and $\mathcal{I}_p$ is a model of $\mathcal{T}$, it follows that $(\sigma(a^{\mathcal{I}_p^{\mathcal{R}}}), \sigma(x_{aP_1P}^{\mathcal{I}_p^{\mathcal{R}}})) \in P^{\mathcal{I}_p}$.

For every x_{P_1} that occurs in $\mathcal{R}$ and $A \in \mathsf{N_C}$, if $x_{P_1}^{\mathcal{I}_p^{\mathcal{R}}} \in A^{\mathcal{I}_p^{\mathcal{R}}}$, since $\mathcal{I}_p^{\mathcal{R}}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models A(x_{P_1})$. Let $\{\alpha\}$ be a cause for $A(x_{P_1})$. By construction, either $\alpha = A'(x_{P_1})$ with $A' \in \mathsf{N_{RC}}$ and $\mathcal{T} \models \exists P_1^- \sqsubseteq A'$, or $\alpha = P_2(x_{P_1}, x_{P_1P_2})$ with P_2 rigid and $\mathcal{T} \models \exists P_1^- \sqsubseteq \exists P_2$. Since $\sigma(x_{P_1}^{\mathcal{I}_p^{\mathcal{R}}}) = x^{\mathcal{I}_p}$ is such that there exists i such that $P_1(y, x) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$ and $\mathcal{I}_i$ is a model of $\mathcal{T}$, then $A'(x) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$ (resp. there is some $P_2(x, z) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_i)$). Therefore $A'(x) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_p)$ (resp. there is some $P_2(x, z) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_p)$). It follows that $\sigma(x_{P_1}^{\mathcal{I}_p^{\mathcal{R}}}) \in A^{\mathcal{I}_p}$ because $\mathcal{I}_p$ is a model of $\mathcal{T}$.

For every $x_{P_1P_2}$ that occurs in $\mathcal{R}$ and $A \in \mathsf{N_C}$, if $x_{P_1P_2}^{\mathcal{I}_p^{\mathcal{R}}} \in A^{\mathcal{I}_p^{\mathcal{R}}}$, since $\mathcal{I}_p^{\mathcal{R}}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models A(x_{P_1P_2})$. Let $\{\alpha\}$ be a cause for $A(x_{P_1P_2})$. By construction, $\alpha = P_2(x_{P_1}, x_{P_1P_2})$, P_2 is rigid, and $\mathcal{T} \models \exists P_1^- \sqsubseteq \exists P_2$. Since $\sigma(x_{P_1P_2}^{\mathcal{I}_p^{\mathcal{R}}}) = x^{\mathcal{I}_p}$ such that there exists $P_2(y, x) \in \mathit{chase}^{\mathcal{K}}_{\text{rig}}(\mathcal{A}_p)$ (with $y^{\mathcal{I}_p^{\mathcal{R}}} = \sigma(x_{P_1}^{\mathcal{I}_p^{\mathcal{R}}})$) and $\mathcal{I}_p$ is a model of $\mathcal{T}$, then

$\sigma(x_{P_1P_2}^{\mathcal{I}_p^{\mathcal{R}}}) \in A^{\mathcal{I}_p}$.

Finally, for every $x, y \notin \mathsf{N}_\mathsf{I}^{\mathcal{K}}$ that occur in $\mathcal{R}$ and role P, if $(x^{\mathcal{I}_p^{\mathcal{R}}}, y^{\mathcal{I}_p^{\mathcal{R}}}) \in P^{\mathcal{I}_p^{\mathcal{R}}}$, since $\mathcal{I}_p^{\mathcal{R}}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle \models P(x, y)$. Let $\{\alpha\}$ be a cause for $P(x, y)$. By construction $x = x_{P_1}$, $y = x_{P_1P_2}$, $\alpha = P_2(x_{P_1}, x_{P_1P_2})$, and P_2 is rigid, so as previously, $(\sigma(x_{P_1}^{\mathcal{I}_p^{\mathcal{R}}}), \sigma(x_{P_1P_2}^{\mathcal{I}_p^{\mathcal{R}}})) \in P^{\mathcal{I}_p}$.

Claim 3. σ can be extended to a homomorphism σ' of $\mathcal{I}_p^{\mathcal{R}}$ into $\mathcal{I}_p$:

Since $\mathcal{I}_p^{\mathcal{R}}$ is the canonical model of $\langle \mathcal{T}, (\mathcal{A}_p \cup \mathcal{R})\rangle$, $\mathcal{I}_p$ is a model of $\mathcal{T}$, and σ preserves the concept or role memberships, we can extend σ to a homomorphism σ' of $\mathcal{I}_p^{\mathcal{R}}$ into $\mathcal{I}_p$ by mapping the anonymous part of $\mathcal{I}_p^{\mathcal{R}}$ rooted in $x^{\mathcal{I}_p^{\mathcal{R}}} \in \{x^{\mathcal{I}_p^{\mathcal{R}}} \mid x \in \mathsf{N}_\mathsf{I}^{\mathcal{K}}$ or occurs in $\mathcal{R}\}$ to the part of $\mathcal{I}_p$ rooted in $\sigma(x^{\mathcal{I}_p^{\mathcal{R}}})$.

It follows from Claim 3 that $\sigma' \circ \pi$ is a homomorphism of $\exists \vec{y}\, \psi(\vec{y})$ into $\mathcal{I}_p$. We have thus shown that if $\mathcal{K}_{\mathcal{R}}, p \models \exists \vec{y}\, \psi(\vec{y})$ then $\mathcal{I}_p \models \exists \vec{y}\, \psi(\vec{y})$. ☐

Lemma 5. *If $q = \exists \vec{y}\, \psi(\vec{y})$ is such that $\mathsf{N}_\mathsf{I}^{q} \subseteq \mathsf{N}_\mathsf{I}^{\mathcal{K}}$, if $\mathcal{I}_p \models q$ then $\mathcal{K}_{\mathcal{R}}, p \models q$.*

Proof. Assume that $\mathcal{I}_p \models \exists \vec{y}\, \psi(\vec{y})$, i.e., there exists a homomorphism π of $\exists \vec{y}\, \psi(\vec{y})$ into $\mathcal{I}_p$. Let $\mathcal{I}_p^{\mathcal{R}} = (\Delta^{\mathcal{I}_p^{\mathcal{R}}}, \cdot^{\mathcal{I}_p^{\mathcal{R}}})$ be a model of $\langle \mathcal{T}, (\mathcal{A}_i \cup \mathcal{R})\rangle$. We define a mapping $h_p^{\mathcal{R}}$ from $\{x^{\mathcal{I}_p} \mid x \in \mathsf{N}_\mathsf{I}^{\mathcal{K}} \cup \Gamma_N, x$ occurs in $\mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_p)\}$ into $\Delta^{\mathcal{I}_p^{\mathcal{R}}}$ (we assume that Δ and $\Delta^{\mathcal{I}_p^{\mathcal{R}}}$ are disjoint) as follows:

- for every $a \in \mathsf{N}_\mathsf{I}^{\mathcal{K}}$, $h_p^{\mathcal{R}}(a^{\mathcal{I}_p}) = a^{\mathcal{I}_p^{\mathcal{R}}}$
- for every $x_{aP_1}^{i_1}$ with $i_1 \neq p$ and P_1 is rigid, $h_p^{\mathcal{R}}(x_{aP_1}^{i_1\mathcal{I}_p}) = x_{aP_1}^{\mathcal{I}_p^{\mathcal{R}}}$
- for every $x_{aP_1\dots P_l}^{i_1\dots i_l}$ with $l > 1$, such that every $i_j \neq p$, and P_l is rigid and P_{l-1} is not rigid, $h_p^{\mathcal{R}}(x_{aP_1\dots P_l}^{i_1\dots i_l\mathcal{I}_p}) = x_{P_{l-1}P_l}^{\mathcal{I}_p^{\mathcal{R}}}$
- for every $x_{aP_1\dots P_l}^{i_1\dots i_l}$ with $l > 1$, such that every $i_j \neq p$, and P_l is rigid and P_{l-1} is rigid, $h_p^{\mathcal{R}}(x_{aP_1\dots P_l}^{i_1\dots i_l\mathcal{I}_p}) = y$ where $(h_p^{\mathcal{R}}(x_{aP_1\dots P_{l-1}}^{i_1\dots i_{l-1}\mathcal{I}_p}), y) \in P_l^{\mathcal{I}_p^{\mathcal{R}}}$ (if there are several such y, choose one of them randomly).
- for every $x_{aP_1\dots P_l}^{i_1\dots i_l}$ such that every $i_j \neq p$, and P_l not rigid, $h_p^{\mathcal{R}}(x_{aP_1\dots P_l}^{i_1\dots i_l\mathcal{I}_p}) = x_{P_l}^{\mathcal{I}_p^{\mathcal{R}}}$
- for every $x_{aP_1\dots P_l}^{i_1\dots i_l}$ such that there exists $i_j = p$, $h_p^{\mathcal{R}}(x_{aP_1\dots P_l}^{i_1\dots i_l\mathcal{I}_p}) = y$ where $(h_p^{\mathcal{R}}(x_{aP_1\dots P_{l-1}}^{i_1\dots i_{l-1}\mathcal{I}_p}), y) \in P_l^{\mathcal{I}_p^{\mathcal{R}}}$ (if there are several such y, choose one of them randomly).

Claim 1. $h_p^{\mathcal{R}}$ is well defined:

- Case $x_{aP_1}^{i_1}$ with $i_1 \neq p$ and P_1 is rigid, $h_p^{\mathcal{R}}(x_{aP_1}^{i_1\mathcal{I}_p}) = x_{aP_1}^{\mathcal{I}_p^{\mathcal{R}}}$:
  The individual x_{aP_1} appears in $\mathcal{R}$ because $x_{aP_1}^{i_1} \in \Gamma_N$ only if $\exists x P_1(a, x)$ is entailed by some $\langle \mathcal{T}, \mathcal{A}_j\rangle$ by (P1) and (P8).
- Case $x_{aP_1\dots P_l}^{i_1\dots i_l}$ with $l > 1$, such that every $i_j \neq p$, and P_l is rigid and P_{l-1} is not rigid, $h_p^{\mathcal{R}}(x_{aP_1\dots P_l}^{i_1\dots i_l\mathcal{I}_p}) = x_{P_{l-1}P_l}^{\mathcal{I}_p^{\mathcal{R}}}$:
  The individual $x_{P_{l-1}P_l}$ appears in $\mathcal{R}$ because P_l is rigid, P_{l-1} is not rigid, and since

$x^{i_1\dots i_l}_{aP_1\dots P_l} \in \Gamma_N$ then by (P4) $\mathcal{T} \models \exists P^-_{l-1} \sqsubseteq \exists P_l$, and by (P2) and (P12) there is some j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models \exists xy P_{l-1}(x, y)$.

