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Infinitely Valued Gödel Semantics for Expressive Description Logics

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Abstract

Fuzzy Description Logics (FDLs) combine classical Description Logics with the semantics of Fuzzy Logics in order to represent and reason with vague knowledge. Most FDLs using truth values from the interval $[0, 1]$ have been shown to be undecidable in the presence of a negation constructor and general concept inclusions. One exception are those FDLs whose semantics is based on the infinitely valued Gödel t-norm (G). We extend previous decidability results for the FDL G-$\mathcal{ALC}$ to deal with complex role inclusions, nominals, inverse roles, and qualified number restrictions. Our novel approach is based on a combination of the known crispification technique for finitely valued FDLs and an automata-based procedure for reasoning in G-$\mathcal{ALC}$.

1 Introduction

Description Logics (DLs) are a well-studied family of knowledge representation formalisms [1]. They constitute the logical backbone of the standard Semantic Web ontology language OWL 2,[1] and its profiles, and have been successfully applied to represent the knowledge of many and diverse application domains, particularly in the bio-medical sciences. DLs describe the domain knowledge using *concepts* (such as $\mathsf{Patient}$) that represent sets of individuals, and *roles* ($\mathsf{hasChild}$) that represent connections between individuals. *Ontologies* are collections of axioms formulated over these concepts and roles, which restrict their possible interpretations. The typical axioms considered in DLs are *assertions*, like $\mathsf{alice}{:}\mathsf{Patient}$, providing knowledge about specific individuals; *general concept inclusions (GCIs)*, such as $\mathsf{Patient} \sqsubseteq \mathsf{Human}$, which express general relations between concepts; and *role inclusions* $\mathsf{hasChild}\,\mathsf{hasChild} \sqsubseteq \mathsf{hasGrandchild}$ between (chains of) roles. Different DLs are characterized by the constructors allowed to formulate complex concepts, roles, and axioms.

$\mathcal{ALC}$ [30] is a prototypical DL of intermediate expressivity that contains the concept constructors conjunction ($\mathsf{Patient} \sqcap \mathsf{Female}$), negation ($\neg\mathsf{Smoker}$), existential restriction over a role ($\exists\mathsf{hasChild}.\mathsf{HeavySmoker}$), and value restriction ($\forall\mathsf{hasChild}.\mathsf{Male}$), and allows assertions and GCIs. The DL underlying the standard language OWL 2 DL is called $\mathcal{SROIQ}$ and additionally provides, among others, role inclusions, number restrictions ($\geqslant 3\,\mathsf{hasChild}.\mathsf{Adult}$), nominals ($\{\mathsf{alice}\}$), and inverse roles ($\mathsf{hasChild}^-$). The complexity of common reasoning problems, such as consistency of ontologies or subsumption between concepts, has been extensively studied for these DLs, and ranges from ExpTime to 2-NExpTime [26, 29, 33].

Fuzzy Description Logics (FDLs) have been introduced as extensions of classical

[1] `http://www.w3.org/TR/owl2-overview/`

DLs to represent and reason with vague knowledge. The main idea is to use truth values from the interval $[0, 1]$ instead of only *true* and *false*. In this way, one can give a more fine-grained semantics to inherently vague concepts like $\mathsf{LowFrequency}$ or $\mathsf{HighConcentration}$, which can be found in biomedical ontologies like SNOMED CT,[2] and Galen.[3] Different FDLs are characterized not only by the constructors they allow, but also by the way these constructors are interpreted. To interpret conjunction in complex concepts like

$$\exists \mathsf{hasHeartRate}.\mathsf{LowFrequency} \sqcap \exists \mathsf{hasBloodAlcohol}.\mathsf{HighConcentration},$$

a popular approach is to use so-called *t-norms* [27]. The semantics of the other logical constructors can then be derived from these t-norms in a principled way, as suggested in [20]. Following the principles of mathematical fuzzy logic, existential and value restrictions are interpreted as suprema and infima of truth values, respectively. However, to avoid problems with infinitely many truth values, reasoning in fuzzy DLs is often restricted to so-called *witnessed models* [21], in which these suprema (infima) are required to be maxima (minima); i.e. the degree is witnessed by at least one domain element.

Unfortunately, most FDLs become undecidable when the logic allows to use GCIs and negation under witnessed model semantics [2, 13, 18]. One of the few exceptions are FDLs using the *Gödel* t-norm, which is defined as $\min\{x, y\}$, to interpret conjunctions [12]. In the absence of an involutive negation constructor and negated assertions, such FDLs are even trivially equivalent to classical DLs [13]. However, in the presence of the involutive negation, reasoning becomes more complicated. Despite not being as well-behaved as finitely valued FDLs, which use a finite total order of truth values instead of the infinite interval $[0, 1]$, it was shown using an automata-based approach that reasoning in Gödel extensions of $\mathcal{ALC}$ exhibits the same complexity as in the classical case, i.e. it is ExpTime-complete [12]. A major drawback of this approach is that it always has an exponential runtime, even when the input ontology has a simple form.

In the present paper, we present a combination of the automata-based construction for $\mathcal{ALC}$ from [12] and automata-based algorithms and reduction techniques developed for more expressive finitely valued FDLs [6,10,11,14,15,31]. We exploit the forest model property of classical DLs [17, 25] to encode order relationships between concepts in a fuzzy interpretation in a manner similar to the Hintikka trees from [12]. However, instead of using automata to determine the existence of such trees, we reduce the fuzzy ontology directly into a classical $\mathcal{ALCOQ}$ ontology, which enables us to use optimized reasoners for classical DLs. In addition to the *cut-concepts* of the form $\boxed{C \geqslant p}$ for a fuzzy concept C and a value p, which are used in the reductions for finitely valued DLs [6,10,31], we employ *order concepts* $\boxed{C \leqslant D}$ expressing relationships between fuzzy concepts. In contrast to the

[2] http://www.ihtsdo.org/snomed-ct/

[3] http://www.opengalen.org/

reductions for finitely valued Gödel FDLs [6, 7], our reduction does not produce an exponential blowup in the nesting depth of concepts in the input ontology.

Although our reduction deals with the Gödel extension of $\mathcal{SROIQ}$, it is not correct if all three constructors nominals ($\mathcal{O}$), inverse roles ($\mathcal{I}$), and number restrictions ($\mathcal{Q}$) are present in the ontology, since then one cannot restrict reasoning to forest-shaped models [32]. However, it is correct for $\mathcal{SRIQ}$, $\mathcal{SROQ}$, and $\mathcal{SROI}$, and we obtain several complexity results that match the currently best known upper bounds for reasoning in (sublogics of) these DLs. In particular, we show that reasoning in Gödel extensions of $\mathcal{SRIQ}$ is 2-ExpTime-complete, and for $\mathcal{SHOI}$ and $\mathcal{SHIQ}$ it is ExpTime-complete.

2 Preliminaries

We consider vague statements taking truth degrees from the infinite interval $[0, 1]$, where the *Gödel t-norm* $\min\{x, y\}$ is used to interpret logical conjunction. The semantics of implications is given by the *residuum* of this t-norm; i.e.,

$$x \Rightarrow y := \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{otherwise.} \end{cases}$$

Note that min is monotone in both arguments, and hence preserves arbitrary infima in suprema, while $\Rightarrow$ is monotone in the second argument and antitone in the first argument. We furthermore have the following useful property.

Proposition 2.1. *For all values $x, x_1, \dots, x_n \in [0, 1]$, we have*

$$\big((x_1 \wedge \dots \wedge x_n) \Rightarrow x\big) = \big(x_1 \Rightarrow \dots (x_n \Rightarrow x) \dots\big).$$

We use both the *residual negation* $x \mapsto (x \Rightarrow 0)$ and the *involutive negation* $x \mapsto (1 - x)$ in the rest of this paper.

We recall some basic definitions from [12]. An *order structure* S is a finite set containing at least the numbers 0, 0.5, and 1, and an involutive unary operation $\mathsf{inv}\colon S \to S$ such that $\mathsf{inv}(x) = 1 - x$ for all $x \in S \cap [0, 1]$. A *total preorder* over S is a transitive and total binary relation $\preccurlyeq \subseteq S \times S$. For $x, y \in S$, we write $x \equiv y$ if $x \preccurlyeq y$ and $y \preccurlyeq x$. Notice that $\equiv$ is an equivalence relation on S. The total preorders considered in [12] have to satisfy additional properties, e.g. that 0 and 1 are always the least and greatest elements, respectively. These properties can be found in our reduction in the axioms of $\mathsf{red}(\mathcal{U})$ (see Section 4).

We now define the fuzzy description logic G-$\mathcal{SROIQ}$. Let $\mathsf{N_I}$, $\mathsf{N_C}$, and $\mathsf{N_R}$ be three mutually disjoint sets of *individual names*, *concept names*, and *role names*, respectively, where $\mathsf{N_R}$ contains the *universal role* r_u. The set of *(complex) roles* is $\mathsf{N_R^-} := \mathsf{N_R} \cup \{r^- \mid r \in \mathsf{N_R}\}$; the elements of the form r^- are called *inverse* *roles*. Since there are several syntactic restrictions based on which roles appear in which role axioms, we start by defining role hierarchies. A *role hierarchy* $\mathcal{R}_h$ is a finite set of *(complex) role inclusions* of the form $\langle w \sqsubseteq r \geqslant p \rangle$, where $r \neq r_u$ is a role name, $w \in (\mathsf{N}_\mathsf{R}^-)^+$ is a non-empty *role chain* not including the universal role, and $p \in (0,1]$. Such a role inclusion is called *simple* if $w \in \mathsf{N}_\mathsf{R}^-$. We extend the notation $\cdot^-$ to inverse roles r^- and role chains $w = r_1 \dots r_n$ by setting $(r^-)^- := r$ and $w^- := r_n^- \dots r_1^-$.

We recall the regularity condition from [5,23]. Let $\prec$ be a strict partial order on N_R^- such that $r \prec s$ iff $r^- \prec s$. A role inclusion $\langle w \sqsubseteq r \geqslant p \rangle$ is $\prec$-*regular* if

- w is of the form rr or r^-, or
- w is of the form $r_1 \dots r_n$, $rr_1 \dots r_n$, or $r_1 \dots r_n r$, and for all $1 \leqslant i \leqslant n$ it holds that $r_i \prec r$.

An role hierarchy $\mathcal{R}_h$ is *regular* if there is a strict partial order $\prec$ as above such that each role inclusion in $\mathcal{R}_h$ is $\prec$-regular. A role name r is *simple* (w.r.t. $\mathcal{R}_h$) if for each $\langle w \sqsubseteq r \geqslant p \rangle \in \mathcal{R}_h$ we have that w is of the form s or s^- for a simple role s. This notion is well-defined since the regularity condition prevents any cyclic dependencies between role names in $\mathcal{R}_h$. An inverse role r^- is *simple* if r is simple. In the following, we always assume that we have a regular role hierarchy $\mathcal{R}_h$.

Concepts in G-$\mathcal{SROIQ}$ are built from concept names using the constructors listed in the upper part of Table 1, where C, D denote concepts, $p \in [0,1]$, $n \in \mathbb{N}$, $a \in \mathsf{N}_\mathsf{I}$, $r \in \mathsf{N}_\mathsf{R}^-$, and $s \in \mathsf{N}_\mathsf{R}^-$ is a simple role. The restriction to simple roles in at-least restrictions is necessary to ensure decidability [24]. We also use the common DL constructors $\top := \overline{1}$ (top concept), $\bot := \overline{0}$ (bottom concept), $C \sqcup D := \neg(\neg C \sqcap \neg D)$ (disjunction), and ${\leqslant} n\, s.C := \neg({\geqslant}(n+1)\, s.C)$ (at-most restriction).

Notice that in [7], fuzzy at-most restrictions are defined using the residual negation: ${\leqslant} n\, s.C := ({\geqslant}(n+1)\, s.C) \to \bot$. This has the effect that the value of ${\leqslant} n\, r.C$ is always either 0 or 1 (see the semantics below). However, this discrepancy in definitions is not an issue since our reduction can handle both cases. The use of truth constants $\overline{p}$ for $p \in [0,1]$ is not standard in FDLs, but it allows us to simulate *fuzzy nominals* [4] of the form $\{p_1/a_1, \dots, p_n/a_n\}$ with $p_i \in [0,1]$ and $a_i \in \mathsf{N}_\mathsf{I}$, $1 \leqslant i \leqslant n$, via $(\{a_1\} \sqcap \overline{p_1}) \sqcup \dots \sqcup (\{a_n\} \sqcap \overline{p_n})$.

The semantics of G-$\mathcal{SROIQ}$ is based on G-*interpretations* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ over a non-empty *domain* $\Delta^\mathcal{I}$, which assign to each individual name $a \in \mathsf{N}_\mathsf{I}$ an element $a^\mathcal{I} \in \Delta^\mathcal{I}$, to each concept name $A \in \mathsf{N}_\mathsf{C}$ a fuzzy set $A^\mathcal{I} \colon \Delta^\mathcal{I} \to [0,1]$, and to each role name $r \in \mathsf{N}_\mathsf{R}$ a fuzzy binary relation $r^\mathcal{I} \colon \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0,1]$. This interpretation is extended to complex concepts and roles as defined in the last column of Table 1, for all $d, e \in \Delta^\mathcal{I}$.

Table 1: Syntax and semantics of G-$\mathcal{SROIQ}$

Name	Syntax	Semantics ($C^{\mathcal{I}}(d)$ / $r^{\mathcal{I}}(d,e)$)
concept name	A	$A^{\mathcal{I}}(d) \in [0,1]$
truth constant	$\overline{p}$	p
conjunction	$C \sqcap D$	$\min\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\}$
implication	$C \to D$	$C^{\mathcal{I}}(d) \Rightarrow D^{\mathcal{I}}(d)$
negation	$\neg C$	$1 - C^{\mathcal{I}}(d)$
existential restriction	$\exists r.C$	$\sup_{e \in \Delta^{\mathcal{I}}} \min\{r^{\mathcal{I}}(d,e), C^{\mathcal{I}}(e)\}$
value restriction	$\forall r.C$	$\inf_{e \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(d,e) \Rightarrow C^{\mathcal{I}}(e)$
nominal	$\{a\}$	$\begin{cases} 1 & \text{if } d = a^{\mathcal{I}} \\ 0 & \text{otherwise} \end{cases}$
at-least restriction	$\geqslant n\, s.C$	$\sup_{\substack{e_1,\dots,e_n \in \Delta^{\mathcal{I}} \\ \text{pairwise different}}} \min_{i=1}^{n} \min\{s^{\mathcal{I}}(d,e_i), C^{\mathcal{I}}(e_i)\}$
local reflexivity	$\exists s.\mathsf{Self}$	$r^{\mathcal{I}}(d,d)$
role name	r	$r^{\mathcal{I}}(d,e) \in [0,1]$
inverse role	r^-	$r^{\mathcal{I}}(e,d)$
universal role	r_u	1

We restrict all reasoning problems to *witnessed* G-interpretations [21], which intuitively require the suprema and infima in the semantics to be maxima and minima, respectively. Formally, a G-interpretation $\mathcal{I}$ is *witnessed* if, for every $d \in \Delta^{\mathcal{I}}$, $n \geqslant 0$, $r \in \mathsf{N}_\mathsf{R}^-$, simple $s \in \mathsf{N}_\mathsf{R}^-$, and concept C, there are $e, e', e_1, \dots, e_n \in \Delta^{\mathcal{I}}$ such that $e_1, \dots, e_n$ are pairwise different,

$$(\exists r.C)^{\mathcal{I}}(d) = \min\{r^{\mathcal{I}}(d,e), C^{\mathcal{I}}(e)\},$$
$$(\forall r.C)^{\mathcal{I}}(d) = r^{\mathcal{I}}(d,e') \Rightarrow C^{\mathcal{I}}(e'), \text{ and}$$
$$(\geqslant n\, s.C)^{\mathcal{I}}(d) = \min_{i=1}^{n} \min\{s^{\mathcal{I}}(d,e_i), C^{\mathcal{I}}(e_i)\}.$$

As we have seen already in the role inclusions, the axioms of G-$\mathcal{SROIQ}$ extend classical axioms by allowing to state a degree in $(0,1]$ to which the axioms hold. Moreover, we allow to compare the degrees of arbitrary *classical assertions* of the form $a{:}C$ or $(a,b){:}r$ for $a, b \in \mathsf{N}_\mathsf{I}$, $r \in \mathsf{N}_\mathsf{R}$, and a concept C. An *order assertion* [12] is of the form $\langle \alpha \bowtie p \rangle$ or $\langle \alpha \bowtie \beta \rangle$ for classical assertions α, β, $\bowtie \in \{<, \leqslant, =, \geqslant, >\}$, and $p \in [0,1]$. An *ordered ABox* is a finite set of order assertions and *individual (in)equality assertions* of the form $a \approx b$ ($a \not\approx b$) for

$a, b \in \mathsf{N_I}$. A *general concept inclusion (GCI)* is of the form $\langle C \sqsubseteq D \geqslant p\rangle$ for concepts C, D and $p \in (0,1]$. A *TBox* is a finite set of GCIs. A *disjoint role axiom* is of the form $\langle \mathsf{dis}(r,s) \geqslant p\rangle$ for two simple roles $r, s \in \mathsf{N_R^-}$ and $p \in (0,1]$. A *reflexivity axiom* is of the form $\langle \mathsf{ref}(r) \geqslant p\rangle$ for a role $r \in \mathsf{N_R^-}$ and $p \in (0,1]$. An *RBox* $\mathcal{R} = \mathcal{R}_h \cup \mathcal{R}_a$ consists of a role hierarchy $\mathcal{R}_h$ and a finite set $\mathcal{R}_a$ of disjoint role and reflexivity axioms. An *ontology* $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ consists of an ABox $\mathcal{A}$, a TBox $\mathcal{T}$, and an RBox $\mathcal{R}$.

A G-interpretation $\mathcal{I}$ *satisfies* (or is a *model* of)

- an order assertion $\langle \alpha \bowtie \beta\rangle$ if $\alpha^\mathcal{I} \bowtie \beta^\mathcal{I}$ (where $p^\mathcal{I} := p$, $(a{:}C)^\mathcal{I} := C^\mathcal{I}(a^\mathcal{I})$, and $((a,b){:}r)^\mathcal{I} := r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I})$);
- an (in)equality assertion $a \approx b$ ($a \not\approx b$) if $a^\mathcal{I} = b^\mathcal{I}$ ($a^\mathcal{I} \neq b^\mathcal{I}$);
- a GCI $\langle C \sqsubseteq D \geqslant p\rangle$ if $C^\mathcal{I}(d) \Rightarrow D^\mathcal{I}(d) \geqslant p$ holds for all $d \in \Delta^\mathcal{I}$;
- a role inclusion $\langle r_1 \dots r_n \sqsubseteq r \geqslant p\rangle$ if $(r_1 \dots r_n)^\mathcal{I}(d_0, d_n) \Rightarrow r^\mathcal{I}(d_0, d_n) \geqslant p$ holds for all $d_0, d_n \in \Delta^\mathcal{I}$, where

$$(r_1 \dots r_n)^\mathcal{I}(d_0, d_n) := \sup_{d_1,\dots,d_{n-1} \in \Delta^\mathcal{I}} \min_{i=1}^{n} r_i^\mathcal{I}(d_{i-1}, d_i);$$

- a disjoint role axiom $\langle \mathsf{dis}(r,s) \geqslant p\rangle$ if $\min\{r^\mathcal{I}(d,e), s^\mathcal{I}(d,e)\} \leqslant 1 - p$ holds for all $d, e \in \Delta^\mathcal{I}$;
- a reflexivity axiom $\langle \mathsf{ref}(r) \geqslant p\rangle$ if $r^\mathcal{I}(d,d) \geqslant p$ holds for all $d \in \Delta^\mathcal{I}$;
- an ontology if it satisfies all its axioms.

An ontology is *consistent* if it has a (witnessed) model.

We can simulate other common role axioms in G-$\mathcal{SROIQ}$ [7, 22] by those we introduced above:

- transitivity axioms $\langle \mathsf{tra}(r) \geqslant p\rangle$ by $\langle rr \sqsubseteq r \geqslant p\rangle$;
- symmetry axioms $\langle \mathsf{sym}(r) \geqslant p\rangle$ by $\langle r^- \sqsubseteq r \geqslant p\rangle$;
- asymmetry axioms $\langle \mathsf{asy}(s) \geqslant p\rangle$ by $\langle \mathsf{dis}(s, s^-) \geqslant p\rangle$;
- irreflexivity axioms $\langle \mathsf{irr}(s) \geqslant p\rangle$ by $\langle \exists s.\mathsf{Self} \sqsubseteq \neg\overline{p} \geqslant 1\rangle$; and
- negated role assertions $\langle (a,b){:}\neg r \geqslant p\rangle$ by $\langle (a,b){:}r \leqslant 1 - p\rangle$.

