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Abstract

Fuzzy description logics (DLs) have been investigated for over two decades, due to their capacity to formalize and reason with imprecise concepts. Very recently, it has been shown that for several fuzzy DLs, reasoning becomes undecidable. Although the proofs of these results differ in the details of each specific logic considered, they are all based on the same basic idea.

In this report, we formalize this idea and provide sufficient conditions for proving undecidability of a fuzzy DL. We demonstrate the effectiveness of our approach by strengthening all previously-known undecidability results and providing new ones. In particular, we show that undecidability may arise even if only crisp axioms are considered.

1 Introduction

Description logics (DLs) [1] are a family of logic-based knowledge representation formalisms, which can be used to represent the knowledge of an application domain in a formally well-understood way. They have been successfully applied in the definition of medical ontologies, like SNOMED CT[1] and GALEN,[2] but their main breakthrough arguably arrived with the adoption of the DL-based language OWL [19] as the standard ontology language for the semantic web.

Fuzzy variants of description logics have been introduced to deal with applications where concepts cannot be specified in a precise way. For example, in the medical domain a high body temperature is often a symptom for a disease. When trying to represent this knowledge, it makes sense to see High as a fuzzy concept: there is no precise point where a temperature becomes high, but we know that 36°C belongs to this concept with a lower membership than, say 39°C. A more detailed

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[1]http://www.ihtsdo.org/snomed-ct/

[2]http://www.opengalen.org/

description of the use of fuzzy semantics in medical applications can be found in [22].

A great variety of fuzzy DLs can be found in the literature (see [21, 16] for a survey). In fact, fuzzy DLs have several degrees of freedom for defining their expressiveness. In addition to the choice of concept constructors (such as conjunction $\sqcap$ or existential restriction $\exists$), and the type of axioms allowed (like acyclic concept definitions or general concept inclusions), one must also decide how to interpret the different constructors, through a choice of functions over the domain of fuzzy values $[0, 1]$. These functions are typically determined by the choice of a continuous t-norm (like Gödel, Łukasiewicz, and product) that interprets conjunction; however, there exist uncountably many such t-norms, each with different properties. For example, under the product t-norm semantics, existential- ($\exists$) and value-restrictions ($\forall$) are not interdefinable, while under the Łukasiewicz t-norm they are. Even after fixing the underlying t-norm, one can choose whether to interpret negation by the involutive negation operator, or using the residual negation. An additional level of liberty comes from selecting the class of models over which reasoning is considered: either all models, or so-called witnessed models only [18].

Most existing reasoning algorithms have been developed for the Gödel semantics, either by a reduction to crisp reasoning [29, 6], or by a simple adaptation of the known algorithms for crisp DLs [26, 27, 31]. However, methods based on other t-norms have also been explored [7, 8, 9, 30, 25]. Usually, these algorithms reason w.r.t. witnessed models.[3]

Very recently, it was shown that the tableaux-based algorithms for logics with semantics based on t-norms other than the Gödel t-norm and allowing general concept inclusions were incorrect [2, 5]. This raised doubts about the decidability of these logics, and eventually led to a series of undecidability results for fuzzy DLs [2, 3, 4, 14]. All these papers, except [4], focus on one specific fuzzy DL; that is, undecidability is proven for a specific set of constructors, axioms, and underlying semantics. A small generalization is made in [4], where undecidability is shown for a whole family of t-norms–specifically, all t-norms "starting" with the product t-norm–and two variants of witnessed models.

Abstracting from the particularities of each logic, the proofs of undecidability appearing in [2, 3, 4, 14] follow similar ideas. The goal of this paper is to formalize this idea and give a general description of a proof of undecidability, which can be instantiated to different fuzzy DLs. More precisely, we describe a general proof method, based on a reduction from the Post Correspondence Problem, and present sufficient conditions for the applicability of this method to a given fuzzy DL.

We demonstrate the effectiveness of our approach by providing several new unde-

[3]In fact, witnessed models were introduced in [18] to correct the algorithm from [31].

Name	t-norm ($x \otimes y$)	Residuum ($x \Rightarrow y$)
Gödel	$\min\{x, y\}$	$\begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
product	$x \cdot y$	$\begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$
Łukasiewicz	$\max\{x + y - 1, 0\}$	$\min\{1 - x + y, 1\}$

Table 1: Gödel, product and Łukasiewicz t-norms and their residua

cidability results for fuzzy DLs. In particular, we improve the results from [2, 14] by showing that a weaker DL suffices for obtaining undecidability, and the results from [3, 4], by allowing a wider family of t-norms. We also provide the first undecidability results for reasoning w.r.t. general models. An interesting outcome of our study is that, for the product t-norm and any t-norm "starting" with the Łukasiewicz t-norm, undecidability arises even if only crisp axioms are allowed.

2 T-norms and Fuzzy Logic

Fuzzy logics are formalisms introduced to express imprecise or vague information [17]. They extend classical logic by interpreting predicates as fuzzy sets over an interpretation domain. Given a non-empty domain $\mathcal{D}$, a *fuzzy set* is a function $F : \mathcal{D} \to [0, 1]$ from $\mathcal{D}$ into the real unit interval $[0, 1]$, with the intuition that an element $\delta \in \mathcal{D}$ belongs to F with *degree* $F(\delta)$. The interpretation of the logical constructors is based on appropriate truth functions that generalize the properties of the connectives of classical logic to the interval $[0, 1]$. The most prominent truth functions used in the fuzzy logic literature are based on t-norms [20].

A *t-norm* is an associative and commutative binary operator $\otimes : [0, 1] \times [0, 1] \to [0, 1]$ that has 1 as its unit element, and is monotonic, i.e., for every $x, y, z \in [0, 1]$, if $x \leq y$, then $x \otimes z \leq y \otimes z$. If $\otimes$ is a continuous t-norm, then there exists a unique binary operator $\Rightarrow$, called the *residuum*, that satisfies $z \leq x \Rightarrow y$ iff $x \otimes z \leq y$ for every $x, y, z \in [0, 1]$. Three important continuous t-norms are the Gödel, product and Łukasiewicz t-norms, shown in Table 1.

The following are simple consequences of the definition of t-norms and their residua (see [17], Lemma 2.1.6).

Lemma 1. *For every continuous t-norm* $\otimes$ *and* $x, y \in [0, 1]$,

- $x \Rightarrow y = 1$ *iff* $x \leq y$ *and*
- $1 \Rightarrow y = y$.

We say that a t-norm $\otimes$ (a,b)-*contains* the t-norm $\otimes'$, for $0 \leq a < b \leq 1$, if for every $x, y \in [0,1]$ it holds that

$$(a + (b-a)x) \otimes (a + (b-a)y) = a + (b-a)(x \otimes' y).$$

In this case, if $\Rightarrow$ and $\Rightarrow'$ denote the residua of $\otimes$ and $\otimes'$, respectively, then it also holds that

$$(a + (b-a)x) \Rightarrow (a + (b-a)y) = \begin{cases} 1 & \text{if } x \leq y, \\ a + (b-a)(x \Rightarrow' y) & \text{otherwise.} \end{cases}$$

Moreover, for every $x \in [a,b]$ and $y \notin [a,b]$, we have that $x \otimes y = \min\{x,y\}$. Intuitively, this means that $\otimes$ behaves like a scaled-down version of $\otimes'$ in the interval $[a,b]$, and as the Gödel t-norm if one and only one of the arguments belongs to $[a,b]$.

We say that a t-norm *contains* $\otimes'$ if it (a,b)-contains $\otimes'$ for some $0 \leq a < b \leq 1$. A consequence of the Mostert-Shields Theorem [23] is that every continuous t-norm $\otimes$ that is not the Gödel t-norm must contain the product or the Łukasiewicz t-norm. Notice that $\otimes$ may contain both the product and the Łukasiewicz t-norms; in fact, it may even contain infinitely many instances of these t-norms over disjoint intervals. For example, the t-norm defined for every $x, y \in [0,1]$ by

$$x \otimes y = \begin{cases} 2xy & \text{if } x, y \in [0, 0.5] \\ \max\{x + y - 1, 0.5\} & \text{if } x, y \in [0.5, 1] \\ \min(x, y) & \text{otherwise,} \end{cases}$$

$(0, 0.5)$-contains the product t-norm, and $(0.5, 1)$-contains the Łukasiewicz t-norm.

We denote the product and Łukasiewicz t-norms by Π and Ł, respectively. In general, a continuous t-norm that is not the Gödel t-norm may contain several instances of the product and Łukasiewicz t-norms. In the following, we always choose and fix a representative, and use the notation $\Pi^{(a,b)}$ to express that the t-norm (a,b)-contains the product t-norm, and similarly for $\text{Ł}^{(a,b)}$. Since the constructions we provide differ according to the t-norm, it is important to emphasize that we assume that the representative is fixed throughout the whole construction.

Fuzzy logics are sometimes extended with the involutive negation operator, defined as $\sim x := 1 - x$ [33, 15]. It should be noted that if $\otimes$ is the Łukasiewicz t-norm, then the involutive negation can be expressed through the equality $\sim x = x \Rightarrow 0$. However, for any other continuous t-norm $\sim$ is not expressible in terms of $\otimes$ and its residuum $\Rightarrow$.

Name	$\top$	$\bot$	$\sqcap$	$\rightarrow$	$\neg$	$\exists$	$\forall$
$\mathcal{EL}$	$\surd$		$\surd$			$\surd$	
$\mathcal{ELC}$	$\surd$		$\surd$		$\surd$	$\surd$	
$\mathfrak{I}\mathcal{EL}$	$\surd$	$\surd$	$\surd$	$\surd$		$\surd$	
$\mathcal{AL}$	$\surd$		$\surd$			$\surd$	$\surd$
$\mathcal{ALC}$	$\surd$		$\surd$		$\surd$	$\surd$	$\surd$
$\mathfrak{I}\mathcal{AL}$	$\surd$	$\surd$	$\surd$	$\surd$		$\surd$	$\surd$

Table 2: Some relevant DLs and the constructors they allow.

3 Fuzzy Description Logics

Just as classical description logics, fuzzy description logics are based on concepts, which are built from the mutually disjoint sets $\mathsf{N_C}, \mathsf{N_R}$ and $\mathsf{N_I}$ of *concept names*, *role names*, and *individual names*, respectively, using different constructors. A wide variety of constructors can be found in the literature. For this report, we consider only the constructors $\top$ (*top*), $\bot$ (*bottom*), $\sqcap$ (*conjunction*), $\rightarrow$ (*implication*), $\neg$ (*negation*), $\exists$ (*existential restriction*), and $\forall$ (*value restriction*). The motivation for these constructors is that, when restricted to classical semantics, they correspond to the crisp DL $\mathcal{ALC}$.

Definition 2 (concepts)**.** *(Complex) concepts* are built inductively from $\mathsf{N_C}$ and $\mathsf{N_R}$ as follows:

- every concept name $A \in \mathsf{N_C}$ is a concept
- if C, D are concepts and $r \in \mathsf{N_R}$, then $\top$, $\bot$, $C \sqcap D$, $C \rightarrow D$, $\neg C$, $\exists r.C$, and $\forall r.C$ are also concepts.

We will use the expression C^n to denote the n-ary conjunction of a concept C with itself; formally, $C^0 := \top$ and $C^{n+1} := C \sqcap C^n$ for every $n \geq 0$.

