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Gödel Description Logics: Decidability in the Absence of the Finitely-Valued Model Property

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Gödel Description Logics: Decidability in the Absence of the Finitely-Valued Model Property

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Abstract

In the last few years there has been a large effort for analysing the computational properties of reasoning in fuzzy Description Logics. This has led to a number of papers studying the complexity of these logics, depending on their chosen semantics. Surprisingly, despite being arguably the simplest form of fuzzy semantics, not much is known about the complexity of reasoning in fuzzy DLs w.r.t. witnessed models over the Gödel t-norm. We show that in the logic G-$\mathfrak{I}\mathcal{ALC}$, reasoning cannot be restricted to finitely-valued models in general. Despite this negative result, we also show that all the standard reasoning problems can be solved in this logic in exponential time, matching the complexity of reasoning in classical $\mathcal{ALC}$.

1 Introduction

Fuzzy Description Logics (DLs) have been studied as a means of representing vague or imprecise knowledge in a formal and well-understood manner. As for classical DLs [1], knowledge is expressed with the help of *concepts* and *roles*. What distinguishes fuzzy DLs from their classical counterparts are their semantics, which are based on *fuzzy sets*. Fuzzy sets associate every element of the domain of interest with a number from the interval $[0, 1]$, which intuitively represents the *degree* to which the element belongs to the set. The larger its membership degree, the more an element belongs to the set.

When defining a fuzzy DL, one must also decide how to interpret the logical constructors, such as conjunction and implication, to handle the truth degrees. The simplest approach is to use the *minimum* operator to generalize intersection to fuzzy sets. In that way, the degree of membership of a conjunction is interpreted as the minimum of the membership degrees of the conjuncts. This operation, also known as the Gödel t-norm, can be used as a base to interpret all other logical constructors in a formally justified manner [23, 19]. Quantifiers ($\forall$, $\exists$) are interpreted as infima and suprema of sets of truth values. To avoid complications in cases where these sets are infinite, reasoning is usually restricted to witnessed models [21].

The study of fuzzy DLs underwent a large change in recent years, after some relatively inexpressive fuzzy DLs were shown to be undecidable when reasoning w.r.t. general ontologies [3, 4, 16]. Since then, most efforts have focused on finding the limits of decidability, yielding very expressive decidable logics on the one hand [12], and inexpressive undecidable logics on the other [13]. Despite being widely regarded as the simplest t-norm, surprisingly little is known about the decidability of fuzzy DLs based on Gödel semantics. While it is generally believed in the community that—at least w.r.t. witnessed models—these logics are decidable, no proofs exist to support this claim. The only known results for similar fuzzy DLs restrict reasoning *a priori* to a finite subset of $[0, 1]$, in which case a reduction to reasoning in classical DLs yields decidability [7, 8].

All existing approaches for reasoning in fuzzy DLs depend on limiting models to use only finitely many different truth degrees. Indeed, for these approaches to work, one must either (i) restrict the semantics to a finite set of truth degrees [7, 8, 10, 11, 14, 15, 28]; (ii) prove that reasoning can be restricted to a finite set of degrees [6, 12, 27]; or (iii) prove that models can be built from a finite pattern [26, 29]. In all three cases, the proofs of correctness of these algorithms imply the *finitely-valued model property*: an ontology has a model iff it has a model using only finitely many truth values. Conversely, the proofs of undecidability [3, 4, 13, 16] construct a model that uses infinitely many truth degrees. Thus, this *finitely-valued model property* appears to be a good indicator of the decidability of a fuzzy DL.

In this paper, we study the standard reasoning problems for the DL $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$, a fuzzy extension of $\mathcal{ALC}$ based on the Gödel semantics restricted to witnessed models. First, we show that this logic does not have the finitely-valued model property. In fact, we provide very simple consistent ontologies that only have infinitely-valued models (see Section 3). The absence of the finitely-valued model property for these logics is a surprising result in itself, contradicting the common lore of the field. In contrast, we show in Sections 4 and 5 that consistency is decidable in exponential time for this logic. Our algorithm is based on the insight that under Gödel semantics, it is only necessary to know an ordering between the relevant truth degrees, rather the precise values they take. This idea has already been used for deciding validity of formulae in propositional Gödel logic [18]. We then extend our algorithm to also compute best subsumption degrees and best satisfiability degrees w.r.t. an ontology. The last section provides some pointers to future work.

2 Preliminaries

Before introducing fuzzy description logics, we briefly consider the operators of Gödel fuzzy logic and introduce several auxiliary notions that will be useful for the reasoning procedures described in the following sections.

The two basic operators of Gödel fuzzy logic are conjunction and implication, interpreted by the *Gödel t-norm* and *residuum*, respectively. The Gödel t-norm of two fuzzy values $x, y \in [0,1]$ is defined as minimum function $\min(x, y)$. The residuum $\Rightarrow$ is uniquely defined by the equivalence $\min(x, y) \leq z$ iff $y \leq x \Rightarrow z$ for all $x, y, z \in [0,1]$, and can be computed as

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

For a more general introduction to t-norms and t-norm-based fuzzy logics, we refer the reader to [17, 19, 23].

A *total preorder* over a set S is a transitive and total binary relation $\lesssim_* \subseteq S \times S$. For $x, y \in S$, we write $x \equiv_* y$ if $x \lesssim_* y$ and $y \lesssim_* x$. Notice that $\equiv_*$ is an equivalence relation on S. Similarly, we write $x <_* y$ if $x \lesssim_* y$, but not $y \lesssim_* x$. By the symbol $\bowtie$ we denote an arbitrary element of $\{=, \geq, >, \leq, <\}$, and by $\bowtie_*$ the corresponding relation induced by the total preorder $\lesssim_*$, i.e. $\equiv_*$, $\gtrsim_*$, $>_*$, $\lesssim_*$, or $<_*$. Throughout this paper, we will use subscripts to distinguish different total preorders over the same carrier set S.

An *order structure* S is a finite set containing at least the numbers 0, 0.5, and 1, together with an involutive unary operation $\mathrm{inv}\colon S \to S$ such that $\mathrm{inv}(x) = 1 - x$ for all $x \in S \cap [0,1]$. For an order structure S, $\mathrm{order}(S)$ is the set of all total preorders $\lesssim_*$ over S that

- have 0 and 1 as least and greatest element, respectively,
- preserve the order of real numbers on $S \cap [0,1]$, and
- satisfy $x \lesssim_* y$ iff $\mathrm{inv}(y) \lesssim_* \mathrm{inv}(x)$ for all $x, y \in S$.

Given $\lesssim_* \in \mathrm{order}(S)$, the following functions on S that mimic the operators of Gödel fuzzy logic over $[0,1]$ are well-defined since $\lesssim_*$ is total:

$$\min_*(x, y) := \begin{cases} x & \text{if } x \lesssim_* y \\ y & \text{otherwise} \end{cases} \qquad \mathrm{res}_*(x, y) := \begin{cases} 1 & \text{if } x \lesssim_* y \\ y & \text{otherwise} \end{cases}$$

Table 1: Semantics of G-$\mathfrak{I}\mathcal{ALC}$

constructor	syntax	semantics
top concept	$\top$	1
involutive negation	$\neg C$	$1 - C^{\mathcal{I}}(x)$
conjunction	$C \sqcap D$	$\min(C^{\mathcal{I}}(x), D^{\mathcal{I}}(x))$
implication	$C \to D$	$C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$
existential restriction	$\exists r.C$	$\sup_{y\in\Delta^{\mathcal{I}}} \min(r^{\mathcal{I}}(x,y), C^{\mathcal{I}}(y))$
value restriction	$\forall r.C$	$\inf_{y\in\Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) \Rightarrow C^{\mathcal{I}}(y)$

It is easy to see that these operators agree with min and $\Rightarrow$ on the set $S \cap [0,1]$.

The fuzzy description logic G-$\mathfrak{I}\mathcal{ALC}$ is based on concepts and roles, which are interpreted as (fuzzy) unary and binary relations, respectively. Given the mutually disjoint sets $\mathsf{N_I}$, $\mathsf{N_R}$, and $\mathsf{N_C}$ of *individual*, *role*, and *concept names*, respectively, G-$\mathfrak{I}\mathcal{ALC}$ *concepts* are built through the rule

$$C ::= A \mid \top \mid \neg C \mid C \sqcap C \mid C \to C \mid \exists r.C \mid \forall r.C,$$

where $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$. We will call concepts of the form $\exists r.C$ or $\forall r.C$ *quantified concepts*. The semantics of this logic is given by means of interpretations. An *interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$, where $\Delta^{\mathcal{I}}$ is a non-empty *domain*, and $\cdot^{\mathcal{I}}$ is a function that maps every $a \in \mathsf{N_I}$ to an element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, every $A \in \mathsf{N_C}$ to a function $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \to [0,1]$, and every $r \in \mathsf{N_R}$ to a function $r^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to [0,1]$. Intuitively, for every domain element $x \in \Delta^{\mathcal{I}}$ the value $A^{\mathcal{I}}(x)$ represents the degree to which x is a member of A. This function is extended to arbitrary concepts using the Gödel operators as shown in Table 1. Notice that we have not introduced an explicit constructor for the *residual negation* $\ominus x := x \Rightarrow 0$, as it is expressible using the constructors $\top$, $\neg$, and $\to$. Similarly, disjunction can be simulated by $\sqcap$ and $\neg$.

In the literature on fuzzy DLs, interpretations are usually restricted to be *witnessed* [20], which means that existential and value restrictions must be interpreted as maxima and minima, respectively. More formally, an interpretation $\mathcal{I}$ is *witnessed* if for every existential restriction $\exists r.C$ and every $x \in \Delta^{\mathcal{I}}$ there is a *witness* $y \in \Delta^{\mathcal{I}}$ such that $(\exists r.C)^{\mathcal{I}}(x) = \min(r^{\mathcal{I}}(x,y), C^{\mathcal{I}}(y))$, and similarly for value restrictions. We also adopt this restriction here, and for the rest of this paper consider only witnessed interpretations. For brevity, we call them simply *interpretations*.

The knowledge of a domain is represented using axioms that restrict the class of interpretations relevant for the different reasoning tasks.

Definition 1 (axioms)**.** A *crisp assertion* is either a *concept assertion* of the form $a{:}C$ or a *role assertion* of the form $(a,b){:}r$ for a concept C, $r \in \mathsf{N_R}$, and $a, b \in \mathsf{N_I}$.

