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A nice Cycle Rule for Goal-Directed *E*-unification

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Abstract. In this paper we improve a goal-directed E-unification procedure by introducing a new rule, *Cycle*, for the case of collapsing equations, i.e. equations of the type $x \approx v$ where $x \in Var(v)$. In the case of these equations some obviously unnecessary infinite paths of inferences were possible, because it was not known if the inference system was still complete if the inferences were not allowed into positions of x in v. Cycle does not allow such inferences and we prove that the system is complete. Hence we prove that as in other approaches, inferences into variable positions in our goal-directed procedure are not needed.

1 Introduction

E-unification problems appear when one has to decide or find solution to an equation between first-order terms or a set of such equations modulo an equational theory. Preferably, we would like to have a procedure which would be able to enumerate all possible solutions, or a set of most general ones. Alas, this problem is in general undecidable. But trying to solve such problem by *brute force*, e.g. using axioms of congruence relation and Resolution, would produce an infinitely many useless inferences even in decidable cases. E-unification problem is in general semi-decidable, and there exist complete semi-decision procedures to solve it. In view of the undecidability, inventing new, better procedures for a general E-unification, have an important practical aim: to understand the problem to such a degree, as to be able to distinguish useful cases of decidable equational theories. [4], [5], [6] are examples of such results.

The result presented in this paper will hopefully open a way to detect even more decidable equational theories, because the improvement presented here for an E-unification procedure reduces in a dramatic way the degree of a "don't know" type of non-determinism involved. Namely, it prevents some unnecessary infinite sequences of inferences, which were possible in our previous E-unification procedure [7] in case of collapsing goal equations, i.e. the equations of type $x \approx v$, $x \in Var(v)$. The result presented in this paper may be stated as follows: in our goal-directed E-unification procedure, just as in some other, rival approaches, inferences into variable position are not needed.

Most of these other E-unification procedures used nowadays are based on Knuth-Bendix completion of an equational theory, and *narrowing* of goal equations. We would like to argue, that a different approach, *goal-directed*, with the improvements presented in this paper is in some cases better than the former. In contrast to the procedures based on completion, a goal-directed approach to E-unification consists in transforming a set of goal equations, without changing an equational theory, E. Hence the goal-directed approach is better, when completion of E produces many equations that are unnecessary for solving a given goal. Moreover, a goal-directed approach does not require any ordering of terms, whereas any procedure based on completion is sensitive to a choice of an order. Nevertheless up to now, this approach demanded some inferences into variable positions, which generated much of a troublesome, "don't know" kind of non-determinism.

A goal-directed E-unification procedure was first presented by Gallier and Snyder in [2]. Alas, they could not succeed in proving the completeness of their system. The difficulty lay in justifying eager applications of Variable Elimination (Fig. 1). In [8] we have finally proved that their system is in fact complete. Please, look up [9] for the details of the proof.

In [3], Gallier and Snyder replaced eager applications of Variable Elimination with Variable Decomposition (Fig. 1), called by them Root Imitation there, and Root Rewriting, which is in fact our Mutate and Variable Mutate (Fig. 1). They noticed at once three main drawbacks of this system. First, "the possibility of rewriting variables in Root Rewriting", second, having to solve variable-variable equations (equations of the type $x \approx y$) and third, "the potential for infinite recursion" in the Root Imitation if we have to solve an equations of the type $x \approx v$ and x occurs in v. (cf. [3], p. 233).

In [7], we have proved that there is really no need to bother with solving variable-variable equations, since they can be dealt with by techniques of syntactic unification after all other equations are solved.

In view of perceived difficulties, in [3], Gallier and Snyder presented a different goal-directed inference system, namely the one based on *Lazy Paramodulation* inference rule. Lazy Paramodulation has the following form:

$$\frac{\{u \approx v\} \cup G}{\{u_1 \approx s_1, \ldots, u_n \approx s_n, u[t]_\alpha \approx v\} \cup G}$$

where $\{u \approx v\} \cup G$ is a set of goal equations, $f(s_1, \ldots, s_n) \approx t$ is a renaming of an equation in an equational theory E ($f(s_1, \ldots, s_n) \approx t \in E$) and $u|_\alpha = f(u_1, \ldots, u_n)$ or

$$\frac{\{u \approx v\} \cup G}{\{u|_\alpha \approx x \quad u[t]_\alpha \approx v\} \cup G}$$

where $x \approx t \in E$ and $u|_\alpha$ is not a variable, (cf. [3], page 242).

This presentation corresponds to assuming that the leftmost, highest step in a proof of $u\sigma \approx v\sigma$ is at position α, where σ is an E-unifier of $u \approx v$. In this paper, we have a different approach, i.e. we look for the rightmost step in a proof. Apart from this, our Mutate and Variable Mutate is exactly Lazy Paramodulation when $\alpha = \epsilon$.

In fact, Gallier and Snyder conjecture, that their system is still complete if Lazy Paramodulation is restricted so that it applies only when either $\alpha = \epsilon$ or one of u, v is a variable (cf. [3], page 247).

The reader can view the result presented in our paper, as proving their conjecture in the context of our system with Variable Elimination eagerly applied. we introduce a new rule, Cycle, which can be viewed as Lazy Paramodulation with Decomposition on equations of the form $x \approx v$, where x occurs in v. Moreover, we don't need Lazy Paramodulation for other equations of the form $x \approx v$, because for them we use Variable Elimination eagerly.

We use a similar kind of analysis of equational proofs which enabled us also to prove completeness of the inference system with the rule Variable Elimination eagerly applied in [8]. We are using a similar, though simplified, definition of paths and a new transformation on equational proofs to justify our new Cycle rule. We use a smarter way to count paths.

The plan of the paper is the following: after preliminary definitions which describe properties of equational proofs, we present the inference rules for a procedure solving E-unification problems. Next we will present and explain operations on equational proofs and a procedure *Solve* which given a solution, yields an E-equivalent solution for a goal. Then we will prove that a new solution must be smaller than the old one in some respect. Hence, we will define a measure of a goal with an E-solution, which is decreasing with an inference rule of E-unification procedure with a new Cycle rule, applied to a goal and thus enables us to prove the completeness of our procedure.

2 Preliminaries

Reader should consult [1] for standard definitions of term, ground term, substitution, ground substitution, position in a term, subterm. If t is a term, and p a position defined in this term, $t[s]_p$ means a term t with a subterm s at position p. Furthermore, $t[s_1]_{p_1} \ldots [s_n]_{p_n}$ means a term t with subterms $s_1, \ldots, s_n$ at parallel positions $p_1, \ldots, p_n$.

We will consider equations of the form $s \approx t$, where s and t are terms. Throughout this paper these equations are considered to be oriented, so that $s \approx t$ is a different equation than $t \approx s$. Let E be a set of equations, and $u \approx v$ be an equation, then we write $E \models u \approx v$ (or $u =_E v$) if $u \approx v$ is true in any model containing E. We call E an equational theory, and assume that E is closed under symmetry. A goal (E-unification problem) is usually denoted by G and it is a set of equations. $E \models G$ means that $E \models e$ for all e in G.

We will be considering ground terms as ground objects that may or may not have the same syntactic form. In other words we will be concerned with the occurrences of the terms more than their values. A term may be identified by its address in a proof sequence and a position of it as a subterm in a term in the proof. Hence the equality sign between ground terms is treated in a special way.

If u, v are ground terms, by $u = v$, u is understood to be an object identical with v, whereas when syntactic equality is sufficient, it will be denoted by $u == v$. Syntactic inequality will be denoted by $u \neq= v$. The difference between identity and syntactic identity is that the first involves *objects* and the second involves *names*.

We can say that a variable x points to its occurrences in a term u, where each of these occurrences under some ground substitution γ, is a subterm of $u\gamma$ at a position α ($x\gamma = u\gamma|_\alpha$). Different occurrences of the same variables are different objects, though they have the same syntactic form (each one is of the form $x\gamma$). In order to distinguish between different occurrences of the same variable, we will use superscript numbers, usually numbering the occurrences from left to right in order of their appearances in an equational proof. Hence $x\gamma^1$ and $x\gamma^2$ are different occurrences of x in a proof.

Sometimes we will want to state that some subterm has a form (or value) of $x\gamma$, but is not identical to $x\gamma$ (hence is not pointed to by a variable x). This will be indicated by quote marks. Hence $w[\text{“}x\gamma\text{”}]_\alpha$ is different from $w[x\gamma]_\alpha$ since in the second term $x\gamma$ actually occurs at position α, while in the first one there is only a subterm that has the value of $x\gamma$.

If γ is a ground substitution, γ_x means the restriction of this substitution to a variable x. Hence if $\gamma = [x \mapsto a, y \mapsto b, z \mapsto c]$, $\gamma_x = [x \mapsto a]$.

E-unification problem is given as an equational theory E is a set of goal equations G and we want to find a substitution γ such that $E \models G\gamma$. γ is then called a solution. In the completeness proof of our procedure, we will assume that there is a ground substitution γ such that $E \models G\gamma$. This is sufficient in order to show that the procedure computes a complete set of most general solutions for G. But if such substitution γ exists, there is a ground equational proof Π for all equations in $G\gamma$. We define here equational proof in a more classic way than in [8].

Definition 1. *(equational proof)*
Let E be a set of equations. An equational proof of an equation $u \approx v$, where u and v are ground terms, is a series of ground terms, $\Pi = (w_1, w_2, \ldots, w_n)$, such that:

1. $u = w_1$, $v = w_n$,
2. *for each pair (w_i, w_{i+1}) for $1 \leq i \leq (n-1)$, there is an equation $s \approx t \in E$ and a matcher ρ, such that there is a subterm $w_i|_\alpha$ of w_i and a subterm $w_{i+1}|_\alpha$ of w_{i+1}, and $w_i|_\alpha = s\rho$, $w_{i+1}|_\alpha = t\rho$.*

We can write the equational proof as
$u\gamma = w_1 \approx_{[\alpha_1, s_1 \approx t_1, \rho_1]} w_2 \approx_{[\alpha_2, s_2 \approx t_2, \rho_2]} \cdots \approx_{[\alpha_{n-1}, s_{n-1} \approx t_{n-1}, \rho_{n-1}]} w_n = v\gamma$
where u and v are not necessarily ground terms, but γ makes them ground. $[\alpha_i, s_i \approx t_i, \rho_i]$ indicates at what position α_i is the matching subterm, which equation from E was used ($s_i \approx t_i$), and how the variables in this equation were substituted (ρ). Each w_i in the above sequence is called a term in the proof, as distinct from any proper subterms of w_i, which are not counted as terms

in the proof. Since an equational proof is a sequence of ground terms, we will sometimes use the notation borrowed from that for arrays, and $\Pi[i]$ will mean the i'th term in Π.