For an ontology $\mathcal{O}$, we denote by $\mathsf{rol}(\mathcal{O})$ the set of all roles occurring in $\mathcal{O}$, together with their inverses; by $\mathsf{ind}(\mathcal{O})$ the set of all individual names occurring in $\mathcal{O}$, and by $\mathsf{sub}(\mathcal{O})$ the closure under negation of the set of all subconcepts occurring in $\mathcal{O}$. We consider $\neg\neg C$ to be equal to C, and thus $\mathsf{sub}(\mathcal{O})$ is of quadratic size in the size of $\mathcal{O}$. We denote by $\mathcal{V}_\mathcal{O}$ the closure under the involutive negation $x \mapsto 1-x$ of the set of all truth degrees appearing in $\mathcal{O}$ (either in axioms or in truth constants), together with 0, 0.5, and 1. This set is of linear size.

Other common reasoning problems for FDLs, such as concept satisfiability and subsumption can be reduced to consistency [12]: the subsumption between C and D to degree q w.r.t. a TBox $\mathcal{T}$ and an RBox $\mathcal{R}$ is equivalent to the inconsistency of $(\{\langle a{:}C \to D < q\rangle\}, \mathcal{T}, \mathcal{R})$, and the satisfiability of C to degree q w.r.t. $\mathcal{T}$ and $\mathcal{R}$ is equivalent to the consistency of $(\{\langle a{:}C \geqslant q\rangle\}, \mathcal{T}, \mathcal{R})$.

The letter $\mathcal{I}$ in G-$\mathcal{SROIQ}$ denotes the presence of inverse roles and the universal role. If such roles are not allowed, the resulting logic is written as G-$\mathcal{SROQ}$. Likewise, G-$\mathcal{SRIQ}$ indicates the absence of nominals, and G-$\mathcal{SROI}$ that of at-least and at-most restrictions. Replacing the letter $\mathcal{R}$ with $\mathcal{H}$ indicates that RBoxes are restricted to simple role inclusions, ABoxes are restricted to order assertions, and local reflexivity is not allowed; however, the letter $\mathcal{S}$ indicates the presence of transitivity axioms. Hence, in G-$\mathcal{SHOIQ}$ we are allowed to use role inclusions of the forms $\langle r \sqsubseteq s \geqslant p\rangle$ and $\langle rr \sqsubseteq r \geqslant p\rangle$. Disallowing axioms of the first type removes the letter $\mathcal{H}$, while the absence of transitivity axioms is denoted by replacing $\mathcal{S}$ with $\mathcal{ALC}$.

Classical DLs are obtained from the above definitions by restricting the set of truth values to 0 and 1. The semantics of a classical concept C is then viewed as a set $C^\mathcal{I} \subseteq \Delta^\mathcal{I}$ instead of the characteristic function $C^\mathcal{I}\colon \Delta^\mathcal{I} \to \{0,1\}$, and likewise for roles. In this setting, all axioms (also order assertions) are restricted to be of the form $\langle \alpha \geqslant 1\rangle$, and usually this is simply written as α, e.g. $C \sqsubseteq D$ instead of $\langle C \sqsubseteq D \geqslant 1\rangle$. We also use $C \equiv D$ to abbreviate $C \sqsubseteq D$ and $D \sqsubseteq C$. Furthermore, the implication constructor $C \to D$, although usually not included in classical DLs, can be expressed via $\neg C \sqcup D$.

In this paper, we provide a reduction from a G-$\mathcal{SROIQ}$ ontology to a classical $\mathcal{ALCOQ}$ ontology. For all sublogics of G-$\mathcal{SROIQ}$ that do not contain the constructors $\mathcal{O}$, $\mathcal{I}$, and $\mathcal{Q}$ at the same time, the reduction preserves consistency. Before we describe the main reduction, however, we provide a characterization of role hierarchies using (weighted) finite automata.

3 Automata for Complex Role Inclusions

Let $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ be a G-$\mathcal{SROIQ}$ ontology. We extend the idea from [23] of using finite automata to characterize all role chains that imply a given role w.r.t. $\mathcal{R}_h$.

In our setting, we need to use a certain kind of weighted automata [19], which use as input symbols the roles in $\mathsf{rol}(\mathcal{O})$, and compute a weight for any given input word.

Definition 3.1 (WFA). *A* weighted finite automaton (WFA) *is a quadruple* $\mathbf{A} = (Q, q_{\mathsf{ini}}, \mathsf{wt}, q_{\mathsf{fin}})$*, consisting of a non-empty set* Q *of* states*, an* initial state $q_{\mathsf{ini}} \in Q$*, a* transition weight function $\mathsf{wt}\colon Q \times (\mathsf{rol}(\mathcal{O}) \cup \{\varepsilon\}) \times Q \to [0,1]$*, and a* final state $q_{\mathsf{fin}} \in Q$*. Given an input word* $w \in \mathsf{rol}(\mathcal{O})^*$*, a* run *of* $\mathbf{A}$ *on* w *is a non-empty sequence of pairs* $\mathsf{r} = (w_i, q_i)_{0 \leqslant i \leqslant m}$ *such that* $(w_0, q_0) = (w, q_{\mathsf{ini}})$*,* $(w_m, q_m) = (\varepsilon, q_{\mathsf{fin}})$*, and for each* i*,* $1 \leqslant i \leqslant m$*, it holds that* $w_{i-1} = x_i w_i$ *for some* $x_i \in \mathsf{rol}(\mathcal{O}) \cup \{\varepsilon\}$*. The* weight *of such a run is* $\mathsf{wt}(\mathsf{r}) := \min_{i=1}^{m} \mathsf{wt}(q_{i-1}, x_i, q_i)$*. The* behavior *of* $\mathbf{A}$ *on* w *is* $(\|\mathbf{A}\|, w) := \sup_{\mathsf{r} \text{ run of } \mathbf{A} \text{ on } w} \mathsf{wt}(\mathsf{r})$*.*

We often denote by $q \xrightarrow{x,p} q' \in \mathbf{A}$ the fact that $\mathsf{wt}(q, x, q') = p$. Further, for a state q of $\mathbf{A}$, we denote by $\mathbf{A}^q$ the automaton resulting from $\mathbf{A}$ by making q the initial state. The following connection is easy to see by the definition of the behavior of a WFA.

Proposition 3.2. *Let* $\mathbf{A}$ *be a WFA,* $q \xrightarrow{x,p} q' \in \mathbf{A}$*, and* $w \in \mathsf{rol}(\mathcal{O})^*$*. Then* $(\|\mathbf{A}^q\|, xw) \geqslant \min\{p, (\|\mathbf{A}^{q'}\|, w)\}$*.*

A *mirrored copy* $\mathbf{A}^-$ is constructed from $\mathbf{A}$ by exchanging initial and final states, and replacing each transition $q \xrightarrow{x,p} q'$ by $q' \xrightarrow{x^-,p} q$, where $\varepsilon^- := \varepsilon$.

Proposition 3.3. *Let* $\mathbf{A}$ *be a WFA,* $\mathbf{A}'$ *be a mirrored copy of* $\mathbf{A}$*, and* $w \in \mathsf{rol}(\mathcal{O})^*$*. Then* $(\|\mathbf{A}\|, w) = (\|\mathbf{A}'\|, w^-)$*.*

Following [23], we now construct, for each role r, a WFA $\mathbf{A}_r$ that recognizes all role chains that "imply" r w.r.t. $\mathcal{R}_h$ (with associated degrees). This construction proceeds in several steps. The first automaton $\mathbf{A}_r^0$ contains the initial state i_r, the final state f_r, and the transition $i_r \xrightarrow{r,1} f_r$, as well as the following transitions for each $\langle w \sqsubseteq r \geqslant p\rangle \in \mathcal{R}$:

- if $w = rr$, then $f_r \xrightarrow{\varepsilon,p} i_r$;
- if $w = r_1 \dots r_n$ with $r_1 \neq r \neq r_n$, then $i_r \xrightarrow{r_1,1} q_w^1 \xrightarrow{r_2,1} \dots \xrightarrow{r_n,1} q_w^n \xrightarrow{\varepsilon,p} f_r$;
- if $w = rr_1 \dots r_n$, then $f_r \xrightarrow{r_1,1} q_w^1 \xrightarrow{r_2,1} \dots \xrightarrow{r_n,1} q_w^n \xrightarrow{\varepsilon,p} f_r$; and
- if $w = r_1 \dots r_n r$, then $i_r \xrightarrow{r_1,1} q_w^1 \xrightarrow{r_2,1} \dots \xrightarrow{r_n,1} q_w^n \xrightarrow{\varepsilon,p} i_r$,

where all states q_w^i are distinct. Here and in the following, all transitions that are not explicitly mentioned have weight 0.

The WFA $\mathbf{A}_r^1$ is now defined as $\mathbf{A}_r^0$ if there is no role inclusion of the form $\langle r^- \sqsubseteq r \geqslant p\rangle \in \mathcal{R}$; otherwise, $\mathbf{A}_r^1$ is the disjoint union of $\mathbf{A}_r^0$ and a mirrored copy of $\mathbf{A}_r^0$, where i_r is the only initial state, f_r is the only final state, and the following transitions are added for the copy f_r' of f_r and the copy i_r' of i_r: $i_r \xrightarrow{\varepsilon,p} f_r'$, $f_r' \xrightarrow{\varepsilon,p} i_r$, $f_r \xrightarrow{\varepsilon,p} i_r'$, and $i_r' \xrightarrow{\varepsilon,p} f_r$.

Finally, we define the WFA $\mathbf{A}_r$ by induction on $\prec$ as follows:

- if r is minimal w.r.t. $\prec$, then $\mathbf{A}_r := \mathbf{A}_r^1$;
- otherwise, $\mathbf{A}_r$ is the disjoint union of $\mathbf{A}_r^1$ with a copy $\mathbf{A}_s^{1'}$ of $\mathbf{A}_s^1$ for each transition $q \xrightarrow{s,1} q'$ in $\mathbf{A}_r^1$ with $s \neq r$.[4] For each such transition, we add ε-transitions with weight 1 from q to the initial state of $\mathbf{A}_s^{1'}$ and from the final state of $\mathbf{A}_s^{1'}$ to q'.
- The automaton $\mathbf{A}_{r^-}$ is a mirrored copy of $\mathbf{A}_r$.

The difference to the construction in [23] is only the inclusion of the appropriate weights for each considered role inclusion. As shown in [23], the size of each $\mathbf{A}_r$ is bounded exponentially in the length of the longest chain $r_1 \prec \dots \prec r_n$ for which there are role inclusions $\langle u_i r_{i-1} v_i \sqsubseteq r_i \geqslant p_i\rangle \in \mathcal{R}$ for all i, $2 \leqslant i \leqslant n$.

The following lemma describes the promised characterization of the role inclusions in $\mathcal{R}$ in terms of the behavior of the automata $\mathbf{A}_r$. Intuitively, the degree to which the interpretation of w must be included in the interpretation of r is determined by the behavior of $\|\mathbf{A}_r\|$ on w.

Lemma 3.4. *A* G*-interpretation* $\mathcal{I}$ *satisfies all role inclusions in* $\mathcal{R}$ *iff for every* $r \in \mathsf{rol}(\mathcal{O})$*, every* $w \in \mathsf{rol}(\mathcal{O})^+$*, and all* $x, y \in \Delta^{\mathcal{I}}$*, we have*

$$w^{\mathcal{I}}(x,y) \Rightarrow r^{\mathcal{I}}(x,y) \geqslant (\|\mathbf{A}_r\|, w).$$

Proof. If $\mathcal{I}$ violates any $\langle w \sqsubseteq r \geqslant p\rangle \in \mathcal{R}$, then there are $d, e \in \Delta^{\mathcal{I}}$ such that $w^{\mathcal{I}}(d,e) \Rightarrow r^{\mathcal{I}}(d,e) < p$. Since $(\|\mathbf{A}_r\|, w) \geqslant p$ by construction of $\mathbf{A}_r$, we get $w^{\mathcal{I}}(d,e) \Rightarrow r^{\mathcal{I}}(d,e) < (\|\mathbf{A}_r\|, w)$.

For the other direction, assume that $\mathcal{I}$ satisfies $\mathcal{R}$, and let $r \in \mathsf{rol}(\mathcal{O})$, $w \in \mathsf{rol}(\mathcal{O})^+$, and $d, e \in \Delta^{\mathcal{I}}$. We prove the claim by well-founded induction on $\prec$. It suffices to show the claim for all role names r since $\mathbf{A}_{r^-}$ is a mirrored copy of $\mathbf{A}_r$.

If $(\|\mathbf{A}_r\|, w) = 0$ or $w^{\mathcal{I}}(d,e) = 0$, then the claim is trivially satisfied. If both values are > 0, then due to the construction of $\mathbf{A}_r$ there must be

- a word $w' = r_1 \dots r_n \in \mathsf{rol}(\mathcal{O})^+$ such that $r_i \prec r$ or $r_i = r$ holds for all $1 \leqslant i \leqslant n$, and

[4] Note that all transitions labeled with roles have weight 0 or 1.

- words $w_1, \ldots, w_n \in \mathsf{rol}(\mathcal{O})^*$ such that $w = w_1 \ldots w_n$ and
$$(\|\mathbf{A}_r\|, w) = \min\big\{(\|\mathbf{A}_r^1\|, w'), (\|\mathbf{A}_{r_1}\|, w_1), \ldots, (\|\mathbf{A}_{r_n}\|, w_n)\big\} > 0, \qquad (1)$$
where, if $r_i = r$, then $w_i = r$, and thus $(\|\mathbf{A}_{r_i}\|, w_i) = 1$. Since we have $(\|\mathbf{A}_{r_i}\|, w_i) > 0$, $1 \leqslant i \leqslant n$, we know by the construction of $\mathbf{A}_{r_i}$ that all w_i are non-empty.

Since $w = w_1 \ldots w_n$, we have
$$w^{\mathcal{I}}(d, e) = \sup_{d_1,\ldots,d_{n-1} \in \Delta^{\mathcal{I}}} \min_{i=1}^{n} w_i^{\mathcal{I}}(d_{i-1}, d_i),$$
where we set $d_0 := d$ and $d_n := e$. For any such choice of $d_1, \ldots, d_{n-1} \in \Delta^{\mathcal{I}}$, it holds that, if $r_i \prec r$, then $w_i^{\mathcal{I}}(d_{i-1}, d_i) \Rightarrow r_i^{\mathcal{I}}(d_{i-1}, d_i) \geqslant (\|\mathbf{A}_{r_i}\|, w_i)$, by the induction hypothesis. But this also holds for $r_i = r$ since then $w_i = r$. Hence, we obtain
$$\begin{aligned}(w')^{\mathcal{I}}(d, e) &= \sup_{d_1,\ldots,d_{n-1} \in \Delta^{\mathcal{I}}} \min_{i=1}^{n} r_i^{\mathcal{I}}(d_{i-1}, d_i) \\ &\geqslant \sup_{d_1,\ldots,d_{n-1} \in \Delta^{\mathcal{I}}} \min_{i=1}^{n} \min\{w_i^{\mathcal{I}}(d_{i-1}, d_i),\ (\|\mathbf{A}_{r_i}\|, w_i)\} \\ &= \min\{w^{\mathcal{I}}(d, e), (\|\mathbf{A}_{r_1}\|, w_1), \ldots, (\|\mathbf{A}_{r_n}\|, w_n)\}. \qquad (2)\end{aligned}$$
We proceed by a case distinction on the transitivity and symmetry properties of r in $\mathcal{R}$.

1. Assume that no role inclusions of the form $\langle rr \sqsubseteq r \geqslant p\rangle$ or $\langle r^- \sqsubseteq r \geqslant p\rangle$ occur in $\mathcal{R}$. Since $(\|\mathbf{A}_r^1\|, w') > 0$, by construction of $\mathbf{A}_r^1 = \mathbf{A}_r^0$ we know that w' is of the form $w' = u_1 \ldots u_m t v_1 \ldots v_k$ such that
   - $\langle u_i r \sqsubseteq r \geqslant p_i\rangle \in \mathcal{R}$ for all $1 \leqslant i \leqslant m$,
   - either $\langle t \sqsubseteq r \geqslant p\rangle \in \mathcal{R}$ or $t = r$ (and then we set $p := 1$),
   - $\langle r v_j \sqsubseteq r \geqslant p'_j\rangle \in \mathcal{R}$ for all $1 \leqslant j \leqslant k$, and
   - $(\|\mathbf{A}_r^1\|, w') = \min\{p_1, \ldots, p_m, p, p'_1, \ldots, p'_k\}$.

   Hence, we get
$$\begin{aligned}r^{\mathcal{I}}(d, e) &\geqslant \min\{p'_k, (rv_k)^{\mathcal{I}}(d, e)\} \\ &= \min\big\{p'_k, \sup_{e'_k \in \Delta^{\mathcal{I}}} \min\{r^{\mathcal{I}}(d, e'_k), v_k^{\mathcal{I}}(e'_k, e)\}\big\} \\ &\ldots \\ &\geqslant \min\big\{p_1, \ldots, p_m, p, p'_1, \ldots, p'_k, \\ &\qquad \sup_{e_1,\ldots,e_m,e'_1,\ldots,e'_k \in \Delta^{\mathcal{I}}} \min\{u_1^{\mathcal{I}}(d, e_1), \ldots, t^{\mathcal{I}}(e_m, e'_1), \ldots, v_k^{\mathcal{I}}(e'_k, e)\}\big\} \\ &= \min\{(\|\mathbf{A}_r^1\|, w'), (w')^{\mathcal{I}}(d, e)\}\end{aligned}$$
   The claim now follows from this inequation together with (1) and (2).

2. Consider the case that $\langle rr \sqsubseteq r \geqslant p_t \rangle \in \mathcal{R}$, but there is no role inclusion $\langle r^- \sqsubseteq r \geqslant p \rangle \in \mathcal{R}$. Then w' must be of the form

$$w' = \big(u_1^{(1)}, \dots, u_{m_1}^{(1)} t^{(1)} v_1^{(1)}, \dots, v_{k_1}^{(1)}\big) \dots \big(u_1^{(\ell)}, \dots, u_{m_\ell}^{(\ell)} t^{(\ell)} v_1^{(\ell)}, \dots, v_{k_\ell}^{(\ell)}\big)$$

with $\ell \geqslant 1$ and

- $\langle u_i^{(o)} r \sqsubseteq r \geqslant p_i^{(o)} \rangle \in \mathcal{R}$ for all $1 \leqslant o \leqslant \ell$ and $1 \leqslant i \leqslant m_o$,
- for each $1 \leqslant o \leqslant \ell$, either $\langle t^{(o)} \sqsubseteq r \geqslant p^{(o)} \rangle \in \mathcal{R}$ or $t^{(o)} = r$ (and then we set $p^{(o)} := 1$),
- $\langle r v_j^{(o)} \sqsubseteq r \geqslant (p'_j)^{(o)} \rangle \in \mathcal{R}$ for all $1 \leqslant o \leqslant \ell$ and $1 \leqslant j \leqslant k_o$, and
- $(\|\mathbf{A}_r^1\|, w') = \min\{p_t, p_0\}$ if $\ell > 1$, and $(\|\mathbf{A}_r^1\|, w') = p_0$ if $\ell = 1$, where $p_0 := \min\{p_i^{(o)}, p^{(o)}, (p'_j)^{(o)} \mid 1 \leqslant o \leqslant \ell,\ 1 \leqslant i \leqslant m_o,\ 1 \leqslant j \leqslant k_o\}$.

The claim can be obtained by the same arguments as in Case 1. Note that the axiom $\langle rr \sqsubseteq r \geqslant p_t \rangle$ is only needed if $\ell > 1$.