- Case $x^{i_1\dots i_l}_{aP_1\dots P_l}$ with $l > 1$, such that every $i_j \neq p$, and P_l is rigid and P_{l-1} is rigid, $h^{\mathcal{R}}_p(x^{i_1\dots i_l\mathcal{I}_p}_{aP_1\dots P_l}) = y$ where $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$:

  We show that there is always such $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$ by induction on the length $length = l - r$ of the sequence of rigid roles $P_r \dots P_{l-1}$.

  - If $length = 1$, we are in one of the following cases:

  (i) $r > 1$ and $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{P_{l-2}P_{l-1}}$. Then $(x^{\mathcal{I}^{\mathcal{R}}_p}_{P_{l-2}}, x^{\mathcal{I}^{\mathcal{R}}_p}_{P_{l-2}P_{l-1}}) \in P_{l-1}^{\mathcal{I}^{\mathcal{R}}_p}$ because $\mathcal{I}^{\mathcal{R}}_p$ is a model of $\mathcal{R}$. Since $x^{i_1\dots i_l}_{aP_1\dots P_l} \in \Gamma_N$, by (P4) $\mathcal{T} \models \exists P^-_{l-1} \sqsubseteq \exists P_l$, so since $\mathcal{I}^{\mathcal{R}}_p$ is a model of $\mathcal{T}$, there is some $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$.

  (ii) $r = 1$ and $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = h^{\mathcal{R}}_p(x^{i_1\mathcal{I}_p}_{aP_1}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{aP_1}$ is such that $(a^{\mathcal{I}^{\mathcal{R}}_p}, x^{\mathcal{I}^{\mathcal{R}}_p}_{aP_1}) \in P_1^{\mathcal{I}^{\mathcal{R}}_p}$ because $P_1(a, x_{aP_1}) \in \mathcal{R}$. Since $x^{i_1 i_2}_{aP_1P_2} \in \Gamma_N$, $\mathcal{T} \models \exists P^-_1 \sqsubseteq \exists P_2$ by (P4), so since $\mathcal{I}^{\mathcal{R}}_p$ is a model of $\mathcal{T}$, there is some $(x^{\mathcal{I}^{\mathcal{R}}_p}_{aP_1}, y) \in P_2^{\mathcal{I}^{\mathcal{R}}_p}$.

  - Then for $length > 1$, $\mathcal{T} \models \exists P^-_{l-1} \sqsubseteq \exists P_l$ by (P4). It follows that since by induction there is an $(x, h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}})) \in P_{l-1}^{\mathcal{I}^{\mathcal{R}}_p}$, then there is some $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$.

- Case $x^{i_1\dots i_l}_{aP_1\dots P_l}$ such that every $i_j \neq p$, and P_l not rigid, $h^{\mathcal{R}}_p(x^{i_1\dots i_l\mathcal{I}_p}_{aP_1\dots P_l}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{P_l}$:
  Since $\mathcal{T}$ does not contain any role inclusion of the form $P' \sqsubseteq P$ with $P' := R_1 | R^-_1$, $R_1 \in \mathsf{N_R} \backslash \mathsf{N_{RR}}$ and $P := R_2 | R^-_2$, $R_2 \in \mathsf{N_{RR}}$, and P_l is not rigid, there is no P such that $P_l \sqsubseteq P$ and P is rigid. Therefore, since $i_l \neq p$, there is no P such that $P(x^{i_1\dots i_{l-1}}_{aP_1\dots P_{l-1}}, x^{i_1\dots i_l}_{aP_1\dots P_l}) \in \mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_p)$ so $x^{i_1\dots i_l}_{aP_1\dots P_l}$ occurs in $\mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_p)$ only if there is $B := A | \exists R | \exists R^-$ with $A \in \mathsf{N_{RC}}$, $R \in \mathsf{N_{RR}}$ such that $\mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_p) \models B(x^{i_1\dots i_l}_{aP_1\dots P_l})$. By (P3) $\mathcal{T} \models \exists P^-_l \sqsubseteq B$, and by (P2) and (P12) there is some j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models \exists xy P_{l-1}(x, y)$. It follows that x_{P_l} appears in $\mathcal{R}$.

- Case $x^{i_1\dots i_l}_{aP_1\dots P_l}$ such that there exists $i_j = p$, $h^{\mathcal{R}}_p(x^{i_1\dots i_l\mathcal{I}_p}_{aP_1\dots P_l}) = y$ where $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$:

  We show that there is always such $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}), y) \in P_l^{\mathcal{I}^{\mathcal{R}}_p}$ by induction on the length $length = l - r$ of the chain of roles that links $x^{i_1\dots i_l}_{aP_1\dots P_l}$ to the first individual $x^{i_1\dots i_r}_{aP_1\dots P_r}$ such that $i_r = p$:

  - If $length = 0$, then $i_l = p$ and there is no $j < l$ such that $i_j = p$. We are thus in one of the following cases: either (i) $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = a^{\mathcal{I}^{\mathcal{R}}_p}$, or (ii) $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{aP_1}$, or (iii) $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{P_{l-2}P_{l-1}}$, or (iv) $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}})$ is such that $(h^{\mathcal{R}}_p(x^{i_1\dots i_{l-2}\mathcal{I}_p}_{aP_1\dots P_{l-2}}), h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}})) \in P_{l-1}^{\mathcal{I}^{\mathcal{R}}_p}$, or (v) $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{P_{l-1}}$:

  (i) if $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = a^{\mathcal{I}^{\mathcal{R}}_p}$: by definition of $h^{\mathcal{R}}_p$, $x^{i_1\dots i_{l-1}}_{aP_1\dots P_{l-1}} = a$, so $x^{i_1\dots p}_{aP_1\dots P_l} = x^p_{aP_1}$. Since $x^p_{aP_1} \in \Gamma_N$, by (P1) $P_1(a, x^p_{aP_1}) \in \mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_p)$. By (P8) either (a) $\langle \mathcal{T}, \mathcal{A}_p \rangle \models \exists x P_1(a, x)$, so there is some $(a^{\mathcal{I}^{\mathcal{R}}_p}, y) \in P_1^{\mathcal{I}^{\mathcal{R}}_p}$ because $\mathcal{I}^{\mathcal{R}}_p$ is a model of $\langle \mathcal{T}, \mathcal{A}_p \rangle$, or (b) there exists $B := A | \exists R | \exists R^-$ with $A \in \mathsf{N_{RC}}$, $R \in \mathsf{N_{RR}}$, such that $\mathcal{T} \models B \sqsubseteq \exists P_1$ and there exists j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models B(a)$. In the latter case, $\mathcal{R} \models B(a)$ by construction of $\mathcal{R}$, and since $\mathcal{I}^{\mathcal{R}}_p$ is a model of $\mathcal{R}$, $\mathcal{I}^{\mathcal{R}}_p \models B(a)$. Since $\mathcal{I}^{\mathcal{R}}_p$ is a model of $\mathcal{T}$, there is some $(a^{\mathcal{I}^{\mathcal{R}}_p}, y) \in P_1^{\mathcal{I}^{\mathcal{R}}_p}$.

  (ii) if $h^{\mathcal{R}}_p(x^{i_1\dots i_{l-1}\mathcal{I}_p}_{aP_1\dots P_{l-1}}) = x^{\mathcal{I}^{\mathcal{R}}_p}_{aP_1}$: by definition of $h^{\mathcal{R}}_p$, $x^{i_1\dots i_{l-1}}_{aP_1\dots P_{l-1}} = x^{i_1}_{aP_1}$ and P_1 is rigid. By (P1) $P_1(a, x^{i_1}_{aP_1}) \in \mathit{chase}^{\mathcal{K}}_{\mathrm{rig}}(\mathcal{A}_{i_1})$, so by (P8) either (a) $\langle \mathcal{T}, \mathcal{A}_{i_1} \rangle \models \exists x P_1(a, x)$, so

$P_1(a, x_{aP_1}) \in \mathcal{R}$ since P_1 is rigid, or (b) there exists $B := A|\exists R|\exists R^-$ with $A \in \mathsf{N}_{\mathsf{RC}}$, $R \in \mathsf{N}_{\mathsf{RR}}$, such that $\mathcal{T} \models B \sqsubseteq \exists P_1$ and there exists j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models B(a)$. In the latter case $\langle \mathcal{T}, \mathcal{A}_j \rangle \models \exists x P_1(a, x)$, so $P_1(a, x_{aP_1}) \in \mathcal{R}$. In both cases, $(a^{\mathcal{I}_p^{\mathcal{R}}}, x_{aP_1}^{\mathcal{I}_p^{\mathcal{R}}}) \in P_1^{\mathcal{I}_p^{\mathcal{R}}}$ since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\mathcal{R}$. Moreover, since $x_{aP_1 \ldots P_l}^{i_1 \ldots i_l} = x_{aP_1P_2}^{i_1 p} \in \Gamma_N$, by (P4) $\mathcal{T} \models \exists P_1^- \sqsubseteq \exists P_2$, so since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\mathcal{T}$, there is some $(x_{aP_1}^{\mathcal{I}_p^{\mathcal{R}}}, y) \in P_2^{\mathcal{I}_p^{\mathcal{R}}}$.

(iii) if $h_p^{\mathcal{R}}(x_{aP_1 \ldots P_{l-1}}^{i_1 \ldots i_{l-1} \mathcal{I}_p}) = x_{P_{l-2}P_{l-1}}^{\mathcal{I}_p^{\mathcal{R}}}$: by definition of $\mathcal{R}$, since $x_{P_{l-2}P_{l-1}}$ appears in $\mathcal{R}$, $P_{l-1}(x_{P_{l-2}}, x_{P_{l-2}P_{l-1}}) \in \mathcal{R}$, so $(x_{P_{l-2}}^{\mathcal{I}_p^{\mathcal{R}}}, x_{P_{l-2}P_{l-1}}^{\mathcal{I}_p^{\mathcal{R}}}) \in P_{l-1}^{\mathcal{I}_p^{\mathcal{R}}}$. Since $x_{aP_1 \ldots P_l}^{i_1 \ldots i_l} \in \Gamma_N$, by (P4) $\mathcal{T} \models \exists P_{l-1}^- \sqsubseteq \exists P_l$, so there is some $(x_{aP_{l-2}P_{l-1}}^{\mathcal{I}_p^{\mathcal{R}}}, y) \in P_l^{\mathcal{I}_p^{\mathcal{R}}}$.