Since every matcher at each step uses a renamed version of an equation from E, the domain of the matcher is disjoint from the domain of γ and the domains of matchers at all other steps in the proof, we extend γ to γ' such that: $\gamma' = \gamma \cup \rho_1 \cup \ldots \cup \rho_n$. From now on we will assume that γ is an extended version of itself.

For the purposes of the completeness proof in Section 6, we have to extend γ even more. We define general extension of γ.

Definition 2. *(general extension of γ)*
Let γ be a ground substitution. A general extension of γ, $ex(\gamma)$, is defined recursively as follows:

1. *if $\gamma_x = [x \mapsto v]$ and $|v| = 1$ (v is a constant), then $ex(\gamma_x) = \gamma_x$,*
2. *if $\gamma_x = [x \mapsto f(v_1, \ldots, v_n)]$, and $n \geq 1$, then let $\gamma_{y_i} = [y_i \mapsto v_i]$, for $1 \leq i \leq n$, and $ex(\gamma_x) = \gamma_x \cup ex(\gamma_{y_1}) \cup \cdots \cup ex(\gamma_{y_n})$,*
3. *$ex(\gamma) = \bigcup_{x \in Dom(\gamma)} ex(\gamma_x)$*

From now on we consider γ in (Π, γ) as the general extension of itself. We have 3 kinds of variables now: the variables in a goal equation $u \approx v$, called *goal variables*, the variables in $Var(s_i \approx t_i)$, where there is a step in Π, $\Pi[i] \approx_{[\alpha, s_i \approx t_i, \gamma)]} \Pi[i+1]$, called *system variables*, and variables introduced in general extension of γ, called *subterm variables*.

For each of the occurrences of these variables we define *orientation*. Let $u \approx v$ be a goal equation. If $x\gamma$ is an occurrence of a goal variable in u, then this $x\gamma$ has right orientation ($\overrightarrow{x\gamma}$), if $x\gamma$ is an occurrence of a goal variable in v, then this $x\gamma$ has left orientation ($\overleftarrow{x\gamma}$), if $x\gamma$ is an occurrence of a system variable in $\Pi[i] \approx_{[\alpha_i, s_i \approx t_i, \gamma]} \Pi[i+1]$ and $x\gamma$ occurs in $\Pi[i]$, then this $x\gamma$ has left orientation ($\overleftarrow{x\gamma}$), and if $x\gamma$ occurs in $\Pi[i+1]$, then this $x\gamma$ has right orientation ($\overrightarrow{x\gamma}$), if $x\gamma$ is a subterm variable occurrence, hence $x\gamma = y\gamma|_\alpha$, then it has the same orientation as $y\gamma$.

$\Sigma_{w \approx w'}$ is a subproof in a proof Π, if there is a part of Π: $\Pi[i] \approx_{[\alpha_i, s_i \approx t_i, \gamma]} \Pi[i+1] \approx_{[\alpha_{i+1}, s_{i+1} \approx t_{i+1}, \gamma]} \cdots \approx_{[\alpha_{i+k}, s_{i+k} \approx t_{i+k}, \gamma]} \Pi[i+k]$, such that for $i \leq j \leq i+k$, $\alpha_j \geq \alpha$ or $\alpha_k || \alpha$, and $\Sigma_{w \approx w'}$ is $\Pi[i]|_\alpha \approx \Pi[i+1]|_\alpha \approx \cdots \approx \Pi[i+k]|_\alpha$ where $w = \Pi[i]|_\alpha$ and $w' = \Pi[i+k]|_\alpha$.

In a subproof $\Sigma_{w \approx w'}$ we can distinguish *internal* and *external* variables. A variable y is called internal in $\Sigma_{w \approx w'}$ if $y \in Var(s \approx t)$, and there is a step $w_i \approx_{[\alpha, s \approx t, \gamma]} w_{i+1}$ in $\Sigma_{w \approx w'}$.

We will use renamings of subproofs in the paper, but notice that renaming of a subproof is a subproof in which only internal variables are renamed.

With each occurrence of a variable $x\gamma$ in an equational proof, we associate a subproof (called a subproof associated with $x\gamma$), which is the longest subproof starting with $x\gamma$ and going in the direction of the orientation of $x\gamma$. The ground term at the end of the subproof associated with $x\gamma$ is called a term associated with $x\gamma$, $ass(x\gamma)$.

If we have a ground term w and a proof Π of the form $w_1 \approx_{[\alpha_1, s_1 \approx t_1, \gamma]} w_2 \approx_{[\alpha_2, s_2 \approx t_2, \gamma]} \cdots \approx_{[\alpha_{n-1}, s_{n-1} \approx t_{n-1}, \gamma]} w_n$, and $w|_\beta == w_1$, for some position β in w, then we can construct new equational proof Π' of the form: $w[w_1]_\beta \approx_{[\beta\alpha_1, s_1 \approx t_1, \gamma]} w[w_2]_\beta \approx_{[\beta\alpha_2, s_2 \approx t_2, \gamma]} \cdots \approx_{[\beta\alpha_{n-1}, s_{n-1} \approx t_{n-1}, \gamma]} w[w_n]_\beta$. We call this construction *embedding of the proof Π in the term w*.We can attach a proof Π' to a given equational proof Π by embedding it into the last term of Π, if the conditions of the definition are met. Then the new proof obtained in this way is called a *composition of Π and Π'*.

We define a non-redundant equational proof as any proof Π such that there are no two terms $\Pi[i]$ and $\Pi[j]$, with $i \neq j$ and $\Pi[i] == \Pi[j]$ in Π, and all proper subproofs of Π are non-redundant.

A simple procedure of cutting out loops out of subproofs in a proof allows us to obtain a non-redundant proof from any redundant one. We call this *contraction*. From now on, we will assume for all the equational proofs we are going to talk about that they are non-redundant. This property will be preserved in all the constructions which will be defined in the paper.

Since each ground solution γ for a goal G in an equational theory E is always associated with some equational proof Π which is a witness for the solution, we will talk rather about a pair (Π, γ) than about γ alone as a solution for a goal.

3 Transformation Rules

In this section we present the inference system for solving an E-unification problem in any equational theory E. Any procedure based on these rules must be non-terminating in some cases, because the problem is in general undecidable.

In [8] we have proved that the set of rules presented in Figure 1 (with slightly different formulation of Variable Mutate) is complete. An arbitrary selection function selects an equation $u \approx v$ from the set of goal equations for an inference.

Decomposition applies if both u and v are not variables and have the same root symbol. **Mutate** applies if there is an equation $s \approx t$ in E, such that t is not a variable and the root symbol of t is the same as root symbol of v (hence v must not be a variable). **Variable Mutate** applies if there is an equation $s \approx x$ in E and v is not a variable. If v is a variable, and u is not, then **Orient** applies. **Variable Elimination** applies if u is a variable and u does not occur in v. Notice that Variable Elimination is applied eagerly to such an equation, because there is no other rule applicable in this situation. If u and v are identical variables then **Trivial** deletes this equation from the goal.

If an equation of the form $x \approx v$ is selected, v is not a variable and $x \in Var(v)$, then we have a choice. **Mutate** applies, if $s \approx t \in E$ such that root symbols of v and t are the same, or **Variable Mutate** applies, if $s \approx x \in E$, or we can apply **Variable Decomposition**.

Variable Decomposition may lead immediately to infinite sequences of inferences, as in the following simple example:

Decomposition

$$\frac{\{f(s_1,\cdots,s_n)\approx f(t_1,\cdots,t_n)\}\cup G}{\{s_1\approx t_1,\cdots,s_n\approx t_n\}\cup G}$$

where $f(s_1,\cdots,s_n)\approx f(t_1,\cdots,t_n)$ is selected in the goal.

Mutate

$$\frac{\{u\approx f(v_1,\cdots,v_n)\}\cup G}{\{u\approx s, t_1\approx v_1,\cdots,t_n\approx v_n\}\cup G}$$

where $u\approx f(v_1,\cdots,v_n)$ is selected in the goal, and $s\approx f(t_1,\cdots,t_n)\in E$.[a]

Variable Mutate

$$\frac{\{u\approx f(v_1,\cdots,v_n)\}\cup G}{\{u\approx s[x\mapsto f(v_1,\cdots,v_n)]\}\cup G}$$

where $s\approx x\in E$, x is a variable, and $u\approx f(v_1,\cdots,v_n)$ is selected in the goal.

Variable Elimination

$$\frac{\{x\approx v\}\cup G}{\{x\approx v\}\cup G[x\mapsto v]}$$

where $x\notin Var(v)$

Orient

$$\frac{\{t\approx x\}\cup G}{\{x\approx t\}\cup G}$$

where x is a variable.
and t is not a variable.

Variable Decomposition (for cycle)

$$\frac{\{x\approx f(t_1,\cdots,t_n)\}\cup G}{\{x\approx f(x_1,\cdots,x_n)\}\cup(\{x_1\approx t_1,\cdots,x_n\approx t_n\}\cup G)[x\mapsto f(x_1,\cdots,x_n)]}$$

where x is a variable, $x\approx f(t_1,\cdots,t_n)$

Trivial

$$\frac{\{x\approx x\}\cup G}{G}$$

where $x\approx x$ is selected in the goal.

[a] We assume that E is closed under symmetry.

Fig. 1. E-Unification with eager Variable Elimination

$$\cfrac{x \approx fx}{x \approx fx_1 \quad \cfrac{x_1 \approx fx_1}{\ddots}}$$

We need to break the sequence of Variable Decompositions applied to the rightmost equation by applying Mutate or Variable Mutate, but we don't know when, after how many steps, we should do it. Hence at this point non-termination of our procedure does not depend only on semi-decidability of an equational theory, but also on properties of the inference rules.

Therefore we want to replace Variable Decomposition, Mutate and Variable Mutate applied to collapse equations, by another rule called **Cycle**, which will not lead to such immediate infinite paths of inferences. The rule is presented in Figure 2.