3. If $\langle r^- \sqsubseteq r \geqslant p_s \rangle \in \mathcal{R}$, but there is no role inclusion $\langle rr \sqsubseteq r \geqslant p \rangle \in \mathcal{R}$, then w' is of the form $w' = u_1 \dots u_m t v_1 \dots v_k$, where

- $\langle u_i r \sqsubseteq r \geqslant p_i \rangle \in \mathcal{R}$ or $\langle r u_i^- \sqsubseteq r \geqslant p_i \rangle \in \mathcal{R}$ for all $1 \leqslant i \leqslant m$,
- $\langle t \sqsubseteq r \geqslant p \rangle \in \mathcal{R}$, $\langle t^- \sqsubseteq r \geqslant p \rangle \in \mathcal{R}$, $t = r$, or $t = r^-$ (in the latter two cases we set $p := 1$),
- $\langle r v_j \sqsubseteq r \geqslant p'_j \rangle \in \mathcal{R}$ or $\langle v_j^- r \sqsubseteq r \geqslant p'_j \rangle \in \mathcal{R}$ for all $1 \leqslant j \leqslant k$, and
- $(\|\mathbf{A}_r^1\|, w') = \min\{p_s, p_0\}$ if one of the "inverse" cases applies, and $(\|\mathbf{A}_r^1\|, w') = p_0$ otherwise, where $p_0 := \min\{p_1, \dots, p_m, p, p'_1, \dots, p'_k\}$.

The claim can be obtained as in Case 1.

4. If both $\langle rr \sqsubseteq r \geqslant p_t \rangle$ and $\langle r^- \sqsubseteq r \geqslant p_s \rangle$ are present in $\mathcal{R}$, then w' is a non-empty sequence of words of the form described in Case 3, and the claim can be shown as before. □

For the universal role r_u, we define $\mathbf{A}_{r_u}$ as above based on the role inclusions $\langle r_u^- \sqsubseteq r_u \geqslant 1 \rangle$, $\langle r_u r_u \sqsubseteq r_u \geqslant 1 \rangle$, and $\langle r \sqsubseteq r_u \geqslant 1 \rangle$ for all $r \in \mathsf{rol}(\mathcal{O})$. Hence, $\mathbf{A}_{r_u}$ accepts any (non-empty) word $w \in \mathsf{rol}(\mathcal{O})^+$ with degree 1, and it is easy to see that Lemma 3.4 also holds for r_u.

We define the relation $\sqsubseteq_p$ as the "transitive closure" of the simple role inclusions in $\mathcal{R}$: we set $r \sqsubseteq_p s$ iff p is the supremum of the values $\min\{p_1, \dots, p_n\}$ over all sequences $\langle r \sqsubseteq r_1 \geqslant p_1 \rangle, \dots, \langle r_{n-1} \sqsubseteq s \geqslant p_n \rangle$ in $\mathcal{R}$. Note that $r \sqsubseteq_1 r$ because of the empty sequence.

Proposition 3.5. *For a simple role r and $w \in \mathsf{rol}(\mathcal{O})^*$, we have*

$$(\|\mathbf{A}_r\|, w) = \begin{cases} p & \text{if } w = s \in \mathsf{rol}(\mathcal{O}) \text{ and } s \sqsubseteq_p r, \\ 0 & \text{otherwise.} \end{cases}$$

4 The Reduction

We now describe the reduction from $\mathcal{O}$ to a classical $\mathcal{ALCOQ}$ ontology $\mathsf{red}(\mathcal{O})$. This reduction always uses nominals, even in the logic G-$\mathcal{SRIQ}$. However, if number restrictions are not allowed (e.g. in G-$\mathcal{SROI}$), then $\mathsf{red}(\mathcal{O})$ is an $\mathcal{ALCO}$ ontology.

As a first pre-processing step, we eliminate role assertions $(a, b){:}r$ from the ABox by replacing them with the equivalent concept assertions $a{:}\exists r.\{b\}$; this simplifies the following reduction. We now extend the set $\mathsf{sub}(\mathcal{O})$ by the following elements (and their negations):

- We add all nominals $\{a\}$ for $a \in \mathsf{ind}(\mathcal{O})$ to be able to distinguish all named domain elements.
- We further consider all concepts $\exists r.\mathsf{Self}$ with $r \in \mathsf{rol}(\mathcal{O})$ (also for non-simple roles), in order to represent the degrees to which a domain element is connected to itself, e.g. for reflexivity axioms.
- We add all "concepts" of the form $\forall \mathbf{A}_r^q.C$ ($\exists \mathbf{A}_r^q.C$) for all $\forall r.C$ ($\exists r.C$) occurring in $\mathcal{O}$ and all states q of $\mathbf{A}_r$. These concepts help to transfer the constraints imposed by the existential and value restrictions along all role chains that imply the possibly non-simple role r. The semantics of $\forall \mathbf{A}.C$ is defined as follows:

  $$(\forall \mathbf{A}.C)^{\mathcal{I}}(d) := \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{(\|\mathbf{A}\|, w), w^{\mathcal{I}}(d, e)\} \Rightarrow C^{\mathcal{I}}(e),$$

  where $\varepsilon^{\mathcal{I}}(d, e) := 1$ if $d = e$, and $\varepsilon^{\mathcal{I}}(d, e) := 0$ otherwise. Intuitively, it behaves like a value restriction, but instead of considering only the role r, we consider any role chain w, weighted by the behavior of $\mathbf{A}$ on w. Recall that for $\mathbf{A}_r$, this behavior represents the degree to which w implies r w.r.t. $\mathcal{R}_h$ (see Lemma 3.4).

The idea is that in our reduction we do not need to explicitly represent all role connections, but only a "skeleton" of connections which are necessary to satisfy the witnessing conditions for role restrictions. The restrictions for all implied role connections are then handled by the concepts $\forall \mathbf{A}_r.C$ and $\exists \mathbf{A}_r.C$ by simulating the transitions of $\mathbf{A}_r$; each transition corresponds to a role connection to a new domain element. Note that we do not need to introduce concepts of the form $\geqslant n\, \mathbf{A}_r.C$ since all roles in at-least restrictions must be simple, i.e. there can be no role chains of length > 1 that imply them (at least not with a degree > 0).

The main idea of the reduction is that instead of precisely defining the interpretation of all concepts at each domain element, it suffices to consider a total preorder on them. For example, if an axiom restricts the value of $C \to D$ at each domain element to be $\geqslant 0.5$, then we do not have to find the exact values of C and D, but only to ensure that either $C^{\mathcal{I}}(d) \leqslant D^{\mathcal{I}}(d)$ or else $D^{\mathcal{I}}(d) \geqslant 0.5$. This information is encoded by total preorders over the order structure $\mathcal{U}$ that is defined below. The other main insight for our reduction is that we consider only *(quasi-)forest-shaped* models of $\mathcal{O}$ [17]. In such a model, the domain elements identified by individual names serve as the roots of several tree-shaped structures. The roots themselves may be arbitrarily interconnected by roles. Due to nominals, there may also be role connections from any domain element back to the roots. Note that complex role inclusions may actually imply role connections between arbitrary domain elements, but the underlying tree-shaped "skeleton" is what is important for reasoning purposes (for details, see [17] and our correctness proof in [16]). This dependence on forest-shaped models is the reason why our reduction works only for G-$\mathcal{SROI}$, G-$\mathcal{SROQ}$, and G-$\mathcal{SRIQ}$—even classical $\mathcal{ALCOIQ}$ does not have the forest model property [32].

We then define the order structure $\mathcal{U}$ as follows:

$$\begin{aligned}
\mathcal{U}_{\mathcal{A}} &:= \mathcal{V}_{\mathcal{O}} \cup \{a{:}C \mid a \in \mathsf{ind}(\mathcal{O}),\ C \in \mathsf{sub}(\mathcal{O})\} \cup \\
&\qquad\qquad \{(a,b){:}s \mid a,b \in \mathsf{ind}(\mathcal{O}),\ r \in \mathsf{rol}(\mathcal{O}),\ s \in \{r, \neg r\}\} \\
\mathcal{U} &:= \mathcal{U}_{\mathcal{A}} \cup \mathsf{sub}(\mathcal{O}) \cup \mathsf{sub}_{\uparrow}(\mathcal{O}) \cup \\
&\qquad\qquad \{s,\ (a,*){:}s,\ (*,a){:}s \mid a \in \mathsf{ind}(\mathcal{O}),\ r \in \mathsf{rol}(\mathcal{O}),\ s \in \{r, \neg r\}\},
\end{aligned}$$

where $\mathsf{sub}_{\uparrow}(\mathcal{O}) := \{\langle C\rangle_{\uparrow} \mid C \in \mathsf{sub}(\mathcal{O})\}$ and the function inv is defined by $\mathsf{inv}(C) := \neg C$, $\mathsf{inv}(a{:}C) := a{:}\neg C$, $\mathsf{inv}(a,*){:}r := (a,*){:}\neg r$, etc.

Total preorders on assertions in $\mathcal{U}_{\mathcal{A}}$ are used to describe the behavior of the named root elements in the forest-shaped model. For example, if the order is such that $a{:}C > (a,b){:}r$, the intention is that in the corresponding G-model $\mathcal{I}$ of $\mathcal{O}$ the value of C at a is strictly greater that the value of the r-connection from a to b, i.e. we have $C^{\mathcal{I}}(a^{\mathcal{I}}) > r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$. For each domain element of $\mathcal{I}$, total preorders on the elements of $\mathsf{sub}(\mathcal{O})$ describe the degrees of all relevant concepts in a similar way. The elements of $\mathsf{sub}_{\uparrow}(\mathcal{O})$ are used to refer back to degrees of concepts at the unique predecessor element in the tree-shaped parts of the interpretation. For convenience, we also define $\langle p\rangle_{\uparrow} := p$ for all $p \in \mathcal{V}_{\mathcal{O}}$. The elements $r \in \mathsf{rol}(\mathcal{O})$ represent the values of the role connections from the predecessor. The special elements $(*,a){:}r$ and $(a,*){:}r$ are used to describe role connections between arbitrary domain elements (represented by $*$) and the named elements in the roots.

In order to describe such total preorders over $\mathcal{U}$ with a classical $\mathcal{ALCOQ}$ ontology, we use special concept names of the form $\boxed{\alpha \leqslant \beta}$ for $\alpha, \beta \in \mathcal{U}$. This differs from previous reductions for finitely valued FDLs [7,9,31] in that we not only consider *cut-concepts* of the form $\boxed{q \leqslant \alpha}$ with $q \in \mathcal{V}_{\mathcal{O}}$, but also relationships between different concepts.[5] We use the abbreviations $\boxed{\alpha \geqslant \beta} := \boxed{\beta \leqslant \alpha}$, $\boxed{\alpha < \beta} := \neg\boxed{\alpha \geqslant \beta}$, and similarly for $=$ and $>$. Furthermore, we define the complex expressions

[5]For the rest of this paper, the expressions $\boxed{\alpha \leqslant \beta}$ denote DL concept names.

- $\boxed{\alpha \geqslant \min\{\beta,\gamma\}} := \boxed{\alpha \geqslant \beta} \sqcup \boxed{\alpha \geqslant \gamma}$,
- $\boxed{\alpha \leqslant \min\{\beta,\gamma\}} := \boxed{\alpha \leqslant \beta} \sqcap \boxed{\alpha \leqslant \gamma}$,
- $\boxed{\alpha \geqslant \beta \Rightarrow \gamma} := (\boxed{\beta \leqslant \gamma} \to \boxed{\alpha \geqslant 1}) \sqcap (\boxed{\beta > \gamma} \to \boxed{\alpha \geqslant \gamma})$,
- $\boxed{\alpha \leqslant \beta \Rightarrow \gamma} := \boxed{\beta \leqslant \gamma} \sqcup \boxed{\alpha \leqslant \gamma}$,

and extend these notions to $\boxed{\alpha \bowtie \beta \Rightarrow \gamma}$ etc., for $\bowtie \in \{<,=,>\}$, analogously.

In our reduction, we additionally use the special concept name AN to identify the *anonymous* domain elements, i.e. those which are not of the form $b^{\mathcal{I}}$ for any $b \in \mathsf{ind}(\mathcal{O})$. The reduction uses only one role name $\mathfrak{r}$. The reduced ontology $\mathsf{red}(\mathcal{O})$ consists of the parts $\mathsf{red}(\mathcal{U})$, $\mathsf{red}(\mathcal{A})$, $\mathsf{red}(\mathsf{AN})$, $\mathsf{red}(\uparrow)$, $\mathsf{red}(\mathcal{R})$, $\mathsf{red}(\mathcal{T})$, and $\mathsf{red}(C)$ for all $C \in \mathsf{sub}(\mathcal{O})$, which we describe in the following. We want to emphasize that $\mathsf{red}(\mathcal{O})$ is formulated in $\mathcal{ALCOQ}$, whenever $\mathcal{O}$ is in G-$\mathcal{SRIQ}$ or G-$\mathcal{SROQ}$, and in $\mathcal{ALCO}$ if $\mathcal{O}$ is a G-$\mathcal{SROI}$ ontology. This is due to the fact that we always use nominals to distinguish the named from the anonymous part of the forest-shaped model, and the inverse of $\mathfrak{r}$ is not needed in the reduction.

The first part of $\mathsf{red}(\mathcal{O})$ is

$$\begin{aligned}
\mathsf{red}(\mathcal{U}) := {} & \{\boxed{\alpha \leqslant \beta} \sqcap \boxed{\beta \leqslant \gamma} \sqsubseteq \boxed{\alpha \leqslant \gamma} \mid \alpha,\beta,\gamma \in \mathcal{U}\} \cup \\
& \{\top \sqsubseteq \boxed{\alpha \leqslant \beta} \sqcup \boxed{\beta \leqslant \alpha} \mid \alpha,\beta \in \mathcal{U}\} \cup \\
& \{\top \sqsubseteq \boxed{0 \leqslant \alpha} \sqcap \boxed{\alpha \leqslant 1} \mid \alpha \in \mathcal{U}\} \cup \\
& \{\top \sqsubseteq \boxed{\alpha \bowtie \beta} \mid \alpha,\beta \in \mathcal{V}_{\mathcal{O}},\ \alpha \bowtie \beta\} \cup \\
& \{\boxed{\alpha \leqslant \beta} \sqsubseteq \boxed{\mathsf{inv}(\beta) \leqslant \mathsf{inv}(\alpha)} \mid \alpha,\beta \in \mathcal{U}\}.
\end{aligned}$$

These axioms ensure that at each domain element the relation "$\leqslant$" forms a total preorder that is compatible with the values in $\mathcal{V}_{\mathcal{O}}$, and that inv is an antitone operator.

To describe the behavior of all named elements, we use the following axioms:

$$\begin{aligned}
\mathsf{red}(\mathcal{A}) := {} & \{c{:}\boxed{\alpha \bowtie \beta} \mid \langle \alpha \bowtie \beta \rangle \in \mathcal{A}\} \cup \{a \approx b \in \mathcal{A}\} \cup \{a \not\approx b \in \mathcal{A}\} \cup \\
& \{(a,b){:}\mathfrak{r} \mid a,b \in \mathsf{ind}(\mathcal{O})\} \cup \{\boxed{\alpha \bowtie \beta} \sqsubseteq \forall \mathfrak{r}.\boxed{\alpha \bowtie \beta} \mid \alpha,\beta \in \mathcal{U}_{\mathcal{A}}\} \cup \\
& \{a{:}\boxed{a{:}C = C} \mid a \in \mathsf{ind}(\mathcal{O}),\ C \in \mathsf{sub}(\mathcal{O})\} \cup \\
& \{b{:}\boxed{(a,b){:}r = (a,*){:}r} \mid a,b \in \mathsf{ind}(\mathcal{O}),\ r \in \mathsf{rol}(\mathcal{O})\} \cup \\
& \{a{:}\boxed{(a,b){:}r = (*,b){:}r} \mid a,b \in \mathsf{ind}(\mathcal{O}),\ r \in \mathsf{rol}(\mathcal{O})\} \cup \\
& \{\top \sqsubseteq \boxed{(a,a){:}r = a{:}\exists r.\mathsf{Self}} \mid a \in \mathsf{ind}(\mathcal{O}),\ r \in \mathsf{rol}(\mathcal{O})\} \cup \\
& \{\top \sqsubseteq \boxed{(a,b){:}r = (b,a){:}r^-} \mid a,b \in \mathsf{ind}(\mathcal{O}) \cup \{*\},\ r \in \mathsf{rol}(\mathcal{O})\},
\end{aligned}$$

where c is an arbitrary individual name. The first two lines are responsible for enforcing that the ABox is satisfied and that information about the behavior of the named individuals is available throughout the whole model. The remaining axioms describe various equivalences for named individuals, e.g. that $(a,b){:}r$ and $(*,b){:}r$ should have the same value when evaluated at a.

The next axiom defines the concept AN of all *anonymous* elements, i.e. those that are not designated by an individual name:

$$\mathsf{red}(\mathsf{AN}) := \Big\{\neg\mathsf{AN} \equiv \bigsqcup_{a\in\mathsf{ind}(\mathcal{O})}\{a\}\Big\}.$$

The following axioms ensure that the order of a node in a tree-shaped part of the model is known at each of its successors via the elements of $\mathsf{sub}_{\uparrow}(\mathcal{O})$:

$$\mathsf{red}(\uparrow) := \{\boxed{\alpha \bowtie \beta} \sqsubseteq \forall\mathfrak{r}.\big(\mathsf{AN} \to \boxed{\langle\alpha\rangle_{\uparrow} \bowtie \langle\beta\rangle_{\uparrow}}\big) \mid \alpha,\beta \in \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})\}.$$

We now come to the reduction of the RBox:

$$\begin{aligned}\mathsf{red}(\mathcal{R}) := \{\top \sqsubseteq{}& \boxed{(a,b){:}r \Rightarrow (a,b){:}s \geqslant p} \sqcap \boxed{r \Rightarrow s \geqslant p} \sqcap \\ & \boxed{r^- \Rightarrow s^- \geqslant p} \sqcap \boxed{\exists r.\mathsf{Self} \Rightarrow \exists s.\mathsf{Self} \geqslant p} \mid \\ & \qquad \langle r \sqsubseteq s \geqslant p\rangle \in \mathcal{R},\ a,b \in \mathsf{ind}(\mathcal{O}) \cup \{*\}\} \cup \\ \{\top \sqsubseteq{}& \boxed{\min\{(a,b){:}r,(a,b){:}s\} \leqslant 1-p} \sqcap \boxed{\min\{r,s\} \leqslant 1-p} \sqcap \\ & \boxed{\min\{r^-,s^-\} \leqslant 1-p} \sqcap \boxed{\min\{\exists r.\mathsf{Self},\exists s.\mathsf{Self}\} \leqslant 1-p} \mid \\ & \qquad \langle \mathsf{dis}(r,s) \geqslant p\rangle \in \mathcal{R},\ a,b \in \mathsf{ind}(\mathcal{O}) \cup \{*\}\} \cup \\ \{\top \sqsubseteq{}& \boxed{\exists r.\mathsf{Self} \geqslant p} \mid \langle \mathsf{ref}(r) \geqslant p\rangle \in \mathcal{R}\} \cup \\ \{\top \sqsubseteq{}& \boxed{(a,b){:}r_u \geqslant 1} \sqcap \boxed{r_u \geqslant 1} \sqcap \boxed{\exists r_u.\mathsf{Self} \geqslant 1} \mid a,b \in \mathsf{ind}(\mathcal{O}) \cup \{*\}\}\end{aligned}$$

The concepts and axioms concerning the universal role, inverse roles, and (local) reflexivity statements are only included in the reduction if the logic under consideration supports them.

These axioms ensure that the various elements of $\mathcal{U}$ that represent the values of role connections, such as $(a,b){:}r$, $\exists r.\mathsf{Self}$, and r, respect the axioms in $\mathcal{R}$. Although the simple role inclusions $\langle r \sqsubseteq s \geqslant p\rangle$ are handled by the automata $\mathbf{A}_r$, we include them also in $\mathsf{red}(\mathcal{R})$. The reason is that the reduction of at-least restrictions below does not need to use these automata since only simple roles can occur in them.

