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Utilizing Methods of Fluid Dynamics to Model Acoustic Movement

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Utilizing Methods of Fluid Dynamics to Model Acoustic Movement

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Abstract

Acoustic theory is a branch of theoretical physics that attempts to explain the movement of sound through models of fluid dynamics and derivations from the Navier-Stokes equation of fluid movement. Foundational models of acoustic theory aimed to explain how sound moves on a microscopic level but have been unable to find reasonable evidence of how models of particle movement relate to what can be heard on a macroscopic scale. This thesis explores current models and research spawned from original models while attempting to unify and apply micro and macro aspects of acoustics, while finding applications of the unifying theory. There is an emphasis on the proposition and predictions of sound movement within a constantly changing environment. Findings illustrate potential links between the models of fluid dynamics and characteristics of acoustics, while current gaps from both fluid dynamics and links between topics prevent a fully cohesive theory from being established. Despite this, applications of how this theory can be used in live sound were found and lay a foundation for new types of technology and methods to be developed once the theory becomes fully established through continued research.

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Introduction

Acoustics and audio engineering are often, erroneously, considered synonymous; however, they share a definitive synergistic relationship. Acoustics act as a foundational concept for the execution of audio engineering, ranging from creating studios to building arrays for a show. Although acoustics encompasses the study of musical instruments and architectural spaces, the scope of acoustics moves far beyond the studio or event spaces. The role of acoustics is evident in earth sciences, medicine, physiology, and sonar systems. It impacts every aspect of the day from traffic noise to seismic waves. Yet nowhere is it more noticeable than when attending a theater or concert performance and the sound quality of what the audience hears is either positively or negatively impacted by the venue and the environment. Acoustical physics often deals with sound movement, and acoustical engineers tend to focus on morphing the characteristics of sound to be pleasing to an audience. Acoustics play a critical role in the ability to successfully engineer and execute the required sound, as every location and venue presents its own unique set of challenges.

In the field of acoustic treatment, the movement of sound within a space is critical in achieving the best auditory results. Diffusion panels and dampeners inside studios can change the characteristics of the sound. Within a music venue, every aspect of design, ranging from drapes to seating, hardscapes, and carpet, all impact the acoustics of that venue. Each venue, studio, arena, and performance area comes with acoustic challenges that the audio engineer must deal with to help produce quality sound. This is of key importance when planning any space (Ramakrishnan & Dumoulin). The reflections that come with interactions between the sound and acoustic material must be considered when evaluating the desired outcome of the sound being produced. Models such as the Sabine equation and measured values of materials, help create approximate solutions to these acoustic problems, but they are not perfect (Long).

Aside from the various, traditional, acoustic impacts (physical structures, diffusion panels, dampeners, etc), some additional considerations and factors influence the live production of sound. Different environmental factors, like temperature, play a crucial role in how sound travels as well as in the auditory perception of listeners. Sound is variable, and much like how humidity and temperature change how the air feels on the skin, it also impacts the speed and transmission characteristics of sound. System engineers often have to account for temperature changes from crowds, as well as atmospheric changes like air temperature and wind speeds if the show is in an open-air venue (McCarthy). Engineers that work in live sound may use predictive software and run their soundcheck in an empty venue only to find that they must adjust and adapt the sound once the audience is present. In this case, the audience causes changes in the sound in the environment. Sound within any arena can be absorbed by the audience or reflected from other surfaces, requiring further adaptations that may not have been taken into account. Live sound engineers often describe live mixing as a battle with the environment when conditions are less than optimal. The concept of fine-tuning systems in live sound gets "close" to optimal, but is not quite perfectly accurate due to these external factors.

The interpretation of sound can be complicated depending on the environment through which sound is moved. While the model of sound is often interpreted as a wave, which is undeniably true, it is also defined as a disturbance of particles. This disturbance is seen as a pressure change, and the wave interpretation comes from the periodic compression and rarefaction of waves. A variety of factors can change sound qualities. While the wave motion and direction are quantifiable, using Huygens' Principle to describe propagation and Snell's Law deriving refraction, it is difficult to create a set of equations for each venue or studio to determine how best to create the optimal sound. Often generalizations and simplifications of these laws and practices become much more common as a “quick fix” to a problem. There are additional complications that come from the form of diffusion and absorption of sound, specifically with their representation. Both diffusion and absorption tend to use a ray-based model of sound rather than an analysis of waves or particles, in which sound is modeled as an individual line as opposed to waves or a field. This is further complicated with frequency-dependent absorption and diffusion, where the amount that sound spreads is dependent on frequency. Consequently, it is this lack of cohesion in evaluation that causes many questions to remain unanswered within these models and creates fragmented topics and methods within acoustics (Siltanen Lokki, & Savioja). Evaluating the impact of acoustics from every physical perspective is a necessity within all applications of acoustics. There is currently no golden equation nor golden theory to describe sound movement through all of these generalizations, leaving a vacancy in solid explanations of techniques or adaptations. Determining the variety of physical factors that influence the sound and finding a consistent manner to achieve optimal results across environments could improve both live sound and acoustical treatment.