Different DLs are determined by the choice of constructors used. The DL $\mathcal{EL}$ allows only for the constructors $\top, \sqcap$, and $\exists$. $\mathcal{AL}$ additionally allows value restrictions. Following the notation from [13], the letters $\mathcal{C}$ and $\mathfrak{I}$ express that the negation and implication constructors are allowed, respectively. Table 2 summarizes this nomenclature.

The knowledge of a domain is represented using a set of axioms that express the relationships between individuals, roles, and concepts.

Definition 3 (axioms)**.** An *axiom* is one of the following:

- A *general concept inclusion axiom (GCI)* is of the form $C \sqsubseteq D$ for concepts C and D.[4]
- An *assertional axiom (assertion)* is of the form $\langle e : C \triangleright p\rangle$ or $\langle (d, e) : r \triangleright p\rangle$, where C is a concept, r a role name, d, e are individual names, and $\triangleright \in \{\geq, =\}$. This axiom is called a *crisp assertion* if $p = 1$, an *inequality assertion* if $\triangleright$ is $\geq$ and an *equality assertion* if $\triangleright$ is $=$.
- A *crisp role axiom* is of the form $\mathsf{crisp}(r)$ for a role name r.

An *ontology* is a finite set of axioms. It is called a *classical ontology* if it contains only GCIs and crisp assertions.

As with the choice of the constructors, the axioms influence the expressivity of the logic. We always assume that our logics allow at least classical ontologies. Given a DL $\mathcal{L}$, we will use the subscripts $\geq$, $=$, and c to denote that also inequality assertions, equality assertions, and crisp role axioms are allowed, respectively. For instance, $\mathcal{EL}_{\geq,\mathsf{c}}$ denotes the logic $\mathcal{EL}$ where ontologies can additionally contain inequality assertions and crisp role axioms, but not equality assertions.

Compared to classical DLs, fuzzy DLs have an additional degree of freedom in the selection of their semantics since the interpretation of the constructors depends on the t-norm chosen. Given a DL $\mathcal{L}$ and a continuous t-norm $\otimes$, we obtain the fuzzy DL $\otimes$-$\mathcal{L}$ that interprets the constructors as follows.

Definition 4 (semantics)**.** An *interpretation* $\mathcal{I} = (\mathcal{D}^\mathcal{I}, \cdot^\mathcal{I})$ consists of a non-empty *domain* $\mathcal{D}^\mathcal{I}$ and an *interpretation function* $\cdot^\mathcal{I}$ that assigns to every $A \in \mathsf{N_C}$ a fuzzy set $A^\mathcal{I} : \mathcal{D}^\mathcal{I} \to [0, 1]$, to every $r \in \mathsf{N_R}$ a fuzzy binary relation $r^\mathcal{I} : \mathcal{D}^\mathcal{I} \times \mathcal{D}^\mathcal{I} \to [0, 1]$, and to every $e \in \mathsf{N_I}$ an element $e^\mathcal{I} \in \mathcal{D}^\mathcal{I}$ of the domain.

The interpretation function is extended to concepts as follows:

- $\top^\mathcal{I}(x) = 1, \qquad \bot^\mathcal{I}(x) = 0,$
- $(C \sqcap D)^\mathcal{I}(x) = C^\mathcal{I}(x) \otimes D^\mathcal{I}(x),$
- $(C \to D)^\mathcal{I}(x) = C^\mathcal{I}(x) \Rightarrow D^\mathcal{I}(x),$
- $(\neg C)^\mathcal{I}(x) = 1 - C^\mathcal{I}(x),$
- $(\exists r.C)^\mathcal{I}(x) = \sup_{y \in \mathcal{D}^\mathcal{I}}(r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)),$
- $(\forall r.C)^\mathcal{I}(x) = \inf_{y \in \mathcal{D}^\mathcal{I}}(r^\mathcal{I}(x, y) \Rightarrow C^\mathcal{I}(y)).$

[4] One can also consider fuzzy GCIs of the form $\langle C \sqsubseteq D \geq p\rangle$ (see, e.g. [28]). Since our proofs of undecidability do not require these more general axioms, we do not consider them in this report.

We say that an interpretation $\mathcal{I}'$ is an *extension* of $\mathcal{I}$ if it has the same domain as $\mathcal{I}$, agrees with $\mathcal{I}$ on the interpretation of $\mathsf{N_C}$, $\mathsf{N_R}$, and $\mathsf{N_I}$ and additionally defines values for some new concept names not appearing in $\mathsf{N_C}$.

The reasoning problem that we consider in this report is ontology consistency; that is, deciding whether one can find an interpretation satisfying all the axioms in an ontology.

Definition 5 (consistency)**.** An interpretation $\mathcal{I} = (\mathcal{D}^\mathcal{I}, \cdot^\mathcal{I})$ *satisfies* the GCI $C \sqsubseteq D$ if $C^\mathcal{I}(x) \leq D^\mathcal{I}(x)$ for all $x \in \mathcal{D}^\mathcal{I}$. It *satisfies* the assertion $\langle e : C \triangleright p\rangle$ (resp., $\langle (d, e) : r \triangleright p\rangle$) if $C^\mathcal{I}(e^\mathcal{I}) \triangleright p$ (resp., $r^\mathcal{I}(d^\mathcal{I}, e^\mathcal{I}) \triangleright p$). It *satisfies* the crisp role axiom $\mathsf{crisp}(r)$ if $r^\mathcal{I}(x, y) \in \{0, 1\}$ for all $x, y \in \mathcal{D}^\mathcal{I}$. It is a *model* of an ontology $\mathcal{O}$ if it satisfies all the axioms in $\mathcal{O}$.

An ontology is *consistent* if it has a model.

Notice that, according to these semantics, the GCIs $C \sqsubseteq D$ and $D \sqsubseteq C$ are satisfied iff $C^\mathcal{I}(x) = D^\mathcal{I}(x)$ for every $x \in \mathcal{D}^\mathcal{I}$. It thus makes sense to abbreviate them through the expression $C \equiv D$.

In fuzzy DLs, reasoning is often restricted to a special kind of models, called witnessed models [18, 9]. An interpretation $\mathcal{I}$ is called *witnessed* if for every concept C, $r \in \mathsf{N_R}$, and $x \in \mathcal{D}^\mathcal{I}$ there exist $y, y' \in \mathcal{D}^\mathcal{I}$ such that

- $(\exists r.C)^\mathcal{I}(x) = r^\mathcal{I}(x, y) \otimes C^\mathcal{I}(y)$, and
- $(\forall r.C)^\mathcal{I}(x) = r^\mathcal{I}(x, y') \Rightarrow C^\mathcal{I}(y')$.

This means that the suprema and infima in the semantics of existential and value restrictions are actually maxima and minima, respectively. Restricting to this kind of models changes the reasoning problem since there exist consistent ontologies that have no witnessed models [18].

We also consider a weaker notion of witnessing, where witnesses are required only for the existential restrictions $\exists r.\top$ evaluated to 1. Formally, $\mathcal{I}$ is called $\top$-*witnessed* if for every $r \in \mathsf{N_R}$ and $x \in \mathcal{D}^\mathcal{I}$ such that $(\exists r.\top)^\mathcal{I}(x) = 1$, there is a $y \in \mathcal{D}^\mathcal{I}$ with $r^\mathcal{I}(x, y) = 1$. Obviously, every witnessed interpretation is also $\top$-witnessed. We will use the subscripts w and $\top$ to indicate that reasoning is restricted to witnessed and $\top$-witnessed models, respectively. For example, $\otimes_\mathsf{w}$-$\mathcal{ELC}$ expresses the logic $\otimes$-$\mathcal{ELC}$ restricted to witnessed models.

In general, a fuzzy DL is determined by three parameters: the class $\mathcal{L}$ of constructors and axioms it allows, the t-norm $\otimes$ that describes its semantics, and the class of models x over which reasoning is considered. In the following, we will use the expression $\otimes_\mathsf{x}$-$\mathcal{L}$ to denote an arbitrary fuzzy DL.

Before we present our general framework for proving undecidability, it is worth to relate the fuzzy DLs we have introduced according to their expressive power.

For every choice of constructors $\mathcal{L}$ and t-norm $\otimes$, the inequality concept assertion $\langle e : C \geq q\rangle$ can be expressed in $\otimes$-$\mathcal{L}_=$ using the axioms $\langle e : A = q\rangle, A \sqsubseteq C$, where A is a new concept name. If we restrict the semantics to the Łukasiewicz t-norm, since involutive negation can be expressed using the residuum, we obtain that Ł-$\mathcal{ELC}$, Ł-$\mathfrak{IEL}$, Ł-$\mathcal{ALC}$, and Ł-$\mathfrak{IAL}$ are all equivalent [17]. The implication can be expressed by negation and conjunction $(C \to D)^\mathcal{I} = \neg(C \sqcap \neg D)^\mathcal{I}$, and the duality between value and existential restrictions $(\forall r.C)^\mathcal{I} = \neg(\exists r.\neg C)^\mathcal{I}$ holds. However, in general these logics have different expressive power. For instance, if any t-norm different from Łukasiewicz is used, then $(\neg\exists r.\neg C)^\mathcal{I} \neq (\forall r.C)^\mathcal{I}$.

4 Showing Undecidability

We will now describe a general approach for proving that the consistency problem for a fuzzy DL $\otimes_\times$-$\mathcal{L}$ is undecidable. This approach is based on a reduction from the Post correspondence problem which is well known to be undecidable [24].

Definition 6 (PCP)**.** Let $\mathcal{P} = \{(v_1, w_1), \ldots, (v_n, w_n)\}$ be a finite set of pairs of words over the alphabet $\Sigma = \{1, \ldots, s\}$ with $s > 1$. The *Post correspondence problem (PCP)* asks whether there is a finite non-empty sequence $i_1 \ldots i_k \in \{1, \ldots, n\}^+$ such that $v_{i_1} \ldots v_{i_k} = w_{i_1} \ldots w_{i_k}$. If this sequence exists, it is called a *solution* for $\mathcal{P}$.

We will abbreviate $\{1, \ldots, n\}$ by $\mathcal{N}$. For $\nu = i_1 \ldots i_k \in \mathcal{N}^+$, we use the notation $v_\nu = v_{i_1} \ldots v_{i_k}$ and $w_\nu = w_{i_1} \ldots w_{i_k}$.

We can represent an instance $\mathcal{P} = \{(v_1, w_1), \ldots, (v_n, w_n)\}$ of the PCP by its *search tree*, which has one node for every $\nu \in \mathcal{N}^*$, where ε represents the root, and νi is the i-th successor of ν, $i \in \mathcal{N}$. Each node ν in this tree is labelled with the words $v_\nu, w_\nu \in \Sigma^*$, as shown in Figure 1.

We will show how to reduce the PCP to the consistency problem of a fuzzy DL. We present this reduction in two parts. Given an instance $\mathcal{P}$ of the PCP, we first construct an ontology $\mathcal{O}_\mathcal{P}$ that describes the search tree of $\mathcal{P}$ using two designated concept names V, W. More precisely, we will enforce that for every model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P}$ and every $\nu \in \mathcal{N}^*$, there is an $x_\nu \in \mathcal{D}^\mathcal{I}$ such that $V^\mathcal{I}(x_\nu) = \mathsf{enc}(v_\nu)$ and $W^\mathcal{I}(x_\nu) = \mathsf{enc}(w_\nu)$, where $\mathsf{enc} : \Sigma^* \to [0, 1]$ is an injective function that encodes words over Σ into the interval $[0, 1]$ (see Section 4.1).