An *(order) assertion* is of the form $\langle \alpha \bowtie \beta \rangle$, where α is a crisp assertion and β is either a crisp assertion or a value from $[0, 1]$. An interpretation $\mathcal{I}$ *satisfies* an order assertion $\langle \alpha \bowtie \beta \rangle$ if $\alpha^\mathcal{I} \bowtie \beta^\mathcal{I}$, where $(a{:}C)^\mathcal{I} := C^\mathcal{I}(a^\mathcal{I})$, $((a, b){:}r)^\mathcal{I} := r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I})$, and $q^\mathcal{I} := q$ for all $q \in [0, 1]$.

An *(ordered) ABox* is a finite set of order assertions. An ordered ABox is called *local* if it contains no role assertions and only one individual name appears in it. An interpretation is a *model* of an ordered ABox $\mathcal{A}$ if it satisfies all order assertions in $\mathcal{A}$.

A *general concept inclusion (GCI)* is an expression of the form $\langle C \sqsubseteq D \geq q \rangle$ for C, D concepts, and $q \in [0, 1]$. $\mathcal{I}$ is a model of this GCI if $C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x) \geq q$ holds for all $x \in \Delta^\mathcal{I}$. A *TBox* is a finite set of GCIs. An *ontology* is a pair $\mathcal{O} = (\mathcal{A}, \mathcal{T})$, where $\mathcal{A}$ is an (ordered) ABox and $\mathcal{T}$ is a TBox. An interpretation is a *model* of a TBox $\mathcal{T}$ if it satisfies all GCIs in $\mathcal{T}$, and it is a *model* of an ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ if it is a model of both $\mathcal{A}$ and $\mathcal{T}$.

We denote by $\mathsf{sub}(\mathcal{O})$ the closure under negation of the set of all subconcepts appearing in an ontology $\mathcal{O}$. The concepts $\neg\neg C$ and C as equivalent, and thus this set is always finite. We further denote by $\mathcal{V}_\mathcal{O}$ the closure under the operator $x \mapsto 1 - x$ of the set of all truth degrees appearing in $\mathcal{O}$, together with 0, 0.5, and 1. Since this operator is involutive, this set is also always finite. We often denote the elements of $\mathcal{V}_\mathcal{O} \subseteq [0, 1]$ as $0 = q_0 < q_1 < \cdots < q_k = 1$.

As with classical DLs, the most basic reasoning task in G-$\mathfrak{I}\mathcal{ALC}$ is to decide consistency, i.e. whether a given ontology has a (witnessed) model. However, one might also be interested in computing the degree to which a given entailment holds, as defined next.

Definition 2 (reasoning)**.** An ontology $\mathcal{O}$ is *consistent* if it has a model. Given $p \in [0, 1]$, a concept C is *p-satisfiable* w.r.t. $\mathcal{O}$ if there is a model $\mathcal{I}$ of $\mathcal{O}$ and an $x \in \Delta^\mathcal{I}$ with $C^\mathcal{I}(x) \geq p$. The *best satisfiability degree* of C w.r.t. $\mathcal{O}$ is the supremum over all p such that C is p-satisfiable w.r.t. $\mathcal{O}$. Furthermore, C is *p-subsumed* by a concept D w.r.t. $\mathcal{O}$ if all models of $\mathcal{O}$ satisfy the GCI $\langle C \sqsubseteq D \geq p \rangle$. The *best subsumption degree* of C and D w.r.t. $\mathcal{O}$ is the supremum over all p such that C is p-subsumed by D w.r.t. $\mathcal{O}$.

If consistency is decidable, then satisfiability and subsumption can be restricted without loss of generality to ontologies containing an empty ABox. Indeed, if $\mathcal{O}$ is inconsistent, then these two problems are trivial. If $\mathcal{O}$ is consistent, then the ABox assertions cannot contradict the p-satisfiability of C, and therefore C is p-satisfiable w.r.t. $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ iff it is p-satisfiable w.r.t. $(\emptyset, \mathcal{T})$. A similar argument can be made for subsumptions.

We show in Section 5 that ontology consistency has the same complexity in G-$\mathfrak{I}\mathcal{ALC}$ as in classical $\mathcal{ALC}$; it is ExpTime-complete. As a first step, we establish the complexity of consistency for ontologies with local ordered ABoxes

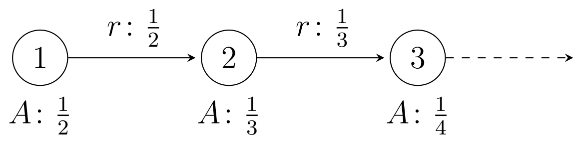

Figure 1: The model $\mathcal{I}_1$ from Example 3

in Section 4, adapting an automata-based technique known from classical and finitely-valued DLs [2, 14]. We later lift these results to the satisfiability and subsumption problems. But first, we briefly illustrate why the naïve approach of simply restricting to finitely-valued reasoning cannot work in this logic.

3 Effects of Restricting to Finitely Many Values

It is an easy observation that any set of truth values that contains 0 and 1 is closed w.r.t. the Gödel connectives. Owing to this observation, it is common to restrict reasoning in fuzzy DLs with Gödel semantics to the finitely many truth values occurring in the ontology [8, 11]. This restriction is also sometimes justified by the "limited precision of computers" [7].

Earlier works have, however, neglected to examine whether the restriction to a fixed finite set of values preserves the semantics of the logic. We now show that this is not the case, even for the simple description logics $\mathsf{G}\text{-}\mathcal{AL}$, which allows only conjunction, existential and value restrictions, and the top concept; and $\mathsf{G}\text{-}\mathfrak{I}\mathcal{EL}$, where concepts are built using existential restrictions, conjunction, implication, and the top concept. We show even stronger results: reasoning in these logics cannot, without loss of generality, be restricted to *finitely-valued* models, i.e. models that only use values from an arbitrary finite subset of $[0, 1]$.

Example 3. Let $\mathcal{T}_1$ be the $\mathsf{G}\text{-}\mathcal{AL}$ TBox

$$\mathcal{T}_1 = \{\langle \forall r.A \sqsubseteq A \geq 1\rangle,\ \langle \exists r.\top \sqsubseteq A \geq 1\rangle\}.$$

We show that $\top$ is *not* 1-subsumed by A w.r.t. the ontology $\mathcal{O} = (\emptyset, \mathcal{T}_1)$, but every finitely-valued model of this ontology also satisfies $\langle \top \sqsubseteq A \geq 1\rangle$.

For the former, we construct a model $\mathcal{I}_1$ of $\mathcal{T}_1$ as follows (see Figure 1). Let $\Delta^{\mathcal{I}_1}$ be the set of all natural numbers. We define $A^{\mathcal{I}_1}(n) := r^{\mathcal{I}_1}(n, n+1) := \frac{1}{n+1}$ for all $n \in \mathbb{N}$ and $r^{\mathcal{I}_1}(n, m) := 0$ if $m \neq n + 1$. It is straightforward to check that this is indeed a witnessed model of $\mathcal{T}_1$ which violates $\langle \top \sqsubseteq A \geq 1\rangle$. Thus, $\top$ is not 1-subsumed by A w.r.t. $\mathcal{O}$. In fact, the best subsumption degree of $\top$ and A w.r.t. $\mathcal{O}$ is 0.

Assume now that there is a witnessed model $\mathcal{I}$ of $\mathcal{T}_1$ that uses only finitely many truth values and that violates $\langle \top \sqsubseteq A \geq 1\rangle$. Since $\mathcal{I}$ uses only finitely many

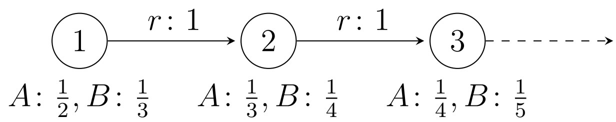

Figure 2: The model $\mathcal{I}_2$ from Example 4

truth values, there exists an element $y \in \Delta^\mathcal{I}$ for which $A^\mathcal{I}(y)$ is minimal, i.e. $A^\mathcal{I}(y) \leq A^\mathcal{I}(x)$ holds for all $x \in \Delta^\mathcal{I}$. Furthermore, since $\mathcal{I}$ violates $\langle \top \sqsubseteq A \geq 1 \rangle$ there must be some $x_0 \in \Delta^\mathcal{I}$ satisfying $A^\mathcal{I}(x_0) < 1$. In particular, this yields $A^\mathcal{I}(y) < 1$.

As $\mathcal{I}$ is witnessed, there must be a $z \in \Delta^\mathcal{I}$ with $(\forall r.A)^\mathcal{I}(y) = r^\mathcal{I}(y,z) \Rightarrow A^\mathcal{I}(z)$. The first axiom of $\mathcal{T}_1$ entails $r^\mathcal{I}(y,z) \Rightarrow A^\mathcal{I}(z) \leq A^\mathcal{I}(y) < 1$, and in particular

$$r^\mathcal{I}(y,z) > A^\mathcal{I}(z). \tag{1}$$

The second axiom from $\mathcal{T}_1$ yields

$$r^\mathcal{I}(y,z) = \min(r^\mathcal{I}(y,z), 1) \leq (\exists r.\top)^\mathcal{I}(y) \leq A^\mathcal{I}(y). \tag{2}$$

From (1) and (2) we obtain $A^\mathcal{I}(y) > A^\mathcal{I}(z)$, contradicting the minimality of $A^\mathcal{I}(y)$. We have thus shown that a witnessed model of $\mathcal{T}_1$ with only finitely many truth values cannot violate $\langle \top \sqsubseteq A \geq 1 \rangle$. That is, $\mathcal{T}_1$ entails $\langle \top \sqsubseteq A \geq 1 \rangle$ when reasoning is restricted to finite sets of values.

This example shows that it is not possible to restrict reasoning in G-$\mathcal{AL}$ to only finitely-valued models without changing the consequences. A similar example shows that this also holds for G-$\mathfrak{IEL}$.

Example 4. Consider the TBox

$$\mathcal{T}_2 = \{\langle B \sqsubseteq A \geq 1 \rangle,\ \langle A{\to}B \sqsubseteq B \geq 1 \rangle,\ \langle \top \sqsubseteq \exists r.\top \geq 1 \rangle,\ \langle \exists r.A \sqsubseteq B \geq 1 \rangle\}.$$

As in the previous example, we show that $\top$ is not 1-subsumed by A w.r.t. $\mathcal{O} := (\emptyset, \mathcal{T}_2)$, but every finitely-valued model of $\mathcal{O}$ satisfies $\langle \top \sqsubseteq A \geq 1 \rangle$.