The GCIs in $\mathcal{T}$ can be translated in a straightforward manner:

$$\mathsf{red}(\mathcal{T}) := \{\top \sqsubseteq \boxed{p \leqslant C \Rightarrow D} \mid \langle C \sqsubseteq D \geqslant p\rangle \in \mathcal{T}\}$$

We now come to the reductions of the concepts. Intuitively, each $\mathsf{red}(C)$ describes the semantics of C in terms of its order relationships to other elements of $\mathcal{U}$. Note that the semantics of the involutive negation $\neg C = \mathsf{inv}(C)$ is already handled by the operator inv (see $\mathsf{red}(\mathcal{U})$ above):

$$\begin{aligned}
\mathsf{red}(\top) &:= \{\top \sqsubseteq \boxed{\top \geqslant 1}\} \\
\mathsf{red}(\{a\}) &:= \{\{a\} \sqsubseteq \boxed{1 \leqslant \{a\}},\ \neg\{a\} \sqsubseteq \boxed{\{a\} \leqslant 0}\} \\
\mathsf{red}(\overline{p}) &:= \{\top \sqsubseteq \boxed{\overline{p} = p}\} \\
\mathsf{red}(\exists r.\mathsf{Self}) &:= \{\top \sqsubseteq \boxed{\exists r.\mathsf{Self} = \exists r^-.\mathsf{Self}}\} \\
\mathsf{red}(\neg C) &:= \emptyset \\
\mathsf{red}(C \sqcap D) &:= \{\top \sqsubseteq \boxed{C \sqcap D = \min\{C, D\}}\} \\
\mathsf{red}(C \to D) &:= \{\top \sqsubseteq \boxed{C \to D = C \Rightarrow D}\}
\end{aligned}$$

The reductions of role restrictions are more involved. In particular, in the case of value and existential restrictions we have to deal with non-simple roles, for which we employ the automata $\mathbf{A}_r$ from the previous section.

$$\begin{aligned}
\mathsf{red}(\forall r.C) := \{\top \sqsubseteq{} & \boxed{(\forall r.C) \leqslant (\forall \mathbf{A}_r.C)}, \\
\mathsf{AN} \sqsubseteq{} & \boxed{(\forall r.C) \geqslant r^- \Rightarrow \langle C\rangle_\uparrow} \sqcup && \text{(I)} \\
& \boxed{(\forall r.C) \geqslant (\exists r.\mathsf{Self}) \Rightarrow C} \sqcup && \text{(S)} \\
& \exists \mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle \forall r.C\rangle_\uparrow \geqslant r \Rightarrow C}\big) \sqcup \\
& \bigsqcup_{a \in \mathsf{ind}(\mathcal{O})} \big(\exists \mathfrak{r}.\{a\} \sqcap \boxed{(\forall r.C) \geqslant (*, a){:}r \Rightarrow a{:}C}\big)\} \cup && \text{(N)} \\
\{a{:}\exists \mathfrak{r}.\big(\big(\mathsf{AN} \sqcap{} & \boxed{\langle \forall r.C\rangle_\uparrow \geqslant r \Rightarrow C}\big) \sqcup \\
& \big(\neg\mathsf{AN} \sqcap \boxed{(a{:}\forall r.C) \geqslant (a, *){:}r \Rightarrow C}\big)\big) \mid a \in \mathsf{ind}(\mathcal{O})\}
\end{aligned}$$

Here and in the following definitions, we label with (I) those concepts or axioms that are contingent on the presence of inverse roles in the source logic. Likewise, terms labeled with (S) are only included if (local) reflexivity statements are allowed, and similarly for (N) and nominals.

The second axiom of $\mathsf{red}(\forall r.C)$ ensures the existence of a witness for $\forall r.C$ at each anonymous domain element. For example, assume that the preorder represented by the concepts $\boxed{\alpha \leqslant \beta}$ at some domain element d satisfies $0.5 < \forall r.C < 1$. The first possibility is that the above axiom creates an anonymous element e that is connected to d via $\mathfrak{r}$, and hence by $\mathsf{red}(\mathsf{AN})$ we know that e satisfies $0.5 < \langle \forall r.C\rangle_\uparrow < 1$. The axiom further requires that $\langle \forall r.C\rangle_\uparrow \geqslant r \Rightarrow C$, which implies that $\langle \forall r.C\rangle_\uparrow \geqslant C$ and $r > C$. We will see below that the reduction of $\forall \mathbf{A}_r.C$ further ensures that $\langle \forall r.C\rangle_\uparrow \leqslant r \Rightarrow C$, and thus we get $\langle \forall r.C\rangle_\uparrow = C$. Hence, e can be seen as an abstract representation of the witness of $\forall r.C$ at d; the precise value of the r-connection between d and e (represented by the element r) is irrelevant, as long as it is strictly greater than the value of C at e. The other disjuncts of this axiom deal with the possibility that d itself, its predecessor, or a named domain element acts as the witness for the value restriction in a similar way. The assertions deal with the case of a named domain element, in which case the first two options above (self and inverse) are subsumed by the last option.

Together with the first axiom of $\mathsf{red}(\forall r.C)$, the following axioms ensure that no other r-successor of d violates the lower bound on $r \Rightarrow C$ given by $\forall r.C$ at d:

$$\mathsf{red}(\forall \mathbf{A}^q.C) := \{\top \sqsubseteq \boxed{(\forall \mathbf{A}^q.C) \leqslant C} \mid q \text{ is final}\} \cup \bigcup_{q \xrightarrow{x,p} q' \in \mathbf{A}} \mathsf{red}_{x,p,q'}(\forall \mathbf{A}^q.C)$$

$$\mathsf{red}_{\varepsilon,p,q'}(\forall \mathbf{A}^q.C) := \{\top \sqsubseteq \boxed{(\forall \mathbf{A}^q.C) \leqslant p \Rightarrow (\forall \mathbf{A}^{q'}.C)}\}$$

$$\mathsf{red}_{s,p,q'}(\forall \mathbf{A}^q.C) :=$$

$$\{\mathsf{AN} \sqsubseteq \boxed{(\forall \mathbf{A}^q.C) \leqslant \min\{p, s^-\} \Rightarrow \langle \forall \mathbf{A}^{q'}.C\rangle_\uparrow}, \quad \text{(I)}$$

$$\top \sqsubseteq \boxed{(\forall \mathbf{A}^q.C) \leqslant \min\{p, \exists s.\mathsf{Self}\} \Rightarrow (\forall \mathbf{A}^{q'}.C)}, \quad \text{(S)}$$

$$\top \sqsubseteq \forall \mathfrak{r}.\big(\mathsf{AN} \rightarrow \boxed{\langle \forall \mathbf{A}^q.C\rangle_\uparrow \leqslant \min\{p, s\} \Rightarrow (\forall \mathbf{A}^{q'}.C)}\big)\} \cup$$

$$\{\exists \mathfrak{r}.\{a\} \sqsubseteq \boxed{(\forall \mathbf{A}^q.C) \leqslant \min\{p, (*,a){:}s\} \Rightarrow a{:}(\forall \mathbf{A}^{q'}.C)},$$

$$\exists \mathfrak{r}.\{a\} \sqsubseteq \boxed{a{:}(\forall \mathbf{A}^q.C) \leqslant \min\{p, (*,a){:}s^-\} \Rightarrow (\forall \mathbf{A}^{q'}.C)} \mid a \in \mathsf{ind}(\mathcal{O})\} \quad \text{(I,N)}$$

Recall that $\mathbf{A}_r$ in particular contains the transition $i_r \xrightarrow{r,1} f_r$ from the initial state i_r to the final state f_r. By the first axiom in $\mathsf{red}(\forall r.C)$ and the third axiom in $\mathsf{red}_{r,1,f_r}(\forall \mathbf{A}_r.C)$, the witness e satisfies $\langle \forall r.C\rangle_\uparrow \leqslant \langle \forall \mathbf{A}_r.C\rangle_\uparrow \leqslant r \Rightarrow (\forall \mathbf{A}_r^{f_r}.C)$ Since f_r is final, we further have $(\forall \mathbf{A}_r^{f_r}.C) \leqslant C$ by $\mathsf{red}(\forall \mathbf{A}_r^{f_r}.C)$, and hence $\langle \forall r.C\rangle_\uparrow \leqslant r \Rightarrow C$, as claimed above. The other axioms in $\mathsf{red}_{r,1,f_r}(\forall \mathbf{A}_r.C)$ deal with the other kinds of possible successors (see above).

Using arbitrary runs through the automaton $\mathbf{A}_r$, we can ensure that no other r-successor of d violates the value restriction. For example, if $r^\mathcal{I}(d, e_1) = 0.3$ and $r^\mathcal{I}(e_1, e_2) = 0.5$ for two other (anonymous) domain elements e_1, e_2, and we further have the role inclusion $\langle rr \sqsubseteq r \geqslant 0.7\rangle$, then we know that $r^\mathcal{I}(d, e_2)$ must be at least 0.5. Although this r-connection is not explicitly represented in our forest-based encoding, concepts of the form $\forall \mathbf{A}_r^q.C$ are appropriately transferred from d via e_1 to e_2 in order to ensure that the value of C at e_2 satisfies $0.5 < (\forall r.C)^\mathcal{I}(d) \leqslant r^\mathcal{I}(d, e_2) \Rightarrow C^\mathcal{I}(e_2)$. In this example, since we know only that $r^\mathcal{I}(d, e_2) \geqslant 0.5$, it must be ensured that $C^\mathcal{I}(e_2) \geqslant r^\mathcal{I}(d, e_2)$.

The reduction for existential restrictions can be defined similarly to that for value restrictions, but replacing $\geqslant$ with $\leqslant$ (and vice versa) and $\Rightarrow$ with min.

We now come to the final component of $\mathsf{red}(\mathcal{O})$.

$$\mathsf{red}({\geqslant} n\, r.C) := \{\mathsf{AN} \sqsubseteq \bigsqcup_{z_i=0}^{1} \bigsqcup_{z_s=0}^{1} \bigsqcup_{m=0}^{n-z_i-z_s} \bigsqcup_{\substack{S \subseteq \mathsf{ind}(\mathcal{O}) \\ |S| = n-m-z_i-z_s}} \bigsqcap \mathsf{red}_{z_i,z_s,m,S,\leqslant}({\geqslant} n\, r.C),$$

$$\mathsf{AN} \sqsubseteq \neg \bigsqcup_{z_i=0}^{1} \bigsqcup_{z_s=0}^{1} \bigsqcup_{m=0}^{n-z_i-z_s} \bigsqcup_{\substack{S \subseteq \mathsf{ind}(\mathcal{O}) \\ |S| = n-m-z_i-z_s}} \bigsqcap \mathsf{red}_{z_i,z_s,m,S,<}({\geqslant} n\, r.C)\} \cup$$

$$\{a{:}{\geqslant}n\,\mathfrak{r}.\big((\mathsf{AN}\sqcap \boxed{\langle{\geqslant}n\,r.C\rangle_\uparrow \leqslant \min\{r,C\}})\sqcup (\neg\mathsf{AN}\sqcap \boxed{(a{:}{\geqslant}n\,r.C) \leqslant \min\{(a,*){:}r,C\}})\big),\ a{:}\neg{\geqslant}n\,\mathfrak{r}.\big((\mathsf{AN}\sqcap \boxed{\langle{\geqslant}n\,r.C\rangle_\uparrow < \min\{r,C\}})\sqcup (\neg\mathsf{AN}\sqcap \boxed{(a{:}{\geqslant}n\,r.C) < \min\{(a,*){:}r,C\}})\big) \mid a\in\mathsf{ind}(\mathcal{O})\},$$

where

$$\begin{aligned}\mathsf{red}_{z_i,z_s,m,S,\lhd}({\geqslant}n\,r.C) := {} & \{\mathsf{AN}\sqcap \boxed{({\geqslant}n\,r.C)\lhd \min\{r^-,\langle C\rangle_\uparrow\}} \mid z_i=1\}\cup\\ & \{\boxed{({\geqslant}n\,r.C)\lhd \min\{\exists r.\mathsf{Self},C\}} \mid z_s=1\}\cup\\ & \{{\geqslant}m\,\mathfrak{r}.(\mathsf{AN}\sqcap \boxed{\langle{\geqslant}n\,r.C\rangle_\uparrow \lhd \min\{r,C\}})\}\cup\\ & \{\exists\mathfrak{r}.(\{a\}\sqcap\neg\{b\}) \mid a,b\in S,\ a\neq b\}\cup\\ & \{\boxed{({\geqslant}n\,r.C)\lhd \min\{(*,a){:}r,a{:}C\}} \mid a\in S\}\end{aligned}$$

If we do not have inverse roles, (local) reflexivity, or nominals, then we fix the numbers z_i, z_s, or m, respectively, to 0, 0, or $n-z_i-z_s$, which effectively eliminates the conjuncts using these constructors from the above axioms.

The reduction of at-least restrictions works similarly to the one of value restrictions: the first axiom ensures the existence of the n required witnesses, while the second one ensures that no n different elements can exceed the value of the at-least restriction. Unfortunately, the number of named successors cannot be counted using a classical at-least restriction in our encoding, since these named successors do not know about the degree of the role connection from an anonymous element; otherwise they would have to store a possibly infinite amount of information since they may have infinitely many anonymous role predecessors. For this reason, the above axioms first guess how many $(m-n)$ and which (S) named elements are connected to the current domain element to the appropriate degrees (given by $(*,a){:}r$). For named elements, however, this guessing is not necessary.

This reduction is correct in the sense that the resulting ontology $\mathsf{red}(\mathcal{O})$ has a classical model iff $\mathcal{O}$ has a G-model. As mentioned before, this holds only for the sublogics $\mathcal{SRIQ}$, $\mathcal{SROQ}$, and $\mathcal{SROI}$ that have the forest model property [17]. However, the correctness is not affected by the presence or absence of (local) reflexivity statements.

4.1 Soundness

We first show that, in $\mathcal{SRIQ}$, $\mathcal{SROQ}$, or $\mathcal{SROI}$, if $\mathsf{red}(\mathcal{O})$ has a classical model, then $\mathcal{O}$ has a G-model.

Since $\mathsf{red}(\mathcal{O})$ contains only the role name $\mathfrak{r}$ and no inverses, and hence is in $\mathcal{ALCOQ}$, we can assume that it has a *quasi-forest model* $\mathcal{I}$ with the following properties [17]:

- $\Delta^{\mathcal{I}} \subseteq \mathsf{ind}(\mathcal{O}) \times \mathbb{N}^*$;
- for each $a \in \mathsf{ind}(\mathcal{O})$, the set $\{u \in \mathbb{N}^* \mid (a,u) \in \Delta^{\mathcal{I}}\}$ is prefix-closed;
- for each $a \in \mathsf{ind}(\mathcal{O})$, we have $a^{\mathcal{I}} = (a, \varepsilon)$;
- for all $a \in \mathsf{ind}(\mathcal{O})$, $u \in \mathbb{N}^*$, and $i \in \mathbb{N}$ with $(a, ui) \in \Delta^{\mathcal{I}}$, we have $((a,u),(a,ui)) \in \mathfrak{r}^{\mathcal{I}}$; and
- whenever $((a,u),(b,u')) \in \mathfrak{r}^{\mathcal{I}}$, then
  a) $a = b$ and $u' = ui$ for some $i \in \mathbb{N}$ or
  b) $u' = \varepsilon$.

We assume here that all named individuals in $\mathsf{ind}(\mathcal{O})$ are interpreted by distinct elements in $\mathcal{I}$. In general, we would have to consider sets of names from $\mathsf{ind}(\mathcal{O})$ as the roots of $\mathcal{I}$. Since this is relevant only for number restrictions and (in)equality assertions, we ignore this in the following and only mention it at the appropriate places.

For any $u = n_1 \dots n_k \in \mathbb{N}^*$ with $k \geqslant 1$, we denote by $u_{\uparrow} := n_1 \dots n_{k-1}$ its predecessor. We denote by $\preccurlyeq_{\mathcal{A}}$ the binary relation on $\mathcal{U}_{\mathcal{A}}$ defined by $\alpha \preccurlyeq_{\mathcal{A}} \beta$ iff $c^{\mathcal{I}} \in \boxed{\alpha \leqslant \beta}^{\mathcal{I}}$ for an arbitrary $c \in \mathsf{ind}(\mathcal{O})$. This is a total preorder due to the axioms in $\mathsf{red}(\mathcal{U})$. We similarly define $\alpha \preccurlyeq^a_u \beta$ iff $(a,u) \in \boxed{\alpha \leqslant \beta}^{\mathcal{I}}$, for all $\alpha, \beta \in \mathcal{U}$. Since $\mathcal{I}$ satisfies $\mathsf{red}(\mathcal{A})$ and all domain elements are connected via $\mathfrak{r}$, we have $\preccurlyeq_{\mathcal{A}} \subseteq \preccurlyeq^a_u$ for all $(a,u) \in \Delta^{\mathcal{I}}$. We further denote by $\equiv_{\mathcal{A}}$ ($\equiv^a_u$) the equivalence relation induced by $\preccurlyeq_{\mathcal{A}}$ ($\preccurlyeq^a_u$).

As a first step in the construction of a G-model of $\mathcal{O}$, we now construct a function $v \colon \mathcal{U}_{\mathcal{A}} \cup (\mathcal{U} \times \Delta^{\mathcal{I}}) \to [0,1]$ such that

(P1) for all $p \in \mathcal{V}_{\mathcal{O}}$, we have $v(p) = p$,

(P2) for all $\alpha, \beta \in \mathcal{U}_{\mathcal{A}}$, we have $v(\alpha) \leqslant v(\beta)$ iff $\alpha \preccurlyeq_{\mathcal{A}} \beta$,

(P3) for all $\alpha \in \mathcal{U}_{\mathcal{A}}$, we have $v(\mathsf{inv}(\alpha)) = 1 - v(\alpha)$,

(P4) for every $C \in \mathsf{sub}(\mathcal{O})$ and all $a \in \mathsf{ind}(\mathcal{O})$, we have $v(a{:}C) = v(C, a, \varepsilon)$,

(P5) for all $(a,u) \in \Delta^{\mathcal{I}}$,

  a) for all $p \in \mathcal{V}_{\mathcal{O}}$, we have $v(p,a,u) = p$,
  b) for all $\alpha, \beta \in \mathcal{U}$, we have $v(\alpha,a,u) \leqslant v(\beta,a,u)$ iff $\alpha \preccurlyeq^a_u \beta$,
  c) for all $\alpha \in \mathcal{U}$, we have $v(\mathsf{inv}(\alpha),a,u) = 1 - v(\alpha,a,u)$, and
  d) if $u \neq \varepsilon$, then for all $C \in \mathsf{sub}(\mathcal{O})$ we have $v(C,a,u_{\uparrow}) = v(\langle C \rangle_{\uparrow}, a, u)$.

We start defining v on $\mathcal{U}_\mathcal{A}$. Let $\mathcal{U}_\mathcal{A}/_{\equiv_\mathcal{A}}$ be the set of all equivalence classes of $\equiv_\mathcal{A}$. Then $\preccurlyeq_\mathcal{A}$ yields a total order $\leqslant_\mathcal{A}$ on $\mathcal{U}_\mathcal{A}/_{\equiv_\mathcal{A}}$. If $0 = p_0 < p_1 < \cdots < p_{k-1} < p_k = 1$ are the elements of $\mathcal{V}_\mathcal{O}$, then since $\mathcal{I}$ satisfies $\mathsf{red}(\mathcal{U})$, we have

$$[0]_\mathcal{A} <_\mathcal{A} [p_1]_\mathcal{A} <_\mathcal{A} \cdots <_\mathcal{A} [p_{k-1}]_\mathcal{A} <_\mathcal{A} [1]_\mathcal{A}$$

w.r.t. this order. For $[\alpha]_\mathcal{A} \in \mathcal{U}_\mathcal{A}/\equiv_\mathcal{A}$, we set $\mathsf{inv}([\alpha]_\mathcal{A}) := [\mathsf{inv}(\alpha)]_\mathcal{A}$, which is well-defined because of the axioms in $\mathsf{red}(\mathcal{U})$. On all $\alpha \in [p]_\mathcal{A}$ for $p \in \mathcal{V}_\mathcal{O}$, we now define $v(\alpha) := p$, which ensures that (P1) holds. For the equivalence classes that do not contain a value from $\mathcal{V}_\mathcal{O}$, note that by $\mathsf{red}(\mathcal{U})$, every such class must be strictly between $[p_i]_\mathcal{A}$ and $[p_{i+1}]_\mathcal{A}$ for some $p_i, p_{i+1} \in \mathcal{V}_\mathcal{O}$. We denote the n_i equivalence classes between $[p_i]_\mathcal{A}$ and $[p_{i+1}]_\mathcal{A}$ as follows:

$$[p_i]_\mathcal{A} <_\mathcal{A} E^i_1 <_\mathcal{A} \cdots <_\mathcal{A} E^i_{n_i} <_\mathcal{A} [p_{i+1}]_\mathcal{A}.$$

For every $\alpha \in E^i_j$, $1 \leqslant j \leqslant n_i$, we set $v(\alpha) := p_i + \frac{j}{n_i+1}(p_{i+1} - p_i)$, which ensures that (P2) is also satisfied. Furthermore, $1 - p_{i+1}$ and $1 - p_i$ are also adjacent in $\mathcal{V}_\mathcal{O}$ and we have $[1-p_{i+1}]_\mathcal{A} <_\mathcal{A} \mathsf{inv}(E^i_{n_i}) <_\mathcal{A} \cdots <_\mathcal{A} \mathsf{inv}(E^i_1) <_\mathcal{A} [1-p_i]_\mathcal{A}$ by the axioms in $\mathsf{red}(\mathcal{U})$. Hence, it follows from the definition of $v(\alpha)$ that (P3) holds.