Once we have encoded the words v_ν and w_ν using V and W, we add axioms that restrict every node to satisfy that $V^\mathcal{I}(x_\nu) \neq W^\mathcal{I}(x_\nu)$. This will be helpful to ensure that $\mathcal{P}$ has a solution if and only if the ontology is inconsistent (see Section 4.2).

Recall that the alphabet Σ consists of the first s positive integers. We can thus view every word in Σ^* as a natural number represented in base $s + 1$. On the

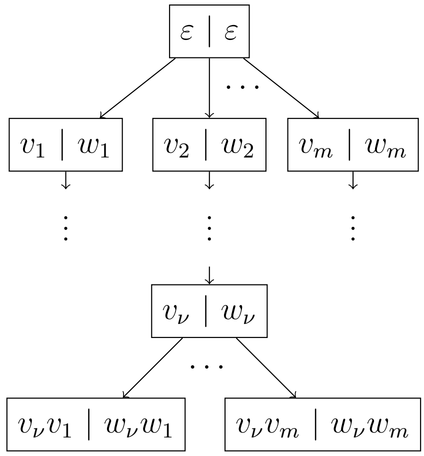

Figure 1: The search tree for an instance $\mathcal{P}$ of the PCP.

other hand, every natural number n has a unique representation in base $s+1$, which can be seen as a word over the alphabet $\Sigma_0 := \Sigma \cup \{0\} = \{0, \ldots, s\}$. This is not a bijection since, e.g. the words 001202 and 1202 represent the same number. However, it is a bijection between the set $\Sigma\Sigma_0^*$ and the positive natural numbers. We will in the following interpret the empty word ε as 0, thereby extending this bijection to $\{\varepsilon\} \cup \Sigma\Sigma_0^*$ and all non-negative integers.

In the following constructions and proofs, we will view elements of Σ_0^* both as words and as natural numbers in base $s+1$. To avoid confusion, we will use the notation $\underline{u}$ to express that u is seen as a word. Thus, for instance, if $s = 3$, then $3 \cdot 2^2 = 30$ (in base 4), but $\underline{3} \cdot \underline{2}^2 = \underline{322}$. Furthermore, $\underline{000}$ is a word of length 3, whereas 000 is simply the number 0. For a word $u = \alpha_1 \cdots \alpha_m$ with $\alpha_i \in \Sigma_0, 1 \leq i \leq m$, we denote as $\overleftarrow{u}$ the word $\alpha_m \cdots \alpha_1 \in \Sigma_0^*$.

Recall that for every $p, q \in [0,1]$, $p = q$ iff $p \Rightarrow q = q \Rightarrow p = 1$ (see Lemma 1). Thus, to decide whether $\mathcal{P}$ has a solution, we have to check whether $\mathsf{enc}(v_\nu) \Rightarrow \mathsf{enc}(w_\nu) < 1$ or $\mathsf{enc}(w_\nu) \Rightarrow \mathsf{enc}(v_\nu) < 1$ holds for every $\nu \in \mathcal{N}^+$. Instead of performing this test directly, we will assume that we can construct a word whose encoding bounds these residua. Clearly, the precise word and encoding must depend on the t-norm used. The needed properties are formalized by the following definition.

Definition 7 (valid encoding function)**.** A function $\mathsf{enc} : \Sigma_0^* \to [0,1]$ is called a *valid encoding function* for $\otimes$ if it is injective on $\{\varepsilon\} \cup \Sigma\Sigma_0^*$ and there exist two words $u_\varepsilon, u_+ \in \Sigma_0^*$ such that for every $\nu \in \mathcal{N}^+$ it holds that

$$v_\nu \neq w_\nu \text{ iff } \min\{\mathsf{enc}(v_\nu) \Rightarrow \mathsf{enc}(w_\nu), \mathsf{enc}(w_\nu) \Rightarrow \mathsf{enc}(v_\nu)\} \leq \mathsf{enc}(\underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|}).$$

For every continuous t-norm $\otimes$ that is not the Gödel t-norm, we will now give a valid encoding function. The precise function depends on whether $\otimes$ contains the product or the Łukasiewicz t-norm. If $\otimes$ is of the form $\Pi^{(a,b)}$, i.e. it (a,b)-contains the product t-norm, then we define $\mathsf{enc}(u) = a + (b-a)2^{-u} \in (a,b]$ for every $u \in \Sigma_0^*$. If $\otimes$ is of the form $\text{Ł}^{(a,b)}$, we use the function $\mathsf{enc}(u) = a+(b-a)(1-0.\overleftarrow{u}) \in (a,b]$.

Lemma 8. *The functions* enc *described above are valid encoding functions.*

Proof. $[\Pi^{(a,b)}]$ Let $v \neq w$ and assume w.l.o.g. that $v < w$. Then $v+1 \leq w$ and hence $2^{-w} \leq 2^{-(v+1)} \leq 2^{-v}/2$. This implies that

$$\mathsf{enc}(v) \Rightarrow \mathsf{enc}(w) = a + (b-a)2^{-w}/2^{-v} \leq a + (b-a)/2 = \mathsf{enc}(1) < 1.$$

Conversely, if $v = w$, then $(\mathsf{enc}(v) \Rightarrow \mathsf{enc}(w)) = 1 = (\mathsf{enc}(w) \Rightarrow \mathsf{enc}(v))$. Thus, the words $u_\varepsilon = 1, u_+ = \varepsilon$ satisfy the condition of Definition 7.

$[\text{Ł}^{(a,b)}]$ Let $k = \max\{|v_i|, |w_i| \mid i \in \mathcal{N}\}$ be the maximal length of a word in the instance $\mathcal{P}$. Then, for every $\nu \in \mathcal{N}^+, |v_\nu| \leq |\nu|k$ and $|w_\nu| \leq |\nu|k$. If $v_\nu \neq w_\nu$, these words must differ in one of the first $|\nu|k$ digits. Thus, either

$$\begin{aligned}\mathsf{enc}(v_\nu) \Rightarrow \mathsf{enc}(w_\nu) &= a + (b-a)\min\{1, 1 + 0.\overleftarrow{v_\nu} - 0.\overleftarrow{w_\nu}\} \\ &= \min\{b, a + (b-a)(1 + 0.\overleftarrow{v_\nu} - 0.\overleftarrow{w_\nu})\} \\ &\leq a + (b-a)(1 - (s+1)^{-|\nu|k}) \\ &= \mathsf{enc}((s+1)^{|\nu|k}) < 1\end{aligned}$$

or $\mathsf{enc}(w_\nu) \Rightarrow \mathsf{enc}(v_\nu) \leq \mathsf{enc}((s+1)^{|\nu|k})$.[5] If $v_\nu = w_\nu$, then both residua yield 1 as result, which is greater than $\mathsf{enc}((s+1)^{|\nu|k})$. Thus, setting $u_\varepsilon = 1$ and $u_+ = \underline{0}^k$ gives the desired result. □

Variants of the above encoding functions and words u_ε, u_+ have been used before to show undecidability of fuzzy description logics based on the product [4] and Łukasiewicz [14] t-norms.

For the rest of this section, enc represents a valid encoding function for $\otimes$.

4.1 Encoding the Search Tree

As a first step for our reduction to the consistency problem in fuzzy DLs, we simulate the search tree for the instance $\mathcal{P}$. We use the concept names V, W to represent the values of the words v_ν and w_ν at the different nodes of the tree. Since we will later use this construction to decide whether a solution exists, we will designate the concept name M to represent the bound $\underline{u_\varepsilon \cdot u_+}^{|\nu|}$ from Definition 7. We will additionally use the concept names V_i, W_i to encode the words v_i, w_i from

[5] The number $(s+1)^{|\nu|k}$ represents $\underline{1} \cdot \underline{0}^{|\nu|k}$ and $(s+1)^{-|\nu|k}$ is equal to $0.\underline{0}^{|\nu|k} \cdot \underline{1}$.

$\mathcal{P}$, and the role names r_i to distinguish the different successors in the search tree. We thus build the interpretation $\mathcal{I}_\mathcal{P} = (\mathcal{N}^*, \cdot^{\mathcal{I}_\mathcal{P}})$, where for every $\nu \in \mathcal{N}^*$ and $i \in \mathcal{N}$,

- $e_0^{\mathcal{I}_\mathcal{P}} = \varepsilon$,
- $V^{\mathcal{I}_\mathcal{P}}(\nu) = \mathsf{enc}(v_\nu)$, $W^{\mathcal{I}_\mathcal{P}}(\nu) = \mathsf{enc}(w_\nu)$,
- $V_i^{\mathcal{I}_\mathcal{P}}(\nu) = \mathsf{enc}(v_i)$, $W_i^{\mathcal{I}_\mathcal{P}}(\nu) = \mathsf{enc}(w_i)$,
- $M^{\mathcal{I}_\mathcal{P}}(\nu) = \mathsf{enc}(\underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|})$, $M_+^{\mathcal{I}_\mathcal{P}}(\nu) = \mathsf{enc}(u_+)$,
- $r_i^{\mathcal{I}_\mathcal{P}}(\nu, \nu i) = 1$ and $r_i^{\mathcal{I}_\mathcal{P}}(\nu, \nu') = 0$ if $\nu' \neq \nu i$.

Since every element of $\mathcal{N}^*$ has exactly one r_i-successor with degree greater than 0, $\mathcal{I}_\mathcal{P}$ is a witnessed interpretation, and hence also $\top$-witnessed.

We want to construct an ontology that can only be satisfied by interpretations that "include" the search tree of $\mathcal{P}$. Given that the interpretation $\mathcal{I}_\mathcal{P}$ represents this tree, we want the logic to satisfy the following property.

Canonical model property ($\mathrm{P}_\triangle$)**:**

$\otimes_\mathsf{x}$-$\mathcal{L}$ has the *canonical model property* if there is an ontology $\mathcal{O}_\mathcal{P}$ such that for every model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P}$ there is a mapping $g : \mathcal{D}^{\mathcal{I}_\mathcal{P}} \to \mathcal{D}^\mathcal{I}$ with

- $A^{\mathcal{I}_\mathcal{P}}(\nu) = A^\mathcal{I}(g(\nu))$, and
- $r_i^\mathcal{I}(g(\nu), g(\nu i)) = 1$

for every $A \in \{V, W, M, M_+\} \cup \bigcup_{j=1}^n \{V_j, W_j\}, \nu \in \mathcal{N}^*$ and $i \in \mathcal{N}$.

Rather than trying to prove this property directly for some fuzzy DL, we provide several simpler properties that together imply the canonical model property. We will often motivate the following constructions using only the concept V and the words v_ν; however, all the arguments apply analogously to W, w_ν and $M, \underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|}$.