Cycle

$$\frac{\{x \approx v[v_1]_{\alpha_1} \cdots [v_k]_{\alpha_k}\} \cup G}{\{x \approx v[x_1]_{\alpha_1} \cdots [x_k]_{\alpha_k}\} \cup \bigcup_{i=1}^{k}\{M(x_i \approx v_i\sigma)\} \cup G\sigma}$$

where $x \approx v[v_1]_{\alpha_1} \cdots [v_k]_{\alpha_k}$ is selected in the goal, $x \in Var(v[v_1]_{\alpha_1} \cdots [v_k]_{\alpha_k})$, each v_i is a non-variable term, which contains at least one occurrence of x, $\sigma = [x \mapsto v[x_1]_{\alpha_1} \cdots [x_k]_{\alpha_k}]$.

$$M(x \approx f(v_1, \ldots, v_n)) = \begin{cases} \{x \approx s, t_1 \approx v_1, \ldots, t_n \approx v_n\}, \\ \text{where } s \approx f(t_1, \ldots, t_n) \in E; \\ \\ x \approx s[y \mapsto f(v_1, \ldots, v_n)], \\ \text{where } s \approx y \in E \text{ and} \end{cases}$$

Fig. 2. Cycle Rule

Cycle applies to a goal with an equation of the type $x \approx v$ selected, where $x \in Var(v)$. There are possibly many occurrences of x in v, hence v can be written as $v[x]_{\beta_1} \cdots [x]_{\beta_l}$, where β_i is a position of i'th occurrence of x in v and there are l occurrences of x in v. All these positions are parallel to each other, hence each β_i and β_j is such that $\beta_i || \beta_j$.

For each occurrence of x in v, we guess a subterm v_i in $v[x]_{\beta_1} \cdots [x]_{\beta_k}$ at a position α_i, containing this x (hence $\alpha_i \leq \beta_i$).

Hence $v[x]_{\beta_1} \cdots [x]_{\beta_l}$ can be represented as $v[v_1]_{\alpha_1} \cdots [v_k]_{\alpha_k}$, where $k \leq l$, because we can guess a subterm v_i containing more than one occurrence of x.

Cycle is a *macro* rule. One can view the effect of Cycle as decomposing a part of the term, by repeated applications of the old Variable Decomposition rule and Variable Elimination, which would allow us to obtain a solved equation $x \approx v[x_1]_{\alpha_1} \cdots [x_k]_{\alpha_k}$ and then performing Mutate or Variable Mutate on the equations $x_i \approx v_i$ obtained by this repeated Variable Decomposition.

The application of Mutate or Variable Mutate is represented by function M in the definition of Cycle. M returns a set of equations obtained from its argument by Mutate or Variable Mutate, depending on the form of equation chosen from E. Notice that only in the context of M, Mutate or Variable Mutate is applied to an equation of the form $x \approx v$.

In other words, Cycle performs Mutate or Variable Mutate at some position defined in a given term v, while *decomposing* symbols at higher positions. Subterms of v which do not contain x (we can call them irrelevant for x) are still inside the term $v[x_1]_{\alpha_1} \cdots [x_k]_{\alpha_k}$ with which x is *eliminated.*

Replacing Variable Decomposition, and Mutate and Variable Mutate as applied to *cyclic* equations by Cycle, reduces nondeterminism of our procedure. We have only finitely many positions in v to guess. Hence e.g. in our previous example we would have to try only two applications of Cycle.

Mutate at root symbol of v (hence we guess that $v[v_1]_\epsilon$), $s \approx f(t) \in E$:

$$\frac{x \approx fx}{x \approx x_1 \; x_1 \approx s \; t \approx x_1}$$

Mutate at $v|_1$, $s \approx f(t) \in E$:

$$\frac{x \approx fx}{x \approx fx_1 \; x_1 \approx s \; t \approx x_1}$$

In another example, where $x \approx fgx$ is our goal and $a \approx gb$ is in E, we have no choice, but to apply Mutate inside the Cycle rule at $fgx|_{<1>}$:

$$\frac{x \approx fgx}{x \approx fx_1 \; x_1 \approx a \; b \approx x_1}$$

4 Operations on Equational Proofs

Before proving completeness of the inference system with Cycle for general E-unification, we have to define operations on equational proofs that can transform such a proof into an E-equivalent equational proof. This means that if (Π, γ) is an E-solution for a goal $u \approx v$, and we obtain (Π', γ') by one of these operations, then $\gamma =_E \gamma'$ and (Π', γ') is also an E-solution for a goal $u \approx v$.

We will define here two such transformations: extending and flattening.

4.1 Extending a proof

If $x\gamma$ is an occurrence of a variable in an equational proof Π, and $\Sigma_{x\gamma \approx v}$ is a subproof associated with this occurrence, then if x has no occurrences in v, we can extend the proof in all places where x has occurrences in Π with the subproof $\Sigma_{x\gamma \approx v}$. We call this operation *extending equational proof with respect to* $x\gamma \approx v$. After this extension we get a new proof Π' and a new substitution γ', hence we write: $(\Pi, \gamma) \overset{[x \to v]}{\longrightarrow} (\Pi', \gamma')$. The formal description of the operation of extending a proof with respect to $x\gamma \approx v$ is the following.

Let (Π, γ) be an equational proof with γ an extended substitution for this proof. Let $x\gamma$ be an occurrence of a variable in Π, $\Sigma_{x\gamma \approx v}$ a subproof associated in Π with this occurrence of x and v does not contain occurrences of x.
An equational proof Π' is exactly as Π with the following modifications.
For each a term w_i in Π, such that $w_i = w_i[x\gamma^k]$, for some occurrence of x in Π, do the following:

1. *If $x\gamma^k$ has right orientation, replace w_i with the following sequence of steps:*

$$w_i[v]_\alpha \approx w_i(\Sigma'_{v \approx \text{“}x\gamma\text{”}}) \approx w_i[\text{“}x\gamma\text{”}]_\alpha$$

   where $w_i(\Sigma'_{v \approx \text{“}x\gamma\text{”}})$ means that a renaming of $\Sigma_{\text{“}x\gamma\text{”} \approx v}$ is reversed and embedded in w_i at position α leftwards. Note that the renamings of internal occurrences of variables and new occurrences of external variables in the renaming of $\Sigma_{\text{“}x\gamma\text{”} \approx v}$ have reversed orientation in the new proof.
2. *If $x\gamma$ has left orientation, replace w_i by the sequence of steps:*

$$w_i[\text{“}x\gamma\text{”}]_\alpha \approx w_i(\Sigma'_{\text{“}x\gamma\text{”} \approx v}) \approx w_i[v]_\alpha$$

   where $w_i(\Sigma'_{\text{“}x\gamma\text{”} \approx v})$ means that a renaming of $\Sigma_{\text{“}x\gamma\text{”} \approx v}$ is embedded in w_i at position α rightwards. The renamings of internal occurrences of variables and new occurrences of external variables in $\Sigma'_{\text{“}x\gamma\text{”} \approx v}$ preserve their orientation in the new proof.

Contract any non-redundant subproofs in the obtained equational proof.

The substitution γ' is then defined in the following way:
$\gamma'_x = [x \mapsto v]$,
if $y\gamma|_\alpha = x\gamma$, then $\gamma'_y = [y \mapsto y\gamma[x\gamma']_\alpha]$,
if $z \notin Dom(\gamma)$, z is a renaming of a variable $z' \in Dom(\gamma)$, that appeared in some $\Sigma'_{x\gamma \approx v}$, then $\gamma'_z = [z \mapsto z'\gamma]$,
for any other variable u, $\gamma'_u = \gamma_u$;

4.2 Flattening a proof

We can flatten an equational proof Π in the following situation.

Suppose there is a subproof $\Sigma_{x_j\gamma\approx x\gamma}$ in Π, where x_j is a subterm variable of $x\gamma$. Hence $x_j\gamma$ is a subterm of another occurrence of x and $\Sigma_{x_j\gamma\approx x\gamma}$ is a subproof of $\Sigma_{x\gamma[x_j\gamma]\approx v[x\gamma]}$. We say that the proof Π is not flat at $x\gamma$. Notice also that then the length of $\Sigma_{x_j\gamma\approx x\gamma}$ must be greater than 0, because $x\gamma$ cannot be syntactically identical with its subterm.

We flatten the proof Π with respect to $x\gamma$ in a recursive way:

1. extend Π with respect to $x\gamma \approx x_j\gamma$ (i. e. $(\Pi,\gamma) \stackrel{[x\to x_j\gamma]}{\longrightarrow} (\Pi',\gamma')$);
2. if there is a subproof $\Sigma_{x_j\gamma'\approx x\gamma'}$ in Π', flatten Π' with respect to $x\gamma'$.

Flattening will always terminate. The reason is that in the new proof Π', $\gamma'_x = [x \mapsto x_j\gamma]$ and since x_j was a subterm variable for $x\gamma$, $|x\gamma'| < |x\gamma|$.

Let us see what exactly happens when Π is extended with $x\gamma \approx x_j\gamma$. Since x_j is a subterm variable of $x\gamma$, there is an occurrence of x, such that $\Sigma_{x_j\gamma\approx x\gamma}$ is a subproof of $\Sigma_{x\gamma[x_j\gamma]\approx v[x\gamma]}$ in Π.

At first extension changes this subproof to the subproof of the form:

"$x_j\gamma$" $\approx (\Sigma_{\text{"}x_j\gamma\text{"}\approx\text{"}x\gamma\text{"}}) \approx$ "$x\gamma[x_j\gamma]$" $\approx v(\Sigma_{\text{"}x_j\gamma\text{"}\approx\text{"}x\gamma\text{"}}) \approx v[$"$x\gamma$"$] \approx v(\Sigma_{\text{"}x\gamma\text{"}\approx\text{"}x_j\gamma\text{"}}) \approx v[$"$x_j\gamma$"$]$

Notice that since there is a subproof $\Sigma_{x_j\gamma\approx x\gamma}$ in Π, all steps in the subproof $\Sigma_{x\gamma[x_j\gamma]\approx v[x\gamma]}$ must be at or bellow position of $x_j\gamma$ in $x\gamma[x_j\gamma]$ (or position of $x\gamma$ in $v[x\gamma]$, which is the same α). Hence $x\gamma[x_j\gamma] == v[x_j\gamma]$. The subproof therefore will be contracted and will have the following form:

"$x_j\gamma$" $\approx (\Sigma_{\text{"}x_j\gamma\text{"}\approx\text{"}x\gamma\text{"}}) \approx v[$"$x_j\gamma$"$]$

The term "$x_j\gamma$" is $x\gamma'$ in the new proof hence in fact this subproof is:

$x\gamma' \approx (\Sigma_{\text{"}x_j\gamma\text{"}\approx\text{"}x\gamma\text{"}}) \approx v[x\gamma']$

Notice that this subproof has the same length as the original $\Sigma_{x\gamma\approx v[x\gamma]}$, but this is not telling us anything about the length of the new equational proof, which in fact may increase in the process of flattening in other places where x occurs.