(iv) if $(h_p^{\mathcal{R}}(x_{aP_1 \ldots P_{l-2}}^{i_1 \ldots i_{l-2} \mathcal{I}_p}), h_p^{\mathcal{R}}(x_{aP_1 \ldots P_{l-1}}^{i_1 \ldots i_{l-1} \mathcal{I}_p})) \in P_{l-1}^{\mathcal{I}_p^{\mathcal{R}}}$: since $x_{aP_1 \ldots P_l}^{i_1 \ldots i_l} \in \Gamma_N$, by (P4) $\mathcal{T} \models \exists P_{l-1}^- \sqsubseteq \exists P_l$, so there is some $(h_p^{\mathcal{R}}(x_{aP_1 \ldots P_{l-1}}^{i_1 \ldots i_{l-1} \mathcal{I}_p}), y) \in P_l^{\mathcal{I}_p^{\mathcal{R}}}$.

(v) if $h_p^{\mathcal{R}}(x_{aP_1 \ldots P_{l-1}}^{i_1 \ldots i_{l-1} \mathcal{I}_p}) = x_{P_{l-1}}^{\mathcal{I}_p^{\mathcal{R}}}$: by (P2) and since $i_l = p$, $P_l(x_{aP_1 \ldots P_{l-1}}^{i_1 \ldots i_{l-1}}, x_{aP_1 \ldots P_l}^{i_1 \ldots p}) \in \mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_p)$. By (P6), since $\mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_p) \models \exists P_l(x_{aP_1 \ldots P_{l-1}}^{i_1 \ldots i_{l-1}})$ and $i_{l-1} \neq p$, there exists $B := A|\exists R|\exists R^-$ with $A \in \mathsf{N}_{\mathsf{RC}}, R \in \mathsf{N}_{\mathsf{RR}}$, such that $\mathcal{T} \models B \sqsubseteq \exists P_l$ and $\mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_{i_{l-1}}) \models B(x_{aP_1 \ldots P_{l-1}}^{i_1 \ldots i_{l-1}})$. By (P3), $\mathcal{T} \models \exists P_{l-1}^- \sqsubseteq B$, so $\mathcal{R} \models B(x_{P_{l-1}})$ (since $x_{P_{l-1}}$ occurs in $\mathcal{R}$ and B is rigid), so $\langle \mathcal{T}, \mathcal{R} \rangle \models \exists x P_l(x_{P_{l-1}}, x)$. Since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\langle \mathcal{T}, \mathcal{R} \rangle$, there is some $(x_{P_{l-1}}^{\mathcal{I}_p^{\mathcal{R}}}, y) \in P_l^{\mathcal{I}_p^{\mathcal{R}}}$.

- Then for *length* > 0, since $x_{aP_1 \ldots P_l}^{i_1 \ldots i_l} \in \Gamma_N$, by (P4) $\mathcal{T} \models \exists P_{l-1}^- \sqsubseteq \exists P_l$. It follows that since by induction there is an $(x, h_p^{\mathcal{R}}(x_{aP_1 \ldots P_{l-1}}^{i_1 \ldots i_{l-1} \mathcal{I}_p})) \in P_{l-1}^{\mathcal{I}_p^{\mathcal{R}}}$, then there is some $(h_p^{\mathcal{R}}(x_{aP_1 \ldots P_{l-1}}^{i_1 \ldots i_{l-1} \mathcal{I}_p}), y) \in P_l^{\mathcal{I}_p^{\mathcal{R}}}$.

Claim 2. $h_p^{\mathcal{R}}$ is a homomorphism of $\mathcal{I}_p$ into $\mathcal{I}_p^{\mathcal{R}}$:

For every $a \in \mathsf{N}_{\mathsf{I}}^{\mathcal{K}}$ and concept A, if $a^{\mathcal{I}_p} \in A^{\mathcal{I}_p}$, i.e., $A(a) \in \mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_p)$, then by (P5), either (i) $\langle \mathcal{T}, \mathcal{A}_p \rangle \models A(a)$, and since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\langle \mathcal{T}, \mathcal{A}_p \rangle$, then $h_p^{\mathcal{R}}(a^{\mathcal{I}_p}) = a^{\mathcal{I}_p^{\mathcal{R}}} \in A^{\mathcal{I}_p^{\mathcal{R}}}$, or (ii) there exists $B := C|\exists R|\exists R^-$ with $C \in \mathsf{N}_{\mathsf{RC}}$, $R \in \mathsf{N}_{\mathsf{RR}}$, such that $\mathcal{T} \models B \sqsubseteq A$ and there exists j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models B(a)$. In the latter case $\mathcal{R} \models B(a)$, so since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\mathcal{R}$, $\mathcal{I}_p^{\mathcal{R}} \models B(a) \models A(a)$. It follows that $h_p^{\mathcal{R}}(a^{\mathcal{I}_p}) = a^{\mathcal{I}_p^{\mathcal{R}}} \in A^{\mathcal{I}_p^{\mathcal{R}}}$. For every pair $a, b \in \mathsf{N}_{\mathsf{I}}^{\mathcal{K}}$ and role P, if $(a^{\mathcal{I}_p}, b^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$, by (P7), similar arguments can be used to prove that $(h_p^{\mathcal{R}}(a^{\mathcal{I}_p}), h_p^{\mathcal{R}}(b^{\mathcal{I}_p})) = (a^{\mathcal{I}_p^{\mathcal{R}}}, b^{\mathcal{I}_p^{\mathcal{R}}}) \in P^{\mathcal{I}_p^{\mathcal{R}}}$.

For every $x_{aP_1 \ldots P_l}^{i_1 \ldots i_l} \in \Gamma_N$, such that $x_{aP_1 \ldots P_l}^{i_1 \ldots i_l \mathcal{I}_p} \in A^{\mathcal{I}_p}$, by (P3), $\mathcal{T} \models \exists P_l^- \sqsubseteq A$, and by construction of $h_p^{\mathcal{R}}$, $h_p^{\mathcal{R}}(x_{aP_1 \ldots P_l}^{i_1 \ldots i_l \mathcal{I}_p}) = y$ such that either (i) there exists $(x, y) \in P_l^{\mathcal{I}_p^{\mathcal{R}}}$, so since $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\mathcal{T}$, $y \in A^{\mathcal{I}_p^{\mathcal{R}}}$, or (ii) $y = x_{P_l}^{\mathcal{I}_p^{\mathcal{R}}}$, P_l is not rigid and for every i_j, $i_j \neq p$. In the latter case by (P6) there exists $B := C|\exists R|\exists R^-$ with $C \in \mathsf{N}_{\mathsf{RC}}$, $R \in \mathsf{N}_{\mathsf{RR}}$, such that $\mathcal{T} \models B \sqsubseteq A$ and $\mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_{i_l}) \models B(x_{aP_1 \ldots P_l}^{i_1 \ldots i_l})$. By (P3) $\mathcal{T} \models \exists P_l^- \sqsubseteq B$, so by construction of $\mathcal{R}$, $\mathcal{R} \models B(x_{P_l})$ and $\langle \mathcal{T}, \mathcal{R} \rangle \models A(x_{P_l})$. It follows that $y \in A^{\mathcal{I}_p^{\mathcal{R}}}$.

For every pair $x, y \in \Gamma_N$ and role P, such that $(x^{\mathcal{I}_p}, y^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$, by (P10) $x = x_{aP_1 \ldots P_l}^{i_1 \ldots i_l}, y = x_{aP_1 \ldots P_l P_{l+1}}^{i_1 \ldots i_l i_{l+1}}$ and $\mathcal{T} \models P_{l+1} \sqsubseteq P$ or $x = x_{aP_1 \ldots P_l P_{l+1}}^{i_1 \ldots i_l i_{l+1}}, y = x_{aP_1 \ldots P_l}^{i_1 \ldots i_l}$ and $\mathcal{T} \models P_{l+1} \sqsubseteq P^-$. We can assume w.l.o.g. that we are in the first case (otherwise we consider $(y^{\mathcal{I}_p}, x^{\mathcal{I}_p}) \in P^{-\mathcal{I}_p}$). If $i_{l+1} = p$, by definition of $h_p^{\mathcal{R}}$, $(h_p^{\mathcal{R}}(x^{\mathcal{I}_p}), h_p^{\mathcal{R}}(y^{\mathcal{I}_p})) \in P_{l+1}^{\mathcal{I}_p^{\mathcal{R}}}$. Otherwise, by (P11), there exists $P' := R|R^-$ with $R \in \mathsf{N}_{\mathsf{RR}}$ such that $\mathcal{T} \models P_{l+1} \sqsubseteq P' \sqsubseteq P$ and $P'(x, y) \in \mathit{chase}_{\mathrm{rig}}^{\mathcal{K}}(\mathcal{A}_{i_{l+1}})$. In this case, there are several possibilities:

(i) P_l is not rigid: given that $\mathcal{T} \models P_{l+1} \sqsubseteq P'$ and P' is rigid, P_{l+1} is rigid by our hypothesis on the TBox. It follows that $h_p^{\mathcal{R}}(y^{\mathcal{I}_p}) = x_{P_l P_{l+1}}^{\mathcal{I}_p^{\mathcal{R}}}$. If there is no $i_j = p$, then $h_p^{\mathcal{R}}(x^{\mathcal{I}_p}) = x_{P_l}^{\mathcal{I}_p^{\mathcal{R}}}$ so since $P_{l+1}(x_{P_1}, x_{P_l P_{l+1}}) \in \mathcal{R}$, $(h_p^{\mathcal{R}}(x^{\mathcal{I}_p}), h_p^{\mathcal{R}}(y^{\mathcal{I}_p})) \in P_{l+1}^{\mathcal{I}_p^{\mathcal{R}}}$. Otherwise there is some $i_j = p$, and $(h_p^{\mathcal{R}}(x^{\mathcal{I}_p}), h_p^{\mathcal{R}}(y^{\mathcal{I}_p})) \in P_{l+1}^{\mathcal{I}_p^{\mathcal{R}}}$ by definition of $h_p^{\mathcal{R}}$.

(ii) P_l is rigid: $h_p^{\mathcal{R}}(y^{\mathcal{I}_p})$ is such that $(h_p^{\mathcal{R}}(x^{\mathcal{I}_p^{\mathcal{R}}}), h_p^{\mathcal{R}}(y^{\mathcal{I}_p^{\mathcal{R}}})) \in P_{l+1}^{\mathcal{I}_p^{\mathcal{R}}}$.