To ensure that the canonical model property holds, we construct the search tree in an inductive way. First, we restrict every model $\mathcal{I}$ to satisfy that $A^{\mathcal{I}_\mathcal{P}}(\varepsilon) = A^\mathcal{I}(e_0^\mathcal{I})$ for every relevant concept name. This makes sure that the root ε of the search tree is properly represented at the individual $g(\varepsilon) := e_0^\mathcal{I}$. Let now $g(\nu)$ be a node satisfying the first property, and $i \in \mathcal{N}$. We need to ensure that there is a node $g(\nu i)$ that also satisfies the property, and $r_i^\mathcal{I}(g(\nu), g(\nu i)) = 1$. We do this in three steps: first, we force the existence of an individual y with $r_i^\mathcal{I}(g(\nu), y) = 1$ and set $g(\nu i) := y$. Then, we compute the value $\mathsf{enc}(v_\nu v_i)$ from $V^\mathcal{I}(g(\nu)) = \mathsf{enc}(v_\nu)$ and $V_i^\mathcal{I}(g(\nu)) = \mathsf{enc}(v_i)$. Finally, we transfer this value to the previously created successor to ensure that $V^{\mathcal{I}}(g(\nu i)) = \mathsf{enc}(v_\nu v_i)$. The value of $V_j^{\mathcal{I}}(g(\nu))$ for every $j \in \mathcal{N}$ is similarly transferred to $V_j^{\mathcal{I}}(g(\nu i))$.

Since the values of V_i, W_i, and M_+ are constant throughout the search tree, we will also present an alternative approach that simply fixes these values for all individuals $x \in \mathcal{D}^{\mathcal{I}}$. This has the advantage that the initialization only has to take care of the simple values $\mathsf{enc}(v_\varepsilon) = \mathsf{enc}(w_\varepsilon) = \mathsf{enc}(\varepsilon)$ and $\mathsf{enc}(u_\varepsilon)$.

Each step of the previous construction will be guaranteed by a property of the underlying logic. These properties, which will ultimately be used to produce the ontology $\mathcal{O}_{\mathcal{P}}$, are described next. For each of the properties, we will give examples of fuzzy DLs satisfying it. It is important to notice that the interpretation $\mathcal{I}_{\mathcal{P}}$ can be extended to a witnessed model of each of the ontologies that we will introduce in the following.

Successor property ($\mathrm{P}_{\rightarrow}$)**:**

$\otimes_{\mathsf{x}}$-$\mathcal{L}$ has the *successor property* if for every role name r there is an ontology $\mathcal{O}_{\exists r}$ such that for every x-model $\mathcal{I}$ of $\mathcal{O}_{\exists r}$ and every $x \in \mathcal{D}^{\mathcal{I}}$ there is a $y \in \mathcal{D}^{\mathcal{I}}$ with $r^{\mathcal{I}}(x, y) = 1$.

Lemma 9. *For every t-norm $\otimes$, $\otimes_{\top}$-$\mathcal{EL}$ and $\otimes$-$\mathcal{EL}_{\mathsf{c}}$ satisfy* $\mathrm{P}_{\rightarrow}$.

Proof. [$\otimes_{\top}$-$\mathcal{EL}$] Consider the ontology $\mathcal{O}_{\exists r} := \{\top \sqsubseteq \exists r.\top\}$. Any model $\mathcal{I}$ of this axiom satisfies $(\exists r.\top)^{\mathcal{I}}(x) = 1$ for every $x \in \mathcal{D}^{\mathcal{I}}$. Since reasoning is restricted to $\top$-witnessed models, there must be a $y \in \mathcal{D}^{\mathcal{I}}$ with $r^{\mathcal{I}}(x, y) = 1$.

[$\otimes$-$\mathcal{EL}_{\mathsf{c}}$] We define $\mathcal{O}_{\exists r} := \{\top \sqsubseteq \exists r.\top, \mathsf{crisp}(r)\}$. In any model of this ontology, r is crisp and we have $(\exists r.\top)^{\mathcal{I}}(x) = 1$ for all $x \in \mathcal{D}^{\mathcal{I}}$. If $r^{\mathcal{I}}(x, y) = 0$ for all $y \in \mathcal{D}^{\mathcal{I}}$, then $(\exists r.\top)^{\mathcal{I}}(x) = \sup_{y \in \mathcal{D}^{\mathcal{I}}} r^{\mathcal{I}}(x, y) \otimes \top^{\mathcal{I}}(y) = 0$, which is a contradiction. Thus, there must be a $y \in \mathcal{D}^{\mathcal{I}}$ with $r^{\mathcal{I}}(x, y) = 1$. □

If a logic satisfies this property, then the ontology

$$\mathcal{O}_{\mathcal{P},\rightarrow} := \bigcup_{i \in \mathcal{N}} \mathcal{O}_{\exists r_i}$$

ensures the existence of an r_i-successor for every node of the search tree and every $i \in \mathcal{N}$.

Concatenation property ($\mathrm{P}_{\circ}$)**:**

$\otimes_{\mathsf{x}}$-$\mathcal{L}$ has the *concatenation property* if for all words $u \in \Sigma_0^*$, and concepts C and C_u, there is an ontology $\mathcal{O}_{C \circ u}$ and a concept name $D_{C \circ u}$ such that for every x-model $\mathcal{I}$ of $\mathcal{O}_{C \circ u}$ and every $x \in \mathcal{D}^{\mathcal{I}}$, if $C_u^{\mathcal{I}}(x) = \mathsf{enc}(u)$ and $C^{\mathcal{I}}(x) = \mathsf{enc}(u')$ for some $u' \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$, then $D_{C \circ u}^{\mathcal{I}}(x) = \mathsf{enc}(\underline{u'u})$.

Lemma 10. *For any continuous t-norm $\otimes$ different from the Gödel t-norm, $\otimes$-$\mathcal{EL}$ satisfies* $\mathrm{P}_\circ$.

Proof. By assumption, $\otimes$ must contain either the product or the Łukasiewicz t-norm in some interval. We divide the proof depending on the representative chosen for the encoding function.

[$\Pi^{(a,b)}$-$\mathcal{EL}$] Since every word in Σ_0^* is seen as a natural number in base $s+1$, for every $u \in \Sigma_0^*$ and $u' \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$, we have $u'(s+1)^{|u|} + u = \underline{u'u}$. We define the ontology

$$\mathcal{O}_{C\circ u} := \{D_{C\circ u} \equiv C^{(s+1)^{|u|}} \sqcap C_u\}.$$

Recall that for every interpretation $\mathcal{I}$ and $x \in \mathcal{D}^\mathcal{I}$, if $C^\mathcal{I}(x) = a + (b-a)p$, then

$$\left(C^m\right)^\mathcal{I}(x) = a + (b-a)p^m.$$

Let now $\mathcal{I}$ be a model of $\mathcal{O}_{C\circ u}$, $x \in \mathcal{D}^\mathcal{I}$, and $u' \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$ with $C_u^\mathcal{I}(x) = \mathsf{enc}(u) = a + (b-a)2^{-u}$ and $C^\mathcal{I}(x) = \mathsf{enc}(u') = a + (b-a)2^{-u'}$. Since $\mathcal{I}$ must satisfy $\mathcal{O}_{C\circ u}$, we have that

$$D_{C\circ u}^\mathcal{I}(x) = a + (b-a)2^{-\left(u'(s+1)^{|u|}+u\right)} = \mathsf{enc}(\underline{u'u}).$$

[$Ł^{(a,b)}$-$\mathcal{EL}$] We define the ontology

$$\mathcal{O}_{C\circ u} := \{C'^{(s+1)^{|u|}} \equiv C,\ D_{C\circ u} \equiv C' \sqcap C_u\}.$$

Let $\mathcal{I}$ be a model of $\mathcal{O}_{C\circ u}$, $x \in \mathcal{D}^\mathcal{I}$, and assume that $C_u^\mathcal{I}(x) = \mathsf{enc}(u)$ and $C^\mathcal{I}(x) = \mathsf{enc}(u') = a + (b-a)(1 - 0.\overleftarrow{u'}) \in (a,b]$ for some $u' \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$. From the first axiom it follows that

$$(C'^{(s+1)^{|u|}})^\mathcal{I}(x) = C^\mathcal{I}(x) = a + (b-a)(1 - 0.\overleftarrow{u'}) \in (a,b].$$

By monotonicity and since $\otimes$ (a,b)-contains the Łukasiewicz t-norm, this implies that (i) $C'^\mathcal{I}(x) > a$ and (ii) $C'^\mathcal{I}(x) \geq b$ iff $C^\mathcal{I}(x) = b$; that is, if u' is the empty word. Recall that, whenever $C'^\mathcal{I}(x) \in [a,b]$ for some interpretation $\mathcal{I}$ and $x \in \mathcal{D}^\mathcal{I}$, then we have

$$\left((C')^m\right)^\mathcal{I}(x) = \max\{a, m\left(C'^\mathcal{I}(x) - b\right) + b\}.$$

If $C^\mathcal{I}(x) < b$, then $C'^\mathcal{I}(x) \in (a,b)$ and

$$a + (b-a)(1 - 0.\overleftarrow{u'}) = C^\mathcal{I}(x) = \max\{a, (s+1)^{|u|}\left(C'^\mathcal{I}(x) - b\right) + b\},$$

and thus

$$C'^\mathcal{I}(x) = a + (b-a)(1 - (s+1)^{-|u|}0.\overleftarrow{u'})$$

and

$$D^{\mathcal{I}}_{C\circ u}(x) = a + (b-a)\max\{0, (1 - 0.\overleftarrow{u}) + (1 - (s+1)^{-|u|}0.\overleftarrow{u'}) - 1\}$$
$$= a + (b-a)(1 - 0.\overleftarrow{u} - (s+1)^{-|u|}0.\overleftarrow{u'}) = \mathsf{enc}(\underline{u'u}).$$

Otherwise, u' is the empty word and $C'^{\mathcal{I}}(x) \geq b$. Since $C^{\mathcal{I}}_u(x) \leq b$, we know that $C'^{\mathcal{I}}(x) \otimes C^{\mathcal{I}}_u(x) = C^{\mathcal{I}}_u(x)$ and thus

$$D^{\mathcal{I}}_{C\circ u}(x) = C^{\mathcal{I}}_u(x) = \mathsf{enc}(u) = \mathsf{enc}(\underline{\varepsilon u}). \qquad \square$$

The goal of this property is to ensure that at every node where $V^{\mathcal{I}}(x) = \mathsf{enc}(u)$ for some $u \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$, and $C^{\mathcal{I}}_{v_i}(x) = v_i$, then $D^{\mathcal{I}}_{V\circ v_i}(x) = \mathsf{enc}(\underline{uv_i})$, and similarly for W, w_i and M, u_+. Thus, we define the ontology

$$\mathcal{O}_{\mathcal{P},\circ} := \bigcup_{i=1}^{n} \left(\mathcal{O}_{V\circ v_i} \cup \mathcal{O}_{W\circ w_i} \cup \mathcal{O}_{M\circ u_+}\right).$$

Notice that by construction, the values of $V^{\mathcal{I}}(x)$ and $W^{\mathcal{I}}(x)$ should always be encodings of words $v_\nu, w_\nu \in \Sigma^* \subseteq \{\varepsilon\} \cup \Sigma\Sigma_0^*$, while $M^{\mathcal{I}}(x)$ might encode words that contain zeros. To simplify the notation, we use the concept names V_i, W_i, M_+ instead of $C_{v_i}, C_{w_i}, C_{u_+}$ in this ontology.