We now construct the values of $v(\alpha, a, \varepsilon)$ using a similar technique. However, we now start by setting $v(\alpha, a, \varepsilon) := v(\alpha)$ for all $\alpha \in [\beta]^a_\varepsilon$ with $\beta \in \mathcal{U}_\mathcal{A}$. To see that this is well-defined, consider the case that $[\beta]^a_\varepsilon = [\beta']^a_\varepsilon$ for two different elements $\beta, \beta' \in \mathcal{U}_\mathcal{A}$. Since $\preccurlyeq^a_\varepsilon$ contains the preorder $\preccurlyeq_\mathcal{A}$, we know that $[\beta]_\mathcal{A} = [\beta']_\mathcal{A}$, and hence $v(\beta) = v(\beta')$ by (P2). We can now arrange all other values between those already fixed as shown above, thereby satisfying (P5)a)–c). Since $a^\mathcal{I} = (a, \varepsilon)$ and $\mathcal{I}$ satisfies $\mathsf{red}(\mathcal{A})$, this construction also ensures that (P4) is satisfied.

We now proceed to define $v(\alpha, a, u)$ by induction on the structure of the tree $\{u \mid (a, u) \in \Delta^\mathcal{I}\}$. Assume that $v(\alpha, a, u_\uparrow)$ has already been defined for all $\alpha \in \mathcal{U}$, and satisfies (P5)a)–c). By assumption, we have $((a, u_\uparrow), (a, u)) \in \mathfrak{r}^\mathcal{I}$, and by $\mathsf{red}(\mathsf{AN})$ we know that $(a, u) \in \mathsf{AN}^\mathcal{I}$. We again use the same construction as before, but this time fixing all values $v(\alpha, a, u) := v(\alpha, a, u_\uparrow)$ for all $\alpha \in \mathcal{U}_\mathcal{A}$ and $v(\alpha, a, u) := v(C, u_\uparrow)$ for all $C \in \mathsf{sub}(\mathcal{O})$ and all $\alpha \in [\langle C\rangle_\uparrow]_u$. This is well-defined by the same arguments as above and the fact that $\mathcal{I}$ satisfies $\mathsf{red}(\uparrow)$. We then fix the remaining values as before. This construction ensures that (P5)a)–d) are satisfied.

Based on v, we now define the G-interpretation $\mathcal{I}_\mathsf{f}$ over the domain $\Delta^{\mathcal{I}_\mathsf{f}} := \Delta^\mathcal{I}$, where $a^{\mathcal{I}_\mathsf{f}} := a^\mathcal{I} = (a, \varepsilon)$ for all $a \in \mathsf{ind}(\mathcal{O})$[6] and $A^{\mathcal{I}_\mathsf{f}}(d) := v(A, d)$ for all $A \in \mathsf{N_C} \cap \mathsf{sub}(\mathcal{O})$ and $d \in \Delta^{\mathcal{I}_\mathsf{f}}$. The values for all other concept names are irrelevant and can be fixed arbitrarily. For all $r \in \mathsf{N_R} \cap \mathsf{rol}(\mathcal{O})$, we first define a

[6]If we are dealing with equivalence classes of individuals as the roots of $\mathcal{I}$, then $a^{\mathcal{I}_\mathsf{f}}$ is interpreted using the equivalence class containing a.

"simple" interpretation $\mathcal{I}_\mathsf{f}^0$ as follows.

$$r^{\mathcal{I}_\mathsf{f}^0}((a,u),(b,u')) := \begin{cases} v(r,a,u') & \text{if } a=b \text{ and } u=u'_{\uparrow}, \\ v(r^-,a,u) & \text{if } a=b \text{ and } u'=u_{\uparrow}, \\ v(\exists r.\mathsf{Self},a,u) & \text{if } (a,u)=(b,u'), \\ v((a,b){:}r) & \text{if } u=u'=\varepsilon \text{ and } a\neq b, \\ v((a,*){:}r,b,u') & \text{if } u=\varepsilon,\ u'\neq\varepsilon, \text{ and } ((b,u'),(a,\varepsilon))\in\mathfrak{r}^{\mathcal{I}}, \\ v((*,b){:}r,a,u) & \text{if } u'=\varepsilon,\ u\neq\varepsilon, \text{ and } ((a,u),(b,\varepsilon))\in\mathfrak{r}^{\mathcal{I}}, \\ 0 & \text{otherwise} \end{cases}$$

In the absence of inverse roles, we set the second and fifth lines to 0; and if (local) reflexivity is not allowed, then the third line is 0; finally, if there are no nominals in our source logic, then the fifth and sixth lines are 0. Due to $\mathsf{red}(\mathcal{A})$ and $\mathsf{red}(\exists r.\mathsf{Self})$, the same expressions as for role names can be used to evaluate inverse roles. In the case that r is simple, this already suffices. Otherwise, we use the automaton $\mathbf{A}_r$ to complete $\mathcal{I}_\mathsf{f}^0$ by additional links as follows: we set

$$r^{\mathcal{I}_\mathsf{f}}(d,e) := \sup_{w\in\mathsf{rol}(\mathcal{O})^+} \min\{(\|\mathbf{A}_r\|,w), w^{\mathcal{I}_\mathsf{f}^0}(d,e)\} \tag{3}$$

for all $d,e\in\Delta^{\mathcal{I}_\mathsf{f}}$. Note that this expression is also valid for simple roles r: by Proposition 3.5, we have $(\|\mathbf{A}_r\|,s)=p$ whenever $s\sqsubseteq_p r$, $(\|\mathbf{A}_r\|,r)=1$, and $(\|\mathbf{A}_r\|,w)=0$ for all other words w, and moreover $\mathsf{red}(\mathcal{R})$ yields

$$\begin{aligned} \min\{(\|\mathbf{A}_r\|,r), r^{\mathcal{I}_\mathsf{f}^0}(d,e)\} &= r^{\mathcal{I}_\mathsf{f}^0}(d,e) \\ &\geqslant \min\{p, s^{\mathcal{I}_\mathsf{f}^0}(d,e)\} \\ &= \min\{(\|\mathbf{A}_r\|,s), s^{\mathcal{I}_\mathsf{f}^0}(d,e)\}. \end{aligned}$$

The expression (3) can also be used to evaluate inverse roles due to the semantics of role chains and Proposition 3.3. Finally, for the universal role r_u, we have $r_u^{\mathcal{I}_\mathsf{f}^0}(d,e)=1$ due to the axioms in $\mathsf{red}(\mathcal{R})$. By the construction of $\mathbf{A}_{u_r}$, we also have $(\|\mathbf{A}_{r_u}\|,r_u)=1$, and hence the expression (3) also holds for the universal role and we have $r_u^{\mathcal{I}_\mathsf{f}}(d,e)=1$ for all $d,e\in\Delta^{\mathcal{I}}$.

We now show by induction on the structure of C that

$$C^{\mathcal{I}_\mathsf{f}}(d) = v(C,d) \text{ for all } C\in\mathsf{sub}(\mathcal{O}) \text{ and } d\in\Delta^{\mathcal{I}}, \tag{4}$$

where we exclude the auxiliary concepts of the form $\forall\mathbf{A}.C$ and $\exists\mathbf{A}.C$.

For nominals $\{a\}$, we have $\{a\}^{\mathcal{I}_\mathsf{f}}(d)=1$ if $d=(a,\varepsilon)$, and $\{a\}^{\mathcal{I}_\mathsf{f}}(d)=0$ otherwise. By $\mathsf{red}(\{a\})$ and (P5)a)–b), in the former case we have $v(\{a\},d)=1$, while in the latter case it holds that $v(\{a\},d)=0$. For local reflexivity concepts $\exists r.\mathsf{Self}$, we have $(\exists r.\mathsf{Self})^{\mathcal{I}_\mathsf{f}}(d)=r^{\mathcal{I}_\mathsf{f}}(d,d)=v(\exists r.\mathsf{Self},d)$ by the definition of $r^{\mathcal{I}_\mathsf{f}}$. For $\neg C$, we have

$$(\neg C)^{\mathcal{I}_\mathsf{f}}(d) = 1 - C^{\mathcal{I}_\mathsf{f}}(d) = 1 - v(C,d) = v(\neg C,d)$$

by the induction hypothesis and (P5)c). The claim for $\top$, $\overline{q}$, $\sqcap$, and $\to$ can also be shown using the induction hypothesis, the semantics of these constructors, and the properties (P5)a)–b) of v.

For value restrictions $\forall r.C$, consider first the case that $d = (a, u)$ with $u \neq \varepsilon$, and hence $d \in \mathsf{AN}^{\mathcal{I}}$. By the second axiom in $\mathsf{red}(\forall r.C)$, one of the following must hold:

- We have $d \in \boxed{(\forall r.C) \geqslant r^- \Rightarrow \langle C \rangle_\uparrow}$, and thus (P5)b) and d) and the induction hypothesis yield

$$\begin{aligned} v(\forall r.C, d) &\geqslant v(r^-, a, u) \Rightarrow v(\langle C \rangle_\uparrow, a, u) \\ &= r^{\mathcal{I}_\mathsf{f}^0}((a, u), (a, u_\uparrow)) \Rightarrow C^{\mathcal{I}_\mathsf{f}}(a, u_\uparrow) \\ &\geqslant r^{\mathcal{I}_\mathsf{f}}(d, (a, u_\uparrow)) \Rightarrow C^{\mathcal{I}_\mathsf{f}}(a, u_\uparrow). \end{aligned}$$

  Hence, $(a, u_\uparrow)$ can take the role of the witness for $\forall r.C$ at d if we can show that the latter implication is $\geqslant v(\forall r.C, d)$ for all successors.

- We have

$$\begin{aligned} v(\forall r.C, d) &\geqslant v(\exists r.\mathsf{Self}, a, u) \Rightarrow v(C, a, u) \\ &\geqslant r^{\mathcal{I}_\mathsf{f}}(d, d) \Rightarrow C^{\mathcal{I}_\mathsf{f}}(d) \end{aligned}$$

  by similar arguments as above. In this case, we can choose d itself as the witness.

- There is an anonymous $\mathfrak{r}$-successor of d that satisfies $\boxed{\langle \forall r.C \rangle_\uparrow \geqslant r \Rightarrow C}$, which must be of the form (a, ui) for $i \in \mathbb{N}$ due to our assumption on the structure of $\mathcal{I}$. We get

$$\begin{aligned} v(\forall r.C, d) &\geqslant v(r, a, ui) \Rightarrow v(C, a, ui) \\ &\geqslant r^{\mathcal{I}_\mathsf{f}}(d, (a, ui)) \Rightarrow C^{\mathcal{I}_\mathsf{f}}(a, ui). \end{aligned}$$

- There is a $b \in \mathsf{ind}(\mathcal{O})$ such that $(d, (b, \varepsilon)) \in \mathfrak{r}^{\mathcal{I}}$, and again we have

$$\begin{aligned} v(\forall r.C, d) &\geqslant v((*, b){:}r, a, u) \Rightarrow v(b{:}C, a, u) \\ &\geqslant r^{\mathcal{I}_\mathsf{f}}(d, (b, \varepsilon)) \Rightarrow C^{\mathcal{I}_\mathsf{f}}(b, \varepsilon) \end{aligned}$$

  due to (P5)b) and d), $\mathsf{red}(\mathcal{A})$, and the induction hypothesis.

The assertions in $\mathsf{red}(\forall r.C)$ similarly ensure the existence of witnesses for $\forall r.C$ at all named domain elements. For the remainder of the claim, consider any $e \in \Delta^{\mathcal{I}_\mathsf{f}}$.

Due to the first axiom in $\mathsf{red}(\forall r.C)$, we have

$$\begin{aligned} r^{\mathcal{I}_\mathsf{f}}(d,e) \Rightarrow C^{\mathcal{I}_\mathsf{f}}(e) &= \Big(\sup_{w\in\mathsf{rol}(\mathcal{O})^+} \min\{(\|\mathbf{A}_r\|,w), w^{\mathcal{I}^0_\mathsf{f}}(d,e)\}\Big) \Rightarrow C^{\mathcal{I}_\mathsf{f}}(e) \\ &= \inf_{w\in\mathsf{rol}(\mathcal{O})^+} \min\{(\|\mathbf{A}_r\|,w), w^{\mathcal{I}^0_\mathsf{f}}(d,e)\} \Rightarrow C^{\mathcal{I}_\mathsf{f}}(e) \\ &\overset{(*)}{\geqslant} v(\forall \mathbf{A}_r.C,d) \\ &\geqslant v(\forall r.C,d). \end{aligned}$$

as required, if we can show $(*)$, i.e. it remains to show that

$$\min\{(\|\mathbf{A}_r\|,w), w^{\mathcal{I}^0_\mathsf{f}}(d_0,d_n)\} \Rightarrow C^{\mathcal{I}_\mathsf{f}}(d_n) \geqslant v(\forall \mathbf{A}_r.C,d_0)$$

holds for all $d_0,d_n \in \Delta^{\mathcal{I}_\mathsf{f}}$ and $w = r_1\dots r_n \in \mathsf{rol}(\mathcal{O})^+$. Since $(\|\mathbf{A}_r\|,w)$ and $w^{\mathcal{I}^0_\mathsf{f}}(d_0,d_n)$ are defined as suprema, it suffices to consider any run $\mathsf{r} = (w_i,q_i)_{0\leqslant i\leqslant m}$ with $(w_0,q_0) = (w,i_r)$, $(w_m,q_m) = (\varepsilon,f_r)$, and transitions $q_{i-1} \xrightarrow{x_i,p_i} q_i$ in $\mathbf{A}_r$, and any sequence $d_1,\dots,d_m \in \Delta^{\mathcal{I}_\mathsf{f}}$. To synchronize these two sequences, we define the mapping $\gamma\colon \{0,\dots,m\} \to \{0,\dots,n\}$, where $\gamma(0) := 0$, and

$$\gamma(i) := \begin{cases} \gamma(i-1) & \text{if } x_i = \varepsilon, \\ \gamma(i-1)+1 & \text{if } x_i \neq \varepsilon. \end{cases}$$

Since $x_1\dots x_m = w = r_1\dots r_n$, we know that γ is surjective and $\gamma(m) = n$. We now show by induction on i that we have

$$v(\forall \mathbf{A}_r.C,d_0) \leqslant \min\Big\{ \min_{j=1}^{i} p_j, \min_{j=1}^{\gamma(i)} r_j^{\mathcal{I}^0_\mathsf{f}}(d_{j-1},d_j)\Big\} \Rightarrow v(\mathbf{A}_r^{q_i},d_{\gamma(i)}) \tag{5}$$

For all i, $0 \leqslant i \leqslant m$. By the axiom $\top \sqsubseteq \boxed{(\forall \mathbf{A}_r^{q_m}.C) \leqslant C}$ in $\mathsf{red}(\forall \mathbf{A}_r^{q_m}.C)$ and the induction hypothesis, the claim for m implies $(*)$.

For $i = 0$, (5) trivially holds. Assume now that it holds for $i-1$. To show the claim for i, by Proposition 2.1 it suffices to show that

$$v(\forall \mathbf{A}_r^{q_{i-1}}.C,d_{\gamma(i-1)}) \leqslant p_i \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C,d_{\gamma(i)}) \tag{6}$$

whenever $x_i = \varepsilon$ (and hence $\gamma(i) = \gamma(i-1)$), and

$$v(\forall \mathbf{A}_r^{q_{i-1}}.C,d_{\gamma(i-1)}) \leqslant \min\{p_i, r_{\gamma(i)}^{\mathcal{I}^0_\mathsf{f}}(d_{\gamma(i-1)},d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C,d_{\gamma(i)}) \tag{7}$$

for all $x_i \neq \varepsilon$ (for which we have $\gamma(i) = \gamma(i-1)+1$).

For the former case, the axioms in $\mathsf{red}_{\varepsilon,p_i,q_i}(\forall \mathbf{A}_r^{q_m}.C)$ and (P5) directly yield the claim (6). If $x_i \notin \varepsilon$, we make a case distinction on $d_{\gamma(i)}$. If $r_{\gamma(i)}^{\mathcal{I}^0_\mathsf{f}}(d_{\gamma(i-1)},d_{\gamma(i)}) = 0$, then the claim is trivially satisfied; otherwise, we must have one of the following cases:

- If $d_{\gamma(i)} = (a, u)$ and $d_{\gamma(i-1)} = (a, u_\uparrow)$, then we have $(d_{\gamma(i-1)}, d_{\gamma(i)}) \in \mathfrak{r}^\mathcal{I}$ and $d_{\gamma(i)} \in \mathsf{AN}^\mathcal{I}$. Hence, the third axiom in $\mathsf{red}_{r_{\gamma(i)},p_i,q_i}(\forall \mathbf{A}_r^{q_{i-1}}.C)$ and (P5) yield

$$
\begin{aligned}
v(\forall \mathbf{A}_r^{q_{i-1}}.C, d_{\gamma(i-1)}) &= v(\langle \forall \mathbf{A}_r^{q_{i-1}}.C\rangle_\uparrow, d_{\gamma(i)}) \\
&\leqslant \min\{p, v(r_{\gamma(i)}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}) \\
&= \min\{p, r_{\gamma(i)}^{\mathcal{I}_\mathsf{f}^0}(d_{\gamma(i-1)}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}),
\end{aligned}
$$

as claimed in (7).

- If $d_{\gamma(i-1)} = (a, u)$ and $d_{\gamma(i)} = (a, u_\uparrow)$, then $d_{\gamma(i-1)} \in \mathsf{AN}^\mathcal{I}$ and inverse roles are allowed. Hence, the first axiom in $\mathsf{red}_{r_{\gamma(i)},p_i,q_i}(\forall \mathbf{A}_r^{q_{i-1}}.C)$ and (P5) yield

$$
\begin{aligned}
v(\forall \mathbf{A}_r^{q_{i-1}}.C, d_{\gamma(i-1)}) &\leqslant \min\{p, v(r_{\gamma(i)}^-, d_{\gamma(i-1)})\} \Rightarrow v(\langle \forall \mathbf{A}_r^{q_i}.C\rangle_\uparrow, d_{\gamma(i-1)}) \\
&= \min\{p, r_{\gamma(i)}^{\mathcal{I}_\mathsf{f}^0}(d_{\gamma(i-1)}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}).
\end{aligned}
$$

- If $d_{\gamma(i-1)} = d_{\gamma(i)}$, then (local) reflexivity is allowed, and we similarly get

$$
\begin{aligned}
v(\forall \mathbf{A}_r^{q_{i-1}}.C, d_{\gamma(i-1)}) &\leqslant \min\{p, v(\exists r_{\gamma(i)}.\mathsf{Self}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}) \\
&= \min\{p, r_{\gamma(i)}^{\mathcal{I}_\mathsf{f}^0}(d_{\gamma(i-1)}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)})
\end{aligned}
$$

by the second axiom in $\mathsf{red}_{r_{\gamma(i)},p_i,q_i}(\forall \mathbf{A}_r^{q_{i-1}}.C)$.

- If $d_{\gamma(i-1)} = (a, \varepsilon)$ and $d_{\gamma(i)} = (b, \varepsilon)$, then $((a, \varepsilon), (b, \varepsilon)) \in \mathfrak{r}^\mathcal{I}$, and thus

$$
\begin{aligned}
v(\forall \mathbf{A}_r^{q_{i-1}}.C, d_{\gamma(i-1)}) &\leqslant \min\{p, v((*, b){:}r_{\gamma(i)}, d_{\gamma(i-1)})\} \Rightarrow v(b{:}(\forall \mathbf{A}_r^{q_i}.C), d_{\gamma(i-1)}) \\
&= \min\{p, r_{\gamma(i)}^{\mathcal{I}_\mathsf{f}^0}(d_{\gamma(i-1)}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)})
\end{aligned}
$$

by the corresponding axiom in $\mathsf{red}_{r_{\gamma(i)},p_i,q_i}(\forall \mathbf{A}_r^{q_{i-1}}.C)$.