Many techniques and manipulation tactics take into account broad-based concepts in acoustical physics, despite atomic properties such as heat and temperature which affect the sound. When analyzing both the physics involved and the techniques used, the concept of acoustics acting as a disturbance of the medium through which it travels should be considered (Kumar, Azharudeen, Pothuri, Subramani). Acoustic theory uses the foundation of fluid dynamics to determine sound movement through a fluid medium. Topics relating to the nature of atomic particles and their relation to temperature and thermal energy further contribute to changes within the field. Through the current model, attempts to unify both perspectives of acoustic movement have been made. The current model fails to account for the broader scale, while successfully focusing on the atomic scale. The compelling question is whether there is any way to unify these topics on the atomic and the broad scale.

The inability to unify the atomic and broader perspectives in acoustics has led to additional gaps in what is being used in practice. The current model for Acoustic Theory has its foundation in fluid dynamics. The original theory was presented by Epstein and Carhart in *The Absorption of Sound in Suspensions and Emulsions*, where the equations of the acoustic field are modeled on the Navier-Stokes equation used in fluid dynamics. Addendums to these equations have been made based on a variety of principles that separate acoustics from fluid dynamics. They are focused heavily on boundary conditions and diffraction problems while negating other properties of sound. Further research has been conducted by Allan D. Pierce, who created a more cohesive explanation for the mathematical foundations of acoustic theory in *Basic Linear Acoustics.* Pierce expanded upon the original model presented by Epstein and Carhart. Pierce's work has become one of the most highly respected textbooks in acoustics education. Pierce's rigor in exploring the physical and mathematical aspects of sound to which he applied equations of continuum mechanics in fluids, particularly the acoustic properties of water and air, furthered the ability to quantify sound within a variety of mediums. Additional research on frequency-domain effects of acoustic movement was investigated by Axel Kierkegaard. His research regarding frequency intensity within small acoustic ducts enabled the examination of possible extrapolation to larger acoustic ducts and fields.

Some research highlights the gaps that continue to exist in acoustic research. A model for using the diffusion equation for acoustics proposed a possible solution for computing non-constant diffusion coefficients. The potential solutions were inconclusive and led to more questions (Raúl Pagán Muñoz, Juan Miguel Navarro Ruiz, and Maarten Hornikx). While not conclusive, the questions that come from such research informs future research. Supplementary research on the diffusion equation model, however, found that the model had potential under the constraints of constant diffusion coefficients but was not able to be applied to situations where the coefficient was not constant (Juan Miguel Navarro, José Escolano, Jose Lopez).

While progress has been made, few solutions on how to utilize the plethora of information to unify the micro and macro aspects of acoustic theory have emerged. Applications of partial differential equations, multivariable calculus, and fluid dynamics serve as explanations as to what is happening to sound as it travels, but the theory is not fully cohesive yet. The goal of this thesis is to explore both sides of acoustic theory, evaluating the micro and macro properties of acoustics together through mathematics and physics, and propose future research for a unified acoustic theory in which all aspects are properly accounted for.

Mathematical Knowledge

Mathematics serves as a way to study problems within physics; however, what math represents in the field of physics is different from that in the field of general mathematics. Mathematics within the realm of physics is utilized to elucidate and expand physical theories, such as acoustic theory, rather than being used simply as a manner to execute mathematical operations. A rigorous mathematical framework can be used to represent the state of how an object changes, and in turn, can expand and can more accurately define existing theories. While acoustic theory has been studied within mathematical structure, it has yet to incorporate both micro and macro perspectives. Acoustic theory will be examined through the application of a different schema of complex mathematics, and possibly expanded to incorporate both perspectives, thereby improving the application of acoustic theory in real world situations. As such, it is first necessary to define the mathematics that will be used for the purposes of this paper, before defining the physics that is required in examining this. Much of the mathematics used in this thesis covers vector calculus, complex analysis, and partial differential equations, where foundations in ordinary differential equations and multivariable calculus will contribute significantly to the mathematics presented.