A witnessed model $\mathcal{I}_2$ of $\mathcal{T}_2$ can be built as follows (see Figure 2). Let $\Delta^{\mathcal{I}_2}$ be the set of all natural numbers, and define $A^{\mathcal{I}_2}(n) := \frac{1}{n+1}$, $B^{\mathcal{I}_2}(n) := \frac{1}{n+2}$, and $r^{\mathcal{I}_2}(n, n+1) := 1$ for all $n \in \mathbb{N}$ and $r^{\mathcal{I}_2}(n,m) := 0$ if $m \neq n+1$. It is straightforward to check that this is indeed a witnessed model of $\mathcal{T}_2$ that violates $\langle \top \sqsubseteq A \geq p \rangle$ for every $p > 0$; in particular for $p = 1$.

Assume now that there is a witnessed model $\mathcal{I}$ of $\mathcal{T}_2$ that uses only finitely many truth values and that violates $\langle \top \sqsubseteq A \geq 1 \rangle$. Let $y \in \Delta^\mathcal{I}$ be such that $A^\mathcal{I}(y)$ is minimal. As in the previous example, we know that $A^\mathcal{I}(y) < 1$, since $\mathcal{I}$ violates $\langle \top \sqsubseteq A \geq 1 \rangle$, and that there must be some witness $z \in \Delta^\mathcal{I}$ such that $(\exists r.\top)^\mathcal{I}(y) = r^\mathcal{I}(y,z)$.

From the first axiom of $\mathcal{T}_2$ we obtain $B^{\mathcal{I}}(y) \leq A^{\mathcal{I}}(y)$ and thus in particular $B^{\mathcal{I}}(y) < 1$. The second axiom yields

$$A^{\mathcal{I}}(y) \Rightarrow B^{\mathcal{I}}(y) \leq B^{\mathcal{I}}(y) < 1,$$

and therefore $A^{\mathcal{I}}(y) > B^{\mathcal{I}}(y)$. The third axiom entails $1 = (\exists r.\top)^{\mathcal{I}}(y) = r^{\mathcal{I}}(y, z)$. Finally, we obtain from $\langle \exists r.A \sqsubseteq B \geq 1 \rangle$ that

$$\begin{aligned} B^{\mathcal{I}}(y) &\geq \sup_{d \in \Delta^{\mathcal{I}}} \min(r^{\mathcal{I}}(y, d), A^{\mathcal{I}}(d)) \\ &\geq \min(r^{\mathcal{I}}(y, z), A^{\mathcal{I}}(z)) = A^{\mathcal{I}}(z). \end{aligned} \tag{3}$$

From $A^{\mathcal{I}}(y) > B^{\mathcal{I}}(y)$ and (3), we get $A^{\mathcal{I}}(y) > A^{\mathcal{I}}(z)$, a contradiction to minimality of $A^{\mathcal{I}}(y)$. Thus, no witnessed model of $\mathcal{T}_2$ with only finitely many truth values can violate $\langle \top \sqsubseteq A \geq 1 \rangle$.

Recall that a (fuzzy) DL has the *finite model property* if every consistent ontology has a model with finite domain. A simple consequence of the last two examples is that $\mathsf{G}\text{-}\mathcal{AL}$ and $\mathsf{G}\text{-}\mathfrak{IEL}$ do not have the finite model property. Indeed, each $\mathcal{I}_i$ is a model of the ontology $(\{\langle a : A = 0.5 \rangle\}, \mathcal{T}_i)$ if we interpret the individual name a as $a^{\mathcal{I}_i} := 1$. This shows that these ontologies are consistent. However, any finite model $\mathcal{I}$ of $\mathcal{T}_i$ uses only finitely many truth degrees. As shown in the examples, such an interpretation must satisfy $A^{\mathcal{I}}(x) = 1$ for all $x \in \Delta^{\mathcal{I}}$, and hence violate the assertion $\langle a : A = 0.5 \rangle$. We thus obtain the following result.

Theorem 5. *$\mathsf{G}\text{-}\mathcal{AL}$ and $\mathsf{G}\text{-}\mathfrak{IEL}$ do not have the finite model property.*

These results imply that some of the standard techniques used for reasoning in fuzzy DLs cannot be directly applied to any logic that contains $\mathsf{G}\text{-}\mathcal{AL}$ or $\mathsf{G}\text{-}\mathfrak{IEL}$. For example, termination of the tableaux-based approach [26, 29] relies on the existence of finitely many *types* that can describe domain elements by specifying the membership degrees for all relevant concepts, while any sound and complete reduction to crisp reasoning [6, 8] implies the *finitely-valued model property*. One could thus be inclined to believe that consistency in $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$ is also undecidable. In the rest of this paper, we show that this is not the case, providing ExpTime automata-based algorithms that decide consistency, subsumption, and satisfiability.

4 Deciding Local Consistency

In this section, we consider only the special case where the ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ is such that $\mathcal{A}$ is a local ordered ABox which uses only the individual name a. In Section 5, we extend the approach to handle arbitrary ontologies.

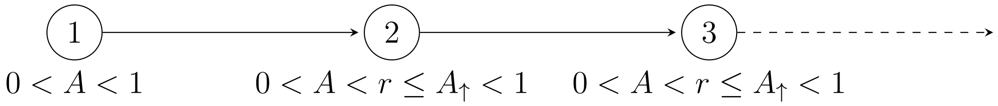

Figure 3: An abstract description of $\mathcal{I}_1$ from Example 3

The algorithm is based on the idea that the axioms and the semantics of the constructors only introduce restrictions on the *order* of the values that models can assign to concepts, not on the values themselves. For example, to satisfy the assertion $\langle a\colon(A \sqcap B) \geq p\rangle$, we need to ensure that $A^{\mathcal{I}}(a^{\mathcal{I}}) \geq p$ and $B^{\mathcal{I}}(a^{\mathcal{I}}) \geq p$. Similarly, $\mathcal{I}$ satisfies the assertion $\langle a\colon(A \to B) = p\rangle$ iff $A^{\mathcal{I}}(a^{\mathcal{I}}) > B^{\mathcal{I}}(a^{\mathcal{I}})$ and $B^{\mathcal{I}}(a^{\mathcal{I}}) = p$. Thus, rather than building a model directly, we first create an abstract representation of a model that encodes for each domain element only the order between concepts. These elements will be arranged in a tree-shaped structure, called *Hintikka tree*. A consequence of the correctness of our approach is then that every consistent ontology with a local ordered ABox has a tree-shaped model.

Example 6. Consider again the TBox $\mathcal{T}_1 = \{\langle\forall r.A \sqsubseteq A \geq 1\rangle, \langle\exists r.\top \sqsubseteq A \geq 1\rangle\}$ from Example 3. When trying to construct a model contradicting $\langle\top \sqsubseteq A \geq 1\rangle$, we start with a domain element satisfying the restriction that the value of A is strictly smaller than 1 (see Figure 3). The second axiom implies that the degree of any outgoing r-connection is bounded by the value of A. Moreover, the first axiom states that the witness of $\forall r.A$ must satisfy A to a degree strictly smaller than the value of the r-connection, and thus strictly smaller than the value of A at the original element.

This yields an abstract description of two domain elements in terms of order relations between values of concepts at the current node and the parent node (denoted by a subscript $\uparrow$). Applying the same argument to the new element yields another element with the same restrictions. However, in order for this construction to yield a model, it is easy to see that the value of A at all considered elements has to be strictly greater than 0—once the value of A is 0, there can be no successors with smaller values for A.

Note that it suffices to consider order relations between concepts of neighboring elements, which are directly connected by some role to a degree greater than 0.

To formally represent the order relationships, we consider the order structure

$$\mathcal{U} := \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O}) \cup \mathsf{sub}_{\uparrow}(\mathcal{O}) \cup \{\lambda, \neg\lambda\}, \tag{4}$$

where $\mathsf{sub}_{\uparrow}(\mathcal{O}) := \{C_{\uparrow} \mid C \in \mathsf{sub}(\mathcal{O})\}$, $\mathrm{inv}(\lambda) := \neg\lambda$, $\mathrm{inv}(C) := \neg C$, and $\mathrm{inv}(C_{\uparrow}) := (\neg C)_{\uparrow}$ for all $C \in \mathsf{sub}(\mathcal{O})$. The idea is that total preorders from $\mathrm{order}(\mathcal{U})$ describe the relationships between all the subconcepts from $\mathcal{O}$ and the truth degrees from $\mathcal{V}_{\mathcal{O}}$ at given domain elements. One can think of such a preorder as the *type* of a domain element, from which a tree-shaped interpretation can be built. As illustrated in Example 6, in order to handle the semantics of the existential and value restrictions, we also need to know the type of the parent node in the tree, as well as the degree of the role relation connecting them. For that reason, we introduce $\mathsf{sub}_\uparrow(\mathcal{O})$ and λ, respectively. More precisely, λ is a special symbol that represents the value of the role relation between a node and its parent, while the elements of $\mathsf{sub}_\uparrow(\mathcal{O})$ refer to the values of the subconcepts of $\mathcal{O}$ at the parent node.

Let n be the number of quantified concepts in $\mathsf{sub}(\mathcal{O})$ and ϕ an arbitrary but fixed bijection between the set of all quantified concepts in $\mathsf{sub}(\mathcal{O})$ and $\{1, \ldots, n\}$. This bijection specifies which quantified concept is witnessed by which successor in the Hintikka tree. For a given role $r \in \mathsf{N_R}$, we denote by Φ_r the set of all indices $\phi(E)$ where $E \in \mathsf{sub}(\mathcal{O})$ is a quantified concept of the form $\exists r.C$ or $\forall r.C$. Our algorithm will try to decide the existence of an n-ary infinite tree whose nodes are labeled with a preorder from $\mathrm{order}(\mathcal{U})$, such that the semantics of the constructors and all the axioms in $\mathcal{O}$ are preserved.

Definition 7 (Hintikka ordering)**.** An element $\lesssim_H \in \mathrm{order}(\mathcal{U})$ is called a *Hintikka ordering* if it satisfies the following conditions for every $C \in \mathsf{sub}(\mathcal{O})$:

- $C = \top$ implies $C \equiv_H 1$,
- if $C = D_1 \sqcap D_2$, then $C \equiv_H \min_H(D_1, D_2)$,
- if $C = D_1 \to D_2$, then $C \equiv_H \mathrm{res}_H(D_1, D_2)$.