- If $d_{\gamma(i-1)} = (a, \varepsilon)$, $d_{\gamma(i)} \in \mathsf{AN}^\mathcal{I}$, and $(d_{\gamma(i)}, d_{\gamma(i-1)}) \in \mathfrak{r}^\mathcal{I}$, then nominals and inverse roles are allowed and

$$
\begin{aligned}
v(\forall \mathbf{A}_r^{q_{i-1}}.C, d_{\gamma(i-1)}) &= v(a{:}\forall \mathbf{A}_r^{q_{i-1}}, d_{\gamma(i)}) \\
&\leqslant \min\{p, v((*, a){:}r_{\gamma(i)}^-, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}) \\
&= \min\{p, r_{\gamma(i)}^{\mathcal{I}_\mathsf{f}^0}(d_{\gamma(i-1)}, d_{\gamma(i)})\} \Rightarrow v(\forall \mathbf{A}_r^{q_i}.C, d_{\gamma(i)}).
\end{aligned}
$$

- Finally, the case that $d_{\gamma(i-1)} \in \mathsf{AN}^\mathcal{I}$, $d_{\gamma(i)} = (b, \varepsilon)$, and $(d_{\gamma(i-1)}, d_{\gamma(i)}) \in \mathfrak{r}^\mathcal{I}$ can be handled as in the previous two cases.

This concludes the proof of (7), and hence that of (5) and of (∗), which shows that (4) holds for all value restrictions. The proof for existential restrictions can be done using dual arguments.

For at-least restrictions $\geqslant n\, r.C$, note first that r must be simple, and hence we have $r^{\mathcal{I}_\mathsf{f}}(d,e) = r^{\mathcal{I}_\mathsf{f}^0}(d,e)$ for all $d,e \in \Delta^{\mathcal{I}_\mathsf{f}}$. We first consider the case that $d \in \mathsf{AN}^{\mathcal{I}}$, i.e. it is of the form (a,u) with $u \neq \varepsilon$. By the first axiom in $\mathsf{red}(\geqslant n\, r.C)$, there are $z_i, z_s \in \{0,1\}$, $m \in \{0,\dots,n-z_i-z_s\}$ and $S \subseteq \mathsf{ind}(\mathcal{O})$ with $|S| = n-m-z_i-z_s$ such that $\mathsf{red}_{z_i,z_s,m,S,\leqslant}(\geqslant n\, r.C)$ is satisfied by d.

- If $z_i = 1$, then inverse roles are allowed and $d \in \boxed{(\geqslant n\, r.C) \leqslant \min\{r^-, \langle C\rangle_\uparrow\}}^{\mathcal{I}}$. Further, we obtain

$$\begin{aligned} v(\geqslant n\, r.C, d) &\leqslant \min\{v(r^-, d), v(\langle C\rangle_\uparrow, d)\} \\ &= \min\{r^{\mathcal{I}_\mathsf{f}}(d,(a,u_\uparrow)), C^{\mathcal{I}_\mathsf{f}}(a,u_\uparrow)\} \end{aligned}$$

  by the induction hypothesis.

- If $z_s = 1$, then local reflexivity is allowed and we have

$$\begin{aligned} v(\geqslant n\, r.C, d) &\leqslant \min\{v(\exists r.\mathsf{Self}, d), v(C, d)\} \\ &= \min\{r^{\mathcal{I}_\mathsf{f}}(d,d), C^{\mathcal{I}_\mathsf{f}^0}(d)\}. \end{aligned}$$

- Consider now any $b \in \mathsf{ind}(\mathcal{O})$. We know that nominals are allowed and $(d,(b,\varepsilon)) \in \mathfrak{r}^{\mathcal{I}}$, and thus

$$\begin{aligned} v(\geqslant n\, r.C, d) &\leqslant \min\{v((*,b){:}r, d), v(b{:}C, d)\} \\ &= \min\{r^{\mathcal{I}_\mathsf{f}}(d,(b,\varepsilon)), C^{\mathcal{I}_\mathsf{f}}(b,\varepsilon)\}. \end{aligned}$$

  Furthermore, all elements of S must be interpreted by different elements in $\mathcal{I}$, and hence also in $\mathcal{I}_\mathsf{f}$, even if we consider sets of individual names in the roots of $\mathcal{I}$.

- Additionally, there are m different elements $e_1,\dots,e_m \in \Delta^{\mathcal{I}}$ such that $(d,e_j) \in \mathfrak{r}^{\mathcal{I}}$ and $e_j \in \mathsf{AN}^{\mathcal{I}}$ for each e_j, and hence they must be of the form (a, ui_j). We obtain, for every j, $1 \leqslant j \leqslant m$,

$$\begin{aligned} v(\geqslant n\, r.C, d) &= v(\langle \geqslant n\, r.C\rangle_\uparrow, e_j) \\ &\leqslant \min\{v(r, e_j), v(C, e_j)\} \\ &= \min\{r^{\mathcal{I}_\mathsf{f}}(d,e_j), C^{\mathcal{I}_\mathsf{f}}(e_j)\}. \end{aligned}$$

Note that all r-successors of d considered above, i.e. $(a,u_\uparrow)$, d, (b,ε), and e_j, $1 \leqslant j \leqslant m$, must be different; in particular, we do not consider nominals and inverse roles at the same time (since obviously $\mathcal{O}$ contains at-least restrictions), and thus even if $u_\uparrow = \varepsilon$, we do not have $a \in S$. Hence, these elements can serve as the witnesses for $\geqslant n\, r.C$ at d (assuming that its value is exactly $v(\geqslant n\, r.C)$, which is shown below). For named domain elements, the witnesses are created by the first kind of assertions in $\mathsf{red}(\geqslant n\, r.C)$, where only two cases need to be considered (named and unnamed successors); note that all unnamed $\mathfrak{r}$-successors of (a, ε) must be of the form (a, i) due to our assumptions on the structure of $\mathcal{I}$.

Assume now again that $d = (a, u) \in \mathsf{AN}^{\mathcal{I}}$ and that the elements found above are not witnesses for $(\geqslant n\, r.C)^{\mathcal{I}_\mathsf{f}}(d) = v(\geqslant n\, r.C, d)$. Then there must be n different elements $e_1, \ldots, e_n \in \Delta^{\mathcal{I}_\mathsf{f}}$ such that

$$v(\geqslant n\, r.C, d) < \min\{r^{\mathcal{I}_\mathsf{f}}(d, e_j), C^{\mathcal{I}_\mathsf{f}}(e_j)\}$$

for all j, $1 \leqslant j \leqslant n$. We show that we can find suitable z_i, z_s, m, and S such that $\mathsf{red}_{z_i,z_s,m,S,<}(\geqslant n\, r.C)$ is satisfied by d, which contradicts our assumption that $\mathcal{I}$ is a model of $\mathsf{red}(\mathcal{O})$.

- If inverse roles are allowed and there is an index j, $1 \leqslant j \leqslant n$, such that $e_j = (a, u_\uparrow)$, then we set $z_i := 1$. By the induction hypothesis and our assumption above, we have

$$\begin{aligned} v(\geqslant n\, r.C, d) &< \min\{r^{\mathcal{I}_\mathsf{f}}(d, e_j), C^{\mathcal{I}_\mathsf{f}}(e_j)\} \\ &= \min\{v(r^-, d), v(C, e_j)\} \\ &= \min\{v(r^-, d), v(\langle C\rangle_\uparrow, d)\}. \end{aligned}$$

- If (local) reflexivity is allowed and there is an index j, $1 \leqslant j \leqslant n$, such that $e_j = d$, then we set $z_s := 1$, and get

$$v(\geqslant n\, r.C, d) < \min\{v(\exists r.\mathsf{Self}, d), v(C, d)\}.$$

- If nominals are allowed, then we collect from the remaining elements those e_j that are equal to (b, ε) for some $b \in \mathsf{ind}(\mathcal{O})$. Let S be the set all of all those individual names. Since they are interpreted by different elements in $\mathcal{I}_\mathsf{f}$, they are also distinct in $\mathcal{I}$, even if we consider sets of individual names in the roots of $\mathcal{I}$. Furthermore, for any $b \in S$, since $r^{\mathcal{I}_\mathsf{f}}(d, (b, \varepsilon)) > v(\geqslant n\, r.C) \geqslant 0$, we have $(d, (b, \varepsilon)) \in \mathfrak{r}^{\mathcal{I}}$ and

$$v(\geqslant n\, r.C, d) < \min\{v((*, b){:}r, d), v(b{:}C, d)\}.$$

- There are exactly $m := n - |S| - z_i - z_s$ remaining elements e_j. If nominals are not allowed, then no e_j can be of the form (b, ε) for $b \in \mathsf{ind}(\mathcal{O})$ since $r^{\mathcal{I}_\mathsf{f}}(d, e_j) > 0$ and d is anonymous. If inverse roles are not allowed, then $e_j \neq (a, u_\uparrow)$ due to the same reason. Similarly, if local reflexivity is not allowed, it cannot be the case that $e_j = d$. Thus, we know for each of the remaining e_j that $e_j = (a, ui_j)$ for some $i_j \in \mathbb{N}$ and

$$v(\langle \geqslant n\, r.C\rangle_\uparrow, e_j) = v(\geqslant n\, r.C, d) < \min\{v(r, e_j), v(C, e_j)\}.$$

  This shows that also the final part of $\mathsf{red}_{z_i,z_s,m,S,<}(\geqslant n\, r.C)$, namely the restriction $\geqslant m\, \mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle \geqslant n\, r.C\rangle_\uparrow < \min\{r, C\}}\big)$ is satisfied.

For named elements $d = (a, \varepsilon)$, we can use a similar argument to contradict the second kind of assertions in $\mathsf{red}(\geqslant n\, r.C)$. Note that there can be no anonymous element e_j satisfying $r^{\mathcal{I}_\mathsf{f}}(d, e_j) > 0$ that is not of the form $e_j = (a, i_j)$ for some $i_j \in \mathbb{N}$, since otherwise we know from the definition of $r^{\mathcal{I}_\mathsf{f}}$ that both inverse roles and nominals must be allowed, which cannot be the case since obviously number restrictions are allowed.

This concludes the proof of (4). It remains to show that $\mathcal{I}_\mathsf{f}$ is a model of $\mathcal{O}$. For every $\langle \alpha \bowtie \beta \rangle \in \mathcal{A}$, we have $v(\alpha) \bowtie v(\beta)$ by $\mathsf{red}(\mathcal{A})$ and (P2). In the case that $\alpha = q \in \mathcal{V}_\mathcal{O}$, we know that $v(\alpha) = q$ by (P5)a); and if $\alpha = a{:}C$, then $v(\alpha) = v(C, a, \varepsilon) = C^{\mathcal{I}_\mathsf{f}}(a, \varepsilon) = C^{\mathcal{I}_\mathsf{f}}(a^{\mathcal{I}_\mathsf{f}})$ by (P4) and (4).[7] Since the same holds for β, we conclude that $\alpha^{\mathcal{I}_\mathsf{f}} \bowtie \beta^{\mathcal{I}_\mathsf{f}}$.

All (in)equality assertions $a \approx b$ ($a \not\approx b$) in $\mathcal{A}$ are satisfied due to $\mathsf{red}(\mathcal{A})$ and the construction of $\mathcal{I}_\mathsf{f}$.

Consider any GCI $\langle C \sqsubseteq D \geqslant p \rangle \in \mathcal{T}$ and $d \in \Delta^\mathcal{I}$. By $\mathsf{red}(\mathcal{T})$ and (P5)b), we have $v(p, d) \leqslant v(C, d) \Rightarrow v(D, d)$. Thus, (4) and (P5)a) yield $C^{\mathcal{I}_\mathsf{f}}(d) \Rightarrow D^{\mathcal{I}_\mathsf{f}}(d) \geqslant p$.

For $\langle \mathsf{ref}(r) \geqslant p \rangle \in \mathcal{R}$, by $\mathsf{red}(\mathcal{R})$ we have $v(\exists r.\mathsf{Self}, d) \geqslant p$ for all $d \in \Delta^{\mathcal{I}_\mathsf{f}}$, and hence $r^{\mathcal{I}_\mathsf{f}}(d, d) \geqslant p$, as required.

For any $\langle \mathsf{dis}(r, s) \geqslant p \rangle \in \mathcal{R}$, r and s are simple, and thus we can restrict our analysis to $r^{\mathcal{I}_\mathsf{f}^0}$ and $s^{\mathcal{I}_\mathsf{f}^0}$. We have $\min\{v((a, b){:}r, c, u), v((a, b){:}s, c, u)\} \leqslant 1 - p$ for all $(c, u) \in \Delta^{\mathcal{I}_\mathsf{f}}$ and $a, b \in \mathsf{ind}(\mathcal{O}) \cup \{*\}$. Hence, $\min\{r^{\mathcal{I}_\mathsf{f}}(a^{\mathcal{I}_\mathsf{f}}, b^{\mathcal{I}_\mathsf{f}}), s^{\mathcal{I}_\mathsf{f}}(a^{\mathcal{I}_\mathsf{f}}, b^{\mathcal{I}_\mathsf{f}})\} \leqslant 1 - p$, where $*^{\mathcal{I}_\mathsf{f}} := (c, u)$. This takes care of all role connections involving named domain elements. Furthermore, we obtain $\min\{v(r, c, u), v(s, c, u)\} \leqslant 1 - p$ in case $u \neq \varepsilon$, and thus $\min\{r^{\mathcal{I}_\mathsf{f}}((c, u_\uparrow), (c, u)), s^{\mathcal{I}_\mathsf{f}}((c, u_\uparrow), (c, u))\} \leqslant 1 - p$. Similarly, if inverse roles are allowed, then $\min\{v(r^-, c, u), v(s^-, c, u)\} \leqslant 1 - p$, and hence $\min\{r^{\mathcal{I}_\mathsf{f}}((c, u), (c, u_\uparrow)), s^{\mathcal{I}_\mathsf{f}}((c, u), (c, u_\uparrow))\} \leqslant 1 - p$. Finally, we know that $\min\{v(\exists r.\mathsf{Self}, c, u), v(\exists s.\mathsf{Self}, c, u)\} \leqslant 1 - p$, which implies that we also have $\min\{r^{\mathcal{I}_\mathsf{f}}((c, u), (c, u)), s^{\mathcal{I}_\mathsf{f}}((c, u), (c, u))\} \leqslant 1 - p$.

For the complex role inclusions in $\mathcal{R}$, by Lemma 3.4 it suffices to show that $w^{\mathcal{I}_\mathsf{f}}(d, e) \Rightarrow r^{\mathcal{I}_\mathsf{f}}(d, e) \geqslant (\|\mathbf{A}_r\|, w)$ holds for all $r \in \mathsf{rol}(\mathcal{O})$, $w \in \mathsf{rol}(\mathcal{O})^+$, and $d, e \in \Delta^{\mathcal{I}_\mathsf{f}}$. We can assume that $w^{\mathcal{I}_\mathsf{f}}(d, e) > 0$ and $(\|\mathbf{A}_r\|, w) > 0$ since otherwise this inequation is trivially satisfied. Let now $w = r_1 \dots r_n$, $n \geqslant 1$. Then we have

$$
\begin{aligned}
w^{\mathcal{I}_\mathsf{f}}(d, e) &= \sup_{d_1, \dots, d_{n-1} \in \Delta^{\mathcal{I}_\mathsf{f}}} \min_{i=1}^{n} r_i^{\mathcal{I}_\mathsf{f}}(d_{i-1}, d_i) \\
&= \sup_{d_1, \dots, d_{n-1} \in \Delta^{\mathcal{I}_\mathsf{f}}} \min_{i=1}^{n} \sup_{w_i \in \mathsf{rol}(\mathcal{O})^+} \min\{(\|\mathbf{A}_{r_i}\|, w_i), w_i^{\mathcal{I}_\mathsf{f}^0}(d_{i-1}, d_i)\} \\
&= \sup_{d_1, \dots, d_{n-1} \in \Delta^{\mathcal{I}_\mathsf{f}}} \sup_{w_1, \dots, w_n \in \mathsf{rol}(\mathcal{O})^+} \min_{i=1}^{n} \min\{(\|\mathbf{A}_{r_i}\|, w_i), w_i^{\mathcal{I}_\mathsf{f}^0}(d_{i-1}, d_i)\},
\end{aligned}
$$

where we set $d_0 := d$ and $d_n := e$. Furthermore, for any choice of elements

[7] Recall that we have eliminated all crisp role assertions from the ABox.

$d_1, \ldots, d_{n-1} \in \Delta^{\mathcal{I}_\mathsf{f}}$ and words $w_1, \ldots, w_n \in \mathsf{rol}(\mathcal{O})^+$, we have

$$\begin{aligned} r^{\mathcal{I}_\mathsf{f}}(d,e) &\geqslant \min\{(\|\mathbf{A}_r\|, w_1 \ldots w_n), (w_1 \ldots w_n)^{\mathcal{I}_\mathsf{f}^0}(d,e)\} \\ &\geqslant \min\left\{(\|\mathbf{A}_r\|, w_1 \ldots w_n), \min_{i=1}^{n} w_i^{\mathcal{I}_\mathsf{f}^0}(d_{i-1}, d_i)\right\} \\ &\geqslant \min\left\{(\|\mathbf{A}_r\|, w), \min_{i=1}^{n} \min\{(\|\mathbf{A}_{r_i}\|, w_i), w_i^{\mathcal{I}_\mathsf{f}^0}(d_{i-1}, d_i)\}\right\} \end{aligned}$$

by the construction of $\mathbf{A}_r$. Hence,

$$\begin{aligned} (\|\mathbf{A}_r\|, w) \Rightarrow r^{\mathcal{I}_\mathsf{f}}(d,e) & \\ &\geqslant \sup_{d_1,\ldots,d_{n-1} \in \Delta^{\mathcal{I}_\mathsf{f}}} \sup_{w_1,\ldots,w_n \in \mathsf{rol}(\mathcal{O})^+} \min_{i=1}^{n} \min\{(\|\mathbf{A}_{r_i}\|, w_i), w_i^{\mathcal{I}_\mathsf{f}^0}(d_{i-1}, d_i)\} \\ &= w^{\mathcal{I}_\mathsf{f}}(d,e), \end{aligned}$$

as required.

4.2 Completeness

Conversely, we now show that, in $\mathcal{SRIQ}$, $\mathcal{SROQ}$, or $\mathcal{SROI}$, if $\mathcal{O}$ has a G-model, then $\mathsf{red}(\mathcal{O})$ has a classical model.

Given a G-model $\mathcal{I}$ of $\mathcal{O}$, we construct the classical interpretation $\mathcal{I}_\mathsf{c}$, whose domain consists of all sequences of the form $ad_1 \ldots d_k$, where

- $a \in \mathsf{N_I}$ and $k \geqslant 0$;
- all d_i are elements of $\Delta^{\mathcal{I}}$;
- if number restrictions are allowed, then we have to put some restrictions on this sequence of domain elements:
  - d_1 is not equal to $b^{\mathcal{I}}$ for any $b \in \mathsf{N_I}$;
  - if reflexivity is allowed, there is no index i such that $d_i = d_{i+1}$;
  - if nominals are allowed, then no d_i is equal to $b^{\mathcal{I}}$ for any $b \in \mathsf{N_I}$; and
  - if inverse roles are allowed, then $d_2 \neq a^{\mathcal{I}}$, and there is no index i such that $d_i = d_{i+2}$.

For ease of presentation, we assume that all individual names are interpreted by distinct elements of $\Delta^{\mathcal{I}}$. In general, however, we would have to consider equivalence classes of individual names as roots for $\mathcal{I}_\mathsf{c}$, where $a, b \in \mathsf{N_I}$ are equivalent iff $a^{\mathcal{I}} = b^{\mathcal{I}}$. Since this is only relevant for number restrictions, we will ignore this for most of the proof and only mention it at the appropriate places.