Since in any case $(h_p^{\mathcal{R}}(x^{\mathcal{I}_p}), h_p^{\mathcal{R}}(y^{\mathcal{I}_p})) \in P_{l+1}^{\mathcal{I}_p^{\mathcal{R}}}$ and $\mathcal{I}_p^{\mathcal{R}}$ is a model of $\mathcal{T}$, $(h_p^{\mathcal{R}}(x^{\mathcal{I}_p}), h_p^{\mathcal{R}}(y^{\mathcal{I}_p})) \in P^{\mathcal{I}_p^{\mathcal{R}}}$.

Finally, if $a \in \mathsf{N_I^{\mathcal{K}}}$ and $x \in \Gamma_N$, $(a^{\mathcal{I}_p}, x^{\mathcal{I}_p}) \in P^{\mathcal{I}_p}$ only if $x = x_{aP_1}^{i_1}$. If $i_1 = p$, by definition of $h_p^{\mathcal{R}}$, $(h_p^{\mathcal{R}}(a^{\mathcal{I}_p}), h_p^{\mathcal{R}}(x^{\mathcal{I}_p})) \in P_1^{\mathcal{I}_p^{\mathcal{R}}}$ and since by (P8) $\mathcal{T} \models P_1 \sqsubseteq P$, $(h_p^{\mathcal{R}}(a^{\mathcal{I}_p}), h_p^{\mathcal{R}}(x^{\mathcal{I}_p})) \in P^{\mathcal{I}_p^{\mathcal{R}}}$. If $i_1 \neq p$, by (P9), there exists P' rigid such that $\mathcal{T} \models P_1 \sqsubseteq P' \sqsubseteq P$, so by our hypothesis on the TBox P_1 is rigid. By (P1) and (P8), there is some j such that $\langle \mathcal{T}, \mathcal{A}_j \rangle \models \exists x P_1(a, x)$, so $P_1(a, x_{aP_1}) \in \mathcal{R}$ so $(h_p^{\mathcal{R}}(a^{\mathcal{I}_p}), h_p^{\mathcal{R}}(x^{\mathcal{I}_p})) = (a^{\mathcal{I}_p^{\mathcal{R}}}, x_{aP_1}^{\mathcal{I}_p^{\mathcal{R}}}) \in P_1^{\mathcal{I}_p^{\mathcal{R}}}$. Thus $(h_p^{\mathcal{R}}(a^{\mathcal{I}_p}), h_p^{\mathcal{R}}(x^{\mathcal{I}_p})) \in P^{\mathcal{I}_p^{\mathcal{R}}}$.

It follows from Claim 2 that $h_p^{\mathcal{R}} \circ \pi$ is a homomorphism of $\exists \vec{y}\, \psi(\vec{y})$ into $\mathcal{I}_p^{\mathcal{R}}$, so we have shown that if $\mathcal{I}_p \models \exists \vec{y}\, \psi(\vec{y})$ then $\mathcal{K}_{\mathcal{R}}, p \models \exists \vec{y}\, \psi(\vec{y})$. $\square$

Now that we have shown that $\mathcal{K}$ and $\mathcal{K}_{\mathcal{R}}$ with $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$ entail the same BCQs, we show by induction on the structure of the BTCQ ϕ that if $\mathsf{N_I^{\phi}} \subseteq \mathsf{N_I^{\mathcal{K}}}$, then $\mathcal{K}, p \models \phi$ iff $\mathcal{K}_{\mathcal{R}}, p \models \phi$ with $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$. It follows that TCQ answering over $\mathcal{K}$ in Case 3 can be done by TCQ answering over $\mathcal{K}_{\mathcal{R}}$ in Case 1 and pruning answers that contain individual names not from $\mathsf{N_I^{\mathcal{K}}}$. Note that a model of $\mathcal{K}_{\mathcal{R}}$ is a model of $\mathcal{K}$ but does not respect rigid predicates in general. We can reduce BTCQ entailment over $\mathcal{K}$ with rigid predicates to BTCQ entailment over $\mathcal{K}_{\mathcal{R}}$ without rigid predicates only because our TCQs do not allow LTL operators to be nested in existential quantifications. This prevents existentially quantified variables to link different time points. Otherwise a query as $\exists xy \Box (R(a, x) \wedge R(x, y))$ with $\mathcal{T} = \{B \sqsubseteq \exists R, \exists R^- \sqsubseteq \exists R\}$, $R \in \mathsf{N_{RR}}$ and $\mathcal{A}_i = \{B(a)\}$ would be entailed from $\mathcal{K}$ but not from $\mathcal{K}_{\mathcal{R}}$ with $\mathsf{N_{RR}} = \emptyset$. Indeed, in this case $\mathcal{R} = \{R(a, x_{aR})\}$, so x_{aR} may have a different R-successor in each interpretation of a model of $\mathcal{K}_{\mathcal{R}}$ and y cannot be mapped to the same object at every time point.

Lemma 6. *If a BTCQ ϕ is such that $\mathsf{N_I^{\phi}} \subseteq \mathsf{N_I^{\mathcal{K}}}$, then $\mathcal{K}, p \models \phi$ iff $\mathcal{K}_{\mathcal{R}}, p \models \phi$ with $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$.*

Proof. By Lemma 2, $\mathcal{K}, p \models \phi$ iff $\mathcal{J}_{\mathcal{K}}, p \models \phi$. We show by induction on the structure of ϕ that $\mathcal{J}_{\mathcal{K}}, p \models \phi$ iff $\mathcal{K}_{\mathcal{R}}, p \models \phi$.

If $\phi = \exists \vec{y}\, \psi(\vec{y})$, since $\mathsf{N_I^{\phi}} \subseteq \mathsf{N_I^{\mathcal{K}}}$, by Lemmas 4 and 5, $\mathcal{J}_{\mathcal{K}}, p \models \phi$ iff $\mathcal{K}_{\mathcal{R}}, p \models \phi$.

Assume that for two BTCQs ϕ_1, ϕ_2 such that $\mathsf{N_I^{\phi_1}} \subseteq \mathsf{N_I^{\mathcal{K}}}$ and $\mathsf{N_I^{\phi_2}} \subseteq \mathsf{N_I^{\mathcal{K}}}$, $\mathcal{J}_{\mathcal{K}}, p \models \phi_i$ iff $\mathcal{K}_{\mathcal{R}}, p \models \phi_i$ $(i \in \{1, 2\})$. Then:

- $\mathcal{J}_{\mathcal{K}}, p \models \phi_1 \wedge \phi_2$ iff $\mathcal{J}_{\mathcal{K}}, p \models \phi_1$ and $\mathcal{J}_{\mathcal{K}}, p \models \phi_2$
  iff $\mathcal{K}_{\mathcal{R}}, p \models \phi_1$ and $\mathcal{K}_{\mathcal{R}}, p \models \phi_2$
  iff $\mathcal{K}_{\mathcal{R}}, p \models \phi_1 \wedge \phi_2$ by Proposition 2 ($\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$)
- $\mathcal{J}_{\mathcal{K}}, p \models \phi_1 \vee \phi_2$ iff $\mathcal{J}_{\mathcal{K}}, p \models \phi_1$ or $\mathcal{J}_{\mathcal{K}}, p \models \phi_2$
  iff $\mathcal{K}_{\mathcal{R}}, p \models \phi_1$ or $\mathcal{K}_{\mathcal{R}}, p \models \phi_2$
  iff $\mathcal{K}_{\mathcal{R}}, p \models \phi_1 \vee \phi_2$ by Proposition 2 ($\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$)

- $\mathcal{J}_\mathcal{K}, p \models \bigcirc\phi_1$ iff $p < n$ and $\mathcal{J}_\mathcal{K}, p+1 \models \phi_1$
  iff $p < n$ and $\mathcal{K}_\mathcal{R}, p+1 \models \phi_1$
  iff $\mathcal{K}_\mathcal{R}, p \models \bigcirc\phi_1$ by Proposition 2 ($\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$)
- $\mathcal{J}_\mathcal{K}, p \models \bullet\phi_1$ iff $p < n$ implies $\mathcal{J}_\mathcal{K}, p+1 \models \phi_1$
  iff $p < n$ implies $\mathcal{K}_\mathcal{R}, p+1 \models \phi_1$
  iff $\mathcal{K}_\mathcal{R}, p \models \bullet\phi_1$ by Proposition 2 ($\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$)
- $\mathcal{J}_\mathcal{K}, p \models \Box\phi_1$ iff for every k, $p \leq k \leq n$, $\mathcal{J}_\mathcal{K}, k \models \phi_1$
  iff for every k, $p \leq k \leq n$, $\mathcal{K}_\mathcal{R}, k \models \phi_1$
  iff $\mathcal{K}_\mathcal{R}, p \models \Box\phi_1$ by Proposition 2 ($\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$)
- $\mathcal{J}_\mathcal{K}, p \models \Diamond\phi_1$ iff there exists k, $p \leq k \leq n$, $\mathcal{J}_\mathcal{K}, k \models \phi_1$
  iff there exists k, $p \leq k \leq n$, $\mathcal{K}_\mathcal{R}, k \models \phi_1$
  iff $\mathcal{K}_\mathcal{R}, p \models \Diamond\phi_1$ by Proposition 2 ($\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$)
- $\mathcal{J}_\mathcal{K}, p \models \phi_1 \mathsf{U} \phi_2$ iff there exists k, $p \leq k \leq n$, $\mathcal{J}_\mathcal{K}, k \models \phi_2$ and for every j, $p \leq j < k$, $\mathcal{J}_\mathcal{K}, j \models \phi_1$
  iff there exists k, $p \leq k \leq n$, $\mathcal{K}_\mathcal{R}, k \models \phi_2$ and for every j, $p \leq j < k$, $\mathcal{K}_\mathcal{R}, j \models \phi_1$
  iff $\mathcal{K}_\mathcal{R}, p \models \phi_1 \mathsf{U} \phi_2$ by Proposition 2 ($\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$)
- $\bigcirc^-\phi_1$, $\bullet^-\phi_1$, $\Box^-\phi_1$, $\Diamond^-\phi_1$, $\phi_1 \mathsf{S} \phi_2$: similar to the corresponding future operators

We conclude by induction that for every BTCQ ϕ such that $\mathsf{N_I}^\phi \subseteq \mathsf{N_I}^\mathcal{K}$, $\mathcal{K}, p \models \phi$ iff $\mathcal{K}_\mathcal{R}, p \models \phi$. □

Theorem 1 states the complexity results for the classical semantics as we will use them for the complexity analysis of the inconsistency-tolerant semantics. They follow from known results and Proposition 4. Note that for data complexity, we will need only the P upper bound implied by the ALogTime-completeness of TCQ answering.