This preorder is *compatible* with the TBox $\mathcal{T}$ if for every GCI $\langle C \sqsubseteq D \geq q \rangle \in \mathcal{T}$ we have $\mathrm{res}_H(C, D) \gtrsim_H q$. It is *compatible* with the ABox $\mathcal{A}$ if for every assertion $\langle a{:}C \bowtie q \rangle$ or $\langle a{:}C \bowtie a{:}D \rangle$ in $\mathcal{A}$, we have $C \bowtie_H q$ or $C \bowtie_H D$, respectively.

The conditions imposed on Hintikka orderings ensure that they preserve the semantics of all the *propositional* constructors. For every quantified concept E, we still need to ensure the existence of a successor that serves as its witness. This is achieved through the bijection ϕ and the Hintikka condition.

Definition 8 (Hintikka condition)**.** A tuple $(\lesssim_0, \lesssim_1, \ldots, \lesssim_n)$ of $n + 1$ Hintikka orderings satisfies the *Hintikka condition* if:

- for every $1 \leq i \leq n$ and all $\alpha, \beta \in \mathcal{V}_\mathcal{O} \cup \mathsf{sub}(\mathcal{O})$, we have $\alpha \lesssim_0 \beta$ iff $\alpha_\uparrow \lesssim_i \beta_\uparrow$, where we set $q_\uparrow := q$ for all $q \in \mathcal{V}_\mathcal{O}$;
- for every $\exists r.D \in \mathsf{sub}(\mathcal{O})$, we have
  - $(\exists r.D)_\uparrow \equiv_i \min_i(\lambda, D)$ for $i = \phi(\exists r.D)$, and
  - $(\exists r.D)_\uparrow \gtrsim_i \min_i(\lambda, D)$ for all $i \in \Phi_r$; and

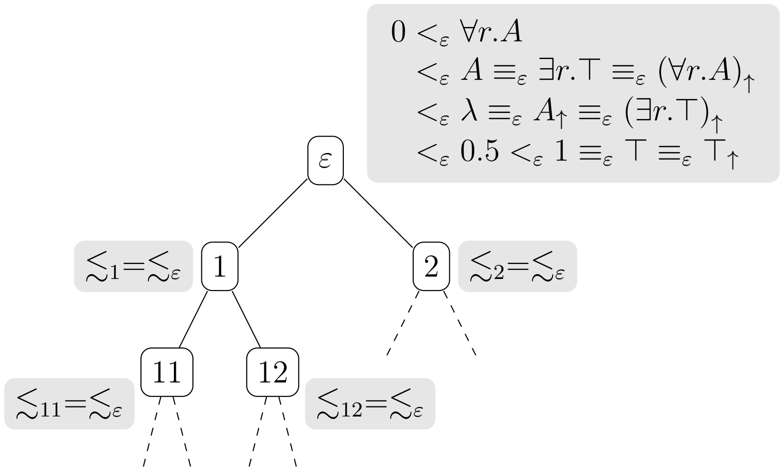

Figure 4: A Hintikka tree for Example 3

- for every $\forall r.D \in \mathsf{sub}(\mathcal{O})$, we have
  - $(\forall r.D)_{\uparrow} \equiv_i \mathrm{res}_i(\lambda, D)$ for $i = \phi(\forall r.D)$, and
  - $(\forall r.D)_{\uparrow} \lesssim_i \mathrm{res}_i(\lambda, D)$ for all $i \in \Phi_r$.

A *Hintikka tree* for $\mathcal{O}$ is an infinite n-ary tree,[1] where every node u is associated with a Hintikka ordering $\lesssim_u$ compatible with $\mathcal{T}$, such that:

- every tuple $(\lesssim_u, \lesssim_{u1}, \dots, \lesssim_{un})$ satisfies the Hintikka condition, and
- $\lesssim_\varepsilon$ is compatible with $\mathcal{A}$.

For instance, Figure 4 shows a Hintikka tree for the TBox $\mathcal{T}_1$ from Example 3, together with the ABox $\mathcal{A} = \{\langle a{:}A < 1\rangle\}$. Notice that in this simple example every node is labeled with the same preorder, which is not true in general. Furthermore, the tree shown in Figure 4 is invariant w.r.t. the choice of ϕ. We now show that the existence of a Hintikka tree for an ontology $\mathcal{O}$ characterizes the consistency of $\mathcal{O}$.

Proposition 9. *If there is a Hintikka tree for $\mathcal{O}$, then $\mathcal{O}$ has a model.*

Proof. Given a Hintikka tree, we construct a model in two steps. In the first step, we recursively define a function $v\colon \mathcal{U} \times \{1,\dots,n\}^* \to [0,1]$ satisfying the following conditions for all nodes u:

(P1) for all values $q \in \mathcal{V}_{\mathcal{O}}$ we have $v(q,u) = q$,

[1]We denote the nodes in an infinite n-ary tree with words from $\{1,\dots,n\}^*$.

(P2) for all $\alpha, \beta \in \mathcal{U}$

$$v(\alpha, u) \leq v(\beta, u) \quad \text{if and only if} \quad \alpha \lesssim_u \beta,$$

(P3) for all $\alpha \in \mathcal{U}$

$$v(\mathrm{inv}(\alpha), u) = 1 - v(\alpha, u),$$

(P4) for all $C \in \mathsf{sub}(\mathcal{O})$ and all $i \in \{1, \ldots, n\}$

$$v(C, u) = v(C_\uparrow, ui).$$

In the second step, we construct, with the help of the function v, an interpretation $\mathcal{I}_v = (\{1, \ldots, n\}^*, \cdot^{\mathcal{I}_v})$ satisfying $C^{\mathcal{I}_v}(u) = v(C, u)$ for all concepts C and all nodes u, and show that $\mathcal{I}_v$ is indeed a model of $\mathcal{O}$.

Step 1 The function v is defined recursively, starting from the root node ε. Let $\mathcal{U}/{\equiv_\varepsilon}$ be the set of all equivalence classes of $\equiv_\varepsilon$. Then $\lesssim_\varepsilon$ yields a total order $\leq_\varepsilon$ on $\mathcal{U}/{\equiv_\varepsilon}$. In particular, since $\lesssim_\varepsilon$ preserves the order of real numbers on $\mathcal{V}_\mathcal{O}$, it holds that $[0]_\varepsilon <_\varepsilon [q_1]_\varepsilon <_\varepsilon [q_2]_\varepsilon <_\varepsilon \cdots <_\varepsilon [q_{k-1}]_\varepsilon <_\varepsilon [1]_\varepsilon$. For an equivalence class $[\alpha]_\varepsilon$, we set $\mathrm{inv}([\alpha]_\varepsilon) := [\mathrm{inv}(\alpha)]_\varepsilon$, which is well-defined since $\lesssim_\varepsilon$ is an element of $\mathrm{order}(\mathcal{U})$.

We first define an auxiliary function $\tilde{v}_\varepsilon \colon \mathcal{U}/{\equiv_\varepsilon} \to [0, 1]$. For all $q \in \mathcal{V}_\mathcal{O}$ we define $\tilde{v}_\varepsilon([q]_\varepsilon) := q$. It remains to define a value for all equivalence classes that do not contain a value from $\mathcal{V}_\mathcal{O}$. Notice that because of the minimality of $[0]_\varepsilon$ and maximality of $[1]_\varepsilon$ every such class must be strictly between $[q_i]_\varepsilon$ and $[q_{i+1}]_\varepsilon$ for two adjacent truth degrees q_i, q_{i+1}. For every $i \in \{0, \ldots, k-1\}$, let ν_i be the number of equivalence classes that are strictly between $[q_i]_\varepsilon$ and $[q_{i+1}]_\varepsilon$. We assume that these classes are denoted by E^i_j such that

$$[q_i]_\varepsilon <_\varepsilon E^i_1 <_\varepsilon E^i_2 <_\varepsilon \cdots <_\varepsilon E^i_{\nu_i} <_\varepsilon [q_{i+1}]_\varepsilon.$$

We then define values $q_i < s^i_1 < s^i_2 < \cdots < s^i_{\nu_i} < q_{i+1}$ as $s^i_j := q_i + \frac{j}{\nu_i+1}(q_{i+1} - q_i)$ and set $\tilde{v}_\varepsilon(E^i_j) := s^i_j$ for every j, $1 \leq j \leq \nu_i$. Finally, we define $v(\alpha, \varepsilon) := \tilde{v}_\varepsilon([\alpha]_\varepsilon)$ for all $\alpha \in \mathcal{U}$. This construction ensures that (P1) and (P2) hold at the node ε. To see that (P3) is also satisfied, note that $1 - q_{i+1}$ and $1 - q_i$ are also adjacent in $\mathcal{V}_\mathcal{O}$ and have exactly the inverses $\mathrm{inv}(E^i_j)$ between them in reversed order.

For the recursion step, assume that we have already defined v for a node u, such that (P1)–(P3) are satisfied at u and let $i \in \{1, \ldots, n\}$. We initialize the auxiliary function $\tilde{v}_{ui} \colon \mathcal{U}/{\equiv_{ui}} \to [0, 1]$ by setting $\tilde{v}_{ui}([q]_{ui}) := q$ for all $q \in \mathcal{V}_\mathcal{O}$ and $\tilde{v}_{ui}([C_\uparrow]_{ui}) := v(C, u)$ for all $C \in \mathsf{sub}(\mathcal{O})$. To see that this is well-defined, consider $[C_\uparrow]_{ui} = [D_\uparrow]_{ui}$, i.e. $C_\uparrow \equiv_{ui} D_\uparrow$. From the Hintikka condition it follows that $C \equiv_u D$, and from (P2) at u we obtain $v(C, u) = v(D, u)$. A similar argument can be used to show that $[q]_{ui} = [C_\uparrow]_{ui}$ implies $v(q, u) = v(C, u)$. For the remaining equivalence classes, we can use a construction analogous to the case for ε by considering the two unique neighboring equivalence classes that contain an element of $\mathcal{V}_\mathcal{O} \cup \mathsf{sub}_\uparrow(\mathcal{O})$. We now define $v(\alpha, ui) := \tilde{v}_{ui}([\alpha]_{ui})$. This construction ensures that (P1)–(P3) hold at ui, and that (P4) holds for u.