Nevertheless, flattening must terminate, because the term we substitute for x in each new proof is strictly smaller than the term in the previous proof.

In the end, the subproof $\Sigma_{x_j\gamma\approx x\gamma}$ will have to disappear, because x on the right will have to appear at the lower position than the step in the subproof $\Sigma_{x\gamma\approx v[x\gamma]}$ is taken and this will prevent the subproof associated with the $x\gamma$ on the right to reach a subterm of the $x\gamma$ on the left. It is obvious that if $|x\gamma| = 1$, then there must be a step at the root in $\Sigma_{x\gamma\approx v[x\gamma]}$.

Let us look at a simple example. Let our equational theory E be $\{a \approx fa\}$ and our goal G be $\{x \approx fx,\, x \approx fffa\}$. Let our E-unifier be $\gamma = [x \mapsto fffa, x_1 \mapsto ffa, x_2 \mapsto fa, x_3 \mapsto a]$ where x_1, x_2, x_3 are subterm variables of $x\gamma$, and the equational proof be $\Pi = \{fffa \approx ffffa, fffa = fffa\}$.

If $x \approx fx$ is selected, Π is not flat at $x\gamma^2$, because $x\gamma^2 \approx x_1\gamma$, (or $fffa \approx ffa$). Flattening will proceed in the following stages:

1. $(\Pi, \gamma) \stackrel{[x \mapsto x_1 \gamma]}{\longrightarrow} (\Pi', \gamma')$. Extension will first change the subproof $fffa \approx ffffa$ into $ffa \approx fffa \approx ffffa \approx fffa$ and contraction will shorten it to $ffa \approx fffa$. Hence $\Pi' = \{ffa \approx fffa, ffa \approx fffa\}$. $\gamma' = [x \mapsto ffa, x_1 \mapsto fa, x_2 \mapsto a]$.
   Still, Π' is not flat at $x\gamma'^2$, because there is a subproof $x\gamma'^2 \approx x_1\gamma'$, (or $ffa \approx fa$).
2. $(\Pi', \gamma') \stackrel{[x \mapsto x_1 \gamma']}{\longrightarrow} (\Pi'', \gamma'')$. Extension will first change the subproof $ffa \approx fffa$ into $fa \approx ffa \approx fffa \approx ffa$ and contraction will shorten it to $fa \approx ffa$. Hence $\Pi'' = \{fa \approx ffa, fa \approx ffa \approx fffa\}$. $\gamma'' = [x \mapsto fa, x_1 \mapsto a]$.
   Still, Π'' is not flat at $x\gamma''^2$, because there is a subproof $x\gamma''^2 \approx x_1\gamma''$, (or $fa \approx a$).
3. $(\Pi'', \gamma'') \stackrel{[x \mapsto x_1 \gamma'']}{\longrightarrow} (\Pi''', \gamma''')$. Extension will first change the subproof $fa \approx ffa$ into $a \approx fa \approx ffa \approx fa$ and contraction will shorten it to $a \approx fa$. Hence $\Pi''' = \{a \approx fa, a \approx fa \approx ffa \approx fffa\}$. $\gamma''' = [x \mapsto a]$.
   Obviously, Π''' is flat at $x\gamma'''^2$ and there is a step at the root in the subproof of $x\gamma''' \approx fx\gamma'''$. Flattening ends here, but notice that Π''' has length 4, whereas Π had length 1. But notice also that the subproof $a \approx fa \approx ffa \approx fffa$ can be found in Π in a form of composition of subproofs: $x_3\gamma^1 \approx x_2\gamma^2 == x_2\gamma^1 \approx x_1\gamma^2 == x_1\gamma^1 \approx x\gamma^2 == x\gamma^3$. We will call such composition of subproofs a *path* and show that Π''' has the same set of paths as Π.

5 Solving variable in a proof

In the proof of completeness theorem, we assume that there is a solution for a goal, hence for a given E – equational theory and G – a set of goal equations, there is (Π, γ), such that $E \models G\gamma$ and Π is an equational proof of $G\gamma$ in E.

We will see that Decomposition, Mutate, Orient and Trivial preserve the form of the solution, i.e. if one of these rules is the right rule to apply, then we can assume that the proof of the new goal obtained by this rule is composed of subproofs of the previous one, with the same substitution and the same set of variables involved, i.e. the same proof (or a set of subproofs thereof) and the same substitution are the solution of a new E-unification goal.

In contrast to this, Variable Elimination, Variable Mutate and Cycle may change the form of assumed solution of the goal.Variable Mutate and Cycle may change the form of a solution of the goal, because they are *macro* rules involving Variable Elimination.

In order to reflect the transformations required by Variable Elimination and Cycle, we now define a procedure *Solve x in* $((\Pi, \gamma), U)$ which takes as input a solution for an E-unification goal, i.e. an equational proof and a substitution, and a set of unsolved variables, and returns another solution E-equivalent to the original one and a new set of unsolved variables.

In the following description,we use the notions of maximal occurrence of a variable and an irrelevant subterm variable, which are defined now.

Definition 3. *(maximal occurrence of a variable)*
Let (Π, γ) be a solution of a goal G, with a set of unsolved variables U, such that $U \subseteq Dom(\gamma)$. Let $x \in U$. An occurrence $x\gamma$ of x is called maximal *in in Π with respect to U, if there is no occurrence, $y\gamma$, of an unsolved variable y, such that $x\gamma$ appears in $\Sigma_{y\gamma\approx t}$, where $\Sigma_{y\gamma\approx t}$ is a subproof associated with $y\gamma$.*

Definition 4. *(subterm variable irrelevant for $x\gamma$)*
Let (Π, γ) be a solution of a goal G, and $x \in Dom(\gamma)$, $x\gamma$ an occurrence of x in Π. Let $\Sigma_{x\gamma\approx v}$ be a subproof associated with this occurrence of x, such that v contains some occurrences of x.

A subterm variable x_i defined for $x\gamma$ is called irrelevant *for $x\gamma$, if $\Sigma_{x_i\gamma\approx v_i}$ is a subproof associated with $x_i\gamma$, and v_i is not a subterm of v, or v_i does not contain any occurrences of x.*

We call x_i a maximal irrelevant subterm variable for $x\gamma$, if whenever $x_i\gamma = x_j|_\alpha\gamma$, for any position $\alpha \neq \epsilon$ and x_j a subterm variable of $x\gamma$, x_j is not irrelevant for $x\gamma$.

Let us see an example illustrating the meaning of the previous definition.

Let our $E = \{b \approx c, a \approx gf(a,b)\}$ and $G = \{x \approx f(gx,c)\}$.

Let $\Pi = \{f(a,b) \approx f(gf(a,b),b) \approx f(gf(a,b),c)\}$ and $\gamma = [x \mapsto f(a,b), x_1 \mapsto a, x_2 \mapsto b]$, where x_1, x_2 are subterm variables for $x\gamma$.

Then according to the definition, x_1 is relevant for $x\gamma^1$, but x_2 is irrelevant. Notice that when Cycle is applied to $x \approx f(gx,c)$, we get the following goal: $\{x \approx f(x_1,c)\} \cup M(x_1 \approx g(f(x_1,c)))$. Hence irrelevant subterm variables are "solved" as if automatically in Cycle.

Solve x in $((\Pi, \gamma), U)$
Let (Π, γ) be a solution of a goal G. $U \subseteq Dom(\gamma)$, is a set of variables called unsolved. x is a variable in U, such that there is at least one maximal occurrence of x, $x\gamma$ in Π. Choose an occurrence of x, $x\gamma$ which is maximal in (Π, γ) with respect to U. Let $\Sigma_{x\gamma\approx v}$ be a subproof associated with $x\gamma$.

1. *If $x \notin Var(v)$,* extend Π with $\Sigma_{x\gamma\approx v}$, $(\Pi, \gamma) \stackrel{[x\to v]}{\longrightarrow} (\Pi', \gamma')$.
   Return (Π', γ') and U', where $U' = (U - \{x, x_1, x_2, \ldots, x_k\}) \cup (Dom(\gamma') - Dom(\gamma))$, where $x_1, \ldots, x_k$ are all subterm variables defined for $x\gamma$;
2. *If $x \in Var(v)$, and there is an occurrence of x, $x\gamma^k$ in v, such that the proof Π is not flat at $x\gamma^k$, then* flatten Π with respect to $x\gamma^k$. *Let the result be (Π', γ').*
   Rename variable x with a new variable z in (Π', γ'), $(\Pi', \gamma')[x \mapsto z]$. Return $(\Pi', \gamma')[x \mapsto z]$ and U', where $U' = (U - \{x, x_1, x_2, \ldots, x_k\}) \cup \{z\} \cup (Dom(\gamma') - Dom(\gamma))$, where $x_1, \ldots, x_k$ are subterm variables defined for positions in $x\gamma$ which are no more defined for $z\gamma'$;
3. *If $x \in Var(v)$, and Π is flat at all occurrences of x in v, then for each subterm variable x_i maximal irrelevant for $x\gamma$ extend Π with $\Sigma_{x_i\gamma\approx v_i}$.*
   Return the result, (Π', γ') and $U' = (U - \{x, x_1, x_2, \ldots, x_k\}) \cup (Dom(\gamma') - Dom(\gamma))$, where $x_1, \ldots, x_k$ are all subterm variables irrelevant for $x\gamma$.

Notice that by choosing maximal occurrence of x, we are making sure that no inner variables in $\Sigma_{x\gamma\approx v}$ are solved before x. Since these variables are renamed in the process of extension, definition of U' would had to be somehow changed, if we had allowed to solve those variables first.(Should these new variables in U' be counted as solved or not?) But fortunately, we can safely restrict ourselves to maximal occurrences of variables, since only these occurrences are playing role in Variable Elimination. Namely, if x is going to be eliminated from an unsolved part of a goal, G, because $x \approx v \in G$, $x\gamma$ must be a maximal occurrence of x in an equational proof of this goal.

In order to use Solve in the completeness proof, we have to show in what sense its result, $((\Pi', \gamma'), U')$, is smaller than its input $((\Pi, \gamma), U)$. For this we define *paths* in an equational proof.