Theorem 1. *If $\mathcal{T}$ does not entail any role inclusion of the form $P_1 \sqsubseteq P_2$ with $P_1 := R_1|R_1^-$, $R_1 \in \mathsf{N_R} \backslash \mathsf{N_{RR}}$ and $P_2 := R_2|R_2^-$, $R_2 \in \mathsf{N_{RR}}$, then consistency checking is in* P *w.r.t. combined complexity and TCQ answering is in* P *w.r.t. data complexity, and* NP*-complete w.r.t. combined complexity.*

Proof. It has been shown in [12] that TCQ answering is in ALogTime $\subseteq$ P w.r.t. data complexity.

The NP membership of TCQ answering in Case 1 ($\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$) for combined complexity follows from the rewritability results of [10]. We describe how to guess a certificate that $\mathcal{K}, p \models \phi$ that can be checked in P. A certificate consists of:

- a sequence of functions $(\nu_i)_{0 \leq i \leq n}$ that associate to each BCQ q of ϕ true or false, and
- for each BCQ q of ϕ and time point i, if $\nu_i(q) = \mathsf{true}$: a rewriting q' of q that holds in $\mathcal{A}_i$ together with the rewriting steps that produce q' from q and $\mathcal{T}$, and a variable assignment that maps q' in $\mathcal{A}_i$.

There are polynomially many pairs of a time point and a BCQ, and the number of steps required to produce each q' from q is polynomial, so the certificate has a polynomial size and checking that each q' is indeed a rewriting of q and holds in $\mathcal{A}_i$ can be done in polynomial time. Moreover verifying that the propositional LTL formula obtained by replacing the BCQs by propositional variables is satisfied by the sequence of truth assignments that assign the propositional abstraction of q to $\nu_i(q)$ is in P because the formula does not contain negation.

	AR	IAR	brave	AR	IAR	brave
Case 1 ($\mathsf{N_{RC}} = \emptyset, \mathsf{N_{RR}} = \emptyset$)	coNP-c	in P	in P	Π_2^p-c	NP-c	NP-c
Case 2 ($\mathsf{N_{RC}} \neq \emptyset, \mathsf{N_{RR}} = \emptyset$)	coNP-c	in P	NP-c	Π_2^p-c	NP-c	NP-c
Case 3* ($\mathsf{N_{RC}} \neq \emptyset, \mathsf{N_{RR}} \neq \emptyset$)	coNP-c	in P	NP-c	Π_2^p-c	NP-c	NP-c

Figure 1: Data [left] and combined [right] complexity of BTCQ entailment over DL-Lite$_{\mathcal{R}}$ TKBs under the different semantics. *: only with rigid specializations of rigid roles

For the NP upper bound of BTCQ entailment in Cases 2 and 3 (if $\mathcal{T}$ does not contain any role inclusion of the form $P_1 \sqsubseteq P_2$ with $P_1 := R_1 | R_1^-$, $R_1 \in \mathsf{N_R} \setminus \mathsf{N_{RR}}$ and $P_2 := R_2 | R_2^-$, $R_2 \in \mathsf{N_{RR}}$), we compute $\mathcal{R}$ in polynomial time then check whether ϕ is entailed from $\mathcal{K}_{\mathcal{R}}$ with $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$.

The NP-hardness comes from the atemporal case. □

We have shown that disallowing negations in the TCQ makes the combined complexity of TCQ answering drop from PSPACE to NP and that rigid concepts and roles can be handled by adding a set of assertions that captures all relevant information about rigid assertions to each ABox of the TKB.

4.2 Complexity of inconsistency-tolerant TCQ answering

We now turn our attention to the inconsistency-tolerant semantics.

Theorem 2. *The results in Figure 1 hold.*

We break the proof of Theorem 2 in several propositions. First, the following lemma shows that verifying that a sequence of ABoxes is a repair of $\mathcal{K}$ is in P.

Lemma 7. *Verifying that a sequence of ABoxes $(\mathcal{A}_i')_{0 \leq i \leq n}$ is a repair of $\mathcal{K}$ can be done in* P.

Proof. We show that $(\mathcal{A}_i')_{0 \leq i \leq n}$ is a repair of $\mathcal{K}$ as follows (consistency checking is in P, cf. Theorem 1):

- For every i, check that $\mathcal{A}_i' \subseteq \mathcal{A}_i$,
- Check that $(\mathcal{A}_i')_{0 \leq i \leq n}$ is $\mathcal{T}$-consistent,
- For every $(\alpha, j) \in (\mathcal{A}_i)_{0 \leq i \leq n} \setminus (\mathcal{A}_i')_{0 \leq i \leq n}$, check that $(\mathcal{A}_i')_{0 \leq i \leq n} \cup \{(\alpha, j)\}$ is $\mathcal{T}$-inconsistent.

□

The complexity results for AR semantics follow straightforwardly from Lemma 7 and the complexity of TCQ answering under classical semantics.

Proposition 5. *AR TCQ answering is co*NP*-complete w.r.t. data complexity, and* Π_2^p*-complete w.r.t. combined complexity.*

Proof. For the upper bounds, we show that a BTCQ ϕ is not entailed under AR semantics from a TKB $\mathcal{K}$ by guessing a repair $(\mathcal{A}_i')_{0 \leq i \leq n}$ of $\mathcal{K}$ that does not entail ϕ. Checking that $(\mathcal{A}_i')_{0 \leq i \leq n}$

is a repair can be done in P by Lemma 7, and checking that $\langle\mathcal{T},(\mathcal{A}'_i)_{0\leq i\leq n}\rangle\not\models\phi$ is in P w.r.t. data complexity and coNP-complete w.r.t. combined complexity (Theorem 1).

The lower bounds come from the atemporal case [20, 9]. □

For IAR semantics, we show that the intersection of the repairs can be computed in polynomial time because in DL-Lite$_\mathcal{R}$ TKBs the size of the conflicts is at most two. The complexity of IAR TCQ answering is then the same as that of the classical semantics.

Proposition 6. *IAR TCQ answering is in* P *w.r.t. data complexity, and* NP*-complete w.r.t. combined complexity.*

Proof. For the upper bounds, we compute the conflicts of $\mathcal{K}$ in P by checking the consistency of every timed-assertion and pair of timed-assertions, then answer the query in P w.r.t. data complexity, NP w.r.t. combined complexity, over the TKB from which they have been removed. Indeed, we show that the intersection of the repairs of $\mathcal{K}$ is obtained by removing the conflicts of $\mathcal{K}$. If a timed-assertion (α,i) is inconsistent it cannot be in a repair, and if (α,i) is consistent, if there exists (β,j) consistent such that $\{(\alpha,i),(\beta,j)\}$ is inconsistent, (α,i) is not in the repairs that contain (β,j). In the other direction, if (α,i) does not appear in some repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$, since the repairs are maximal, $(\mathcal{A}'_i)_{0\leq i\leq n}\cup\{(\alpha,i)\}$ is inconsistent so (α,i) is in some conflict of $\mathcal{K}$.

The lower bound comes from CQ entailment in the atemporal case. □

For brave semantics, the combined complexity follows from Lemma 7 and Theorem 1.

Proposition 7. *Brave TCQ answering is* NP*-complete w.r.t. combined complexity.*

Proof. For the upper bound, we show that a BTCQ ϕ is entailed under brave semantics from a TKB $\mathcal{K}$ by guessing a repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$ that entails ϕ together with a certificate that $\langle\mathcal{T},(\mathcal{A}'_i)_{0\leq i\leq n}\rangle\models\phi$ (cf. Theorem 1). Checking that $(\mathcal{A}'_i)_{0\leq i\leq n}$ is a repair can be done in P by Lemma 7, and checking the certificate that $\langle\mathcal{T},(\mathcal{A}'_i)_{0\leq i\leq n}\rangle\models\phi$ is in P as in proof of Theorem 1.

The lower bound comes from CQ entailment in the atemporal case. □

The data complexity of brave semantics is less straightforward. Indeed, the data complexity upper bound for brave CQ answering relies on the fact that the size of the minimal sets of assertions that support the query is bounded by the query size, which is not true in the temporal setting (e.g., consider $\phi=\Box A(a)$, which needs n assertions to be entailed). Moreover, while brave BCQ entailment is tractable in the atemporal setting, we show that if rigid concepts are allowed, brave BTCQ entailment is NP-hard.

Proposition 8. *If* $\mathsf{N_{RC}}\neq\emptyset$*, then brave TCQ answering is* NP*-complete w.r.t. data complexity.*

Proof. The upper bound comes from the combined complexity.

We show the lower bound by reduction from SAT. Let $\varphi=C_1\wedge...\wedge C_n$ be a CNF formula over variables $x_1,...,x_m$. We define the following problem of BTCQ entailment under brave semantics, with two rigid concepts T and F. Let $\mathcal{K}=\{\mathcal{T},(\mathcal{A}_i)_{1\leq i\leq n}\}$ be such that:

$$\mathcal{T}=\{\exists Pos\sqsubseteq Sat,\ \exists Neg\sqsubseteq Sat,\ \exists Pos^-\sqsubseteq T,\ \exists Neg^-\sqsubseteq F,\ T\sqsubseteq\neg F\}$$
$$\mathcal{A}_i=\{Pos(c,x_j)\mid x_j\in C_i\}\ \cup\ \{Neg(c,x_j)\mid\neg x_j\in C_i\}\ \text{ for } 1\leq i\leq n$$

Let $\phi=\Box^- Sat(c)$. We show that φ is satisfiable iff $\mathcal{K},n\models_{\text{brave}}\phi$. Indeed, since T and F are rigid, a repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$ is such that each x_j has only Pos or Neg incoming edges in

$(\mathcal{A}'_i)_{0\leq i\leq n}$. We can thus define a valuation ν of the variables such that $\nu(x_j) = \mathsf{true}$ if $(\mathcal{A}'_i)_{0\leq i\leq n}$ does not contain any timed-assertion of the form $(\mathit{Neg}(c, x_j), k)$, $\nu(x_j) = \mathsf{false}$ otherwise. The clause C_i is satisfied by ν iff there exists x_j such that either $x_j \in C_i$ and $\nu(x_j) = \mathsf{true}$ or $\neg x_j \in C_i$ and $\nu(x_j) = \mathsf{false}$, so iff there exists x_j such that either $\mathit{Pos}(c, x_j) \in \mathcal{A}'_i$ or $\mathit{Neg}(c, x_j) \in \mathcal{A}'_i$, so iff $\langle \mathcal{T}, (\mathcal{A}'_i)_{0\leq i\leq n}\rangle, i \models \mathit{Sat}(c)$. It follows that φ is satisfiable iff there exists a repair $(\mathcal{A}'_i)_{0\leq i\leq n}$ of $\mathcal{K}$ that entails ϕ at time point n. $\square$

It remains to show that in Case 1, brave TCQ answering can be done in polynomial time. We describe a method for brave BTCQ entailment when $\mathsf{N}_{\mathsf{RC}} = \mathsf{N}_{\mathsf{RR}} = \emptyset$ that proceeds by type elimination over a set of tuples built from the query and that represent the TCQs that are entailed at each time point. First, we define the structure on which the method operates. We consider the set $L(\phi)$ of *leaves* of ϕ, that is, the set of all BCQs in ϕ, and the set $F(\phi)$ of *subformulas* of ϕ. In what follows, we identify the BCQs of $L(\phi)$ and the BTCQs of $F(\phi)$ with their *propositional abstractions*: if we write that a KB or a TKB entails some elements of $L(\phi)$ or $F(\phi)$, we consider them as BCQs or BTCQs, and if we write that some elements of $L(\phi)$ or $F(\phi)$ entail others, we consider the elements of $L(\phi)$ as propositional variables and those of $F(\phi)$ as propositional LTL formulas built over these variables.