Step 2 Given a Hintikka tree and a function v that satisfies (P1)–(P4), we define the interpretation $\mathcal{I}_v = (\{1, \dots, n\}^*, \cdot^{\mathcal{I}_v})$ as follows. For every concept name $A \in \mathsf{N_C}$ and all domain elements u, we set

$$A^{\mathcal{I}_v}(u) := \begin{cases} v(A, u) & \text{if } A \in \mathsf{sub}(\mathcal{O}), \\ 0 & \text{otherwise.} \end{cases}$$

For every role name $r \in \mathsf{N_R}$ and all domain elements u, we likewise define

$$r^{\mathcal{I}_v}(u, w) := \begin{cases} v(\lambda, ui) & \text{if } w = ui \text{ with } i \in \Phi_r, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we define $a^{\mathcal{I}_v} := \varepsilon$ for the individual name a. We show by induction on the structure of C that

$$C^{\mathcal{I}_v}(u) = v(C, u) \text{ for all } C \in \mathsf{sub}(\mathcal{O}), u \in \{1, \dots, n\}^* \tag{5}$$

holds. If $C \in \mathsf{N_C}$, this follows from the definition of $\mathcal{I}_v$. If $C = \top$, we get $\top \equiv_u 1$ since $\lesssim_u$ is a Hintikka ordering. From (P1) and (P2) it follows that $v(\top, u) = v(1, u) = 1$, and thus $\top^{\mathcal{I}_v}(u) = 1 = v(\top, u)$.

In the case that $C = \neg D$, we have

$$C^{\mathcal{I}_\nu}(u) = 1 - D^{\mathcal{I}_\nu}(u) = 1 - v(D, u) = v(C, u)$$

by induction hypothesis and (P3).

Consider $C = D_1 \sqcap D_2$. Because $\lesssim_u$ is a Hintikka ordering, we obtain

$$\begin{aligned} C \equiv_u \min\nolimits_u(D_1, D_2) &= \begin{cases} D_1 & \text{if } D_1 \lesssim_u D_2 \\ D_2 & \text{if } D_2 <_u D_1 \end{cases} \\ &\overset{\text{(P2)}}{=} \begin{cases} D_1 & \text{if } v(D_1, u) \leq v(D_2, u) \\ D_2 & \text{if } v(D_2, u) < v(D_1, u). \end{cases} \end{aligned}$$

By (P2), $v(C, u) = v(D_1, u)$ if $v(D_1, u) \leq v(D_2, u)$, and $v(C, u) = v(D_2, u)$ otherwise. Thus, we obtain

$$\begin{aligned} v(C, u) &= \min(v(D_1, u), v(D_2, u)) \\ &= \min(D_1^{\mathcal{I}_v}(u), D_2^{\mathcal{I}_v}(u)) = C^{\mathcal{I}_v}(u) \end{aligned}$$

from the induction hypothesis. The case of $D_1 \to D_2$ can be treated similarly.

Let $C = \exists r.D$. For $i_0 = \phi(\exists r.D)$, from the Hintikka condition it follows that $(\exists r.D)_\uparrow \equiv_{ui_0} \min_{ui_0}(\lambda, D)$. As in the case for $C = D_1 \sqcap D_2$ above, (P2) yields

$$v((\exists r.D)_\uparrow, ui_0) = \min(v(\lambda, ui_0), v(D, ui_0)).$$

Using (P4) and the induction hypothesis, we obtain

$$v(\exists r.D, u) = \min(r^{\mathcal{I}_v}(u, ui_0), D^{\mathcal{I}_v}(ui_0)).$$

Similarly, for $i \in \Phi_r$ we can show

$$v(\exists r.D, u) \geq \min(r^{\mathcal{I}_v}(u, ui), D^{\mathcal{I}_v}(ui)).$$

Thus,

$$\begin{aligned} (\exists r.D)^{\mathcal{I}_v}(u) &= \sup_{w \in \{1,\dots,n\}^*} \min(r^{\mathcal{I}_v}(u, w), D^{\mathcal{I}_v}(w)) \\ &= \max_{i \in \Phi_r} \min(r^{\mathcal{I}_v}(u, ui), D^{\mathcal{I}_v}(ui)) \\ &= v(\exists r.D, u). \end{aligned}$$

The case $C = \forall r.D$ can be treated analogously.

It remains to show that $\mathcal{I}_v$ is indeed a model of $\mathcal{O}$. For every $\langle a{:}C \bowtie q\rangle \in \mathcal{A}$, the Hintikka tree satisfies $C \bowtie_\varepsilon q$, and thus we obtain from (5), (P1), and (P2):

$$C^{\mathcal{I}_v}(a^{\mathcal{I}_v}) = v(C, \varepsilon) \bowtie v(q, \varepsilon) = q,$$

and similarly for assertions of the form $\langle a{:}C \bowtie a{:}D\rangle$.

Now, let $\langle C \sqsubseteq D \geq q\rangle \in \mathcal{T}$ be a GCI and $u \in \{1, \dots, n\}^*$ a domain element of $\mathcal{I}_v$. Since $p \in \mathcal{V}_\mathcal{O}$ and $\lesssim_u$ is compatible with $\mathcal{T}$, it must hold that

$$\begin{aligned} q \lesssim_u \mathrm{res}_u(C, D) &= \begin{cases} 1 & \text{if } C \lesssim_u D \\ D & \text{if } D <_u C \end{cases} \\ &\overset{(P2)}{=} \begin{cases} 1 & \text{if } v(C, u) \leq v(D, u) \\ D & \text{if } v(D, u) < v(C, u). \end{cases} \end{aligned}$$

Thus, (P1) and (P2) yield

$$\begin{aligned} q = v(q, u) &\leq \begin{cases} v(1, u) & \text{if } v(C, u) \leq v(D, u) \\ v(D, u) & \text{if } v(D, u) < v(C, u) \end{cases} \\ &= v(C, u) \Rightarrow v(D, u) \\ &= C^{\mathcal{I}_v}(u) \Rightarrow D^{\mathcal{I}_v}(u). \end{aligned}$$

□

Conversely, every model can be transformed into a Hintikka tree. The idea is to *unravel* the model into an infinite tree, and then abstract from the specific values by just considering the ordering between the elements of $\mathcal{U}$. This idea is formalized next.

Proposition 10. *If $\mathcal{O}$ has a model, then there is a Hintikka tree for $\mathcal{O}$.*

Proof. Let $\mathcal{I}$ be a model of $\mathcal{O}$. We use this model to guide the construction of a Hintikka tree for $\mathcal{O}$. During this construction, we will recursively generate a mapping $g\colon \{1,\dots,n\}^* \to \Delta^{\mathcal{I}}$ specifying which domain elements correspond to the nodes in the tree. This mapping will preserve the following condition:

(P5) For all $\alpha,\beta \in \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$ and all $u \in \{1,\dots,n\}^*$, we have

$$\alpha \lesssim_u \beta \quad \text{if and only if} \quad \alpha^{\mathcal{I}}(g(u)) \leq \beta^{\mathcal{I}}(g(u)),$$

where $q^{\mathcal{I}}(x) := q$ for all $q \in \mathcal{V}_{\mathcal{O}}$ and $x \in \Delta^{\mathcal{I}}$.

We first consider the root node ε of the tree. Recall that the ontology contains a local ABox, using only the individual name a. We define $g(\varepsilon) := a^{\mathcal{I}}$ and the Hintikka ordering $\lesssim_\varepsilon$ as follows for all $\alpha,\beta \in \mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$:

$$\alpha \lesssim_\varepsilon \beta \quad \text{if and only if} \quad \alpha^{\mathcal{I}}(a^{\mathcal{I}}) \leq \beta^{\mathcal{I}}(a^{\mathcal{I}}). \tag{6}$$

We extend this order to the elements in $\mathsf{sub}_{\uparrow}(\mathcal{O}) \cup \{\lambda, \neg\lambda\}$ arbitrarily, in such a way that for all $\alpha,\beta \in \mathcal{U}$ we have $\alpha \lesssim_\varepsilon \beta$ iff $\mathrm{inv}(\beta) \lesssim_\varepsilon \mathrm{inv}(\alpha)$. Such an extension is possible since $\neg$ is interpreted as the involutive negation. It is clear that this defines a total preorder satisfying (P5). In particular, it preserves the natural order on $\mathcal{V}_{\mathcal{O}}$ and has 0 and 1 as least and greatest element, respectively. Thus, it is an element of $\mathrm{order}(\mathcal{U})$.

We show that $\lesssim_\varepsilon$ is a Hintikka ordering. Let $C \in \mathsf{sub}(\mathcal{O})$. If $C = \top$, we have $\top^{\mathcal{I}}(a^{\mathcal{I}}) = 1$, and thus $\top \equiv_\varepsilon 1$. If $C = D \sqcap E$, then

$$\begin{aligned} C^{\mathcal{I}}(a^{\mathcal{I}}) &= \min(D^{\mathcal{I}}(a^{\mathcal{I}}), E^{\mathcal{I}}(a^{\mathcal{I}})) \\ &= \begin{cases} D^{\mathcal{I}}(a^{\mathcal{I}}) & \text{if } D^{\mathcal{I}}(a^{\mathcal{I}}) \leq E^{\mathcal{I}}(a^{\mathcal{I}}) \\ E^{\mathcal{I}}(a^{\mathcal{I}}) & \text{if } E^{\mathcal{I}}(a^{\mathcal{I}}) < D^{\mathcal{I}}(a^{\mathcal{I}}). \end{cases} \end{aligned}$$

Thus, by definition of $\lesssim_\varepsilon$, we get $C \equiv_\varepsilon \min_\varepsilon(D,E)$. Analogous arguments can be used for $C = D \to E$. Furthermore, $\lesssim_\varepsilon$ is compatible with $\mathcal{T}$ since for every $\langle C \sqsubseteq D \geq q\rangle \in \mathcal{T}$ we have $q \leq C^{\mathcal{I}}(a^{\mathcal{I}}) \Rightarrow D^{\mathcal{I}}(a^{\mathcal{I}})$, and thus $q \lesssim_\varepsilon \mathrm{res}_\varepsilon(C,D)$.