Multivariable Calculus

In mathematics, the partial derivative for some function $f(x, y)$ can be denoted as either f_x or $\frac{\partial f}{\partial x}$. In the context of this thesis, the notation of f_x and f_y will be used for vector components, not partial derivatives. The standard form for vector components are represented as f_x and f_y to represent the direction of a vector function in algebra, since this notation lends better to algebraic manipulation. Additional topics included in this thesis use subscript notation to denote other quantities, more specifically for tensor notation (Epstein & Carhart). Because of this, The representation and notation to follow will be represented below.

$$f = < f_x, f_y, f_z >$$

$$f_x \neq \frac{\partial f}{\partial x},\ f_y \neq \frac{\partial f}{\partial y},\ f_z \neq \frac{\partial f}{\partial z}$$

The majority of the math involved for acoustic theory involves vector calculus, multivariable calculus, and partial differential equations. Therefore, it is important to define some operators frequently used in these fields. An operator is a symbol that represents some form of operation, simple ones being addition or subtraction. One of the most common operators found in multivariable calculus is the gradient, or del operator, which is represented as ∇. This operator represents partial derivatives with respect to each variable of a function, of the form (Stewart):

$$\nabla f(x, y, z) = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$

The gradient is most often used in terms of vectors, as it is able to illustrate rates of change based on direction. An important caveat about this notation is the distinction between multiplication and the dot product. As previously mentioned, fluid dynamics and acoustic theory deal with the manipulation of vector and use of vector operations, meaning notation becomes crucial to distinguish these operations. The dot product results in a scalar value, while vector multiplication does not.

$$\nabla f \neq \nabla \cdot f$$

This concept also leads to divergence. Divergence is defined as the sum of the gradient components. The concept of divergence often describes what is happening to a system, and frequently appears in fields like thermodynamics, electromagnetism, and fluid mechanics (Chabay & Sherwood).

$$div\, f = \nabla \cdot f(x, y, z) = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}$$

The divergence operator describes the tendency of a vector to diverge from that point. It can also be defined as the "flux density" of the vector field, representing how much of a field is flowing at some given point or area. This specific concept is important when divergence takes the form:

$$div\, f = 0$$

When this case occurs, it means that either there is no change in a system or the amount of something entering into a system is the same as something exiting a system. Divergence of gradients can also be found, and are represented a del operator squared (∇^2), also referred to as the Laplacian operator, and takes the form (Greiner, Ivrii, & Seco):

$$\nabla^2 f(x, y, z) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Both of these notations are used, but the ∇^2 notation is not only preferred but generally more common. For this reason, del squared will be the notation used. The Laplacian operator often describes how the change of a quantity is changing in mathematics, which has great significance when looking at changes in a physical system. Examples of how the Laplacian operator is used in physics range from the Laplace and Poisson equations in electromagnetism, to Schrodinger's equation in quantum mechanics.

The final operation relating to the state of a system is that of curl. Curl is an operator similar to divergence that measures the amount of rotation that occurs within a system. It is defined as the determinant of a matrix involving the first partial derivatives of a vector f with components defined as f_x, f_y, f_z (Stewart):

$$curl\, f = \nabla \times f$$

$$\nabla \times f = \hat{i} \cdot \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right) + \hat{j} \cdot \left(\frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z}\right) + \hat{k} \cdot \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$< \frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}, \frac{\partial f_z}{\partial x} - \frac{\partial f_x}{\partial z}, \frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y} >$$

Curl defines the rotation of a system by looking at the change with respect to direction. If the curl of a vector f is nonzero, then there is rotation in the system. When the curl is zero, then the vector f is considered to have irrotational flow. As the name implies, irrotational flow means there is no rotation in the system. This is seen as a special case within vector field flow, and takes the form of (Stewart):