Definition 5. *(path starting with a variable occurrence, variables used in a path) Let (Π, γ) be a solution for a goal G, U a set of unsolved variables in $Dom(\gamma)$, $x \in U$ and $x\gamma$ a given variable occurrence in Π. A path in Π starting with $x\gamma$ is a composition of subproofs, $\Sigma_1 \cdots \Sigma_n$, defined in a recursive way:*

1. *if $\Sigma_{x\gamma\approx v}$ is an associated subproof for $x\gamma$, $\Sigma_{x\gamma\approx v}$ is a path starting with $x\gamma$;*
2. *(a) if $\Sigma_1 \cdots \Sigma_n$ is a path in Π starting with $x_1\gamma$ with the last term of the form $v[x_{n+1}\gamma^k]$, x_{n+1} is an external variable in $\Sigma_{x_n\gamma\approx v[x_{n+1}\gamma^k]}$, and if $\Sigma'_1, \ldots, \Sigma'_m$ is a path in Π starting with $x_{n+1}\gamma^i$, and if no variable which is* used *in one path appears as not* used *in the other, then the composition $\Sigma_1 \cdots \Sigma_n \Sigma'_1 \cdots \Sigma'_m$ is also a path in Π starting with $x_1\gamma$ and all variables* used *in the first and second path are* used *in this path;*
   (b) if $\Sigma_1 \cdots \Sigma_n$ is a path in Π starting with $x_1\gamma$ and with the last term of the form $y\gamma|_\alpha$, and if $\Sigma_{y\gamma^k|_\alpha\approx s}$ is a subproof in Π and no variable which is used *in one path appears as not* used *in the other, then $\Sigma_1 \cdots \Sigma_n \Sigma_{y\gamma^k|_\alpha\approx s}$ is also a path in Π starting with $x_1\gamma$ and all variables* used *in the first and second path are* used *in the new path;*

Notice that when Solve is applied to x in (Π, γ, U), where $x\gamma$ is a chosen maximal occurrence of x and $\Sigma_{x\gamma\approx v}$ is a subproof associated with $x\gamma$, some compositions of subproofs in Π become subproofs in Π'. Since x and possibly some of its subterm variables are not in U', we don't have paths starting with these occurrences defined for Π' any more. Notice also that there are no paths in Π', which represent the subproofs where x or some of its subterm variables were used in extension in a different way, i.e. with different variable occurrence and different subproof used in extension.

Consider the restriction in the definition of paths to the external variables in Definition 5.2.2a. Notice that this restriction is just what we need in order to account for what happens in the proof when a variable is eliminated in the goal. Namely, if by Mutate new variables are discovered in the goal, i.e. a step in an equational proof is being explored, then these new variables appear as external in all subproofs represented by the unsolved equations in the goal. If $u \approx v$ is in the goal, the variable occurrences in $u\gamma$ have always opposite orientations to those in $v\gamma$.

We want to show that when Solve x is applied to $((\Pi,\gamma),U)$, a multiset of lengths of paths in Π' defined with respect to U', where $((\Pi',\gamma'),U')$ is a result of Solve x in $((\Pi,\gamma),U)$, is smaller than the multiset of paths in Π defined with respect to U. In order to do this, we have to be careful with the way we count paths. Namely, if a shorter path is a common part of longer paths, we have to count it as separate for each such longer path. In fact, since paths are linear, it is enough to count separately occurrences of a path included in different maximal paths. Hence we define maximal and proper paths and prove that the set of proper paths after solving a variable does not increase.

Definition 6. *(maximal paths) Let (Π,γ) be a solution for a goal, and U a set of unsolved variables in $Dom(\gamma)$. Let P be the set of all paths in Π defined with respect to U. Then M is a set of maximal paths if $M = \{p \in \Pi |$ for no $q \in \Pi$, p is a part of $q\}$.*

Now we define a set of proper paths.

Definition 7. *(proper paths)*
Let (Π,γ), U, P and M be as in the previous definition.

For each $p \in M$, we define $P_p = \{\Sigma | \Sigma$ is a copy of p or there is $q \in P$ such that q is a part of p and Σ is a copy of $q\}$.

A set PP is a multiset of proper paths for Π defined with respect to U if PP is a multiset union of P_p, for all $p \in M$.

As an example of proper paths assume that the following subproofs are in Π: $\Sigma_{x_1\gamma\approx v_1[z\gamma^1]}$, $\Sigma_{x_2\gamma\approx v_2[z\gamma^2]}$, $\Sigma_{z\gamma^3\approx s[y\gamma^1]}$, $\Sigma_{y\gamma^2\approx t_1}$, $\Sigma_{y\gamma^3\approx t_2}$.

There will be the following maximal paths in Π:
$p_1 = \Sigma_{x_1\gamma\approx v_1[z\gamma^1]}\Sigma_{z\gamma^3\approx s[y\gamma^1]}\Sigma_{y\gamma^2\approx t_1}$,
$p_2 = \Sigma_{x_2\gamma\approx v_2[z\gamma^2]}\Sigma_{z\gamma^3\approx s[y\gamma^1]}\Sigma_{y\gamma^2\approx t_1}$,
$p_3 = \Sigma_{x_1\gamma\approx v_1[z\gamma^1]}\Sigma_{z\gamma^3\approx s[y\gamma^1]}\Sigma_{y\gamma^3\approx t_2}$,
$p_4 = \Sigma_{x_2\gamma\approx v_2[z\gamma^2]}\Sigma_{z\gamma^3\approx s[y\gamma^1]}\Sigma_{y\gamma^3\approx t_2}$.

You can see that the subproof, which is also a path, $\Sigma_{z\gamma^3\approx s[y\gamma^1]}$ is repeated in all of them. Hence a copy of this subproof will appear in $P_{p_1}, P_{p_2}, P_{p_3}, P_{p_4}$. Similarly, copy of $\Sigma_{x_1\gamma\approx v_1[z\gamma^1]}\Sigma_{z\gamma^3\approx s[y\gamma^1]}$ will appear in P_{p_1} and P_{p_3}, and so on.

P_{p_1} will consist of the copies of the following paths:
$\Sigma_{x_1\gamma\approx v_1[z\gamma^1]}\Sigma_{z\gamma^3\approx s[y\gamma^1]}\Sigma_{y\gamma^2\approx t_1}$,
$\Sigma_{x_1\gamma\approx v_1[z\gamma^1]}\Sigma_{z\gamma^3\approx s[y\gamma^1]}$,
$\Sigma_{z\gamma^3\approx s[y\gamma^1]}\Sigma_{y\gamma^2\approx t_1}$,
$\Sigma_{x_1\gamma\approx v_1[z\gamma^1]}$, $\Sigma_{z\gamma^3\approx s[y\gamma^1]}$, $\Sigma_{y\gamma^2\approx t_1}$.

We have to show that after Solve if extension applies, a multiset of lengths of proper paths in Π' defined with respect to the new set of unsolved variables, U', is not greater than the one defined for Π with respect to U.

Lemma 1. *Let (Π,γ) be a solution for a goal, and U a set of unsolved variables in $Dom(\gamma)$, $x\gamma$ is a maximal occurrence of x in Π, and $\Sigma_{x\gamma\approx v}$ a subproof associated with $x\gamma$, such that case 1 of the Solve x in $((\Pi,\gamma),U)$ applies.*

Let $((\Pi', \gamma'), U')$ *be a result of Solve in this case, then each proper path defined for* Π *with respect to* U *is a unique renaming of a path defined for* Π' *with respect to* U'.

Proof. Let p be a proper path defined for Π' with respect to U'. Let p starts with $y\gamma'$. Assume that p is maximal.

Assume that $y \in Dom(\gamma)$. Then by definition of path and extension, there is a path in Π, p', such that p is a renaming of p' and p' differs from p only in this that p does not use occurrences of x or its subterm variables any more in places where these variable occurrences were used in p'.

If p' is maximal in Π, then it is unique, maximal path starting with $y\gamma$, and then obviously only one path defined for Π' can be a renaming of p'.

Now, if p' is not maximal in Π, then p' must be a part of a maximal path, q, starting with $x\gamma^i$, for some occurrence of x in Π. p' is part of q ($p' \in P_q$). Even if there may be many different maximal paths starting with $x\gamma^i$, there is only one such path (q) that contains p', for otherwise p could not become maximal in Π'. Hence p is the only renaming of p' in Π'.

Assume now that $y \notin Dom(\gamma)$ (p is maximal in Π'). Then there is an inner variable y' in $\Sigma_{x\gamma\approx v}$, such that $y\gamma'$ is a renaming of $y'\gamma$. By definition of path and extension, there must be a path in Π, p', starting with $y'\gamma$, such that p is a renaming of p'. p' must be either maximal or a part of a path starting with $x\gamma$. If p' is maximal, it is a unique path in Π, of which p is a renaming. If p' is a part of a maximal path q starting with $x\gamma$. (q must be maximal, because otherwise, p would not become maximal in Π'.) Even if there may be many maximal paths starting with $x\gamma$, there will be only one containing p' and there can be only one renaming of p' in Π'.

Now assume that p is not maximal.

In this case, p is a part of a maximal path q in Π' and $p \in P_q$. Then there is a unique (not necessarily maximal) path q' in Π, such that q is a renaming of q' and p' is part of q'. From the previous argument we know that q' is unique for q. Since q' is unique, then p' must also be unique, hence there may be only one renaming of p' in P_q.

If in Solve flattening applies, we have to prove that Solve does not increase the set of paths in Π'.

Lemma 2. *Let* (Π, γ) *be a solution for a goal, and* U *a set of unsolved variables in* $Dom(\gamma)$, $x\gamma$ *is a maximal occurrence of* x *in* Π, *and* $\Sigma_{x\gamma\approx v}$ *a subproof associated with* $x\gamma$, *such that case 2 of the Solve* x *in* $((\Pi, \gamma), U)$ *applies.*

Let $((\Pi', \gamma'), U')$ *be a result of Solve (with flattening of* Π*), then each proper path defined for* Π *with respect to* U *is a unique renaming of a path defined for* Π' *with respect to* U'.

Proof. Let there be an occurrence $x\gamma^k$ in v, such that Π is not flat at $x\gamma^k$.