Assume now that $g(u)$ and $\lesssim_u$ are already defined for a node $u \in \{1,\dots,n\}^*$ such that (P5) is satisfied. For all $i \in \{1,\dots,n\}$, we now define $\lesssim_{ui}$ in such a way that the tuple $(\lesssim_u, \lesssim_{u1}, \dots, \lesssim_{un})$ satisfies the Hintikka condition. For brevity, we consider only the case that $i = \phi(\exists r.D)$; value restrictions can be handled using similar arguments. Since $\mathcal{I}$ is witnessed, there must be a domain element $y_i \in \Delta^{\mathcal{I}}$ such that $(\exists r.D)^{\mathcal{I}}(g(u)) = \min(r^{\mathcal{I}}(g(u), y_i), D^{\mathcal{I}}(y_i))$. Define $g(ui) := y_i$, and $\lesssim_{ui}$ for all $\alpha, \beta \in \mathcal{U}$ by

$$\alpha \lesssim_{ui} \beta \quad \text{if and only if} \quad \alpha^{\mathcal{I}}(g(ui)) \leq \beta^{\mathcal{I}}(g(ui)), \tag{7}$$

where $\lambda^{\mathcal{I}}(g(ui)) := r^{\mathcal{I}}(g(u), g(ui))$ and $(C_{\uparrow})^{\mathcal{I}}(g(ui)) := C^{\mathcal{I}}(g(u))$ for all concepts $C \in \mathsf{sub}(\mathcal{O})$. It is clear that $\lesssim_{ui}$ behaves on $\mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}_{\uparrow}(\mathcal{O})$ exactly as $\lesssim_u$ does on $\mathcal{V}_{\mathcal{O}} \cup \mathsf{sub}(\mathcal{O})$. Following the same arguments used for the root node, it is easy to show that $\lesssim_{ui}$ is actually a Hintikka ordering compatible with $\mathcal{T}$.

We show the Hintikka condition for $(\lesssim_u, \lesssim_{u1}, \dots, \lesssim_{un})$. If $i = \phi(\exists r.D)$, then by construction of g we have $(\exists r.D)^{\mathcal{I}}(g(u)) = \min\left(r^{\mathcal{I}}(g(u), g(ui)), D^{\mathcal{I}}(g(ui))\right)$, and thus

$$((\exists r.D)_{\uparrow})^{\mathcal{I}}(g(ui)) = \min\left(\lambda^{\mathcal{I}}(g(ui)), D^{\mathcal{I}}(g(ui))\right).$$

Using (7), we obtain $(\exists r.D)_{\uparrow} \equiv_{ui} \min_{ui}(\lambda, D)$, as required. Furthermore, for all $i \in \Phi_r$, it holds that

$$\begin{aligned}(\exists r.D)^{\mathcal{I}}(g(u)) &= \sup_{y \in \Delta^{\mathcal{I}}} \min\left(r^{\mathcal{I}}(g(u), y), D^{\mathcal{I}}(y)\right) \\ &\geq \min\left(r^{\mathcal{I}}(g(u), g(ui)), D^{\mathcal{I}}(g(ui))\right),\end{aligned}$$

which similarly shows that $(\exists r.D)_{\uparrow} \gtrsim_{ui} \min_{ui}(\lambda, D)$ holds. Similar arguments apply to the value restrictions in $\mathsf{sub}(\mathcal{O})$.

Finally, for every $\langle a{:}C \bowtie q\rangle \in \mathcal{A}$, we have $C^{\mathcal{I}}(a^{\mathcal{I}}) \bowtie q$, and thus $C \bowtie_{\varepsilon} q$ by definition of $\lesssim_{\varepsilon}$, and similarly for assertions of the form $\langle a{:}C \bowtie a{:}D\rangle$. Hence, the tree defined by $\lesssim_u$, for $u \in \{1, \dots, n\}^*$, is a Hintikka tree for $\mathcal{O}$. □

Propositions 9 and 10 show that Hintikka trees characterize consistency of an ontology with a local ordered ABox. In other words, deciding the existence of a Hintikka tree for $\mathcal{O}$ suffices for deciding consistency of $\mathcal{O}$. We now turn our attention to deciding the existence of such trees, and show that this problem can be solved in exponential time in the size of $\mathcal{O}$. For this, we construct a *looping tree automaton* whose runs correspond exactly to such Hintikka trees. Thus, the automaton accepts a non-empty language iff the ontology $\mathcal{O}$ is consistent.

A *looping automaton* over n-ary trees is a tuple $\mathbf{A} = (Q, I, \Delta)$, consisting of a non-empty set Q of *states*, a subset $I \subseteq Q$ of *initial states*, and a *transition relation* $\Delta \subseteq Q^{n+1}$. A *run* of this automaton is a mapping $\rho \colon \{1, \dots, n\}^* \to Q$ such that (i) $\rho(\varepsilon) \in I$, and (ii) for all $u \in \{1, \dots, n\}^*$, we have $\left(\rho(u), \rho(u1), \dots, \rho(un)\right) \in \Delta$. $\mathbf{A}$ is *non-empty* iff it has a run.

Definition 11. The *Hintikka automaton* for an ontology $\mathcal{O}$ is the looping tree automaton $\mathbf{A}_{\mathcal{O}} := (Q_{\mathcal{O}}, I_{\mathcal{O}}, \Delta_{\mathcal{O}})$, where

- $Q_\mathcal{O}$ is the set of all Hintikka orderings compatible with $\mathcal{T}$,
- $I_\mathcal{O} := \{\lesssim_H \in Q_\mathcal{O} \mid \lesssim_H$ is compatible with $\mathcal{A}\}$, and
- $\Delta_\mathcal{O}$ contains all tuples from $Q_\mathcal{O}^{n+1}$ that satisfy the Hintikka condition.

It is easy to see that the runs of $\mathbf{A}_\mathcal{O}$ are exactly the Hintikka trees for $\mathcal{O}$. Observe that the number of Hintikka orderings for $\mathcal{O}$ is bounded by $2^{|\mathcal{U}|^2}$ and the cardinality of $\mathcal{U} = \mathcal{V}_\mathcal{O} \cup \mathsf{sub}(\mathcal{O}) \cup \mathsf{sub}_\uparrow(\mathcal{O}) \cup \{\lambda, \neg\lambda\}$ is linear in the size of $\mathcal{O}$. Likewise, the arity n of the automaton is bounded by $|\mathsf{sub}(\mathcal{O})|$, which is linear in the size of $\mathcal{O}$. Thus, the size of the Hintikka automaton $\mathbf{A}_\mathcal{O}$ is exponential in the size of $\mathcal{O}$. Since (non-)emptiness of looping tree automata can be decided in polynomial time [30], we obtain an ExpTime-decision procedure for consistency of ontologies with local ordered ABoxes in $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$. Note that concept satisfiability in classical $\mathcal{ALC}$ is already ExpTime-hard w.r.t. general TBoxes [25], and hence we have tight complexity bounds.

Theorem 12. *Consistency in $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$ w.r.t. local ordered ABoxes and witnessed models is* ExpTime*-complete.*

In the following section we remove the restriction to local ordered ABoxes and show that consistency remains ExpTime-complete in the general case.

5 Reducing Consistency to Local Consistency

To decide consistency of $\mathsf{G}\text{-}\mathfrak{I}\mathcal{ALC}$-ontologies containing more that one individual name, we adapt a technique from classical DLs known as *pre-completion* [22]. Intuitively, we are trying to build a forest-shaped model that satisfies the ontology. This model is composed of a finite set of trees, one for each individual name appearing in the ABox, whose roots can be arbitrarily interconnected due to the presence of role assertions. As before, rather than explicitly building such models, we use total preorders to represent them in an abstract manner.

The idea of pre-completion is to extend the input ABox to a full specification of each individual, and then decide consistency w.r.t. the local ABoxes associated with each individual name. In our setting, this amounts to extending the input ABox to a total preorder $\lesssim_\mathcal{A}$. This preorder represents the nucleus of a model of the ontology. To extend this to a full model, we check an (ordered) local consistency condition for each of the individual names, and use $\lesssim_\mathcal{A}$ to combine the resulting interpretations.

More formally, let $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ be an ontology, and let $\mathsf{Ind}(\mathcal{A})$ denote the set of individual names occurring in $\mathcal{A}$. We define the order structure

$$\begin{aligned}\mathcal{W} := \mathcal{V}_\mathcal{O} &\cup \{a{:}C \mid a \in \mathsf{Ind}(\mathcal{A}),\ C \in \mathsf{sub}(\mathcal{O})\} \\ &\cup \{(a,b){:}r \mid a,b \in \mathsf{Ind}(\mathcal{A}),\ r \text{ occurs in } \mathcal{O}\} \\ &\cup \{(a,b){:}\neg r \mid a,b \in \mathsf{Ind}(\mathcal{A}),\ r \text{ occurs in } \mathcal{O}\}\end{aligned}$$

with $\mathrm{inv}(a{:}C) := a{:}\neg C$ and $\mathrm{inv}((a,b){:}r) := (a,b){:}\neg r$.

Definition 13 (pre-completion). A *pre-completion* of $\mathcal{A}$ w.r.t. $\mathcal{T}$ is a total pre-order $\lesssim_\mathcal{A} \in \mathrm{order}(\mathcal{W})$ such that:

a) for every $a \in \mathsf{Ind}(\mathcal{A})$ and all $C \in \mathsf{sub}(\mathcal{O})$,
   - if $C = \top$, then $a{:}C \equiv_\mathcal{A} 1$,
   - if $C = D_1 \sqcap D_2$, then $a{:}C \equiv_\mathcal{A} \min_\mathcal{A}(a{:}D_1, a{:}D_2)$,
   - if $C = D_1 \to D_2$, then $a{:}C \equiv_\mathcal{A} \mathrm{res}_\mathcal{A}(a{:}D_1, a{:}D_2)$;

b) for every $\exists r.C \in \mathsf{sub}(\mathcal{O})$ and $a,b \in \mathsf{Ind}(\mathcal{A})$, we have

$$a{:}\exists r.C \gtrsim_\mathcal{A} \min_\mathcal{A}((a,b){:}r, b{:}C);$$

c) for every $\forall r.C \in \mathsf{sub}(\mathcal{O})$ and $a,b \in \mathsf{Ind}(\mathcal{A})$, we have

$$a{:}\forall r.C \lesssim_\mathcal{A} \mathrm{res}_\mathcal{A}((a,b){:}r, b{:}C);$$

d) for all $a \in \mathsf{Ind}(\mathcal{A})$ and every GCI $\langle C \sqsubseteq D \geq q\rangle \in \mathcal{T}$, we have

$$\mathrm{res}_\mathcal{A}(a{:}C, a{:}D) \gtrsim_\mathcal{A} q; \text{ and}$$

e) for every assertion $\langle \alpha \bowtie \beta\rangle \in \mathcal{A}$, we have $\alpha \bowtie_\mathcal{A} \beta$.