We now set $a^{\mathcal{I}_\mathsf{c}} := a$ for all $a \in \mathsf{N_I}$, and

$$\mathfrak{r}^{\mathcal{I}_\mathsf{c}} := \{(\varrho, \varrho d) \mid \varrho d \in \Delta^{\mathcal{I}_\mathsf{c}}\} \cup \{(a,b) \mid a,b \in \mathsf{N_I}\} \cup \begin{cases} \{(\varrho, a) \mid \varrho \in \Delta^{\mathcal{I}_\mathsf{c}}, a \in \mathsf{N_I}\} & \text{if nominals are present,} \\ \emptyset & \text{otherwise.} \end{cases}$$

We denote by $\mathsf{tail}(ad_1 \ldots d_k)$ the element d_k if $k > 0$, and $a^{\mathcal{I}}$ if $k = 0$. Similarly, we set $\mathsf{prev}(ad_1 \ldots d_k)$ to d_{k-1} if $k > 1$, and to $a^{\mathcal{I}}$ if $k = 1$. For any $\alpha \in \mathcal{U}$ and $\varrho \in \Delta^{\mathcal{I}_\mathsf{c}}$, we define

$$\alpha^{\mathcal{I}}(\varrho) := \begin{cases} C^{\mathcal{I}}(\mathsf{tail}(\varrho)) & \text{if } \alpha = C \in \mathsf{sub}(\mathcal{O}); \\ C^{\mathcal{I}}(\mathsf{prev}(\varrho)) & \text{if } \alpha = \langle C \rangle_\uparrow \text{ for } C \in \mathsf{sub}(\mathcal{O}); \\ q & \text{if } \alpha = q \in \mathcal{V}_\mathcal{O}; \\ -1 & \text{if } \alpha = r \text{ and } \varrho \in \mathsf{N_I}. \\ r^{\mathcal{I}}(\mathsf{prev}(\varrho), \mathsf{tail}(\varrho)) & \text{if } \alpha = r \text{ and } \varrho \notin \mathsf{N_I}; \\ C^{\mathcal{I}}(a^{\mathcal{I}}) & \text{if } \alpha = a{:}C; \\ r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) & \text{if } \alpha = (a,b){:}r; \\ r^{\mathcal{I}}(a^{\mathcal{I}}, \mathsf{tail}(\varrho)) & \text{if } \alpha = (a,*){:}r; \\ r^{\mathcal{I}}(\mathsf{tail}(\varrho), a^{\mathcal{I}}) & \text{if } \alpha = (*,a){:}r; \\ 1 - \mathsf{inv}(\alpha)^{\mathcal{I}}(\varrho) & \text{if } \alpha \text{ involves a negated role } \neg r, \end{cases}$$

where

$$(\forall \mathbf{A}.C)^{\mathcal{I}}(d) := \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{(\|\mathbf{A}\|, w), w^{\mathcal{I}}(d,e)\} \Rightarrow C^{\mathcal{I}}(e),$$

where $\varepsilon^{\mathcal{I}}(d,e) := 1$ if $d = e$, and $\varepsilon^{\mathcal{I}}(d,e) := 0$ otherwise. Note that $\alpha^{\mathcal{I}}(\varrho)$ is not defined for $\alpha \in \mathsf{sub}_\uparrow(\mathcal{O})$ if $\varrho \in \mathsf{N_I}$. We fix these values $\alpha^{\mathcal{I}}(\varrho)$ arbitrarily, in such a way that for all $\alpha, \beta \in \mathcal{U}$ we have $\alpha^{\mathcal{I}}(\varrho) \leqslant \beta^{\mathcal{I}}(\varrho)$ iff $\mathsf{inv}(\beta)^{\mathcal{I}}(\varrho) \leqslant \mathsf{inv}(\alpha)^{\mathcal{I}}(\varrho)$. We now define $\mathsf{AN}^{\mathcal{I}_\mathsf{c}} := \Delta^{\mathcal{I}_\mathsf{c}} \setminus \mathsf{N_I}$ and, for all concept names $\boxed{\alpha \leqslant \beta}$ with $\alpha, \beta \in \mathcal{U}$,

$$\boxed{\alpha \leqslant \beta}^{\mathcal{I}_\mathsf{c}} := \{\varrho \mid \alpha^{\mathcal{I}}(\varrho) \leqslant \beta^{\mathcal{I}}(\varrho)\}.$$

It is easy to see that we also have $\varrho \in \boxed{\alpha \bowtie \beta}^{\mathcal{I}_\mathsf{c}}$ iff $\alpha^{\mathcal{I}}(\varrho) \bowtie \beta^{\mathcal{I}}(\varrho)$ for all other order expressions $\bowtie$, and that $\mathcal{I}_\mathsf{c}$ satisfies $\mathsf{red}(\mathsf{AN})$ and $\mathsf{red}(\mathcal{U})$. We now show that $\mathcal{I}_\mathsf{c}$ satisfies the remaining parts of $\mathsf{red}(\mathcal{O})$.

For any $\langle \alpha \bowtie \beta \rangle \in \mathcal{A}$, we have $\alpha^{\mathcal{I}} \bowtie \beta^{\mathcal{I}}$ since $\mathcal{I}$ satisfies $\mathcal{A}$. From this it follows that $\alpha^{\mathcal{I}}(c) \bowtie \beta^{\mathcal{I}}(c)$ for any $c \in \mathsf{ind}(\mathcal{O})$. All (in)equalities $a \approx b$ ($a \not\approx b$) in $\mathsf{red}(\mathcal{A})$ are satisfied if we consider the generalized construction with equivalence classes of individual names as roots for $\mathcal{I}_\mathsf{c}$. It is straightforward to verify the remaining axioms in $\mathsf{red}(\mathcal{A})$.

For any GCI $\langle C \sqsubseteq D \geqslant p\rangle \in \mathcal{T}$ and every $\varrho \in \Delta^{\mathcal{I}_\mathsf{c}}$, we know that

$$C^{\mathcal{I}}(\varrho) \Rightarrow D^{\mathcal{I}}(\varrho) = C^{\mathcal{I}}(\mathsf{tail}(\varrho)) \Rightarrow D^{\mathcal{I}}(\mathsf{tail}(\varrho)) \geqslant p,$$

and hence $\varrho \in \boxed{p \leqslant C \Rightarrow D}^{\mathcal{I}_\mathsf{c}}$.

For $\mathsf{red}(\uparrow)$, consider any $\varrho \in \boxed{\alpha \bowtie \beta}^{\mathcal{I}_\mathsf{c}}$ and $\varrho' \in \mathsf{AN}$ with $(\varrho, \varrho') \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$. Then ϱ' must be of the form ϱd for some $d \in \Delta^{\mathcal{I}}$, and we have

$$\langle\alpha\rangle_{\uparrow}^{\mathcal{I}}(\varrho d) = \alpha^{\mathcal{I}}(\mathsf{tail}(\varrho)) \bowtie \beta^{\mathcal{I}}(\mathsf{tail}(\varrho)) = \langle\beta\rangle_{\uparrow}^{\mathcal{I}}(\varrho d),$$

and hence $\varrho' \in \boxed{\langle\alpha\rangle_{\uparrow} \bowtie \langle\beta\rangle_{\uparrow}}^{\mathcal{I}_\mathsf{c}}$.

For $\mathsf{red}(\mathcal{R})$, consider first a role inclusion of the form $\langle r \sqsubseteq s \geqslant p\rangle \in \mathcal{R}$. Then $r^{\mathcal{I}}(\mathsf{prev}(\varrho), \mathsf{tail}(\varrho)) \Rightarrow s^{\mathcal{I}}(\mathsf{prev}(\varrho), \mathsf{tail}(\varrho)) \geqslant p$ for any $\varrho \in \Delta^{\mathcal{I}_\mathsf{c}} \setminus \mathsf{N_I}$; furthermore, every $a \in \mathsf{N_I}$ also satisfies $\boxed{r \Rightarrow s \geqslant p}$ since $-1 \leqslant -1$. A similar argument can be made for $\boxed{(a,b){:}r \Rightarrow (a,b){:}s \geqslant p}$ and $\boxed{\exists r.\mathsf{Self} \Rightarrow \exists s.\mathsf{Self} \geqslant p}$, and for the reduction of disjoint role axioms. For every $\langle \mathsf{ref}(r) \geqslant p\rangle \in \mathcal{R}$ and every $d \in \Delta^{\mathcal{I}}$, we have $(\exists r.\mathsf{Self})^{\mathcal{I}}(d) = r^{\mathcal{I}}(d,d) \geqslant p$. Finally, the three concepts for the universal role are obviously also satisfied at every domain element.

It remains to consider the axioms in $\mathsf{red}(C)$ for $C \in \mathsf{sub}(\mathcal{O})$. The reductions for $\top$, $\{a\}$, $\overline{q}$, $\exists r.\mathsf{Self}$, $\sqcap$, and $\to$ obviously reflect the semantics of these constructors and are easy to verify.

We now consider the axioms in $\mathsf{red}(\forall r.C)$. By Lemma 3.4, we have

$$\begin{aligned}(\forall \mathbf{A}_r.C)^{\mathcal{I}}(d) &= \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{(\|\mathbf{A}_r\|, w), w^{\mathcal{I}}(d,e)\} \Rightarrow C^{\mathcal{I}}(e) \\ &\geqslant \inf_{e \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(d,e) \Rightarrow C^{\mathcal{I}}(e) \\ &= (\forall r.C)^{\mathcal{I}}(d)\end{aligned}$$

for all $d \in \Delta^{\mathcal{I}}$. Lemma 3.4 talks only about $w \in \mathsf{rol}(\mathcal{O})^+$, but it holds also for $w = \varepsilon$ since then $(\|\mathbf{A}_r\|, w) = 0$ due to the construction of $\mathbf{A}_r$. Hence, the axiom $\top \sqsubseteq \boxed{(\forall r.C) \leqslant (\forall \mathbf{A}_r.C)}$ is satisfied by $\mathcal{I}_\mathsf{c}$. We consider the axiom

$$\begin{aligned}\mathsf{AN} \sqsubseteq\ & \boxed{(\forall r.C) \geqslant r^- \Rightarrow \langle C\rangle_{\uparrow}} \sqcup \\ & \boxed{(\forall r.C) \geqslant (\exists r.\mathsf{Self}) \Rightarrow C} \sqcup \\ & \exists \mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle\forall r.C\rangle_{\uparrow} \geqslant r \Rightarrow C}\big) \sqcup \\ & \bigsqcup_{a \in \mathsf{ind}(\mathcal{O})} \big(\exists \mathfrak{r}.\{a\} \sqcap \boxed{(\forall r.C) \geqslant (*, a){:}r \Rightarrow a{:}C}\big),\end{aligned}$$

where the first disjunct is only present if inverse roles are considered, likewise for the second disjunct and (local) reflexivity, and the last disjunct is contingent on the presence of nominals. Let further $\varrho \in \mathsf{AN}^{\mathcal{I}_\mathsf{c}}$, i.e. $\varrho = ad_1 \dots d_k$ with $k \geqslant 1$. Since $\mathcal{I}$ is witnessed, there is an $e \in \Delta^{\mathcal{I}}$ with $(\forall r.C)^{\mathcal{I}}(d_k) = r^{\mathcal{I}}(d_k, e) \Rightarrow C^{\mathcal{I}}(e)$. If $\varrho e \in \Delta^{\mathcal{I}_\mathsf{c}}$, then $(\varrho, \varrho e) \in \mathfrak{r}^{\mathcal{I}}$ and $\varrho e \in \mathsf{AN}^{\mathcal{I}_\mathsf{c}}$. Furthermore,

$$\langle\forall r.C\rangle_{\uparrow}^{\mathcal{I}}(\varrho e) = r^{\mathcal{I}}(d_k, e) \Rightarrow C^{\mathcal{I}}(e) = r^{\mathcal{I}}(\varrho e) \Rightarrow C^{\mathcal{I}}(\varrho e),$$

and hence $\varrho e \in \boxed{\langle\forall r.C\rangle_\uparrow \geqslant r \Rightarrow C}^{\mathcal{I}_\mathsf{c}}$. Otherwise, i.e. in the case that $\varrho e \notin \Delta^{\mathcal{I}_\mathsf{c}}$, there are three cases to consider:

- Reflexivity is allowed and $e = d_k$. Then
$$(\forall r.C)^{\mathcal{I}}(\varrho) = r^{\mathcal{I}}(d_k, d_k) \Rightarrow C^{\mathcal{I}}(d_k) = (\exists r.\mathsf{Self})^{\mathcal{I}}(\varrho) \Rightarrow C^{\mathcal{I}}(\varrho).$$
- Nominals are allowed and $e = b^{\mathcal{I}}$ for some $b \in \mathsf{N_I}$. Then we have $(\varrho, b) \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$ and
$$(\forall r.C)^{\mathcal{I}}(\varrho) = r^{\mathcal{I}}(d_k, b^{\mathcal{I}}) \Rightarrow C^{\mathcal{I}}(b^{\mathcal{I}}) = ((*, b){:}r)^{\mathcal{I}}(\varrho) \Rightarrow (b{:}C)^{\mathcal{I}}(\varrho).$$
- Inverse roles are allowed and either (i) $k > 1$ and $d_{k-1} = b^{\mathcal{I}}$ or (ii) $k = 1$ and $a = b$. In both cases, we have $\mathsf{prev}(\varrho) = b^{\mathcal{I}}$, and hence
$$(\forall r.C)^{\mathcal{I}}(\varrho) = r^{\mathcal{I}}(k_m, \mathsf{prev}(\varrho)) \Rightarrow C^{\mathcal{I}}(\mathsf{prev}(\varrho)) = (r^-)^{\mathcal{I}}(\varrho) \Rightarrow \langle C\rangle_\uparrow^{\mathcal{I}}(\varrho).$$

Consider now the axiom
$$a{:}\exists\mathfrak{r}.\big((\mathsf{AN} \sqcap \boxed{\langle\forall r.C\rangle_\uparrow \geqslant r \Rightarrow C}) \sqcup (\neg\mathsf{AN} \sqcap \boxed{(a{:}\forall r.C) \geqslant (a,*){:}r \Rightarrow C})\big)$$
for some $a \in \mathsf{ind}(\mathcal{O})$. Since $\mathcal{I}$ is witnessed, there is an element $e \in \Delta^{\mathcal{I}}$ such that $(\forall r.C)^{\mathcal{I}}(a^{\mathcal{I}}) = r^{\mathcal{I}}(a^{\mathcal{I}}, e) \Rightarrow C^{\mathcal{I}}(e)$.

- If $e = b^{\mathcal{I}}$ for some $b \in \mathsf{N_I}$, then we have $(a, b) \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$ and $b \notin \mathsf{AN}^{\mathcal{I}_\mathsf{c}}$. Furthermore, $(a{:}\forall r.C)^{\mathcal{I}}(b) = (\forall r.C)^{\mathcal{I}}(a^{\mathcal{I}}) = r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}}) \Rightarrow C^{\mathcal{I}}(b^{\mathcal{I}})$, which is equal to $((a,*){:}r)^{\mathcal{I}}(b) \Rightarrow C^{\mathcal{I}}(b)$, and hence we have $b \in \boxed{(a{:}\forall r.C) \geqslant (a,*){:}r \Rightarrow C}^{\mathcal{I}_\mathsf{c}}$.
- If $e \neq b^{\mathcal{I}}$ for all $b \in \mathsf{N_I}$, then we have $ae \in \Delta^{\mathcal{I}_\mathsf{c}}$, and thus $(a, ae) \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$. Moreover, $ae \in \mathsf{AN}^{\mathcal{I}_\mathsf{c}}$ and $\langle\forall r.C\rangle_\uparrow^{\mathcal{I}}(ae) = (\forall r.C)^{\mathcal{I}}(a^{\mathcal{I}}) = r^{\mathcal{I}}(a^{\mathcal{I}}, e) \Rightarrow C^{\mathcal{I}}(e)$, which is equal to $r^{\mathcal{I}}(ae) \Rightarrow C^{\mathcal{I}}(ae)$.

For $\mathsf{red}(\forall\mathbf{A}^q.C)$, we first consider the axiom $\top \sqsubseteq \boxed{(\forall\mathbf{A}^q.C) \leqslant C}$ for a final state q of $\mathbf{A}$. We have
$$(\forall\mathbf{A}^q.C)^{\mathcal{I}}(d) \leqslant \min\{(\|\mathbf{A}^q\|, \varepsilon), \varepsilon^{\mathcal{I}}(d,d)\} \Rightarrow C^{\mathcal{I}}(d) = C^{\mathcal{I}}(d)$$
for all $d \in \Delta^{\mathcal{I}}$, and hence $\mathcal{I}_\mathsf{c}$ satisfies this axiom. For any transition $q \xrightarrow{\varepsilon,p} q'$ in $\mathbf{A}^q$, we have to satisfy the axiom $\top \sqsubseteq \boxed{(\forall\mathbf{A}^q.C) \leqslant p \Rightarrow (\forall\mathbf{A}^{q'}.C)}$. By Proposition 3.2, we get
$$\begin{aligned}(\forall\mathbf{A}^q.C)^{\mathcal{I}}(d) &= \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{(\|\mathbf{A}^q\|, w), w^{\mathcal{I}}(d,e)\} \Rightarrow C^{\mathcal{I}}(e)\\ &\leqslant \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{p, (\|\mathbf{A}^{q'}\|, w), w^{\mathcal{I}}(d,e)\} \Rightarrow C^{\mathcal{I}}(e)\\ &= p \Rightarrow \Big(\inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{(\|\mathbf{A}^{q'}\|, w), w^{\mathcal{I}}(d,e)\} \Rightarrow C^{\mathcal{I}}(e)\Big)\\ &= p \Rightarrow (\forall\mathbf{A}^{q'}.C)^{\mathcal{I}}(d).\end{aligned}$$

Consider now the axiom $\mathsf{AN} \sqsubseteq \boxed{(\forall \mathbf{A}^q.C) \leqslant \min\{p, s^-\} \Rightarrow \langle \forall \mathbf{A}^{q'}.C \rangle_{\uparrow}}$ for a transition $q \xrightarrow{r,p} q'$ in $\mathbf{A}$, and any $\varrho \in \mathsf{AN}^{\mathcal{I}_c}$, which must be of the form $ad_1 \dots d_k$ with $k \geqslant 1$. We have

$$\begin{aligned}
&(\forall \mathbf{A}^q.C)^{\mathcal{I}}(\varrho) \\
&\leqslant \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{(\|\mathbf{A}^q\|, sw), (sw)^{\mathcal{I}}(d_k, e)\} \Rightarrow C^{\mathcal{I}}(e) \\
&\leqslant \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\left\{p, (\|\mathbf{A}^{q'}\|, w), \sup_{e' \in \Delta^{\mathcal{I}}} \min\{s^{\mathcal{I}}(d_k, e'), w^{\mathcal{I}}(e', e)\}\right\} \Rightarrow C^{\mathcal{I}}(e) \\
&\leqslant \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{p, (\|\mathbf{A}^{q'}\|, w), s^{\mathcal{I}}(d_k, \mathsf{prev}(\varrho)), w^{\mathcal{I}}(\mathsf{prev}(\varrho), e)\} \Rightarrow C^{\mathcal{I}}(e) \\
&= \min\{p, (s^-)^{\mathcal{I}}(\varrho)\} \Rightarrow \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{(\|\mathbf{A}^{q'}\|, w), w^{\mathcal{I}}(\mathsf{prev}(\varrho), e)\} \Rightarrow C^{\mathcal{I}}(e) \\
&= \min\{p, (s^-)^{\mathcal{I}}(\varrho)\} \Rightarrow \langle \forall \mathbf{A}^{q'}.C \rangle_{\uparrow}^{\mathcal{I}}(\varrho).
\end{aligned}$$

by Propositions 2.1 and 3.2. For $\top \sqsubseteq \boxed{(\forall \mathbf{A}^q.C) \leqslant \min\{p, \exists s.\mathsf{Self}\} \Rightarrow (\forall \mathbf{A}^{q'}.C)}$ and any $d \in \Delta^{\mathcal{I}}$, we get

$$\begin{aligned}
&(\forall \mathbf{A}^q.C)^{\mathcal{I}}(d) \\
&\leqslant \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{p, (\|\mathbf{A}^{q'}\|, w), s^{\mathcal{I}}(d, d), w^{\mathcal{I}}(d, e)\} \Rightarrow C^{\mathcal{I}}(e) \\
&= \min\{p, (\exists s.\mathsf{Self})^{\mathcal{I}}(d)\} \Rightarrow (\forall \mathbf{A}^{q'}.C)^{\mathcal{I}}(d)
\end{aligned}$$

by similar arguments. For $\top \sqsubseteq \forall \mathfrak{r}.\big(\mathsf{AN} \to \boxed{\langle \forall \mathbf{A}^q.C \rangle_{\uparrow} \leqslant \min\{p, s\} \Rightarrow (\forall \mathbf{A}^{q'}.C)}\big)$, consider any $\varrho, \varrho' \in \Delta^{\mathcal{I}_c}$ with $(\varrho, \varrho') \in \mathfrak{r}^{\mathcal{I}_c}$ and $\varrho' \in \mathsf{AN}^{\mathcal{I}_c}$. Thus, we must have $\varrho' = \varrho d$ for some $d \in \Delta^{\mathcal{I}}$, and we know that $\mathsf{prev}(\varrho d) = \mathsf{tail}(\varrho)$. We obtain

$$\begin{aligned}
&\langle \forall \mathbf{A}^q.C \rangle_{\uparrow}^{\mathcal{I}}(\varrho d) \\
&\leqslant \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{p, (\|\mathbf{A}^{q'}\|, w), s^{\mathcal{I}}(\mathsf{tail}(\varrho), d), w^{\mathcal{I}}(d, e)\} \Rightarrow C^{\mathcal{I}}(e) \\
&= \min\{p, s^{\mathcal{I}}(\varrho d)\} \Rightarrow (\forall \mathbf{A}^{q'}.C)^{\mathcal{I}}(\varrho d).
\end{aligned}$$

Consider now the axiom $\exists \mathfrak{r}.\{a\} \sqsubseteq \boxed{(\forall \mathbf{A}^q.C) \leqslant \min\{p, (*, a){:}s\} \Rightarrow a{:}(\forall \mathbf{A}^{q'}.C)}$ for any $a \in \mathsf{ind}(\mathcal{O})$, and $\varrho \in \Delta^{\mathcal{I}_c}$ with $(\varrho, a) \in \mathfrak{r}^{\mathcal{I}_c}$. We get

$$\begin{aligned}
&(\forall \mathbf{A}^q.C)^{\mathcal{I}}(\varrho) \\
&\leqslant \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{p, (\|\mathbf{A}^{q'}\|, w), s^{\mathcal{I}}(\mathsf{tail}(\varrho), a^{\mathcal{I}}), w^{\mathcal{I}}(a^{\mathcal{I}}, e)\} \Rightarrow C^{\mathcal{I}}(e) \\
&= \min\{p, ((*, a){:}s)^{\mathcal{I}}(\varrho)\} \Rightarrow (a{:}(\forall \mathbf{A}^{q'}.C))^{\mathcal{I}}(\varrho).
\end{aligned}$$

Finally, for $\exists \mathfrak{r}.\{a\} \sqsubseteq \boxed{a{:}(\forall \mathbf{A}^q.C) \leqslant \min\{p, (*, a){:}s^-\} \Rightarrow (\forall \mathbf{A}^{q'}.C)}$ and any $a \in \mathsf{ind}(\mathcal{O})$ and $\varrho \in \Delta^{\mathcal{I}_c}$ with $(\varrho, a) \in \mathfrak{r}^{\mathcal{I}_c}$, we obtain

$$\begin{aligned}
&(a{:}(\forall \mathbf{A}^q.C))^{\mathcal{I}}(\varrho) \\
&\leqslant \inf_{w \in \mathsf{rol}(\mathcal{O})^*} \inf_{e \in \Delta^{\mathcal{I}}} \min\{p, (\|\mathbf{A}^{q'}\|, w), s^{\mathcal{I}}(a^{\mathcal{I}}, \mathsf{tail}(\varrho)), w^{\mathcal{I}}(\mathsf{tail}(\varrho), e)\} \Rightarrow C^{\mathcal{I}}(e) \\
&= \min\{p, ((*, a){:}s^-)^{\mathcal{I}}(\varrho)\} \Rightarrow (\forall \mathbf{A}^{q'}.C)^{\mathcal{I}}(\varrho).
\end{aligned}$$

This concludes the analysis of the reduction of value restrictions.