Transfer property ($\mathrm{P}_{\rightsquigarrow}$)**:**

$\otimes_{\mathsf{x}}$-$\mathcal{L}$ has the *transfer property* if for all concepts C, D and role names r there is an ontology $\mathcal{O}_{C \overset{r}{\rightsquigarrow} D}$ such that for every x-model $\mathcal{I}$ of $\mathcal{O}_{C \overset{r}{\rightsquigarrow} D}$ and every $x, y \in \mathcal{D}^{\mathcal{I}}$, if $r^{\mathcal{I}}(x,y) = 1$ and $C^{\mathcal{I}}(x) = \mathsf{enc}(u)$ for some $u \in \Sigma_0^*$, then $C^{\mathcal{I}}(x) = D^{\mathcal{I}}(y)$.

Lemma 11. *For every t-norm $\otimes$, $\otimes$-$\mathcal{AL}$ and $\otimes$-$\mathcal{ELC}$ satisfy* $\mathrm{P}_{\rightsquigarrow}$.

Proof. Notice first that for any model $\mathcal{I}$ of the $\otimes$-$\mathcal{EL}$ axiom $\exists r.D \sqsubseteq C$ and all $x, y \in \mathcal{D}^{\mathcal{I}}$ with $r^{\mathcal{I}}(x,y) = 1$ it holds that

$$D^{\mathcal{I}}(y) = r^{\mathcal{I}}(x,y) \otimes D^{\mathcal{I}}(y) \leq (\exists r.D)^{\mathcal{I}}(x) \leq C^{\mathcal{I}}(x).$$

We now add a restriction ensuring that also $D^{\mathcal{I}}(y) \geq C^{\mathcal{I}}(x)$ holds, depending on the expressivity of the logic used.

[$\otimes$-$\mathcal{AL}$] The axiom $C \sqsubseteq \forall r.D$ restricts every model $\mathcal{I}$ to satisfy that if $r^{\mathcal{I}}(x,y) = 1$, then

$$C^{\mathcal{I}}(x) \leq (\forall r.D)^{\mathcal{I}}(x) \leq r^{\mathcal{I}}(x,y) \Rightarrow D^{\mathcal{I}}(y) = D^{\mathcal{I}}(y).$$

Thus, the ontology $\mathcal{O}_{C \overset{r}{\rightsquigarrow} D} := \{C \sqsubseteq \forall r.D, \exists r.D \sqsubseteq C\}$ satisfies the condition.

[$\otimes$-$\mathcal{ELC}$] If $\mathcal{I}$ is a model of $\exists r.\neg D \sqsubseteq \neg C$ and $r^{\mathcal{I}}(x,y) = 1$, then

$$1 - D^{\mathcal{I}}(y) = r^{\mathcal{I}}(x,y) \otimes (1 - D^{\mathcal{I}}(y)) \leq (\exists r.\neg D)^{\mathcal{I}}(x) \leq 1 - C^{\mathcal{I}}(x),$$

and thus we can define $\mathcal{O}_{C \overset{r}{\rightsquigarrow} D} := \{\exists r.\neg D \sqsubseteq \neg C, \exists r.D \sqsubseteq C\}$. $\square$

To ensure that the values of $\mathsf{enc}(\underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|})$, $\mathsf{enc}(u_+)$, $\mathsf{enc}(v_{\nu i})$, and $\mathsf{enc}(v_j)$ for every $j \in \mathcal{N}$ are transfered from x to the successor y_i for every $i \in \mathcal{N}$, we use the ontology

$$\mathcal{O}_{\mathcal{P},\rightsquigarrow} := \bigcup_{i \in \mathcal{N}} \mathcal{O}_{D_{M \circ u_+} \overset{r_i}{\rightsquigarrow} M} \cup \mathcal{O}_{M_+ \overset{r_i}{\rightsquigarrow} M_+} \cup \mathcal{O}_{D_{V \circ v_i} \overset{r_i}{\rightsquigarrow} V} \cup \mathcal{O}_{D_{W \circ w_i} \overset{r_i}{\rightsquigarrow} W}$$
$$\cup \bigcup_{i,j \in \mathcal{N}} \mathcal{O}_{V_j \overset{r_i}{\rightsquigarrow} V_j} \cup \mathcal{O}_{W_j \overset{r_i}{\rightsquigarrow} W_j}.$$

Initialization property ($\mathrm{P}_{\mathsf{ini}}$)**:**

$\otimes_{\mathsf{x}}$-$\mathcal{L}$ has the *initialization property* if for every concept C, individual name e, and $u \in \Sigma_0^*$ there is an ontology $\mathcal{O}_{C(e)=u}$ such that $C^{\mathcal{I}}(e^{\mathcal{I}}) = \mathsf{enc}(u)$ for every x-model $\mathcal{I}$ of $\mathcal{O}_{C(e)=u}$.

Lemma 12. *For every t-norm $\otimes$, $\otimes$-$\mathcal{EL}_=$ and $\otimes$-$\mathcal{ELC}_\geq$ satisfy* $\mathrm{P}_{\mathsf{ini}}$.

Proof. [$\otimes$-$\mathcal{EL}_=$] If the equality assertion $\langle e : C = \mathsf{enc}(u)\rangle$ is satisfied by $\mathcal{I}$, then $C^{\mathcal{I}}(e^{\mathcal{I}}) = \mathsf{enc}(u)$.

[$\otimes$-$\mathcal{ELC}_\geq$] We use the ontology $\{\langle e : C \geq \mathsf{enc}(u)\rangle, \langle e : \neg C \geq 1 - \mathsf{enc}(u)\rangle\}$. The first axiom expresses that $C^{\mathcal{I}}(e^{\mathcal{I}}) \geq \mathsf{enc}(u)$, while the second requires that $1 - C^{\mathcal{I}}(e^{\mathcal{I}}) \geq 1 - \mathsf{enc}(u)$, i.e. $C^{\mathcal{I}}(e^{\mathcal{I}}) \leq \mathsf{enc}(u)$, holds. □

To initialize the search tree, we need to fix an individual name e_0 at which V and W are both interpreted as the encoding of the empty word and M as the encoding of u_ε. Moreover, we need that M_+ encodes u_+ and every V_i and W_i encodes the word v_i, w_i, respectively. We thus define the ontology

$$\mathcal{O}_{\mathcal{P},\mathsf{ini}} := \mathcal{O}_{M(e_0)=u_\varepsilon} \cup \mathcal{O}_{M_+(e_0)=u_+} \cup \mathcal{O}_{V(e_0)=\varepsilon} \cup \mathcal{O}_{W(e_0)=\varepsilon}$$
$$\cup \bigcup_{i=1}^{n} \left(\mathcal{O}_{V_i(e_0)=v_i} \cup \mathcal{O}_{W_i(e_0)=w_i}\right).$$

In some cases where the initialization property cannot be guaranteed, it suffices to consider a weaker version, where only two words need to be initialized. Together with a property guaranteeing constant concepts, this weak initialization property can also lead to undecidability.

Weak initialization property ($\mathrm{P}^{w}_{\mathsf{ini}}$)**:**

$\otimes_{\mathsf{x}}$-$\mathcal{L}$ has the *weak initialization property* if for every concept C, individual name e, and $u \in \{\varepsilon, u_\varepsilon\}$ there is an ontology $\mathcal{O}_{C(e)=u}$ such that $C^{\mathcal{I}}(e^{\mathcal{I}}) = \mathsf{enc}(u)$ holds for every x-model $\mathcal{I}$ of $\mathcal{O}_{C(e)=u}$.

Notice that the only difference between $\mathrm{P}_{\mathsf{ini}}$ and $\mathrm{P}^{w}_{\mathsf{ini}}$ is that the former allows encoding *every* word, while the latter only requires the empty word and u_ε.

Lemma 13. *The logic* Π-$\mathcal{ELC}$ *satisfies* $\mathrm{P}^{w}_{\mathsf{ini}}$.

Proof. We have $\mathsf{enc}(\varepsilon) = 1$ and hence the crisp assertion $\langle e : C \geq 1\rangle$ yields the desired condition for ε. For $u_\varepsilon = 1$, we use the axiom $C \equiv \neg C$, which in particular restricts $C^{\mathcal{I}}(e^{\mathcal{I}}) = 1 - C^{\mathcal{I}}(e^{\mathcal{I}})$ to be $0.5 = \mathsf{enc}(1)$. □

For any logic satisfying $\mathrm{P}^{w}_{\mathsf{ini}}$, any model of the ontology

$$\mathcal{O}^{w}_{\mathcal{P},\mathsf{ini}} := \mathcal{O}_{V(e_0)=\varepsilon} \cup \mathcal{O}_{W(e_0)=\varepsilon} \cup \mathcal{O}_{M(e_0)=u_\varepsilon},$$

must contain an individual encoding the values of V, W and M at the root of the search tree of $\mathcal{P}$. Note that the construction for Π-$\mathcal{ELC}$ works since we know that $u_+ = \varepsilon$, i.e. the value of M is constant.

Constant property ($\mathrm{P}_=$)**:**

$\otimes_{\mathsf{x}}$-$\mathcal{L}$ has the *constant property* if for every concept name C and word $u \in \Sigma_0^*$ there is an ontology $\mathcal{O}_{C=u}$ such that for every x-model of $\mathcal{O}_{C=u}$ and every $x \in \mathcal{D}^{\mathcal{I}}$ we have $C^{\mathcal{I}}(x) = \mathsf{enc}(u)$.

Lemma 14. *The logic* Π-$\mathcal{ELC}$ *satisfies* $\mathrm{P}_=$.

Proof. Consider the ontology

$$\mathcal{O}_{C=u} := \{H \equiv \neg H, C \equiv H^u\}.$$

From the first axiom it follows that for every model $\mathcal{I}$ of this ontology and $x \in \mathcal{D}^{\mathcal{I}}$, we have $H^{\mathcal{I}}(x) = 1 - H^{\mathcal{I}}(x)$, and thus $H^{\mathcal{I}}(x) = 0.5 = 2^{-1}$. Thus, from the second axiom, $C^{\mathcal{I}}(x) = (2^{-1})^u = 2^{-u} = \mathsf{enc}(u)$. □

We use this property to define the ontology

$$\mathcal{O}_{\mathcal{P},=} := \mathcal{O}_{M_+=u_+} \cup \bigcup_{i=1}^{n} \mathcal{O}_{V_i=v_i} \cup \mathcal{O}_{W_i=w_i}.$$

If we combine the different properties as described at the beginning of this section, we obtain the canonical model property.

Theorem 15. *If a logic* $\otimes_{\mathsf{x}}$-$\mathcal{L}$ *satisfies the properties* $\mathrm{P}_{\circ}$, $\mathrm{P}_{\mathsf{ini}}$, $\mathrm{P}_{\rightarrow}$, *and* $\mathrm{P}_{\rightsquigarrow}$, *then it also satisfies* $\mathrm{P}_{\triangle}$.

Proof. We show that the ontology $\mathcal{O}_{\mathcal{P}} := \mathcal{O}_{\mathcal{P},\mathsf{ini}} \cup \mathcal{O}_{\mathcal{P},\circ} \cup \mathcal{O}_{\mathcal{P},\rightarrow} \cup \mathcal{O}_{\mathcal{P},\rightsquigarrow}$ satisfies the conditions from the definition of $\mathrm{P}_{\triangle}$. For a model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}}$, we construct the function $g : \mathcal{N}^* \rightarrow \mathcal{D}^{\mathcal{I}}$ inductively as follows.