This definition generalizes the local conditions of Definitions 7 and 8 to handle several named individuals simultaneously. The main difference is that we do not create witnesses for the quantified concepts here. This will be taken care of by testing the following local ordered ABoxes for consistency.

For a pre-completion $\lesssim_\mathcal{A}$ and $a \in \mathsf{Ind}(\mathcal{A})$, we define the local ordered ABox $\mathcal{A}_a$ as the set of all order assertions $\langle \alpha \bowtie \beta\rangle$ over a and $\mathsf{sub}(\mathcal{O})$ for which $\alpha \bowtie_\mathcal{A} \beta$ holds.[2] That is,

$$\begin{aligned}\mathcal{A}_a := \{\langle a{:}C \bowtie q\rangle \mid C \in \mathsf{sub}(\mathcal{O}),\ q \in \mathcal{V}_\mathcal{O},\ a{:}C \bowtie_\mathcal{A} q\} \cup \\ \{\langle a{:}C \bowtie a{:}D\rangle \mid C,D \in \mathsf{sub}(\mathcal{O}),\ a{:}C \bowtie_\mathcal{A} a{:}D\}.\end{aligned}$$

[2] It actually suffices to consider only $\bowtie \in \{>,=,<\}$.

Lemma 14. *An ontology $\mathcal{O} = (\mathcal{A}, \mathcal{T})$ is consistent if and only if there is a pre-completion $\lesssim_\mathcal{A}$ of $\mathcal{A}$ w.r.t. $\mathcal{T}$ such that, for every $a \in \mathsf{Ind}(\mathcal{A})$, the ontology $\mathcal{O}_a := (\mathcal{A}_a, \mathcal{T})$ is consistent.*

Proof. Let $\mathcal{I}$ be a model of $\mathcal{O}$. We define the total preorder $\lesssim_\mathcal{A}$ by

$$\alpha \lesssim_\mathcal{A} \beta \text{ iff } \alpha^\mathcal{I} \leq \beta^\mathcal{I},$$

where we set $((a,b){:}\neg r)^\mathcal{I} := 1 - r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I})$. In particular, $\lesssim_\mathcal{A}$ preserves the natural order on $\mathcal{V}_\mathcal{O}$ and has 0 and 1 as least and greatest element, respectively. Furthermore, it satisfies $\alpha \lesssim_\mathcal{A} \beta$ iff $\mathrm{inv}(\beta) \lesssim_\mathcal{A} \mathrm{inv}(\alpha)$ for all $\alpha, \beta \in \mathcal{W}$, i.e. it is an element of $\mathrm{order}(\mathcal{W})$.

Since $\mathcal{I}$ satisfies $\mathcal{A}$, for every $\langle \alpha \bowtie \beta \rangle \in \mathcal{A}$, we have $\alpha^\mathcal{I} \bowtie \beta^\mathcal{I}$, which shows that the preorder $\lesssim_\mathcal{A}$ satisfies Condition e) of Definition 13. For Condition b), consider some $a, b \in \mathsf{Ind}(\mathcal{A})$ and $\exists r.C \in \mathsf{sub}(\mathcal{O})$. By the semantics of $\exists$, we have that $(\exists r.C)^\mathcal{I}(a^\mathcal{I}) \geq \min(r^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}), C^\mathcal{I}(b^\mathcal{I}))$, which already shows the claim. The remaining conditions of Definition 13 can be shown using similar arguments. Finally, it is easy to see that $\mathcal{I}$ is also a model of $(\mathcal{A}_a, \mathcal{T})$ for each $a \in \mathsf{Ind}(\mathcal{A})$.

Conversely, let $\lesssim_\mathcal{A}$ be a pre-completion of $\mathcal{A}$ w.r.t. $\mathcal{T}$ and each $(\mathcal{A}_a, \mathcal{T})$ be consistent. By Proposition 10, there are Hintikka trees for $(\mathcal{A}_a, \mathcal{T})$ that consist of Hintikka orderings $\lesssim_u^a$ for all $u \in \{1, \dots, n\}^*$, where n is the number of existential and value restrictions in $\mathsf{sub}(\mathcal{O})$. Similar to the proof of Proposition 9, we first construct a function $v \colon \mathcal{W} \cup (\mathsf{Ind}(\mathcal{A}) \times \mathcal{U} \times \{1, \dots, n\}^*) \to [0,1]$ such that

- for all values $q \in \mathcal{V}_\mathcal{O}$, we have $v(q) = q$,
- for all $\alpha, \beta \in \mathcal{W}$, we have $v(\alpha) \leq v(\beta)$ iff $\alpha \lesssim_\mathcal{A} \beta$,
- for all $\alpha \in \mathcal{W}$, we have $v(\mathrm{inv}(\alpha)) = 1 - v(\alpha)$,
- for every $C \in \mathsf{sub}(\mathcal{O})$ and all $a \in \mathsf{Ind}(\mathcal{A})$, we have $v(a{:}C) = v(a, C, \varepsilon)$,
- for all $u \in \{1, \dots, n\}^*$ and all $a \in \mathsf{Ind}(\mathcal{A})$,
  - for all values $q \in \mathcal{V}_\mathcal{O}$, we have $v(a, q, u) = q$,
  - for all $\alpha, \beta \in \mathcal{U}$, we have $v(a, \alpha, u) \leq v(a, \beta, u)$ iff $\alpha \lesssim_u^a \beta$,
  - for all $\alpha \in \mathcal{U}$, we have $v(a, \mathrm{inv}(\alpha), u) = 1 - v(a, \alpha, u)$, and
  - for all concepts $C \in \mathsf{sub}(\mathcal{O})$ and all $i \in \{1, \dots, n\}$, we have that $v(a, C, u) = v(a, C_\uparrow, ui)$.

We will then use this function to define a model of $\mathcal{O}$.

Using the technique from the proof of Proposition 9, we first define v on $\mathcal{W}$. On the set $\mathcal{W}/{\equiv_\mathcal{A}}$ of all equivalence classes of $\equiv_\mathcal{A}$, $\lesssim_\mathcal{A}$ induces a total order $<_\mathcal{A}$ such that $[0]_\mathcal{A} <_\mathcal{A} [q_1]_\mathcal{A} <_\mathcal{A} \cdots <_\mathcal{A} [q_{k-1}]_\mathcal{A} <_\mathcal{A} [1]_\mathcal{A}$. We first define the auxiliary function $\tilde{v}_\mathcal{A} \colon \mathcal{W}/{\equiv_\mathcal{A}} \to [0,1]$, starting with $\tilde{v}_\mathcal{A}([q]_\mathcal{A}) := q$ for each $q \in \mathcal{V}_\mathcal{O}$. For every $i \in \{0, \dots, k-1\}$, let now $E^i_1, \dots, E^i_{\nu_i}$ be all equivalence classes strictly between $[q_i]_\mathcal{A}$ and $[q_{i+1}]_\mathcal{A}$ such that

$$[q_i]_\mathcal{A} <_\mathcal{A} E^i_1 <_\mathcal{A} \cdots <_\mathcal{A} E^i_{\nu_i} <_\mathcal{A} [q_{i+1}]_\mathcal{A}.$$

We set $\tilde{v}_\mathcal{A}(E^i_j) := q_i + \frac{j}{\nu_i+1}(q_{i+1} - q_i)$, for all j, $1 \leq j \leq \nu_i$, and then define $v(\alpha) := \tilde{v}_\mathcal{A}([\alpha]_\mathcal{A})$ for all $\alpha \in \mathcal{W}$.

For each $a \in \mathsf{Ind}(\mathcal{A})$ and $C \in \mathsf{sub}(\mathcal{O})$, we now set $v(a, C, \varepsilon) := v(a{:}C)$. The values of $v(a, \alpha, \varepsilon)$ for elements $\alpha \in \mathsf{sub}_\uparrow(\mathcal{O}) \cup \{\lambda, \neg\lambda\}$ are irrelevant for the desired properties and can be fixed arbitrarily, as long as we have $v(a, \alpha, \varepsilon) \leq v(a, \beta, \varepsilon)$ iff $\alpha \lesssim^a_\varepsilon \beta$ and $v(a, \mathrm{inv}(\alpha), \varepsilon) = 1 - v(a, \alpha, u)$ for all $\alpha, \beta \in \mathcal{U}$, e.g. using the technique from above. The definition of $v(a, \alpha, u)$ can now proceed as in the proof of Proposition 9 based on the Hintikka trees for $(\mathcal{A}_a, \mathcal{T})$. This construction ensures that v has the desired properties.

We now define the interpretation $\mathcal{I}$ as follows:

- $\Delta^\mathcal{I} := \mathsf{Ind}(\mathcal{A}) \times \{1, \dots, n\}^*$,
- $a^\mathcal{I} := (a, \varepsilon)$ for each $a \in \mathsf{Ind}(\mathcal{A})$,
- $A^\mathcal{I}(a, u) := v(a, A, u)$ for all $a \in \mathsf{Ind}(\mathcal{A})$, concept names $A \in \mathsf{sub}(\mathcal{O})$, and $u \in \{1, \dots, n\}^*$, and
- $r^\mathcal{I}((a, u), (b, u')) :=$

$$\begin{cases} v(a, \lambda, ui) & \text{if } a = b \text{ and } u' = ui \text{ with } i \in \Phi_r, \\ v((a,b){:}r) & \text{if } u = u' = \varepsilon \text{ and } r \text{ occurs in } \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

The interpretation of the remaining individual and concept names is irrelevant and can be fixed arbitrarily. As in Proposition 9, we can show by induction on the structure of C that $C^\mathcal{I}(a, u) = v(a, C, u)$ holds for all $C \in \mathsf{sub}(\mathcal{O})$, $a \in \mathsf{Ind}(\mathcal{A})$, and $u \in \{1, \dots, n\}^*$. The claim for $\top$, $\neg C$, $C \sqcap D$, and $C \to D$ follows as before from Condition a) of Definition 13 and the fact that each $\lesssim^a_u$ is a Hintikka ordering.