Definition 8. A *justification structure* J for the BTCQ ϕ in the TKB $\mathcal{K}$ is a set of tuples of the form $(i, L_{\text{now}}, F_{\text{now}}, F_{\text{prev}}, F_{\text{next}})$, where $0 \leq i \leq n$, $L_{\text{now}} \subseteq L(\phi)$, $F_{\text{now}} \subseteq F(\phi)$, $F_{\text{prev}} \subseteq F(\phi)$, and $F_{\text{next}} \subseteq F(\phi)$.

Note that the size of a justification structure for ϕ in $\mathcal{K} = \langle \mathcal{T}, (\mathcal{A}_i)_{0\leq i\leq n}\rangle$ is linearly bounded in n and independent of the size of the ABoxes. A tuple $(i, L_{\text{now}}, F_{\text{now}}, F_{\text{prev}}, F_{\text{next}})$ is *justified in* J iff it fulfils all of the following conditions:

1. $\langle \mathcal{T}, \mathcal{A}_i\rangle \models_{\text{brave}} \bigwedge_{q\in L_{\text{now}}} q$
2. If $i > 0$, there exists $(i-1, L'_{\text{now}}, F'_{\text{now}}, F'_{\text{prev}}, F'_{\text{next}}) \in J$ such that
   $F_{\text{prev}} = F'_{\text{now}}$ and $F_{\text{now}} = F'_{\text{next}}$
3. If $i < n$, there exists $(i+1, L'_{\text{now}}, F'_{\text{now}}, F'_{\text{prev}}, F'_{\text{next}}) \in J$ such that
   $F_{\text{next}} = F'_{\text{now}}$ and $F_{\text{now}} = F'_{\text{prev}}$
4. For every $\psi \in L(\phi)$, if $F_{\text{now}} \models \psi$, then $\psi \in L_{\text{now}}$
5. For every $\psi \in F(\phi)$, if $F_{\text{now}} \models \psi$, then $\psi \in F_{\text{now}}$
6. For every $\psi \in F(\phi)$, if $\bigwedge_{q\in L_{\text{now}}} q \wedge \bigcirc^{-}(\bigwedge_{\chi\in F_{\text{prev}}} \chi) \wedge \bigcirc(\bigwedge_{\chi\in F_{\text{next}}} \chi) \models \psi$, then $\psi \in F_{\text{now}}$
7. For every $\psi, \psi' \in F(\phi)$:
   if $\psi \vee \psi' \in F_{\text{now}}$, then either $\psi \in F_{\text{now}}$ or $\psi' \in F_{\text{now}}$
   if $\Diamond\psi \in F_{\text{now}}$, then either $\psi \in F_{\text{now}}$ or $\Diamond\psi \in F_{\text{next}}$
   if $\Diamond^{-}\psi \in F_{\text{now}}$, then either $\psi \in F_{\text{now}}$ or $\Diamond^{-}\psi \in F_{\text{prev}}$
   if $\psi'\mathsf{U}\psi \in F_{\text{now}}$, then either $\psi \in F_{\text{now}}$ or $\psi' \in F_{\text{now}}$ and $\psi'\mathsf{U}\psi \in F_{\text{next}}$
   if $\psi'\mathsf{S}\psi \in F_{\text{now}}$, then either $\psi \in F_{\text{now}}$ or $\psi' \in F_{\text{now}}$ and $\psi'\mathsf{S}\psi \in F_{\text{prev}}$
8. If $i = n$,
   $\forall\psi \in F(\phi)$ of the form $\bullet\varphi$, $\psi \in F_{\text{now}}$
   $\forall\psi \in F(\phi)$ of the form $\bigcirc\varphi$, $\psi \notin F_{\text{now}}$
   $\forall\psi \in F(\phi)$ of the form $\Diamond\varphi, \Box\varphi, \varphi'\mathsf{U}\varphi$, $\psi \in F_{\text{now}}$ iff $\varphi \in F_{\text{now}}$

9. If $i = 0$,
   $\forall \psi \in F(\phi)$ of the form $\bullet^- \varphi$, $\psi \in F_{\text{now}}$
   $\forall \psi \in F(\phi)$ of the form $\bigcirc^- \varphi$, $\psi \notin F_{\text{now}}$
   $\forall \psi \in F(\phi)$ of the form $\Diamond^- \varphi, \Box^- \varphi, \varphi' \mathsf{S} \varphi$, $\psi \in F_{\text{now}}$ iff $\varphi \in F_{\text{now}}$

We give the intuition behind the elements of the tuples fulfilling these conditions. The first element i is the time point we are considering, L_{now} is a set of BCQs whose conjunction is entailed under brave semantics by $\langle \mathcal{T}, \mathcal{A}_i \rangle$ (Condition 1), and F_{now} is the set of formulas that can be entailed together with L_{now}, depending on what is entailed in the previous and next time points, this information being stored in F_{prev} and F_{next} respectively (Condition 6). Conditions 2 and 3 ensure that there is a sequence of tuples representing every time point from 0 to n such that this information is coherent between consecutive tuples. Condition 4 expresses that L_{now} is exactly the set of BCQs contained in F_{now} and Condition 5 that F_{now} is maximal in the sense that it contains its consequences. Condition 7 enforces that F_{now}, F_{prev} and F_{next} respect the semantics of LTL operators and Conditions 8 and 9 enforce this semantics at the ends of the finite sequence.

A justification structure J is *correct* if every tuple is justified, and ϕ is *justified at time point* p *by* J if there is $(p, L_{\text{now}}, F_{\text{now}}, F_{\text{prev}}, F_{\text{next}}) \in J$ such that $\phi \in F_{\text{now}}$. We show that ϕ is entailed from $\mathcal{K}$ at time point p under brave semantics iff there is a correct justification structure for ϕ in $\mathcal{K}$ that justifies ϕ at time point p. The main idea is to link the tuples of a sequence $((i, L_{\text{now}}, F_{\text{now}}, F_{\text{prev}}, F_{\text{next}}))_{0 \leq i \leq n}$ to a consistent TKB $\mathcal{K}' = \langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle$ such that for every i, $\mathcal{C}_i \subseteq \mathcal{A}_i$ and $\langle \mathcal{T}, \mathcal{C}_i \rangle \models \bigwedge_{q \in L_{\text{now}}} q$. We show that there is such a $\mathcal{K}'$ such that $\mathcal{K}', p \models \phi$ iff there is such a sequence of tuples that is a correct justification structure for ϕ in $\mathcal{K}$ and justifies ϕ at time point p.

Lemma 8. *If* $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$ *and there is a correct justification structure* J *for* ϕ *in* $\mathcal{K}$ *that justifies* ϕ *at time point* p*, then* $\mathcal{K}, p \models_{brave} \phi$*.*

Proof. In order to show $\mathcal{K}, p \models_{\text{brave}} \phi$, we determine a cause $(\mathcal{C}_i)_{0 \leq i \leq n}$ for ϕ. To do this, we first select a sequence of tuples from J as follows:

1. The tuple $(p, L^p_{\text{now}}, F^p_{\text{now}}, F^p_{\text{prev}}, F^p_{\text{next}})$ is such that $\phi \in F^p_{\text{now}}$.
2. If the tuple $(i, L^i_{\text{now}}, F^i_{\text{now}}, F^i_{\text{prev}}, F^i_{\text{next}})$ was selected and $0 < i \leq p$, select a tuple $(i - 1, L^{i-1}_{\text{now}}, F^{i-1}_{\text{now}}, F^{i-1}_{\text{prev}}, F^{i-1}_{\text{next}})$ such that $F^{i-1}_{\text{now}} = F^i_{\text{prev}}$ and $F^{i-1}_{\text{next}} = F^i_{\text{now}}$.
3. If the tuple $(i, L^i_{\text{now}}, F^i_{\text{now}}, F^i_{\text{prev}}, F^i_{\text{next}})$ was selected and $p \leq i < n$, select a tuple $(i + 1, L^{i+1}_{\text{now}}, F^{i+1}_{\text{now}}, F^{i+1}_{\text{prev}}, F^{i+1}_{\text{next}})$ such that $F^{i+1}_{\text{now}} = F^i_{\text{next}}$ and $F^{i+1}_{\text{prev}} = F^i_{\text{now}}$.

Because J is correct and justifies ϕ at time point p, such a sequence can always be selected. Based on this sequence, we construct a sequence of ABoxes $(\mathcal{C}_i)_{0 \leq i \leq n}$ by taking for each tuple $(i, L^i_{\text{now}}, F^i_{\text{now}}, F^i_{\text{prev}}, F^i_{\text{next}})$ a cause $\mathcal{C}_i \subseteq \mathcal{A}_i$ for $\bigwedge_{q \in L^i_{\text{now}}} q$. Such a cause exists because $\langle \mathcal{T}, \mathcal{A}_i \rangle \models_{\text{brave}} \bigwedge_{q \in L^i_{\text{now}}} q$ by Condition 1. Since each $\mathcal{C}_i$ is consistent and rigid predicates are not allowed, the TKB $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle$ is consistent.