We first set $g(\varepsilon) := e_0^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of $\mathcal{O}_{\mathcal{P},\mathsf{ini}}$, we have that $V^{\mathcal{I}}(g(\varepsilon)) = V^{\mathcal{I}}(e_0^{\mathcal{I}}) = \mathsf{enc}(\varepsilon) = V^{\mathcal{I}_{\mathcal{P}}}(\varepsilon)$, and likewise for W, M, M_+, V_i, and W_i for all $i \in \mathcal{N}$.

Let now ν be such that $g(\nu)$ has already been defined and $V^{\mathcal{I}}(g(\nu)) = \mathsf{enc}(v_\nu)$, $V_i^{\mathcal{I}}(g(\nu)) = \mathsf{enc}(v_i)$. $\mathcal{I}$ being a model of $\mathcal{O}_{\mathcal{P},\circ}$ ensures that $D^{\mathcal{I}}_{V \circ v_i} = \mathsf{enc}(v_{\nu i})$. Since $\mathcal{I}$ satisfies $\mathcal{O}_{\mathcal{P},\rightarrow}$, for each $i \in \{1, \ldots, n\}$ there must be an element $y_i \in \mathcal{D}^{\mathcal{I}}$ with $r_i^{\mathcal{I}}(g(\nu), y_i) = 1$. Define now $g(\nu i) := y_i$. The restrictions imposed by $\mathcal{O}_{\mathcal{P},\rightsquigarrow}$ ensure that $V^{\mathcal{I}}(g(\nu i)) = D^{\mathcal{I}}_{V \circ v_i}(g(\nu)) = \mathsf{enc}(v_{\nu i}) = V^{\mathcal{I}_{\mathcal{P}}}(\nu i)$ and $V_i^{\mathcal{I}}(g(\nu i)) = \mathsf{enc}(v_i) = V_i^{\mathcal{I}_{\mathcal{P}}}(\nu i)$ for all $i \in \mathcal{N}$, and analogously for W, W_i and M, M_+. □

From this theorem and Lemmata 9 to 12, we obtain the following result.

Corollary 16. *If* $\otimes$ *is a continuous t-norm, but not the Gödel t-norm, then the logics* $\otimes_{\top}$-$\mathcal{AL}_{=}$, $\otimes$-$\mathcal{AL}_{=,\mathsf{c}}$, $\otimes_{\top}$-$\mathcal{ELC}_{\geq}$, *and* $\otimes$-$\mathcal{ELC}_{\geq,\mathsf{c}}$ *satisfy* $\mathrm{P}_{\triangle}$.

An alternative way of obtaining the canonical model property is with the weak initialization property together with $\mathrm{P}_{=}$. The proof of this is analogous to that of Theorem 15, using the ontology $\mathcal{O}_{\mathcal{P}} := \mathcal{O}^{w}_{\mathcal{P},\mathsf{ini}} \cup \mathcal{O}_{\mathcal{P},=} \cup \mathcal{O}_{\mathcal{P},\circ} \cup \mathcal{O}_{\mathcal{P},\rightarrow} \cup \mathcal{O}_{\mathcal{P},\rightsquigarrow}$.

Theorem 17. *If* $\otimes_{\mathsf{x}}$-$\mathcal{L}$ *satisfies the properties* $\mathrm{P}_{\circ}$, $\mathrm{P}^{w}_{\mathsf{ini}}$, $\mathrm{P}_{=}$, $\mathrm{P}_{\rightarrow}$, *and* $\mathrm{P}_{\rightsquigarrow}$, *then it also satisfies* $\mathrm{P}_{\triangle}$.

With the help of Lemmata 9 to 14, we now obtain the following result.

Corollary 18. *The logics* $\Pi_{\top}$-$\mathcal{ELC}$ *and* Π-$\mathcal{ELC}_{\mathsf{c}}$ *satisfy* $\mathrm{P}_{\triangle}$.

It is a simple task to verify that the interpretation $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of the ontology $\mathcal{O}_{\mathcal{P}}$ in all the cases described. We only need to assume that one uses a unique new concept name for every auxiliary concept name appearing in the different ontologies. In fact, the values of these auxiliary concept names at each node ν are uniquely determined by the values of the concept names V, W, V_i, W_i, M, M_+ in ν. Moreover, since every ν has exactly one r_i-successor with degree greater than 0 for every $i \in \mathcal{N}$, it follows that $\mathcal{I}_{\mathcal{P}}$ can be extended to a witnessed model of $\mathcal{O}_{\mathcal{P}}$.

We now describe how the property $\mathrm{P}_{\triangle}$ can be used to prove undecidability of a fuzzy DL. The main idea is to add axioms to $\mathcal{O}_{\mathcal{P}}$ so that every model $\mathcal{I}$ is restricted to satisfy $V^{\mathcal{I}}(g(\nu)) \neq W^{\mathcal{I}}(g(\nu))$ for every $\nu \in \mathcal{N}^+$, thus obtaining an ontology that is consistent if and only if $\mathcal{P}$ has no solution.

4.2 Finding a Solution

For the rest of this section, we assume that $\otimes_\mathsf{x}$-$\mathcal{L}$ satisfies $\mathrm{P}_\triangle$ and for any given model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P}$, g denotes the function mapping the nodes of $\mathcal{I}_\mathcal{P}$ to nodes in $\mathcal{I}$ given by the property. Furthermore, we assume that $\mathcal{I}_\mathcal{P}$ can be extended to a model of $\mathcal{O}_\mathcal{P}$. These assumptions have been shown to hold for a variety of fuzzy DLs in the previous section.

The key to showing undecidability of $\otimes_\mathsf{x}$-$\mathcal{L}$ is to be able to express the restriction that V and W encode different words at every non-root node $\nu \in \mathcal{N}^+$ of the search tree. Since enc is a valid encoding function, and the concept name M encodes the word $\underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|}$ at every $\nu \in \mathcal{N}^*$, it suffices to check whether, for all $\nu \in \mathcal{N}^+$, either $(V \to W)^{\mathcal{I}_\mathcal{P}}(\nu) \leq M^{\mathcal{I}_\mathcal{P}}(\nu)$ or $(V \to W)^{\mathcal{I}_\mathcal{P}}(\nu) \leq M^{\mathcal{I}_\mathcal{P}}(\nu)$ (recall Definition 7). This can easily be done in every logic that allows for the implication constructor $\to$. However, this constructor is not necessary in general to show undecidability.

Solution property ($\mathrm{P}_{\neq}$):

A logic $\otimes_\mathsf{x}$-$\mathcal{L}$ satisfying $\mathrm{P}_\triangle$ has the *solution property* if there is an ontology $\mathcal{O}_{V \neq W}$ such that

1. For every x-model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P} \cup \mathcal{O}_{V\neq W}$ and every $\nu \in \mathcal{N}^+$,

$$\min\{V^\mathcal{I}(g(\nu)) \Rightarrow W^\mathcal{I}(g(\nu)),\ W^\mathcal{I}(g(\nu)) \Rightarrow V^\mathcal{I}(g(\nu))\} \leq M^\mathcal{I}(g(\nu)).$$

2. If for every $\nu \in \mathcal{N}^+$ we have

$$\min\{V^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu),\ W^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu)\} \leq M^{\mathcal{I}_\mathcal{P}}(\nu),$$

then $\mathcal{I}_\mathcal{P}$ can be extended to a model of $\mathcal{O}_\mathcal{P} \cup \mathcal{O}_{V\neq W}$.

Lemma 19. *Let $\otimes$ be a continuous t-norm $\otimes$ different from the Gödel t-norm and $\mathcal{L}$ contain either $\mathfrak{I}\mathcal{AL}$ or $\mathcal{ELC}$. If $\otimes_\mathsf{x}$-$\mathcal{L}$ satisfies $\mathrm{P}_\triangle$ and $\mathcal{I}_\mathcal{P}$ can be extended to a model of $\mathcal{O}_\mathcal{P}$, then $\otimes_\mathsf{x}$-$\mathcal{L}$ satisfies $\mathrm{P}_{\neq}$.*

Proof. We divide the proof according to the constructors allowed.

[$\mathfrak{I}\mathcal{AL}$] Let

$$\mathcal{O}_{V\neq W} := \{\top \sqsubseteq \forall r_i.(((V \to W) \sqcap (W \to V)) \to M) \mid i \in \mathcal{N}\}.$$

This ontology is satisfied by $\mathcal{I}$ iff for every $x, y \in \mathcal{D}^\mathcal{I}$ and every $i \in \mathcal{N}$ we have $r_i^\mathcal{I}(x,y) \Rightarrow (((V \to W) \sqcap (W \to V))^\mathcal{I}(y) \Rightarrow M^\mathcal{I}(y)) = 1$. Let now $\mathcal{I}$ be an x-model of $\mathcal{O}_\mathcal{P} \cup \mathcal{O}_{V\neq W}$. Since at least one of $(V \to W)^\mathcal{I}(g(\nu i))$, $(W \to V)^\mathcal{I}(g(\nu i))$ must be

1 and $r_i^{\mathcal{I}}(g(\nu), g(\nu i)) = 1$ for every $\nu \in \mathcal{N}^*$ and $i \in \mathcal{N}$, we have $\min\{V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)), W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu))\} \leq M^{\mathcal{I}}(g(\nu))$ for every $\nu \in \mathcal{N}^+$.

For the second condition, consider an extension $\mathcal{I}$ of $\mathcal{I}_{\mathcal{P}}$ that satisfies $\mathcal{O}_{\mathcal{P}}$ and assume that it violates $\mathcal{O}_{V\neq W}$. Thus, there are $\nu \in \mathcal{N}^*$, $i \in \mathcal{N}$ such that

$$1 = \top^{\mathcal{I}_{\mathcal{P}}}(\nu) > (\forall r_i.(((V \to W) \sqcap (W \to V)) \to M))^{\mathcal{I}_{\mathcal{P}}}(\nu).$$

Since νi is the only r_i-successor of ν, this implies that

$$\begin{aligned} M^{\mathcal{I}_{\mathcal{P}}}(\nu i) &< (V^{\mathcal{I}_{\mathcal{P}}}(\nu i) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(\nu i)) \otimes (W^{\mathcal{I}_{\mathcal{P}}}(\nu i) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(\nu i)) \\ &\leq \min\{V^{\mathcal{I}_{\mathcal{P}}}(\nu i) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(\nu i), W^{\mathcal{I}_{\mathcal{P}}}(\nu i) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(\nu i)\}. \end{aligned}$$

[$\mathcal{ELC}$] Consider the ontologies

$$\begin{aligned} \mathcal{O}_{\mathsf{aux}} &:= \{X \sqsubseteq X \sqcap X, \top \sqsubseteq \neg(X \sqcap \neg X)\} \cup \\ &\quad\ \{\langle e_0 : \neg Y \geq 1\rangle\} \cup \{\exists r_i.\neg Y \sqsubseteq \bot \mid 1 \leq i \leq n\}, \\ \mathcal{O}_{V\neq W} &:= \mathcal{O}_{\mathsf{aux}} \cup \\ &\quad\ \{Y \sqcap X \sqcap V \sqsubseteq Y \sqcap X \sqcap W \sqcap M, \quad (1) \\ &\quad\ Y \sqcap \neg X \sqcap W \sqsubseteq Y \sqcap \neg X \sqcap V \sqcap M\}. \quad (2) \end{aligned}$$

Every model of $\mathcal{O}_{\mathsf{aux}}$ has to satisfy that every r_i-successor with degree 1 must belong to Y with degree 1 too, for every $1 \leq i \leq n$. In particular, this means that for every model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{\mathsf{aux}}$ and every $\nu \in \mathcal{N}^+$, we have $Y^{\mathcal{I}}(g(\nu)) = 1$. The first axiom ensures that for every $x \in \mathcal{D}^{\mathcal{I}}$, $X^{\mathcal{I}}(x) \leq X^{\mathcal{I}}(x) \otimes X^{\mathcal{I}}(x)$, and hence, $X^{\mathcal{I}}(x)$ must be an idempotent element w.r.t. $\otimes$. In particular, this means that $(X \sqcap \neg X)^{\mathcal{I}}(x) = \min\{X^{\mathcal{I}}(x), 1 - X^{\mathcal{I}}(x)\}$ [20], and from the second axiom it follows that $X^{\mathcal{I}}(x) \in \{0, 1\}$.