Flattening of Π with respect to $x\gamma$ must terminate. Hence we can use induction on number of extensions used in this process. Assume therefore, that $(\Pi', \gamma'), U'$ were obtained by extension from $(\Pi'', \gamma''), U''$. $(\Pi'', \gamma'') \stackrel{[x \mapsto x_i\gamma'']}{\longrightarrow}$

(Π', γ') where $x_i\gamma''$ is an occurrence of subterm variable for $x\gamma''$, and $\Sigma_{x\gamma''^k \approx x_i\gamma''}$ is a subproof associated with $x\gamma''^k$. By induction assumption we know that for each proper path in Π'' defined with respect to U'', there is a unique proper path in Π, defined with respect to U.

Assume that p is a proper path defined for Π' with respect to U'.If p starts with $y\gamma'$, where y is not x_1, then the argument from the proof of the previous lemma applies.

Hence assume that p starts with $z\gamma'$ (which is renamed $x\gamma'$). From the definition of extension and flattening, we know that $z\gamma' = x_i\gamma''$, where x_i is a subterm variable in $Dom(\gamma'')$ for $x\gamma''$.

Hence by the definition of path and extension, p is a renaming of a proper path in Π'', p', which starts with $x_i\gamma''$. The argument similar to the one used in the proof of the previous lemma shows that p' is a unique such path, which p is a renaming of.

Lemma 3. *Let (Π, γ) be a solution for a goal, and U a set of unsolved variables in $Dom(\gamma)$, $x\gamma$ is a maximal occurrence of x in Π, and $\Sigma_{x\gamma \approx v}$ a subproof associated with $x\gamma$, such that case 3 of the Solve x in $((\Pi, \gamma), U)$ applies.*

Let $((\Pi', \gamma'), U')$ be a result of Solve (with flattening of Π), then each proper path defined for Π with respect to U is a unique renaming of a path defined for Π' with respect to U'.

Proof. Proof of this lemma is the same as that of Lemma 1, with a not that p differs from p' in this that it does not uses occurrences of $x\gamma$ or some of its subterm variables in places where p' had to use them. But some of the subterm variables are still unsolved in Π'.

Now we show that actually Solve x in $(\Pi, \gamma), U$ decreases multiset of lengths of proper paths in a new solution and hence we can take it as a measure for a solution.

Definition 8. *(measure of a solution)*
Let (Π, γ) be a solution of a goal, and U a set of unsolved variables in $Dom(\gamma)$. Let PP be a set of proper paths defined for Π with respect to U.

The measure of the solution (Π, γ) with U is $M((\Pi, \gamma), U)$ a multiset of lengths of paths in PP.

Lemma 4. *Let (Π, γ) be a solution of a goal, and U a set of unsolved variables in $Dom(\gamma)$. Let PP be a set of proper paths defined for Π with respect to U. Let $x \in U$, be a variable with a maximal occurrence $x\gamma$ in Π chosen in such a way that Solve x in $(\Pi, \gamma), U$ applies. Let $(\Pi', \gamma'), U'$ be a result of Solve and PP' a set of proper paths defined for Π' with respect to U'. Then $M((\Pi, \gamma), U) > M((\Pi', \gamma'), U')$.*

Proof. Let $\Sigma_{x\gamma \approx v}$ be a subproof associated with $x\gamma$ in Π. If Solve applies when $x\gamma$ is chosen, we have 3 cases to consider. In the fist case, when $x \notin Var(v)$, x is no longer in U', and by Lemma 1, all paths in PP' are renamings of unique paths in PP. But whereas at least one path starting with $x\gamma$ is in PP, there is no such path in PP'.

In the second case, x is replaced by z in U'. By Lemma 2, we know that all paths in PP' are renamings of unique paths in PP. But since $z\gamma'$ is strictly smaller than $x\gamma$, some subterm variables of $x\gamma$ are no longer in U'. Hence the proper paths starting with these subterm variables will no longer be defined in PP'.

In the third case, where $x \in Var(v)$, but Π is flat at all occurrences of x in v, by Lemma 3,we know that all paths in PP' are renamings of unique paths in PP. But x and possibly some subterm variables are no longer in U', hence the paths starting with these variable occurrences are no longer in PP'.

6 Completeness

We will prove completeness of the inference system presented in Figure 3, where M in a definition of Cycle is defined by:

$$M(x \approx f(v_1,\ldots,v_n)) = \begin{cases} \{x \approx s,\, t_1 \approx v_1,\ldots,t_n \approx v_n\}, \text{where} \\ s \approx f(t_1,\ldots,t_n) \in E; \\ \\ x \approx s[y \mapsto f(v_1,\ldots,v_n)], \text{where } s \approx y \in E \end{cases}$$

Notice that now neither Mutate nor Variable Mutate is applicable to an equation of the type $x \approx v$. If such an equation is selected, Variable Elimination, Orient, Cycle or Trivial applies and either of these rules applies eagerly.

This said, it must also be pointed out that there is a "don't know" type non-determinism involved in an application of Cycle, because we don't know which is the right place to "divide" the term on the right in a goal equation of the type $x \approx v$ if x occurs in v. Nevertheless, we have only finitely many positions to choose from.

We prove that in any equational theory E, a given goal G such that $E \models G\sigma$, may be transformed by applications of rules in Figure 3 applied to equations which are not *solved*, into a *solved form* with which we can define an E-unifier more general than σ. The solved form of an equation and of a goal is defined in the following way.

Definition 9. *(solved equation and solved goal)*
Let G be a set of equations. An equation $x \approx t \in G$ is in a solved form*, if x is a variable, $x \notin Var(t)$ and $x \notin Var(G \backslash \{x \approx t\})$.*

G is in a solved form if all equations in G are in solved form.

If G is in the solved form, then we define a substitution $\theta_G = [x_1 \mapsto t_1,\ldots,x_n \mapsto t_n]$. Obviously, θ_G is the most general unifier of G.

If G is a set of goal equations, an inference performed on G with one of the rules of Figure 3 is denoted by $G \rightarrow G'$, where G' is the result of this inference. The transitive, reflexive closure of $\rightarrow$ is written as $\xrightarrow{*}$.

Decomposition

$$\frac{\{f(s_1,\cdots,s_n)\approx f(t_1,\cdots,t_n)\}\cup G}{\{s_1\approx t_1,\cdots,s_n\approx t_n\}\cup G}$$

where $f(s_1,\cdots,s_n)\approx f(t_1,\cdots,t_n)$ is selected in the goal.

Mutate

$$\frac{\{u\approx f(v_1,\cdots,v_n)\}\cup G}{\{u\approx s, t_1\approx v_1,\cdots,t_n\approx v_n\}\cup G}$$

where $u\approx f(v_1,\cdots,v_n)$ is selected in the goal, u is not a variable and $s\approx f(t_1,\cdots,t_n)\in E$.

Variable Mutate

$$\frac{\{u\approx f(v_1,\cdots,v_n)\}\cup G}{\{u\approx s[x\mapsto f(v_1,\cdots,v_n)]\}\cup G}$$

where $s\approx x\in E$, x is a variable, u is not variable and $u\approx f(v_1,\cdots,v_n)$ is selected in the goal.

Variable Elimination

$$\frac{\{x\approx v\}\cup G}{\{x\approx v\}\cup G[x\mapsto v]}$$

where $x\notin Var(v)$

Orient

$$\frac{\{t\approx x\}\cup G}{\{x\approx t\}\cup G}$$

where x is a variable.
and t is not a variable.

Cycle

$$\frac{\{x\approx v[v_1]_{\alpha_1}\cdots[v_k]_{\alpha_k}\}\cup G}{\{x\approx v[x_1]_{\alpha_1}\cdots[x_k]_{\alpha_k}\}\cup\bigcup_{i=1}^{k}\{M(x_i\approx v_i\sigma)\}\cup G\sigma}$$

where $x\approx v[v_1]_{\alpha_1}\cdots[v_k]_{\alpha_k}$ is selected in the goal,
$x\in Var(v[v_1]_{\alpha_1}\cdots[v_k]_{\alpha_k})$, each v_i is a non-variable term, which contains at least one occurrence of x, $\sigma=[x\mapsto v[x_1]_{\alpha_1}\cdots[x_k]_{\alpha_k}]$.

Trivial

$$\frac{\{x\approx x\}\cup G}{G}$$

where $x\approx x$ is selected in the goal.

Fig. 3. E-Unification with nice Cycle rule

In order to prove completeness, we will need the measure of a goal G, of which we will show that it may be decreased by application of an inference rule if G is E-unifiable and not in solved form.

Definition 10. *(measure of a goal)*
Let E be an equational theory, and G, an unsolved part of a goal G', such that there is a ground substitution γ, for which $E \models G'\gamma$ and hence there is a solution (Π', γ) of G' and Π a subproof of Π', such that (Π, γ) is a solution of G, and all variables in $Var(G)$ are unsolved in (Π, γ).

The measure of G' with respect to (Π', γ) is a 4-tuple (m, n, o, p), where $m = M(\Pi, \gamma)$, n is the length of Π, o is the size of terms in $G\gamma$, p is the number of equations in G, of the form $t \approx x$, where x is a variable and t is not a variable.

Notice that the measure of a goal is in fact a measure of its *unsolved* part. Measures for different goals are compared with respect to lexicographic order.

Theorem 1. *Let E be a set of equations, such that $E \models G\gamma$ for some ground substitution γ. Then there is H, a set of equations in the solved form, such that $G \xrightarrow{*} H$ and $\theta_H[Var(G)] \leq_E \gamma$.*

Proof. If G is already in the solved form, then $\theta_G \leq_E \gamma$.

If G is not in solved form, then there is an unsolved part of G, which consists of all unsolved equations in G. Only unsolved equations in G may be selected for inference. Assume that $u \approx v$ was selected for an inference. If $E \models G\gamma$, there must be an equational proof Π of $G\gamma$. We will call (Π, γ) an **actual solution** of G. There must be a subproof in Π, of $u\gamma \approx v\gamma$, $\Sigma_{u\gamma \approx v\gamma}$ and $u\gamma$, $v\gamma$ are the extreme terms in this subproof, i.e. there is no subproof in Π at position of $u\gamma$ or $v\gamma$ containing $\Sigma_{u\gamma \approx v\gamma}$ as its proper part. It is important to show in each of the following cases, that our rules preserve this property, since we use Solve in justifying completeness of some of them, and Solve is defined with respect to associated subproofs which are the subproofs of maximal length starting with some variable occurrence in Π. Hence if $x \approx v$ is selected and $\Sigma_{x\gamma \approx v\gamma}$ is its subproof in Π, we want to be sure that $\Sigma_{x\gamma \approx v\gamma}$ is a subproof associated with $x\gamma$ (and hence maximal subproof starting with $x\gamma$). We can also assume that all solved variables in G are solved in Π, i.e. not in U, and all unsolved variables in G are unsolved in Π, i,e. there are in U.