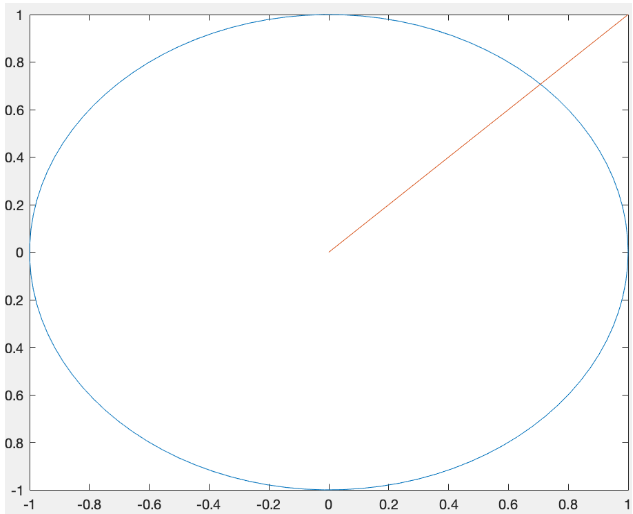
$$\frac{\partial f_z}{\partial y} = \frac{\partial f_y}{\partial z}, \ \frac{\partial f_z}{\partial x} = \frac{\partial f_x}{\partial z}, \ \frac{\partial f_y}{\partial x} = \frac{\partial f_x}{\partial y}$$

Additional concepts that are worth mentioning are those of a conservative vector field and the potential function. While distinct concepts, they are mathematically related through irrotational flow. If the curl of a vector function is zero, then the vector field is a conservative vector field. Furthermore, a conservative field is represented as the gradient of some function, called the potential function.

Complex Variables

Complex variables and complex graphing is essential for later in this text. Complex graphing follows a different set of conditions than graphing real values. Cartesian coordinates follow the notation of $(x, y)$for $x, y \in \mathbb{R}$, where any ordered pair of numbers represents a point in the xy plane. Complex graphing follows a similar convention, but does not yield a one-dimensional result. For any complex number $x + iy$, the coordinate system is defined as (x, iy),where the complex number $x + iy$ represents a line from the origin to the number $x + iy$. With this convention defined, the importance of Euler's identity becomes more apparent (Brown & Churchill).

$$e^{i\theta} = cos(\theta) + i \cdot sin(\theta)$$



Pictured above: $e^{i\theta}$ graphed in blue and $1 + i$ plotted in red

The identity functions as a way to draw a unit circle in the complex plane. By having $cos(\theta)$represent x,and $i \cdot sin(\theta)$ represent $i \cdot y$. This means that for any θ, Euler's identity represents any point on the unit circle that is θ radians away from 1 in the counterclockwise direction. This understanding becomes crucial when dealing with frequencies in the auditory spectrum, as its application assists in illustrating frequency strength.

Differential Equations

A differential equation is defined as an equation that involves one or more variables, as well as their rates of change. The important distinction that makes differential equations unique is that the solution must be a function, rather than an individual value. Ordinary differential equations appear frequently in science, ranging from current equations for circuits to the kinematic equations of classical mechanics (Graef, Henderson, Kong, et al.). Ordinary differential equations generally deal with differential equations of one variable, like time or distance. Partial differential equations take a similar form but require two or more variables, as well as their partial derivatives (Greiner Ivrii & Seco). These differential equations often represent how some value changes with respect to space and time. For the sake of understanding what partial differential equations represent, analyzing a common partial differential equation and what it represents is imperative. One of the most common examples for illustrating partial differential equations is the heat equation, due to the more simplistic nature and common uses in mathematics. The heat equation represents the change in thermal energy of a material, defined as (Carlslaw & Jaeger):

$$\frac{\partial T}{\partial t} = k\nabla^2 T$$

$$\frac{\partial T}{\partial t} = k \cdot \left(\frac{\partial^2 T}{dx^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right), \text{ for } T(x, y, z, t)$$

For the heat equation, $T(x, y, z, t)$ represents temperature in three dimensions for some time t, and k represents the thermal conductivity of any material, regarded as a constant value. The easiest way to envision how the heat equation works is to start by looking at a metal bar. The function of the temperature $T(x)$ represents the temperature T of the metal bar at every point x. The bar will reach thermal equilibrium as time continues, and all T values will become closer to being uniform across all x (Van Wylen & Sonntag). A time variable t is therefore necessary for $T(x)$ in order to evaluate the change in temperature of the metal bar over time, and $T(x)$ then becomes $T(x, t)$. The notation of $\frac{\partial T}{\partial t}(x, t)$ represents how the function T changes with respect to time, while $\frac{\partial T}{\partial x}(x, t)$ represents how the function T changes with respect to space (Carlslaw & Jaeger). The expression $\frac{\partial^2 T}{\partial x^2}(x, t)$ represents differentials of differentials, or how the change of a value is changing. For areas that have these higher rates of change, the movement of these values towards equilibrium will be more drastic. Visually, thinking about a three-dimensional plot of