Let now $\mathcal{I}$ be a model of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V\neq W}$ and $\nu \in \mathcal{N}^+$. If $X^{\mathcal{I}}(g(\nu)) = 1$, then axiom (1) states that $V^{\mathcal{I}}(g(\nu)) \leq W^{\mathcal{I}}(g(\nu)) \otimes M^{\mathcal{I}}(g(\nu))$. We consider which representative was chosen for the encoding function:

[$\Pi^{(a,b)}$] Since $W^{\mathcal{I}}(g(\nu)) = \mathsf{enc}(w_\nu) > a$ and $M^{\mathcal{I}}(g(\nu)) = \mathsf{enc}(1) < b$, we have $W^{\mathcal{I}}(g(\nu)) \otimes m' > W^{\mathcal{I}}(g(\nu)) \otimes M^{\mathcal{I}}(g(\nu)) \geq V^{\mathcal{I}}(g(\nu))$ for any $m' > M^{\mathcal{I}}(g(\nu))$.

[$\mathsf{Ł}^{(a,b)}$] Since the length of w_ν is bounded by $|\nu|k$ and

$$W^{\mathcal{I}}(g(\nu)) \otimes M^{\mathcal{I}}(g(\nu)) = a + (b-a)\max\{0, 1 - 0.\overleftarrow{w_\nu} - (0.\underline{0}^{|\nu|k} \cdot \underline{1})\},$$

we have $W^{\mathcal{I}}(g(\nu)) \otimes M^{\mathcal{I}}(g(\nu)) = a + (b-a)(1 - 0.\overleftarrow{w_\nu} - (0.\underline{0}^{|\nu|k} \cdot \underline{1})) \in (a, b)$. Thus, $W^{\mathcal{I}}(g(\nu)) \otimes m' > W^{\mathcal{I}}(g(\nu)) \otimes M^{\mathcal{I}}(g(\nu)) \geq V^{\mathcal{I}}(g(\nu))$ for any $m' > M^{\mathcal{I}}(g(\nu))$.

In both cases, since

$$W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu)) = \sup\{z \in [0,1] \mid W^{\mathcal{I}}(g(\nu)) \otimes z \leq V^{\mathcal{I}}(g(\nu))\},$$

we have $W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu)) \leq M^{\mathcal{I}}(g(\nu))$. Similarly, if $X^{\mathcal{I}}(g(\nu)) = 0$, then axiom (2) yields $V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)) \leq M^{\mathcal{I}}(g(\nu))$.

To show the second point of $\mathrm{P}_{\neq}$, consider an extension $\mathcal{I}$ of $\mathcal{I}_{\mathcal{P}}$ that satisfies $\mathcal{O}_{\mathcal{P}}$, which exists by assumption. We show that $\mathcal{I}$ can be further extended to a model of $\mathcal{O}_{V\neq W}$. We first set $Y^{\mathcal{I}}(\nu) = 1$ for every $\nu \in \mathcal{N}^+$ and $X^{\mathcal{I}}(\varepsilon) = Y^{\mathcal{I}}(\varepsilon) = 0$.

To find the remaining values for X, consider any $\nu \in \mathcal{N}^+$. By assumption, we know that

$$\min\{V^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(\nu),\ W^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(\nu)\} \leq M^{\mathcal{I}_{\mathcal{P}}}(\nu) < 1.$$

One of the two residua must be equal to 1. If $V^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(\nu) = 1$ and $W^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(\nu) \leq M^{\mathcal{I}_{\mathcal{P}}}(\nu)$, then we set $X^{\mathcal{I}}(\nu) = 1$, which trivially satisfies axiom (2) at ν. By definition of the residuum, this implies that $W^{\mathcal{I}_{\mathcal{P}}}(\nu) \otimes m' > V^{\mathcal{I}_{\mathcal{P}}}(\nu)$ for all $m' > M^{\mathcal{I}_{\mathcal{P}}}(\nu)$. Since $\otimes$ is continuous and monotone, this means that $V^{\mathcal{I}_{\mathcal{P}}}(\nu) \leq W^{\mathcal{I}_{\mathcal{P}}}(\nu) \otimes M^{\mathcal{I}_{\mathcal{P}}}(\nu)$, i.e. axiom (1) is also satisfied at ν.

If the other residuum is equal to 1, we set $X^{\mathcal{I}}(\nu) = 0$ and use dual arguments to show that axioms (1) and (2) are satisfied at ν. We have thus constructed an extension of $\mathcal{I}$ that also satisfies $\mathcal{O}_{V\neq W}$. □

If a fuzzy DL satisfies the property $\mathrm{P}_{\neq}$, then consistency of ontologies is undecidable.

Theorem 20. *Let $\otimes_{\mathsf{x}}$-$\mathcal{L}$ satisfy $\mathrm{P}_{\neq}$. Then $\mathcal{P}$ has a solution iff $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V\neq W}$ is inconsistent.*

Proof. If $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V\neq W}$ is inconsistent, then in particular no extension of $\mathcal{I}_{\mathcal{P}}$ can satisfy this ontology. By $\mathrm{P}_{\neq}$, there is a $\nu \in \mathcal{N}^+$ such that both $V^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(\nu)$ and $W^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(\nu)$ are greater than $M^{\mathcal{I}_{\mathcal{P}}}(\nu)$. By Definition 7 and since $M^{\mathcal{I}_{\mathcal{P}}}(\nu) = \mathsf{enc}(\underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|})$, we have $\mathsf{enc}(v_\nu) = V^{\mathcal{I}_{\mathcal{P}}}(\nu) = W^{\mathcal{I}_{\mathcal{P}}}(\nu) = \mathsf{enc}(w_\nu)$, i.e. $\mathcal{P}$ has a solution.

Assume now that $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V\neq W}$ has a model $\mathcal{I}$. By $\mathrm{P}_{\neq}$, for every $\nu \in \mathcal{N}^+$, we have $V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)) \leq M^{\mathcal{I}}(g(\nu)) = \mathsf{enc}(\underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|})$ or $W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu)) \leq \mathsf{enc}(\underline{u_\varepsilon} \cdot \underline{u_+}^{|\nu|})$. By $\mathrm{P}_{\triangle}$, it follows that $\mathsf{enc}(v_\nu) = V^{\mathcal{I}}(g(\nu)) \neq W^{\mathcal{I}}(g(\nu)) = \mathsf{enc}(w_\nu)$, and thus $v_\nu \neq w_\nu$ for all $\nu \in \mathcal{N}^+$, i.e. $\mathcal{P}$ has no solution. □

Together with Corollaries 16 and 18, we obtain the following results.

Corollary 21. *For every continuous t-norm different from the Gödel t-norm, ontology consistency is undecidable in the logics $\otimes_\top$-$\mathfrak{IAL}_=$, $\otimes$-$\mathfrak{IAL}_{=,\mathsf{c}}$, $\otimes_\top$-$\mathcal{ELC}_\geq$, $\otimes$-$\mathcal{ELC}_{\geq,\mathsf{c}}$, $\Pi_\top$-$\mathcal{ELC}$, and Π-$\mathcal{ELC}_\mathsf{c}$.*

Since every extension of $\mathcal{I}_{\mathcal{P}}$ is witnessed, from these results it also follows that ontology consistency in the logics $\otimes_{\mathsf{w}}$-$\mathfrak{I}\mathcal{AL}_{=}$, $\otimes_{\mathsf{w}}$-$\mathcal{ELC}_{\geq}$, and Π_{w}-$\mathcal{ELC}$ is undecidable.

5 Undecidability of $Ł^{(0,b)}$-$\mathfrak{I}\mathcal{EL}$

We now consider the fuzzy DLs $Ł^{(0,b)}_{\top}$-$\mathfrak{I}\mathcal{EL}$ and $Ł^{(0,b)}$-$\mathfrak{I}\mathcal{EL}_{\mathsf{c}}$ for $b > 0$ and show that consistency in these logics is also undecidable. The t-norms $(0, b)$-containing the Łukasiewicz t-norm cover an important family of t-norms, known as the Mayor-Torrens t-norms that have been studied in the literature [20].

In this setting, we will use a slightly different encoding function to the one presented in the previous section for t-norms containing the Łukasiewicz t-norm. We encode a word $u \in \Sigma_0^*$ by $\mathsf{enc}(u) = b(0.\overleftarrow{u})$. The proof that this is indeed a valid encoding uses similar arguments to the case for $Ł^{(a,b)}$ of Lemma 8.

Assume that $v_\nu \neq w_\nu$. Then these words must differ in one of the first $|\nu|k$ digits, and thus either

$$\begin{aligned}\mathsf{enc}(v_\nu) \Rightarrow \mathsf{enc}(w_\nu) &= b\min\{1, 1 - 0.\overleftarrow{v_\nu} + 0.\overleftarrow{w_\nu}\}\\ &\leq b(1 - (s+1)^{-|\nu|k})\\ &= \mathsf{enc}(\underline{\varepsilon} \cdot \underline{s}^{|\nu|k})\end{aligned}$$

or $\mathsf{enc}(w_\nu) \Rightarrow \mathsf{enc}(v_\nu) \leq \mathsf{enc}(\underline{\varepsilon} \cdot \underline{s}^{|\nu|k}) < 1$. Conversely, if $v_\nu = w_\nu$, then both residua are 1. Thus, the words $u_\varepsilon = \varepsilon$ and $u_+ = \underline{s}^k$ satisfy the condition of Definition 7.

We will use Theorem 17 to show that the logics $Ł^{(0,b)}_{\top}$-$\mathfrak{I}\mathcal{EL}$ and $Ł^{(0,b)}$-$\mathfrak{I}\mathcal{EL}_{\mathsf{c}}$ satisfy the canonical model property. Thus, we need to prove that they satisfy $\mathrm{P}_{\rightarrow}$, $\mathrm{P}_{\circ}$, $\mathrm{P}_{\rightsquigarrow}$, $\mathrm{P}^{w}_{\mathsf{ini}}$, and $\mathrm{P}_{=}$. By Lemma 9, they satisfy the successor property. It now suffices to show that $Ł^{(0,b)}$-$\mathfrak{I}\mathcal{EL}$ satisfies the rest of the properties.