Obviously, if $x \approx v$ is selected for an inference, $x\gamma$ is a maximal node in Π with respect to U.

For the proof, we have to consider all possible forms of an unsolved goal equation $u \approx v$ selected for an inference. We will show that in all these cases, there is an inference rule from Figure 3, such that it is applicable to the selected equation and this application decreases the measure for the new goal. Hence we show that $G \to G'$, and measure of G' is strictly smaller than that of G. Moreover we show that if $E \models G\gamma$, then also $E \models G'\gamma'$, where $\gamma =_E \gamma'[Var(G)]$. Then by induction hypothesis $G' \xrightarrow{*} H$ and $\theta_H[Var(G)] \leq_E \gamma$. Hence also $G \xrightarrow{*} H$ and $\theta_H[Var(G)] \leq_E \gamma$.

Hence it is enough to consider now the following possible forms of a selected equation in a goal.

1. Assume that neither u nor v is a variable.
   Let $\Sigma_{u\gamma\approx v\gamma}$ be a subproof in Π of $u\gamma \approx v\gamma$.
   Assume also that there is no step at the root in $\Sigma_{u\gamma\approx v\gamma}$. Hence u and v must have the same root symbols.
   The right rule to apply in this case is **Decomposition**. In the new goal $u \approx v$ is replaced by equations $s_1 \approx t_1, \ldots, s_n \approx t_n$. There is a subproof in Π for each $s_i\gamma \approx t_i\gamma$, $i \in \{1, \ldots, n\}$, and if $u\gamma$, $v\gamma$ were the extreme terms in $\Sigma_{u\gamma\approx v\gamma}$, $s_i\gamma$, $t_i\gamma$ are extreme terms in the respective subproofs. $E \models \{s_1\gamma \approx t_1\gamma, \ldots, s_n\gamma \approx t_n\gamma\}$. The sum of the lengths of the subproofs is equal to the length of the original subproof of $u\gamma \approx v\gamma$, but $\Sigma_{i=1}^{n}(|s_i\gamma| + |t_i\gamma|) < |u\gamma| + |v\gamma|$.
   Let (m, n, o, p) be the measure of the goal before Decomposition and (m', n', o', p') after Decomposition. $m' = m$, $n' = n$ and $o' < o$.
2. Assume that u and v are as in case 1. Assume also that there is a step at the root in $\Sigma_{u\gamma\approx v\gamma}$.
   $\Sigma_{u\gamma\approx v\gamma}$ has the form: $u\gamma \approx \cdots \approx w_i \approx_{[\epsilon, s\approx t, \gamma]} w_{i+1} \approx \cdots \approx v\gamma$. Let us choose i in such a way, that this is the rightmost root step in this subproof and assume that t is not a variable.
   Then there is no root step between w_{i+1} and $v\gamma$. Since the i'th step is at the root position, $s\gamma = w_i$ and $t\gamma = w_{i+1}$. Since there is no step at the root between $t\gamma$ and $v\gamma$, and t is not a variable, t and v must have the same root symbol and thus we can at once decompose them, obtaining possible empty set of equations: $t_1 \approx v_1, \ldots, t_n \approx v_n$, such that for each $i \in \{1, \ldots, n\}$, $t_i\gamma \approx v_i\gamma$ has a subproof in Π, and moreover $t_i\gamma$, $v_i\gamma$ are extreme subterms in their respective subproofs. Hence in this case **Mutate** is applicable, and we see that $E \models \{u\gamma \approx s\gamma, t_1\gamma \approx v_1\gamma, \ldots, t_n\gamma \approx v_n\gamma\}$.
   Let (m, n, o, p) be the measure of the goal before Mutate and (m', n', o', p') after Mutate. $m' = m$ and $n' < n$.
3. Assume that u and v are the same as in case 2, but now t is a variable. In this case **Variable Mutate** is applicable.
   As in the previous case we see that: $E \models u\gamma \approx s\gamma$ and $E \models t\gamma \approx v\gamma$. Both $u\gamma \approx s\gamma$ and $t\gamma \approx v\gamma$ have subproofs in Π, and $\Sigma_{t\gamma\approx v\gamma}$ is a subproof associated with $t\gamma$.. Solve t in $((\Pi, \gamma), U)$ gives us a new E-equivalent solution, (Π', γ'), such that Π' is an equational proof of the goal G.
   Since t is a system variable used in a root step $s\gamma \approx t\gamma$, beside $t\gamma$, t may appear only in $s\gamma$. Hence solving t does not changes subproofs of Π for any of the other equations in the goal. Only $\Sigma_{u\gamma\approx v\gamma}$ is affected. Therefore $E \models (\{u \approx s[t \mapsto v]\} \cup G_1)\gamma'$ and all equations in the new goal have subproofs in Π'. Hence we assume (Π', γ') as our new actual solution.
   Let (m, n, o, p) be the measure of the goal before Variable Mutate and (m', n', o', p') after Variable Mutate. $m' < m$.
4. Assume that u is a variable x, v is not a variable and $x \in Var(v)$, hence v can be written as $v[x]_{\alpha_1} \ldots [x]_{\alpha_n}$. In this case **Cycle** applies eagerly.
   Let $\Sigma_{x\gamma\approx v[x]_{\alpha_1}\ldots[x]_{\alpha_n}\gamma}$ be a subproof in Π of $x\gamma \approx v[x]_{\alpha_1} \ldots [x]_{\alpha_n}\gamma$.
   Since x has an occurrence in $v[x]_{\alpha_1} \ldots [x]_{\alpha_n}\gamma$, the subproof $\Sigma_{x\gamma\approx v[x]_{\alpha_1}\ldots[x]_{\alpha_n}\gamma}$ must have length greater than 0.

(a) Assume that there is a step at the root in $\Sigma_{x\gamma \approx v[x]_{\alpha_1} \dots [x]_{\alpha_n}\gamma}$.
Notice that in this case, since v is not a variable, if $x\gamma$ is chosen in Solve x in $((\Pi, \gamma), U)$, then case 2 applies, because Π is flat at any occurrence of x in v. Notice also that there are no irrelevant subterm variables for $x\gamma$. Hence the effect of Solve is just a removal of x from U.
$\Sigma_{x\gamma \approx v[x]_{\alpha_1} \dots [x]_{\alpha_n}\gamma}$ has the form: $x\gamma \approx \dots \approx w_i \approx_{[\epsilon, s \approx t, \gamma]} w_{i+1} \approx \dots \approx v[x]_{\alpha_1} \dots [x]_{\alpha_n}\gamma$. We choose i in such a way, that this is the rightmost root step in this subproof. Obviously, $E \models (\{x \approx s, t \approx v\} \cup G)\gamma$ and each of these equations has a subproof in Π.
If (Π', γ') is obtained by Solve, we have a set of unsolved variables $U' = U - \{x\}$. We change our actual solution to $(\Pi'', \gamma'') = (\Pi'[x \mapsto z, \gamma'[x \mapsto z])$, where z is a new variable, added in $Dom(\gamma'')$, in such a way that $\gamma''_z = [z \mapsto x\gamma']$. Now $U'' = U' \cup \{z\}$.
Notice that this renaming of x is needed here only in order to keep one Cycle rule for all relevant cases. Notice also that although Solve decreased the measure for the goal under solution (Π', γ'), but since we renamed x with a new unsolved variable in (Π'', γ''), (Π'', γ'') has the same multiset of paths as (Π, γ) and in fact it is a renaming of it.
$E \models (\{x \approx z, z \approx s, t \approx v[x \mapsto z]\} \cup G[x \mapsto z])\gamma''$ and all these equations have subproofs in Π''. Except for the renaming nothing can change in Π''. Hence $\Sigma_{z\gamma \approx v[z]_{\alpha_1} \dots [z]_{\alpha_n}\gamma''}$ is just a renaming of $\Sigma_{x\gamma \approx v[x]_{\alpha_1} \dots [x]_{\alpha_n}\gamma}$, with the same step at the root $s\gamma'' \approx t\gamma''$.

i. Assume that t is not variable. Since v is not a variable either and there is no step at the root between $t\gamma''$ and $v[x \mapsto z]\gamma''$, $v = f(v_1, \dots, v_n)$ and $t = f(t_1, \dots, t_n)$ and $E \models (\{t_1 \approx v_1, \dots, t_n \approx v_n\})\gamma''$, where all of these equations have subproofs in Π''.
This is exactly what we need because in this case $M(z \approx v[z]_{\alpha_1} \dots [z]_{\alpha_n}) = \{z \approx s, t_1 \approx v_1, \dots, t_n \approx v_n\}$, where $\{t_1 \approx v_1, \dots, t_n \approx v_n\}$ is an effect of decomposing $t \approx v[z]_{\alpha_1} \dots [z]_{\alpha_n}$.
Hence if $G \to G'$ by Cycle, and $E \models G\gamma$, then $E \models G\gamma''$ and $\gamma = \gamma''[Var(G)]$.
Let (m, n, o, p) be the measure of the goal before Cycle and (m', n', o', p') after Cycle. $m' = m$ and $n' < n$.

ii. Assume now that t is a variable. Then $M(z \approx v[z]_{\alpha_1} \dots [z]_{\alpha_n}) = \{z \approx s[t \to v[z]_{\alpha_1} \dots [z]_{\alpha_n}]\}$.
As in the previous case, we know that $E \models (\{x \approx z, z \approx s, t \approx f(v_1, \dots, v_n)\gamma''\}$, where $x \approx z$ is solved and all of these equations have subproofs in Π'', such that the respective terms are the extreme terms of these subproofs. Also if $E \models G_1\gamma$, then also $E \models G_1[x \mapsto z]\gamma''$, where $G_1 = G \backslash \{x \approx f(v_1, \dots, v_n)\}$.
Since t is a system variable used in the step $s\gamma'' \approx t\gamma''$, then besides $t\gamma''$, t may only appear in $s\gamma''$ in the goal. Solve t in $((\Pi'', \gamma''), U'')$ (case 1 applies) with $t\gamma''$ the chosen maximal occurrence of t, yields an equivalent solution (Π''', γ''') with t removed from U'''. Notice that we don't need to keep the equation $t \approx f(v_1, \dots, v_n)$ in the

goal, because t is not in the set of goal variables, and hence we don't need it in the solved form of G in order to define a solution.
$E \models (\{x \approx z, z \approx s[t \mapsto f(v_1,\ldots,v_n)]\}\gamma'''\}$ and t is eliminated from the goal. $\gamma =_E \gamma'''$.
Let (m,n,o,p) be the measure of the goal before Cycle and (m',n',o',p') after Cycle. $m' < m$.