We prove that $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle, p \models \phi$, by proving that $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle, p \models F^p_{\text{now}}$. To do this, we consider the sets of LTL formulas $F^{i,d}_{\text{now}} = \{\psi \mid \psi \in F^i_{\text{now}}, \mathsf{degree}(\psi) \leq d\}$ where $\mathsf{degree}(\psi)$ is the maximal number of nested LTL operators in ψ and prove by induction on d that for all $0 \leq i \leq n$, for all $\psi \in F^{i,d}_{\text{now}}$, $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle, i \models \psi$, i.e., $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle, i \models F^{i,d}_{\text{now}}$.

For $d = 0$, $F^{i,0}_{\text{now}}$ contains only conjunctive queries of the form $\exists \vec{y}\, \varphi(\vec{y})$. Since for every $\psi \in L(\phi)$, if $F^i_{\text{now}} \models \psi$ then $\psi \in L^i_{\text{now}}$ (Condition 4), $F^{i,0}_{\text{now}} \subseteq L^i_{\text{now}}$. Then since $\langle \mathcal{T}, \mathcal{C}_i \rangle \models \bigwedge_{q \in L^i_{\text{now}}} q$, it follows that $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle, i \models F^{i,0}_{\text{now}}$.

Assume that for all $0 \leq i \leq n$, $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models F^{i,d}_{\text{now}}$. Let $\psi \in F^{i,d+1}_{\text{now}}$ for some $0 \leq i \leq n$. If $\psi \in F^{i,d}_{\text{now}}$, then $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \psi$. Otherwise, $\mathsf{degree}(\psi) = d+1$ and ψ is of one of the following forms:

- $\psi = \psi_1 \wedge \psi_2$ where $\mathsf{degree}(\psi_1) \leq d$, $\mathsf{degree}(\psi_2) \leq d$: since $\psi \in F^i_{\text{now}}$, then $F^i_{\text{now}} \models \psi_1$ and $F^i_{\text{now}} \models \psi_2$, so by Condition 5, $\psi_1 \in F^i_{\text{now}}$ and $\psi_2 \in F^i_{\text{now}}$. It follows that $\psi_1 \in F^{i,d}_{\text{now}}$ and $\psi_2 \in F^{i,d}_{\text{now}}$, so $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \psi_1$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \psi_2$. Hence $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \psi_1 \wedge \psi_2$.

- $\psi = \psi_1 \vee \psi_2$ where $\mathsf{degree}(\psi_1) \leq d$, $\mathsf{degree}(\psi_2) \leq d$: since $\psi \in F^i_{\text{now}}$, then by Condition 7 either $\psi_1 \in F^i_{\text{now}}$ or $\psi_2 \in F^i_{\text{now}}$. It follows that $\psi_1 \in F^{i,d}_{\text{now}}$ or $\psi_2 \in F^{i,d}_{\text{now}}$, so $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \psi_1$ or $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \psi_2$. Hence $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \psi_1 \vee \psi_2$.

- $\psi = \bigcirc\psi_1$ where $\mathsf{degree}(\psi_1) \leq d$: by Condition 8, $i < n$ because there cannot be a formula of the form $\psi = \bigcirc\psi_1$ in F^n_{now}. Since $\bigcirc\psi_1 \in F^i_{\text{now}} = F^{i+1}_{\text{prev}}$, we have that $\bigwedge_{q \in L^{i+1}_{\text{now}}} q \wedge \bigcirc^-(\bigwedge_{\chi \in F^{i+1}_{\text{prev}}} \chi) \wedge \bigcirc(\bigwedge_{\chi \in F^{i+1}_{\text{next}}} \chi) \models \bigcirc^-\bigcirc\psi_1 \models \psi_1$, so by Condition 6, $\psi_1 \in F^{i+1}_{\text{now}}$. Hence $\psi_1 \in F^{i+1,d}_{\text{now}}$ so $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i+1 \models \psi_1$, so $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \bigcirc\psi_1$.

- $\psi = \bigcirc^-\psi_1$ where $\mathsf{degree}(\psi_1) \leq d$: proof similar to $\bigcirc$.

- $\psi = \bullet\psi_1$ where $\mathsf{degree}(\psi_1) \leq d$: if $i < n$, since $\bullet\psi_1 \in F^i_{\text{now}} = F^{i+1}_{\text{prev}}$, we have that $\bigwedge_{q \in L^{i+1}_{\text{now}}} q \wedge \bigcirc^-(\bigwedge_{\chi \in F^{i+1}_{\text{prev}}} \chi) \wedge \bigcirc(\bigwedge_{\chi \in F^{i+1}_{\text{next}}} \chi) \models \bigcirc^-\bullet\psi_1 \models \psi_1$, so by Condition 6, $\psi_1 \in F^{i+1}_{\text{now}}$. Hence $\psi_1 \in F^{i+1,d}_{\text{now}}$ so $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i+1 \models \psi_1$, so $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \bullet\psi_1$. Otherwise, if $i = n$, $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, n \models \bullet\psi_1$ by definition of $\bullet$.

- $\psi = \bullet^-\psi_1$ where $\mathsf{degree}(\psi_1) \leq d$: proof similar to $\bullet$.

- $\psi = \Box\psi_1$ where $\mathsf{degree}(\psi_1) \leq d$: we show that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \Box\psi_1$ by descending induction on i.

  For $i = n$, if $\Box\psi_1 \in F^n_{\text{now}}$ then $\psi_1 \in F^n_{\text{now}}$ by Condition 8, so $\psi_1 \in F^{n,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, n \models \psi_1$, which implies that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, n \models \Box\psi_1$.

  For $i < n$, we assume that if $\Box\psi_1 \in F^{i+1}_{\text{now}}$ then $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i+1 \models \Box\psi_1$. Then since $\Box\psi_1 \in F^i_{\text{now}} = F^{i+1}_{\text{prev}}$, we have that $\bigwedge_{q \in L^{i+1}_{\text{now}}} q \wedge \bigcirc^-(\bigwedge_{\chi \in F^{i+1}_{\text{prev}}} \chi) \wedge \bigcirc(\bigwedge_{\chi \in F^{i+1}_{\text{next}}} \chi) \models \bigcirc^-\Box\psi_1 \models \Box\psi_1$, so by Condition 6, $\Box\psi_1 \in F^{i+1}_{\text{now}}$, so by assumption $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i+1 \models \Box\psi_1$. Moreover, since $\Box\psi_1 \in F^i_{\text{now}}$, then $F^i_{\text{now}} \models \psi_1$, so $\psi_1 \in F^i_{\text{now}}$ by Condition 5. Hence $\psi_1 \in F^{i,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \psi_1$. It follows that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \Box\psi_1$.

- $\psi = \Box^-\psi_1$ where $\mathsf{degree}(\psi_1) \leq d$: proof similar to $\Box$.

- $\psi = \Diamond\psi_1$ where $\mathsf{degree}(\psi_1) \leq d$: we show that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \Diamond\psi_1$ by descending induction on i.

  For $i = n$, if $\Diamond\psi_1 \in F^n_{\text{now}}$ then $\psi_1 \in F^n_{\text{now}}$ by Condition 8, so $\psi_1 \in F^{n,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, n \models \psi_1$, which implies that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, n \models \Diamond\psi_1$.

  For $i < n$, we assume that if $\Diamond\psi_1 \in F^{i+1}_{\text{now}}$ then $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i+1 \models \Diamond\psi_1$. Then, since $\Diamond\psi_1 \in F^i_{\text{now}}$, by Condition 7, either (i) $\psi_1 \in F^i_{\text{now}}$, $\psi_1 \in F^{i,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \psi_1$ so $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \Diamond\psi_1$, or (ii) $\Diamond\psi_1 \in F^i_{\text{next}} = F^{i+1}_{\text{now}}$, so by assumption $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i+1 \models \Diamond\psi_1$. It follows that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \Diamond\psi_1$.

- $\psi = \Diamond^-\psi_1$ where $\mathsf{degree}(\psi_1) \leq d$: proof similar to $\Diamond$.

- $\psi = \psi_1 \mathsf{U} \psi_2$ where $\mathsf{degree}(\psi_1) \leq d$, $\mathsf{degree}(\psi_2) \leq d$: we show that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, i \models \psi_1 \mathsf{U} \psi_2$ by descending induction on i.

  For $i = n$, if $\psi_1 \mathsf{U} \psi_2 \in F^n_{\text{now}}$ then $\psi_2 \in F^n_{\text{now}}$ by Condition 8, so $\psi_2 \in F^{n,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, n \models \psi_2$, which implies that $\langle \mathcal{T}, (\mathcal{C}_i)_{0\leq i\leq n}\rangle, n \models \psi_1 \mathsf{U} \psi_2$.

For $i < n$, we assume that if $\psi_1 \mathsf{U} \psi_2 \in F^{i+1}_{\text{now}}$ then $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle, i{+}1 \models \psi_1 \mathsf{U} \psi_2$. Then since $\psi_1 \mathsf{U} \psi_2 \in F^i_{\text{now}}$, by Condition 7, either (i) $\psi_2 \in F^i_{\text{now}}$, $\psi_2 \in F^{i,d}_{\text{now}}$ and $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle, i \models \psi_2$ so $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle, i \models \psi_1 \mathsf{U} \psi_2$, or (ii) $\psi_1 \in F^i_{\text{now}}$, $\psi_1 \in F^{i,d}_{\text{now}}$, so $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle, i \models \psi_1$, and $\psi_1 \mathsf{U} \psi_2 \in F^i_{\text{next}} = F^{i+1}_{\text{now}}$, so by assumption $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle, i + 1 \models \psi_1 \mathsf{U} \psi_2$, thus $\langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle, i \models \psi_1 \mathsf{U} \psi_2$.

- $\psi = \psi_1 \mathsf{S} \psi_2$ where $\mathsf{degree}(\psi_1) \leq d$, $\mathsf{degree}(\psi_2) \leq d$: proof similar to U.