Consider now an existential restriction $\exists r.C \in \mathsf{sub}(\mathcal{O})$ and the domain element (a, ε) for some $a \in \mathsf{Ind}(\mathcal{A})$. By the Hintikka condition and the induction hypothesis, we obtain that $v(a, \exists r.C, u) = \min\big(r^\mathcal{I}((a, \varepsilon), (a, i_0)), C^\mathcal{I}(a, i_0)\big)$, where $i_0 = \phi(\exists r.C)$, as in the proof of Proposition 9. Likewise, we get that $v(a, \exists r.C, u) \geq \min(r^\mathcal{I}((a, \varepsilon), (a, i)), C^\mathcal{I}(a, i))$ holds for all $i \in \Phi_r$. Finally, for each $b \in \mathsf{Ind}(\mathcal{A})$, we have $v(a, \exists r.C, u) \geq \min(r^\mathcal{I}((a, \varepsilon), (b, \varepsilon)), C^\mathcal{I}(b, \varepsilon))$ by Condition b) of Definition 13. Since (a, ε) does not have any other relevant r-successors, this shows the claim for $\exists r.C$ at (a, ε). At the other domain elements, it can be shown as for Proposition 9. Similar arguments apply for any $\forall r.C \in \mathsf{sub}(\mathcal{O})$.

Finally, the fact that $\mathcal{I}$ is actually a model of $\mathcal{O}$ is ensured by compatibility of all Hintikka orderings with $\mathcal{T}$ and Conditions e) and d) of Definition 13. $\square$

Note that the cardinality of order($\mathcal{W}$) is exponential in the size of $\mathcal{O}$, and all elements of order($\mathcal{W}$) are of polynomial size. We can thus enumerate order($\mathcal{W}$), check for each element whether it satisfies Definition 13 in polynomial time, and then execute the polynomially many local consistency tests as described by Lemma 14. This yields the following complexity result.

Corollary 15. *Consistency in* G-$\mathfrak{I}\mathcal{ALC}$ *w.r.t. witnessed models is* ExpTime-*complete.*

6 Satisfiability and Subsumption

We have described an exponential-time algorithm for deciding consistency of G-$\mathfrak{I}\mathcal{ALC}$ ontologies. We now direct our attention at other standard reasoning problems in (fuzzy) DLs; namely, deciding concept satisfiability and subsumption, and computing the best truth degrees to which these hold. Recall from Section 2 that we can restrict our attention to ontologies with an empty ABox.

Let now $\mathcal{O} = (\emptyset, \mathcal{T})$ be an ontology. It is easy to see that p-subsumption and p-satisfiability w.r.t. $\mathcal{O}$ can be reduced in polynomial time to consistency w.r.t. local ABoxes. More precisely, for any two concepts C, D and $p \in [0, 1]$,

- C is p-satisfiable w.r.t. $\mathcal{O}$ iff $(\{\langle a : C \geq p \rangle\}, \mathcal{T})$ is consistent, and
- C is p-subsumed by D w.r.t. $\mathcal{O}$ iff $(\{\langle a : C \to D < p \rangle\}, \mathcal{T})$ is inconsistent,

where a is an arbitrary individual name. We thus obtain the following result from Theorem 12.

Theorem 16. *Satisfiability and subsumption in* G-$\mathfrak{I}\mathcal{ALC}$ *w.r.t. witnessed models are* ExpTime-*complete.*

We now consider the problems of computing the *best* satisfiability and subsumption degrees. We first show that the local consistency checks required for deciding p-satisfiability and p-subsumption only depend on the position of p relative to the values occurring in $\mathcal{T}$, but not on the precise value of p. To prove this, we again use the preorders of the previous sections, and in particular Hintikka trees.

Lemma 17. *Let* $p, p' \in (q_i, q_{i+1})$ *for two adjacent values* $q_i, q_{i+1} \in \mathcal{V}_\mathcal{O}$*, and* C *be a concept. Then* $(\{\langle a : C \bowtie p \rangle\}, \mathcal{T})$ *is consistent iff* $(\{\langle a : C \bowtie p' \rangle\}, \mathcal{T})$ *is consistent.*

Proof. By Propositions 9 and 10, both consistency conditions are equivalent to the existence of Hintikka trees, albeit over different order structures. We denote by $\mathcal{U}_p$ the order structure defined in (4) over the set $\mathcal{V}_p := \mathcal{V}_\mathcal{O} \cup \{p, 1-p\}$, and by $\mathcal{U}_{p'}$ the one over $\mathcal{V}_{p'} := \mathcal{V}_\mathcal{O} \cup \{p', 1-p'\}$. Observe that the bijection $\iota\colon \mathcal{V}_p \to \mathcal{V}_{p'}$ that simply maps p to p' and $1-p$ to $1-p'$ and leaves the other values as they are, can be extended to a bijection between $\mathcal{U}_p$ and $\mathcal{U}_{p'}$ by defining it as the identity on all elements outside of $\mathcal{V}_p$. Furthermore, it is compatible with the involutive operator inv, i.e. we have $\iota(\text{inv}(\alpha)) = \text{inv}(\iota(\alpha))$ for all $\alpha \in \mathcal{U}_p$.

We now lift this bijection to the sets order($\mathcal{U}_p$) and order($\mathcal{U}_{p'}$) by setting, for any $\lesssim_p \in \text{order}(\mathcal{U}_p)$, $\alpha \lesssim_{p'} \beta$ iff $\iota(\alpha) \lesssim_p \iota(\beta)$ for all $\alpha, \beta \in \mathcal{U}_{p'}$. It is easy to see that this defines an element of order($\mathcal{U}_{p'}$) and that every element of order($\mathcal{U}_{p'}$) can be obtained in this way (simply apply the inverse of ι). In particular, $\lesssim_{p'}$ preserves the order of the real numbers on $\mathcal{V}_{p'}$ since p and p' are in the same relative position w.r.t. the elements of $\mathcal{V}_\mathcal{O}$. Furthermore, we have $\iota(\min_p(\alpha, \beta)) = \min_{p'}(\iota(\alpha), \iota(\beta))$ and $\iota(\text{res}_p(\alpha, \beta)) = \text{res}_{p'}(\iota(\alpha), \iota(\beta))$.

Moreover, if $\lesssim_p$ is a Hintikka ordering, then $\lesssim_{p'}$ is also a Hintikka ordering, and vice versa, since this notion only depends on the order between the concepts in $\mathcal{U}_p/\mathcal{U}_{p'}$. Compatibility with $\mathcal{T}$ is also equivalent for the two preorders. Similarly, by definition of $\lesssim_{p'}$, $\lesssim_p$ is compatible with $\{\langle a\colon C \bowtie p\rangle\}$ iff $C \bowtie_p p$ iff $C \bowtie_{p'} p'$ iff $\lesssim_{p'}$ is compatible with $\{\langle a\colon C \bowtie p'\rangle\}$.

From the above arguments and similar ones for the Hintikka condition, it follows that there is a Hintikka tree for $(\{\langle a\colon C \bowtie p\rangle\}, \mathcal{T})$ iff there is a Hintikka tree for $(\{\langle a\colon C \bowtie p'\rangle\}, \mathcal{T})$, which concludes the proof. □

This shows that subsumption between C and D or satisfiability of C either holds for all values in an interval (q_i, q_{i+1}), or for none of them.

Corollary 18. *For any two concepts C and D, the best subsumption degree of C and D w.r.t. $\mathcal{O}$ is always in $\mathcal{V}_\mathcal{O}$. Likewise, the best satisfiability degree of C w.r.t. $\mathcal{O}$ is always in $\mathcal{V}_\mathcal{O}$.*

Since the best subsumption degree p of C and D is always a subsumption degree, i.e. C is p-subsumed by D, it suffices to check subsumption w.r.t. the values from $\mathcal{V}_\mathcal{O}$ in order to determine the best subsumption degree. Thus, we only have to execute linearly many (in-)consistency checks to compute the best subsumption degree.

However, it is possible that C is p-satisfiable for every $p \in (q_i, q_{i+1})$, but it is not q_{i+1}-satisfiable. Therefore, to compute the best satisfiability degree, we have to check satisfiability for all values $\frac{q_i+q_{i+1}}{2}$. The best satisfiability degree is then the largest q_{i+1} for which this check succeeds (or 0 if it never succeeds). Again, this means that we have to execute linearly many consistency checks to compute the best satisfiability degree.

By combining these reductions with Theorem 12, we obtain the following corollary.

Corollary 19. *In* G-$\mathcal{ALC}$ *w.r.t. witnessed models, best subsumption and satisfiability degrees can be computed in exponential time.*

7 Conclusions

We have studied the standard reasoning problems for the fuzzy DL G-$\mathfrak{I}\mathcal{ALC}$ w.r.t. witnessed model semantics. The contributions of the paper are twofold. First, we have shown that, contrary to popular belief, reasoning in this logic cannot be restricted to reasoning over finitely-valued models without affecting its consequences. In particular, this implies that the algorithms based on maintaining only a finite set of truth degrees [7, 8] are incomplete for the general semantics. Moreover, this also implies that the logic does not have the finite model property, and hence standard tableau-based approaches cannot terminate [9, 29, 5].

As the second contribution of the paper, we showed that all standard reasoning problems can be solved in exponential time. To achieve this, we developed an automaton that decides the existence of a Hintikka tree, which is an abstract representation of a model of a given ontology. The main insight needed for this approach is that we can abstract from the precise truth degrees assigned by an interpretation, and focus only on their ordering.

As an added benefit, our formalism allows us to express order assertions like $\langle \mathsf{ana}\!:\!\mathsf{Tall} > \mathsf{bob}\!:\!\mathsf{Tall} \rangle$, intuitively stating that Ana is taller than Bob, without needing to specify the precise degrees to which ana and bob belong to the concept Tall. This is similar to concrete domains [24], which can even compare values at unnamed domain elements. But concrete domains allow only for atomic attributes, whereas order assertions can also contain complex concepts.

As we have developed an automata-based algorithm, it is natural to ask whether previous automata-based approaches [2, 14] can be adapted to this setting in order to handle the expressivity up to G-$\mathfrak{I}\mathcal{SCHI}$, or provide better upper-bounds for reasoning w.r.t. *acyclic* TBoxes. We will study this problem in future work. We also plan to adapt these ideas into a tableau-based algorithm which is more suitable for implementation.

Recall that we have restricted our framework to reasoning w.r.t. *witnessed* models only. Indeed, this restriction is fundamental for our proof of Proposition 10. One open question is whether consistency of G-$\mathfrak{I}\mathcal{ALC}$ ontologies w.r.t. *general* models is still decidable. We conjecture that it is, and in fact remains in ExpTime.

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