(b) Now, let assume that there is no step at the root in $\Sigma_{x\gamma \approx v[x]_{\alpha_1}\ldots[x]_{\alpha_n}\gamma}$. Notice that for each occurrence of x at position α_i in $v[x]_{\alpha_1}\ldots[x]_{\alpha_n}\gamma$, there must be a step at a position higher, equal or lower than α_i, because otherwise $x\gamma$ would have to be syntactically identical with its subterm.

i. Assume that the proof Π is flat at all positions of x in $v[x]_{\alpha_1}\ldots[x]_{\alpha_n}\gamma$. Hence we know that for each occurrence of x in $v[x]_{\alpha_1}\ldots[x]_{\alpha_n}\gamma$, there is a subterm v_l at a position β_l in $v[x]_{\alpha_1}\ldots[x]_{\alpha_n}\gamma$, such that there is a step in the subproof at this position and x occurs in v_l. We know also that $\beta_l \neq \epsilon$, because we assumed that there is no step at the root.
Let us choose for each occurrence of x in $v[x]_{\alpha_1}\ldots[x]_{\alpha_n}\gamma$ highest β_l for which there is such a subterm in $v[x]_{\alpha_1}\ldots[x]_{\alpha_n}\gamma$ and there is a step at this position in the subproof.
Then the subproof $\Sigma_{x\gamma \approx v[x]_{\alpha_1}\ldots[x]_{\alpha_n}\gamma}$ can be viewed a composition of subproofs $\Sigma_1 \ldots \Sigma_k$ embedded at parallel positions, $\beta_1,\ldots,\beta_k$, of $x\gamma$.
Notice also that there must be at least one step at the root in each of the subproofs. In Cycle, we are guessing the right positions $\beta_1,\ldots,\beta_k$. Hence $v[x]_{\alpha_1}\ldots[x]_{\alpha_n}\gamma$ can be presented as $v[v_1]_{\beta_1}\ldots[v_k]_{\beta_k}\gamma$.
We know that Solve x in $((\Pi,\gamma),U)$, case 2 applies. It yields $((\Pi',\gamma'),U')$ with x and subterm variables irrelevant for $x\gamma$ solved. In Π', $\Sigma_{x\gamma' \approx v[v_1]_{\beta_1}\ldots[v_k]_{\beta_k}\gamma'}$ is such that $x\gamma' = v[x_1]_{\beta_1}\ldots[x_k]_{\beta_k}\gamma'$, where $x_1,\ldots,x_k$ are subterm variables defined for subterms of $x\gamma$ at positions $\beta_1,\ldots,\beta_k$ respectively.
It is then obvious that $E \models (\{x \approx v[x_1]_{\beta_1}\ldots[x_k]_{\beta_k}\} \cup \bigcup_{i=1}^{k}\{x_i \approx v_i\} \cup G)\gamma'$. Moreover, for each $x_i\gamma \approx v_i\gamma$ there is a subproof Σ_i in Π' with at least one step at the root. We choose the rightmost such step in each Σ_i. We know also that v_i is not a variable, because the proof Π was flat at each occurrence of x and v_i contains at least one occurrence of x. Let $v_i = f(v'_1,\ldots,v'_m)$.
Hence each such Σ_i has the form $x\gamma' \approx \cdots \approx w_i \approx_{[\epsilon,s\approx t,\gamma']} w_{i+1} \approx \cdots \approx f(v'_1,\ldots,v'_m)\gamma'$, where there are no steps at the root between $t\gamma' = w_{i+1}$ and $f(v'_1,\ldots,v'_m)\gamma'$.
Depending on whether t is a variable or not, $M(x_i \approx f(v'_1,\ldots,v'_m))$ yields $\{x_i \approx s, t_1 \approx v'_1,\ldots,t_m \approx v'_m\}$, when $t = f(t_1,\ldots,t_m)$ and $\{x_i \approx s[t \mapsto f(v_1,\ldots,v_m)]$ in the case where t is a variable.
Here again we have to analyze both these cases separately.
If t is not a variable the analysis similar to that in point 2 of this proof assures us that $E \models M(x_i \approx f(v'_1,\ldots,v'_m))\gamma'$ and each equation in $M(x_i \approx f(v'_1,\ldots,v'_m))$ has a subproof in Π' such that it's terms are

extreme terms in this subproof are extreme terms of a given equation.

Now, if t is a variable, as in point 3, we change our actual solution for the goal to (Π'', γ'') with t solved, given by Solve t in (Π', γ'). At each such step the set of lengths of proper paths is decreased. Hence if (m, n, o, p) is the measure of the goal before Cycle and (m', n', o', p') after Cycle. $m' < m$.

ii. Assume now that there is a position α_i in $v[x]_{\alpha_1} \ldots [x]_{\alpha_n}\gamma$ such that Π is not flat at $x\gamma^i$ at this position. If $x\gamma$ is chosen for Solve x in $((\Pi, \gamma), U)$, it gives us a new, E-equivalent solution (Π', γ') such that the subproof $\Sigma_{x\gamma' \approx v[x]_{\alpha_1} \ldots [x]_{\alpha_n}\gamma'}$ has a step at the root.
Then the analysis of case 4.4(a)i of this proof applies and if (m, n, o, p) is the measure of the goal before Cycle and (m', n', o', p') after Cycle, $m' < m$.

5. Assume that v is a variable and u is not a variable. Then **Orient** applies eagerly. Obviously, Orient preserves the set of E-unifiers for $u \approx v$. Let (m, n, o, p) be the measure of the goal before Orient and (m', n', o', p') after Orient. $m' \leq m, n' \leq n, o' \leq o$ and $p' < p$.
6. Assume that $x \approx v$ was selected for an inference and $x \notin Var(v)$. In this case **Variable Elimination** applies eagerly.
Then $E \models x\gamma \approx v\gamma$ and there is a subproof $\Sigma_{x\gamma \approx v\gamma}$ in the proof Π such that $x\gamma$ and $v\gamma$ are the extreme terms of $\Sigma_{x\gamma \approx v\gamma}$ and hence this is the subproof associated with $x\gamma$. If x is unsolved in the goal G, x is also unsolved in Π.
Solve x in (Π, γ), if $x\gamma$ is chosen as the maximal occurrence of x, yields a new, E-equivalent solution (Π', γ') with x no longer U'.
Since $E \models G\gamma$, also $E \models G\gamma'$ and (Π', γ') is the proof of $G\gamma'$. We change the actual solution to (Π', γ') and take it as the basis of completeness argument of further inferences. Since $x\gamma' = v\gamma'$, $E \models G_1[x \mapsto v]\gamma'$, where $G_1 = G \backslash \{x \approx v\}$ and because of extension, all equations in this part of the goal have subproofs in Π'.
Let (m, n, o, p) be the measure of the goal before Variable Elimination and (m', n', o', p') after Variable Elimination. $m' < m$.
7. Assume that u and v are occurrences of the same variable x. Since a non-redundant proof of $x\gamma \approx x\gamma$ has length 0, we can get rid of this equation in the goal by eagerly applying **Trivial**.
Let (m, n, o, p) be the measure of the goal before Trivial and (m', n', o', p') after Trivial. $m' = m$, $n' = n$ and $o' < o$.

Let us look at one more example of application of Cycle.
Let $E = \{c \approx d, gy \approx f(gy, c)\}$ and the goal is $G = \{x \approx f(x, d)\}$.
Let $\Pi = \{f(ga, c) \approx_{[<1>, gy \approx f(gy,c), [y \mapsto a]]} f(f(ga, c), c) \approx_{[<2>, c \approx d, []]} f(f(ga, c), d)\}$ with $\gamma = [x \mapsto f(ga, c), y \mapsto a, x_1 \mapsto ga, x_2 \mapsto c, x_3 \mapsto a]$. x_1, x_2, x_3 are subterm variables for $x\gamma$.

Since Π is not flat at $x\gamma^2$, the construction behind the Cycle will change our actual solution to (Π', γ') such that

$\Pi' = \{ga \approx_{[\epsilon, gy \approx f(gy,c), [y \mapsto a]]} f(ga, c) \approx_{[<2>, c \approx d, []]} f(ga, d)\}$ with $\gamma' = [x \mapsto ga, y \mapsto a, x_1 \mapsto a, z \mapsto x\gamma']$. x_1 is a subterm variable for $x\gamma'$ and z is additional variable, renaming of x.

Notice that there is a step at the root in Π', and thus Π' justifies the conclusion of Cycle: $\{x \approx z\} \cup M(z \approx f(z, d)) = \{x \approx z\} \cup \{z \approx gy, z \approx gy, d \approx c\}$.

7 Conclusion

We have proved that the goal-directed procedure based on inference rules in Figure 3 and an arbitrary selection function is complete.

In contrast to the proof of completeness of Gallier and Snyder's Lazy Paramodulation, we did not take a detour through a possibility of unfailing completion of a theory E, assumption that there is a solution with a reduced substitution for the variables in the goal and then showing that our rules can simulate the inferences in a completed E as it is done in [3].

In the case of collapsing goal equations, Solve allows us as if to "reduce" the E-unifier for the goal only when we need it, but this is a different kind of reduction than the one assumed in [3].

In general our proof uses a straightforward analysis of what happens in the realm of equational proofs if one of our inference rules is applied without even mentioning any ordering on ground terms substituted for variables, except for the fact that we can reduce their size measured by number of symbols if we need this. The possibility of proving our result without taking recourse to simulating inferences in some other system, shows also that the selection function involved in choosing equations for inferences, may be arbitrary and thus generates only the "don't care" kind of non-determinism. Our system is then strongly independent of the selection rule. The weak independence of Gallier-Snyder's Lazy Paramodulation follows from proofs in [10], and can be proved straightforward by the same analysis as in our paper.

The fact that we can use a similar style of proof in [8] and here shows that this is a robust way of looking at properties of goal-directed E-unification systems. We believe that it will enable us to search for more cases of decidable equational theories and efficient practical applications of E-unification.

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