□

Lemma 9. *If $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$ and $\mathcal{K}, p \models_{brave} \phi$, then there is a justification structure for ϕ in $\mathcal{K}$ that is correct and justifies ϕ at time point p.*

Proof. Assume $\mathcal{K}, p \models_{\text{brave}} \phi$, and let $\mathcal{K}' = \langle \mathcal{T}, (\mathcal{C}_i)_{0 \leq i \leq n} \rangle$, $\mathcal{C}_i \subseteq \mathcal{A}_i$ such that $\mathcal{K}'$ is consistent and $\mathcal{K}', p \models \phi$. Based on $\mathcal{K}'$, we construct a justification structure J for ϕ in $\mathcal{K}$ that justifies ϕ at time point p. The elements of the tuples $(i, L^i_{\text{now}}, F^i_{\text{now}}, F^i_{\text{prev}}, F^i_{\text{next}})$ are selected as follows:

1. L^i_{now} is the largest subset of $L(\phi)$ such that $\mathcal{K}', i \models \bigwedge_{q \in L^i_{\text{now}}} q$,
2. F^i_{now} is the largest subset of $F(\phi)$ such that $\mathcal{K}', i \models F^i_{\text{now}}$,
3. $F^i_{\text{prev}} = F^{i-1}_{\text{now}}$ for $i > 0$, and
4. $F^i_{\text{next}} = F^{i+1}_{\text{now}}$ for $i < n$
5. $F^0_{\text{prev}} = F^n_{\text{next}} = \emptyset$

We show that J is correct and justifies ϕ at time point p. Since $\mathcal{K}', p \models \phi$, then $\phi \in F^p_{\text{now}}$ so ϕ is justified by J at time point p.

It remains to show that J is correct, i.e., that every tuple of J satisfies the nine conditions of the definition of justified tuples. Conditions 1, 2, 3 and 4 follow straightforwardly from the construction. Condition 5 is satisfied because if $\psi \in F(\phi)$ is such that $\psi \notin F^i_{\text{now}}$, then $\mathcal{K}', i \not\models \psi$ so $F^i_{\text{now}} \not\models \psi$.

For Condition 6, we show that for every $\psi \in F(\phi)$, for every $0 \leq i \leq n$, if $\bigwedge_{q \in L^i_{\text{now}}} q \wedge \bigcirc^-(\bigwedge_{\chi \in F^i_{\text{prev}}} \chi) \wedge \bigcirc(\bigwedge_{\chi \in F^i_{\text{next}}} \chi) \models \psi$, then $\mathcal{K}', i \models \psi$, so $\psi \in F^i_{\text{now}}$. Since $\mathcal{K}'$ entails every CQ of L^i_{now} at time point i, every TCQ of F^i_{prev} at time point $i-1$, and every TCQ of F^i_{next} at time point $i+1$, then every TCQ that corresponds to a formula entailed by L^i_{now}, $\bigcirc^-(\bigwedge_{\chi \in F^i_{\text{prev}}} \chi)$ or $\bigcirc(\bigwedge_{\chi \in F^i_{\text{next}}} \chi)$ is entailed from $\mathcal{K}'$ at time point i. Hence, if $\bigwedge_{q \in L^i_{\text{now}}} q \wedge \bigcirc^-(\bigwedge_{\chi \in F^i_{\text{prev}}} \chi) \wedge \bigcirc(\bigwedge_{\chi \in F^i_{\text{next}}} \chi) \models \psi$, then $\mathcal{K}', i \models \psi$.

For Condition 7, since $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, by Proposition 2, for all BTCQs ψ, ψ':

- if $\mathcal{K}', i \models \psi \vee \psi'$, then $\mathcal{K}', i \models \psi$ or $\mathcal{K}', i \models \psi'$, so if $\psi \vee \psi' \in F^i_{\text{now}}$ then either $\psi \in F^i_{\text{now}}$, or $\psi' \in F^i_{\text{now}}$.
- if $\mathcal{K}', i \models \Diamond \psi$, then $\mathcal{K}', i \models \psi$ or $\mathcal{K}', i + 1 \models \Diamond \psi$, so if $\Diamond \psi \in F^i_{\text{now}}$ then either $\psi \in F^i_{\text{now}}$, or $\Diamond \psi \in F^{i+1}_{\text{now}} = F^i_{\text{next}}$.
- if $\mathcal{K}', i \models \Diamond^- \psi$, then $\mathcal{K}', i \models \psi$ or $\mathcal{K}', i - 1 \models \Diamond^- \psi$, so if $\Diamond^- \psi \in F^i_{\text{now}}$ then either $\psi \in F^i_{\text{now}}$, or $\Diamond^- \psi \in F^{i-1}_{\text{now}} = F^i_{\text{prev}}$.
- if $\mathcal{K}', i \models \psi \mathsf{U} \psi'$, then $\mathcal{K}', i \models \psi'$ or $\mathcal{K}', i \models \psi$ and $\mathcal{K}', i + 1 \models \psi \mathsf{U} \psi'$, so if $\psi \mathsf{U} \psi' \in F^i_{\text{now}}$ then either $\psi' \in F^i_{\text{now}}$, or $\psi \in F^i_{\text{now}}$ and $\psi \mathsf{U} \psi' \in F^i_{\text{next}}$.

- if $\mathcal{K}', i \models \psi \mathsf{S} \psi'$, then $\mathcal{K}', i \models \psi'$ or $\mathcal{K}', i \models \psi$ and $\mathcal{K}', i-1 \models \psi \mathsf{S} \psi'$, so if $\psi \mathsf{S} \psi' \in F^i_{\text{now}}$ then either $\psi' \in F^i_{\text{now}}$, or $\psi \in F^i_{\text{now}}$ and $\psi \mathsf{S} \psi' \in F^i_{\text{prev}}$.

The proof of Condition 8 is as follows:

- if $\psi \in F(\phi)$ is of the form $\newmoon\varphi$, $\mathcal{K}', n \models \psi$ so $\psi \in F^n_{\text{now}}$
- if $\psi \in F(\phi)$ is of the form $\ocircle\varphi$, $\mathcal{K}', n \not\models \psi$ so $\psi \notin F^n_{\text{now}}$
- if $\varphi \in F^n_{\text{now}}$, then $\mathcal{K}', n \models \varphi$ so $\mathcal{K}', n \models \Diamond\varphi$, $\mathcal{K}', n \models \Box\varphi$ and $\mathcal{K}', n \models \varphi' \mathsf{U} \varphi$. It follows that if they belong to $F(\phi)$, then $\Diamond\varphi \in F^n_{\text{now}}$, $\Box\varphi \in F^n_{\text{now}}$ and $\varphi' \mathsf{U} \varphi \in F^n_{\text{now}}$.

  For the other direction

  - if $\Diamond\varphi \in F^n_{\text{now}}$, $\mathcal{K}', n \models \Diamond\varphi$ so $\mathcal{K}', n \models \varphi$ and $\varphi \in F^n_{\text{now}}$
  - if $\Box\varphi \in F^n_{\text{now}}$, $\mathcal{K}', n \models \Box\varphi$ so $\mathcal{K}', n \models \varphi$ and $\varphi \in F^n_{\text{now}}$
  - if $\varphi' \mathsf{U} \varphi \in F^n_{\text{now}}$, $\mathcal{K}', n \models \varphi' \mathsf{U} \varphi$ so $\mathcal{K}', n \models \varphi$ and $\varphi \in F^n_{\text{now}}$

We prove Condition 9 similarly to Condition 8.

We have thus shown that every tuple in J is justified, so J is correct and justifies ϕ at p. ☐

The data complexity of brave TCQ answering in Case 1 follows from the characterization of brave BTCQ entailment with justification structures.

Proposition 9. *If $\mathsf{N_{RC}} = \mathsf{N_{RR}} = \emptyset$, then brave TCQ answering is in* P *w.r.t. data complexity.*

Proof. We start with a justification structure J for ϕ in $\mathcal{K}$ that contains all possible tuples. We then remove the unjustified tuples as follows: (i) remove every tuple that does not satisfy Conditions 1, 4, 5, 6, 7, 8 or 9, and (ii) repeat the following steps until a fix-point has been reached: iterate over the tuples from time point 0 to n, eliminating those which do not satisfy Condition 3, then from n to 0 eliminating those which do not satisfy Condition 2. For the resulting justification structure, we check whether it contains a tuple $(p, L_{\text{now}}, F_{\text{now}}, F_{\text{prev}}, F_{\text{next}})$ such that $\phi \in F_{\text{now}}$. If yes, we return "*entailed at time point p*", otherwise, we return "*not entailed at time point p*". Since the size of J is linear in n, this process requires at most quadratically many steps. The verification that a given tuple is justified requires polynomial time w.r.t. data complexity (the verification of Condition 3 or Condition 2 is linear in n and only the brave entailment of a BCQ from a DL-Lite$_\mathcal{R}$ KB for Condition 1 depends on the size of the ABox), so the complete procedure runs in polynomial time w.r.t. data complexity. ☐

Our complexity analysis of the three semantics for DL-Lite$_\mathcal{R}$ shows that, encouragingly, only brave semantics in the cases where rigid predicates are allowed has a higher data complexity than in the atemporal case, and that the combined complexity is not impacted by the temporal reasoning.

5 Conclusion and Future Work

We extended the AR, IAR and brave semantics to the setting of temporal query answering in description logics. We first showed that in the case where rigid predicates are not allowed, TCQ answering under IAR semantics can be achieved by combining algorithms developed for TCQ answering under the classical semantics with algorithms for CQ answering under IAR semantics over atemporal KBs. We also showed that in some cases, the same applies to AR semantics and that in any case, this method provides a sound approximation of AR answers. Since this is not true for brave semantics and we believe that this semantics can be relevant, for instance in the application of situation recognition, it would be useful to characterize the queries for which this method would be correct. Indeed, for many pairs of TBox and query, the minimal subsets of the TKB such that the query can be mapped into them cannot be inconsistent, for instance if no pair of predicates that may be involved at the same time point appears in a NI entailed by the TBox (e.g., if $\mathcal{T} = \{A \sqsubseteq \neg C, B \sqsubseteq \neg C\}$ and $\phi = \exists x A(x) \wedge \Diamond(\exists x B(x) \wedge \bigcirc(\exists x C(x)))$, for ϕ being entailed at time point p, $\exists x A(x)$ should hold at p, $\exists x B(x)$ at time point $i \geq p$ and $\exists x C(x)$ at $i+1 > p$, so there cannot be a conflict between the C and the A or B timed-assertions used to satisfy the different CQs).

Our second contribution is a complexity analysis of the three semantics for DL-Lite$_{\mathcal{R}}$, depending on which predicates are allowed to be rigid. Encouragingly, only brave semantics in the cases where rigid predicates are allowed has a higher data complexity than in the atemporal case. We also showed that for the classical semantics, rigid predicates can be handled by adding a set of assertions to each ABox of the TKB, proving that disallowing negations in the query makes the combined complexity of TCQ answering drop from PSPACE to NP. Practical algorithms for inconsistency-tolerant query answering with rigid predicates remain to be found. In particular, note that adding the set of assertions $\mathcal{R}$ to every ABox to reduce Cases 2 or 3 to Case 1 works only for the classical semantics.

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