Concatenation property We can use the ontology

$$\mathcal{O}_{\mathcal{P},\circ} := \{(C' \to \bot)^{(s+1)^{|u|}} \equiv C \to \bot,\ D_{C\circ u} \to \bot \equiv (C' \to \bot) \sqcap (C_u \to \bot)\}$$

to concatenate words represented by C with the constant word u. This can be shown analogously to the case for $Ł^{(a,b)}$-$\mathcal{EL}$ of Lemma 10.

Transfer property Suppose we want to transfer the value of C through the role r to D. If $C^{\mathcal{I}}(x) = \mathsf{enc}(w)$ for some $w \in \Sigma^*$, then $C^{\mathcal{I}}(x) < b$, and thus for every model $\mathcal{I}$ of $\exists r.(D \to \bot) \sqsubseteq C \to \bot$ if $r^{\mathcal{I}}(x, y) = 1$ then

$$b - C^{\mathcal{I}}(x) = (C \to \bot)^{\mathcal{I}}(x) \geq (\exists r.(D \to \bot))^{\mathcal{I}}(x) \geq (D \to \bot)^{\mathcal{I}}(y).$$

This holds if (i) $D^{\mathcal{I}}(y) \geq b > C^{\mathcal{I}}(x)$ or (ii) $D^{\mathcal{I}}(y) < b$ and $b - D^{\mathcal{I}}(y) \geq b - C^{\mathcal{I}}(x)$. Thus, we can use $\mathcal{O}_{C \overset{r}{\rightsquigarrow} D} := \{\exists r.(D \to \bot) \sqsubseteq C \to \bot, \exists r.D \sqsubseteq C\}$ to satisfy the property.

Weak initialization property Since we only have to be able to initialize the value $\mathsf{enc}(u_\varepsilon) = \mathsf{enc}(\varepsilon) = 0$, we can use the simple axiom $\langle e : C \to \bot \geq 1\rangle$.

Constant property It remains to show how we can restrict the value of a concept C to always be $\mathsf{enc}(u)$ for some word $u \in \Sigma_0^*$. For this, we employ the ontology

$$\mathcal{O}_{C=u} := \{H^{(s+1)^{|u|}} \equiv (H^{(s+1)^{|u|}}) \to \bot,\ C \to \bot \equiv H^{2\overleftarrow{u}}\}.$$

If an interpretation $\mathcal{I}$ satisfies the first axiom, then for every $x \in \mathcal{D}^{\mathcal{I}}$ we have $-b = 2(s+1)^{|u|}(H^{\mathcal{I}}(x) - b)$; that is $H^{\mathcal{I}}(x) = b - \frac{b}{2(s+1)^{|u|}}$. From the second axiom it follows that

$$(C \to \bot)^{\mathcal{I}}(x) = \max\left\{0, 2\overleftarrow{u}\left(-\frac{b}{2(s+1)^{|u|}}\right) + b\right\}.$$

Since $\frac{\overleftarrow{u}}{(s+1)^{|u|}} = 0.\overleftarrow{u} < 1$, we obtain $(C \to \bot)^{\mathcal{I}}(x) = b - b(0.\overleftarrow{u}) = b - \mathsf{enc}(u)$. Since $\mathsf{enc}(u) < b$, we have $(C \to \bot)^{\mathcal{I}}(x) > 0$, and thus $C^{\mathcal{I}}(x) < b$ and $(C \to \bot)^{\mathcal{I}}(x) = b - C^{\mathcal{I}}(x)$. From this, we obtain that $C^{\mathcal{I}}(x) = \mathsf{enc}(u)$.

It is easy to see how to extend $\mathcal{I}_{\mathcal{P}}$ to a model of the ontology $\mathcal{O}_{\mathcal{P}}$ that results from the above definitions. By Theorem 17, the logics $\text{Ł}_{\top}^{(0,b)}$-$\mathfrak{I}\mathcal{EL}$ and $\text{Ł}^{(0,b)}$-$\mathfrak{I}\mathcal{EL}_{\mathsf{c}}$ satisfy the canonical model property. Before we can apply Theorem 20 to show undecidability of these logics, we need to show that the solution property also holds.

Lemma 22. *The logics* $\text{Ł}_{\top}^{(0,b)}$-$\mathfrak{I}\mathcal{EL}$ *and* $\text{Ł}^{(0,b)}$-$\mathfrak{I}\mathcal{EL}_{\mathsf{c}}$ *satisfy* $\mathrm{P}_{\neq}$.

Proof. Consider the ontology

$$\mathcal{O}_{V \neq W} := \{\exists r_i.((((V \to W) \sqcap (W \to V)) \to M) \to \bot) \sqsubseteq \bot \mid i \in \mathcal{N}\}.$$

In any model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}_{V \neq W}$ it holds that for every $\nu \in \mathcal{N}^+$,

$$((((V \to W) \sqcap (W \to V)) \to M) \to \bot)^{\mathcal{I}}(g(\nu)) = 0,$$

and thus, $(((V \to W) \sqcap (W \to V)) \to M)^{\mathcal{I}}(g(\nu)) \geq b$. In the case that $V^{\mathcal{I}}(g(\nu)) \leq W^{\mathcal{I}}(g(\nu))$, then

$$\min\{b, b - (W \to V)^{\mathcal{I}}(g(\nu)) + M^{\mathcal{I}}(g(\nu))\} \geq b,$$

	$\mathfrak{IEL}$	$\mathfrak{IAL}$	$\mathcal{ELC}$
classical	Ł$^{(0,b)}$	Ł$^{(0,b)}$	Π, Ł
$\geq$	Ł$^{(0,b)}$	Ł$^{(0,b)}$	$\otimes$
$=$	Ł$^{(0,b)}$	$\otimes$	$\otimes$

Table 3: A summary of the results.

which implies $(W \to V)^{\mathcal{I}}(g(\nu)) \leq M^{\mathcal{I}}(g(\nu))$. Similarly, if $W^{\mathcal{I}}(g(\nu)) \leq V^{\mathcal{I}}(g(\nu))$, then $(V \to W)^{\mathcal{I}}(g(\nu)) \leq M^{\mathcal{I}}(g(\nu))$. In both cases, we have

$$\min\{V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)), W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu))\} \leq M^{\mathcal{I}}(g(\nu)).$$

To show the second condition of $\mathrm{P}_{\neq}$, consider an extension $\mathcal{I}$ of $\mathcal{I}_{\mathcal{P}}$ that satisfies $\mathcal{O}_{\mathcal{P}}$ and assume that it violates $\mathcal{O}_{V \neq W}$. Then there must be $\nu \in \mathcal{N}^*$ and $i \in \mathcal{N}$ such that

$$((((V \to W) \sqcap (W \to V)) \to M) \to \bot)^{\mathcal{I}}(\nu i) > 0,$$

which implies that

$$(V \to W)^{\mathcal{I}}(\nu i) \otimes (W \to V)^{\mathcal{I}}(\nu i) > M^{\mathcal{I}}(\nu i).$$

By monotonicity of $\otimes$, both $V^{\mathcal{I}}(\nu i) \Rightarrow W^{\mathcal{I}}(\nu i)$ and $W^{\mathcal{I}}(\nu i) \Rightarrow V^{\mathcal{I}}(\nu i)$ must be greater than $M^{\mathcal{I}}(\nu i)$, contradicting the assumption. □

This shows that consistency in Ł$_{\top}^{(0,b)}$-$\mathfrak{IEL}$ and Ł$^{(0,b)}$-$\mathfrak{IEL}_{\mathsf{c}}$ is undecidable. As before, undecidability of Ł$_{\mathsf{w}}^{(0,b)}$-$\mathfrak{IEL}$ follows from the same arguments since every extension of $\mathcal{I}_{\mathcal{P}}$ is witnessed.

Notice that Ł$^{(0,1)}$-$\mathfrak{IEL}$ corresponds to the logic Ł-$\mathfrak{IEL}$ and that Ł-$\mathfrak{IEL}$ and Ł-$\mathcal{ELC}$ are equivalent. Thus, the logic Ł-$\mathcal{ELC}$ is also undecidable.

6 Conclusions

We have presented a framework for showing undecidability of consistency in fuzzy description logics and have successfully applied this framework to numerous fuzzy DLs. Table 3 summarizes the obtained undecidability results. Every cell represents a combination of constructors and axioms. The entry in a cell denotes the largest family of t-norms for which we have shown undecidability of the resulting fuzzy DL with either $\top$-witnessed or witnessed models or crisp role axioms. Here, $\otimes$ represents all continuous t-norms different from the Gödel t-norm.

Our results strictly strengthen all previously known undecidability results for fuzzy DLs in several ways. First, in all previous works, ontologies required more expressive *fuzzy GCIs* of the form $\langle C \sqsubseteq D \geq q\rangle$. To be more precise, previous work has shown that the fuzzy DLs

- Π_{w}-$\mathcal{ALC}_{\geq}$ (with some additional axioms) [2],

- $\Pi_{\mathsf{w}}^{(0,b)}$-$\mathfrak{I}\mathcal{AL}_{=}$ [4], and

- $Ł_{\mathsf{w}}$-$\mathcal{ELC}_{\geq}$ [14]

extended with fuzzy GCIs are undecidable. For the first and last case, we were able to show that classical ontologies suffice to get undecidability. We find these results especially interesting, since they show that it is the underlying semantics, and not the expressivity of the axioms, that yields undecidability. In the second case, we extended the class of t-norms for which the logic is undecidable to cover all continuous t-norms, except the Gödel t-norm.

The decision problem considered in this report, ontology consistency, is the hardest standard reasoning problem in the sense that the other reasoning problems (like concept satisfiability or subsumption between concepts) can be reduced to it, but a converse reduction is not possible using only the constructors of $\mathfrak{I}\mathcal{ALC}$. It is thus natural to ask whether these other problems are also undecidable. Our proofs of undecidability w.r.t. classical ontologies (first row of Table 3) use a set of GCIs and a set of crisp concept assertions using all the same individual name. It thus follows that concept satisfiability in $Ł^{(0,b)}$-$\mathfrak{I}\mathcal{EL}$, Π-$\mathcal{ELC}$ and $Ł$-$\mathcal{ELC}$ is undecidable w.r.t. $\top$-witnessed and witnessed models, and w.r.t. general models if crisp role axioms are allowed. If no GCIs are allowed, then the problem is decidable in $\otimes$-$\mathcal{ELC}$, for any continuous t-norm $\otimes$ [18]. We will continue studying the decidability of these reasoning problems in different fuzzy DLs.

To the best of our knowledge, we have presented the first undecidability results w.r.t. general models. Crisp role axioms allow us to eliminate the restriction to witnessed models. Crisp roles have been considered before in some applications where it does not make sense to view certain relations as fuzzy (e.g. in the `fuzzyDL` reasoner[6] or [32]).

In the future, we will continue studying the problem of reasoning w.r.t. general models, and consider also reasoning in other classes of models like finite, or strongly witnessed models, for which only a few undecidability results exist [4]. We also want to find decidable classes of fuzzy DLs, beyond the simple restrictions to finitely many fuzzy values [11, 12, 10] or to acyclic terminologies [5].

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