If number restrictions are present, we also have to satisfy $\mathsf{red}(\geqslant n\, r.C)$. Consider the first axiom and any $ad_1 \dots d_k \in \Delta^{\mathcal{I}_c}$ with $k \geqslant 1$. Since $\mathcal{I}$ is witnessed, there must be n different elements $e_1, \dots, e_n \in \Delta^{\mathcal{I}}$ such that

$$(\geqslant n\, r.C)^{\mathcal{I}}(d_k) = \min_{j=1}^{n} \min\{r^{\mathcal{I}}(d_k, e_j), C^{\mathcal{I}}(e_j)\}.$$

If reflexivity is allowed and we have $e_j = d_k$ for one of these elements, we set $z_s := 1$. If inverse roles are allowed and we have (i) $k > 1$ and $e_j = d_{k-1}$ or (ii) $k = 1$ and $e_j = a^{\mathcal{I}}$, we set $z_i := 1$. Note that the previous two elements identified for z_s and z_i must be different since otherwise we would have $d_k = d_{k-1}$ or $d_1 = a^{\mathcal{I}}$. If nominals are allowed, we define S to be the set of all individual names $b \in \mathsf{ind}(\mathcal{O})$ for which $b^{\mathcal{I}}$ is among the remaining elements from $e_1, \dots, e_n$; otherwise we set $S := \emptyset$. We thus have $m := n - z_i - z_s - |S|$ remaining elements e_j, and have uniquely identified one of the disjuncts of the axiom. We now show that for each of these elements e_j the corresponding conjunct in this disjunct is satisfied, thus showing that the whole axiom is satisfied.

- If $z_s = 1$, let e_j be the element equal to d_k. We have

  $$\begin{aligned}(\geqslant n\, r.C)^{\mathcal{I}}(\varrho) &\leqslant \min\{r^{\mathcal{I}}(d_k, d_k), C^{\mathcal{I}}(d_k)\} \\ &= \min\{(\exists r.\mathsf{Self})^{\mathcal{I}}(\varrho), C^{\mathcal{I}}(\varrho)\},\end{aligned}$$

  and thus the conjunct $\boxed{(\geqslant n\, r.C) \leqslant \min\{\exists r.\mathsf{Self}, C\}}$ is satisfied by ϱ.

- If $z_i = 1$, let e_j be the element equal to d_{k-1} or $a^{\mathcal{I}}$. Then $\mathsf{prev}(\varrho) = e_j$ and

  $$\begin{aligned}(\geqslant n\, r.C)^{\mathcal{I}}(\varrho) &\leqslant \min\{r^{\mathcal{I}}(d_k, e_j), C^{\mathcal{I}}(e_j)\} \\ &= \min\{(r^-)^{\mathcal{I}}(\varrho), \langle C\rangle_{\uparrow}^{\mathcal{I}}(\varrho)\},\end{aligned}$$

  validating the conjunct $\boxed{(\geqslant n\, r.C) \leqslant \min\{r^-, \langle C\rangle_{\uparrow}\}}$.

- Consider any $a \in S$. Then nominals are present, and thus $(\varrho, a) \in \mathfrak{r}^{\mathcal{I}}$ and

  $$\begin{aligned}(\geqslant n\, r.C)^{\mathcal{I}}(\varrho) &\leqslant \min\{r^{\mathcal{I}}(d_k, a^{\mathcal{I}}), C^{\mathcal{I}}(a^{\mathcal{I}})\} \\ &= \min\{((*, a){:}r)^{\mathcal{I}}(\varrho), (a{:}C)^{\mathcal{I}}(\varrho)\},\end{aligned}$$

  which corresponds to $\boxed{(\geqslant n\, r.C) \leqslant \min\{(*, a){:}r, a{:}C\}}$.

- For any e_j not corresponding to any of the previous cases, we know by the construction of $\Delta^{\mathcal{I}_c}$ that $\varrho e_j \in \Delta^{\mathcal{I}_c}$, and hence $(\varrho, \varrho e_j) \in \mathfrak{r}^{\mathcal{I}_c}$ and

  $$\begin{aligned}\langle \geqslant n\, r.C\rangle_{\uparrow}^{\mathcal{I}}(\varrho e_j) &\leqslant \min\{r^{\mathcal{I}}(d_k, e_j), C^{\mathcal{I}}(e_j)\} \\ &= \min\{r^{\mathcal{I}}(\varrho e_j), C^{\mathcal{I}}(\varrho e_j)\}.\end{aligned}$$

  Since there are m different such elements e_j, the corresponding elements ϱe_j are also different, and $\geqslant m\, \mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle \geqslant n\, r.C\rangle_{\uparrow} \leqslant \min\{r, C\}}\big)$ is satisfied by ϱ.

For the second axiom in $\mathsf{red}(\geqslant n\, r.C)$, assume to the contrary that there is a $\varrho = ad_1 \dots d_k \in \Delta^{\mathcal{I}_\mathsf{c}}$, and numbers z_i (if there are inverse roles), z_s (if reflexivity is allowed), $0 \leqslant m \leqslant n - z_i - z_s$, and a set $S \subseteq \mathsf{ind}(\mathcal{O})$ of cardinality $n - m - z_i - z_s$ (which is 0 unless there are nominals) such that the corresponding conjunction is satisfied by ϱ in $\mathcal{I}_\mathsf{c}$.

- If $z_i = 1$, then $\varrho \in \mathsf{AN}^{\mathcal{I}_\mathsf{c}}$, i.e. we have $k \geqslant 1$, and

$$\begin{aligned}(\geqslant n\, r.C)^{\mathcal{I}}(\mathsf{tail}(\varrho)) &= (\geqslant n\, r.C)^{\mathcal{I}}(\varrho) \\ &< \min\{(r^-)^{\mathcal{I}}(\varrho), \langle C\rangle^{\mathcal{I}}_{\uparrow}(\varrho)\} \\ &= \min\{r^{\mathcal{I}}(\mathsf{tail}(\varrho), \mathsf{prev}(\varrho)), C^{\mathcal{I}}(\mathsf{prev}(\varrho))\}.\end{aligned}$$

- If $z_s = 1$, then

$$\begin{aligned}(\geqslant n\, r.C)^{\mathcal{I}}(\mathsf{tail}(\varrho)) &< \min\{(\exists r.\mathsf{Self})^{\mathcal{I}}(\varrho), C^{\mathcal{I}}(\varrho)\} \\ &= \min\{r^{\mathcal{I}}(\mathsf{tail}(\varrho), \mathsf{tail}(\varrho)), C^{\mathcal{I}}(\mathsf{tail}(\varrho))\}.\end{aligned}$$

- For each $a \in S$, we have

$$\begin{aligned}(\geqslant n\, r.C)^{\mathcal{I}}(\mathsf{tail}(\varrho)) &< \min\{((*, a){:}r)^{\mathcal{I}}(\varrho), (a{:}C)^{\mathcal{I}}(\varrho)\} \\ &= \min\{r^{\mathcal{I}}(\mathsf{tail}(\varrho), a^{\mathcal{I}}), C^{\mathcal{I}}(a^{\mathcal{I}})\}.\end{aligned}$$

  Furthermore, even if we consider equivalence classes of individual names as roots for $\mathcal{I}_\mathsf{c}$, all $a \in S$ are interpreted by different domain elements.

- Additionally, there are m different $\mathfrak{r}$-successors $\varrho_1, \dots, \varrho_m$ that all satisfy AN, i.e. are of the form $\varrho e_1, \dots, \varrho e_m$ for different elements $e_1, \dots, e_m \in \Delta^{\mathcal{I}}$, and, for all $1 \leqslant j \leqslant m$,

$$\begin{aligned}(\geqslant n\, r.C)^{\mathcal{I}}(\mathsf{tail}(\varrho)) &= \langle \geqslant n\, r.C\rangle^{\mathcal{I}}_{\uparrow}(\varrho e_j) \\ &< \min\{r^{\mathcal{I}}(\varrho e_j), C^{\mathcal{I}}(\varrho e_j)\} \\ &= \min\{r^{\mathcal{I}}(\mathsf{tail}(\varrho), e_j), C^{\mathcal{I}}(e_j)\}.\end{aligned}$$

Due to the construction of $\Delta^{\mathcal{I}_\mathsf{c}}$, the above elements of $\Delta^{\mathcal{I}}$ ($\mathsf{prev}(\varrho)$, $\mathsf{tail}(\varrho)$, $a^{\mathcal{I}}$ for $a \in S$, and e_j, $1 \leqslant j \leqslant m$) must be different. But this contradicts the semantics of $(\geqslant n\, r.C)^{\mathcal{I}}(\mathsf{tail}(\varrho))$.

For the first kind of assertions in $\mathsf{red}(\geqslant n\, r.C)$, consider any $a \in \mathsf{ind}(\mathcal{O})$. Since $\mathcal{I}$ is witnessed, there must be n different elements $e_1, \dots, e_n \in \Delta^{\mathcal{I}}$ such that

$$(\geqslant n\, r.C)^{\mathcal{I}}(a^{\mathcal{I}}) = \min_{j=1}^{n} \min\{r^{\mathcal{I}}(a^{\mathcal{I}}, e_j), C^{\mathcal{I}}(e_j)\}.$$

For each e_j, $1 \leqslant j \leqslant n$, we make a case distinction on whether it is named or not.

- If $e_j = b^\mathcal{I}$ for some $b \in \mathsf{ind}(\mathcal{O})$, then we have $(a,b) \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$, $b \in (\neg\mathsf{AN})^{\mathcal{I}_\mathsf{c}}$, and

$$\begin{aligned}(a{:}{\geqslant}n\, r.C)^\mathcal{I}(b) &= ({\geqslant}n\, r.C)^\mathcal{I}(a^\mathcal{I}) \\ &\leqslant \min\{r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}), C^\mathcal{I}(b^\mathcal{I})\} \\ &= \min\{((a,*){:}r)^\mathcal{I}(b), C^\mathcal{I}(b)\}.\end{aligned}$$

- If $e_j \neq b^\mathcal{I}$ for all $b \in \mathsf{ind}(\mathcal{O})$, then we have $ae_j \in \Delta^{\mathcal{I}_\mathsf{c}}$, and thus $(a, ae_j) \in \mathfrak{r}^{\mathcal{I}_\mathsf{c}}$. Furthermore,

$$\begin{aligned}\langle{\geqslant}n\, r.C\rangle_\uparrow^\mathcal{I}(ae_j) &= ({\geqslant}n\, r.C)^\mathcal{I}(a^\mathcal{I}) \\ &\leqslant \min\{r^\mathcal{I}(a^\mathcal{I}, e_j), C^\mathcal{I}(e_j)\} \\ &= \min\{r^\mathcal{I}(ae_j), C^\mathcal{I}(ae_j)\}.\end{aligned}$$

Since different elements of $\Delta^\mathcal{I}$ induce different elements of $\Delta^{\mathcal{I}_\mathsf{c}}$, this shows that the required at-least restriction is satisfied by a.

For the second kind of assertions in $\mathsf{red}({\geqslant}n\, r.C)$, assume to the contrary that there are n different $\mathfrak{r}$-successors $\varrho_1, \dots, \varrho_n$ of a that are either anonymous and satisfy $\boxed{\langle{\geqslant}n\, r.C\rangle_\uparrow < \min\{r, C\}}$, or named and satisfy $\boxed{(a{:}{\geqslant}n\, r.C) < \min\{(a,*){:}r, C\}}$.

- If ϱ_j satisfies AN, then it must be of the form ae_j for some $e_j \in \Delta^\mathcal{I}$ and we have

$$\begin{aligned}({\geqslant}n\, r.C)^\mathcal{I}(a^\mathcal{I}) &= \langle{\geqslant}n\, r.C\rangle_\uparrow^\mathcal{I}(ae_j) \\ &< \min\{r^\mathcal{I}(ae_j), C^\mathcal{I}(ae_j)\} \\ &= \min\{r^\mathcal{I}(a^\mathcal{I}, e_j), C^\mathcal{I}(e_j)\}.\end{aligned}$$

- If ϱ_j does not satisfy AN, then it is of the form b and we obtain

$$\begin{aligned}({\geqslant}n\, r.C)^\mathcal{I}(a^\mathcal{I}) &= (a{:}{\geqslant}n\, r.C)^\mathcal{I}(b) \\ &< \min\{((a,*){:}r)^\mathcal{I}(b), C^\mathcal{I}(b)\} \\ &= \min\{r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}), C^\mathcal{I}(b^\mathcal{I})\}.\end{aligned}$$

  If we consider equivalence classes of individual names as roots for $\mathcal{I}_\mathsf{c}$, all such $b^\mathcal{I}$ are different.

This again contradicts the semantics of $({\geqslant}n\, r.C)^\mathcal{I}(a^\mathcal{I})$.

This concludes the proof of the following result.

Lemma 4.1. *In* G-$\mathcal{SRIQ}$, G-$\mathcal{SROQ}$, *or* G-$\mathcal{SROI}$, $\mathcal{O}$ *has a* G-*model iff* $\mathsf{red}(\mathcal{O})$ *has a classical model.*

4.3 Complexity

We now analyze the complexity of the reduction. As in [23], the construction of the automata $\mathbf{A}_r$ causes an exponential blowup in the size of $\mathcal{R}$, which cannot be avoided [26]. Independent of this, our reduction also involves an exponential blowup in the (binary encoding of) the largest number n involved in a number restriction in $\mathcal{O}$, and in the number of individual names occurring in $\mathcal{O}$, since the number of disjuncts in each GCI from $\mathsf{red}({\geqslant} n\, r.C)$ is linear in $n \cdot 2^{|\mathsf{ind}(\mathcal{O})|}$. Note, however, that this blowup only occurs if we consider both nominals and number restrictions. Hence, we obtain the following complexity results.

Theorem 4.2. *Deciding consistency is*

- 2-ExpTime-*complete in* G-$\mathcal{SRIQ}$*,*
- *in* 2-ExpTime *in* G-$\mathcal{SROI}$ *and* G-$\mathcal{SROQ}$*, and*
- ExpTime-*complete in all FDLs between* G-$\mathcal{ALC}$ *and* G-$\mathcal{SHOI}$ *or* G-$\mathcal{SHIQ}$.

Proof. The consistency of the $\mathcal{ALCOQ}$ ontology $\mathsf{red}(\mathcal{O})$ is decidable in exponential time in the size of $\mathsf{red}(\mathcal{O})$ [17]. The first upper bounds thus follow from the fact that the size of $\mathsf{red}(\mathcal{O})$ is exponential in the size of $\mathcal{O}$. 2-ExpTime-hardness holds already for G-$\mathcal{SRIQ}$ without involutive negation and only assertions of the form $\langle \alpha \geqslant p \rangle$ since in this case reasoning in G-$\mathcal{SRIQ}$ is equivalent to reasoning in classical $\mathcal{SRIQ}$ [13, 26].

Without complex role inclusions, i.e. restricting to simple role inclusions and transitivity axioms, the size of the automata $\mathbf{A}_r$ is polynomial in the size of $\mathcal{R}$ [23]. The other exponential blowup can be avoided by disallowing nominals or number restrictions. Hence, for G-$\mathcal{SHOI}$ and G-$\mathcal{SHIQ}$, the size of $\mathsf{red}(\mathcal{O})$ is polynomial in the size of $\mathcal{O}$, and the lower bound follows again from the reduction in [13] and ExpTime-hardness of consistency in classical $\mathcal{ALC}$ [29]. □

To the best of our knowledge, it is still open whether consistency in $\mathcal{SROI}$ and $\mathcal{SROQ}$ is actually 2-ExpTime-hard, even in the classical case [17, 28]; the best known lower bound is given by the ExpTime-hardness for $\mathcal{ALC}$ [29]. We also leave open the precise complexity of G-$\mathcal{SHOQ}$, which is ExpTime-complete in the classical case [17, 29].

5 Conclusions

Using a combination of techniques developed for infinitely valued Gödel extensions of $\mathcal{ALC}$ [12] and for finitely valued Gödel extensions of $\mathcal{SROIQ}$ [6, 7], we derived several tight complexity bounds for consistency in sublogics of G-$\mathcal{SROIQ}$.

Our reduction is more practical than the automata-based approach in [12] and does not exhibit the exponential blowup of the reduction from [6, 7]. However, it introduces a new kind of exponential blowup in the size of the binary encoding of numbers in number restrictions and the number of individual names occurring in the ontology. Beyond the complexity results, an important benefit of our approach is that it does not need the development of a specialized fuzzy DL reasoner, but can use any state-of-the-art reasoner for classical $\mathcal{ALCOQ}$ without modifications. For that reason, this new reduction aids in closing the gap between efficient classical and fuzzy DL reasoners.

A promising direction for future research is to integrate our reduction directly into a classical tableaux procedure. Observe that the axioms in $\mathsf{red}(C)$ are already closely related to the rules employed in (classical and fuzzy) tableaux algorithms (see, e.g. [3, 8, 23]). Such a tableaux procedure would need to deal with total preorders in each node, possibly using an external solver.

On the theoretical side, we want to extend our result to prove 2-NExpTime-completeness of reasoning in G-$\mathcal{SROIQ}$. As a prerequisite, we would have to eliminate the dependency on the forest-shaped structure of interpretations. It may be possible to adapt the tableaux rules from [22] for this purpose. It also remains open whether consistency in G-$\mathcal{SHOQ}$ is ExpTime-complete, as for its classical counterpart.

As done previously in [7], we can also combine our reduction with the one for infinitely valued Zadeh semantics. Although Zadeh semantics is not based on t-norms, it nevertheless is one of the most widely used semantics for fuzzy applications. It also has some properties that make it closer to the classical semantics, and hence is a natural choice for simple applications.

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