(x, T, t) helps further illustrate what the heat equation represents. Let x represent points on the length of the rod, T represents the temperature of the rod and t represents time. The plot would show a curve of the temperature T at all points x in the (x, T) plane, while the (x, T, t) plane shows a the graph of a place that represents the temperature change towards thermal equilibrium along the t axis. The idea of identifying different types of changes is the same for any number of dimensions. In this example, defined functions of temperature are rarely ever found in applied situations. The equation is not necessarily presented as a problem to be solved, but as a description of an object (Van Wylen & Sunntag). This idea of using mathematics to represent the state of a system rather than an equation to solve is a key concept for partial differential equations. This helps conceptualize the physics that is happening in this text.

Physics

Acoustic Theory

It is necessary to be familiar with aerostatics and fluid mechanics. Sound represents pressure changes in an environment, a foundational concept to aerostatics is the proportion of the change in pressure, where the change in pressure is proportional to acceleration due to gravity and density. The important caveat about this law is that this only applies to fluids at rest (Schobieri). When the change in pressure is 0, it means that the fluid has a uniform density. This is more for an understanding of what is happening due to changes in pressure, and the relationship between density and pressure changes.

The foundation of fluid dynamics, and by extension acoustic theory, is the Navier-Stokes equation. The Navier-Stokes equation represents the flow of a fluid, but the original form of the equation has many different notations. For the derivations most commonly used in this writing, the Navier-Stokes equation is defined as (Peyret & Taylor):

$$\rho \cdot (\frac{\partial v}{\partial t} + v \cdot \nabla v) + \nabla p = 0$$

$$\frac{\partial \rho}{\partial t} + (\nabla \cdot \rho v) = 0$$

$$\rho \frac{\partial u}{\partial t} + \rho v \frac{\partial u}{\partial x} + p \frac{du}{dx} - \Psi_{\eta} - \frac{\partial}{\partial x} \cdot (k \frac{\partial T}{\partial x}) = 0$$

The values represented in the Navier-Stokes equation are u as the fluid energy, ρ as the fluid density, p as fluid pressure, and v as fluid velocity. These equations represent basic fluid motion in multiple directions over a period of time. Of note, the Navier-Stokes equations are regarded as being notoriously difficult to solve, specifically with complicated or chaotic flow. This is partially due to the nature of fluid dynamics becoming increasingly complex as specificity of examination increases. However, the importance of such specificities cannot be understated, as anomalies and inconsistencies tend to appear once the smallest particle scales are questioned.

An additional concept to introduce is that of perturbation. By definition, perturbation is synonymous with a disturbance, specifically within a system of equilibrium (Pierce). Perturbation is a common topic within fluid dynamics, often representing differences in pressure for the case of fluid dynamics. Given this definition, the concept of sound is nearly synonymous with perturbation, as it represents disturbances in fluid pressure (Epstein & Carhart). Mathematically, perturbations are denoted as a system in which there is an initial value and a change of the system over space and time (Peyret & Taylor). In the example of a perturbation in pressure, the system defining the perturbation is denoted with an initial pressure (p_0) and a function of the perturbation of pressure, also known as the perturbed pressure. For example, if a perturbation took place in three dimensions, it would take the form $p = p_0 + p(x, y, z, t)$. This definition is what allows for the application of concepts from fluid dynamics into acoustic theory, and are the fundamentals from Epstein & Carhart's writings.

The basic foundation of acoustic theory begins by defining the movement of the acoustic field. The first proposed equation of the acoustic field was of the form (Epstein & Carhart):

$$\frac{\partial \rho}{\partial t} + \rho(\nabla \cdot v) = 0$$

ρ represents density, and v represents the velocity function of the sound. The implications of this equation illustrate conservation in any defined system, stating that the change in density of a medium due to sound is proportional to the initial density multiplied by the total changes in the velocity of the sound field. This means that the system is closed. Sound disturbs two main components when traveling through a medium: pressure and density. As defined before, a change in pressure is related to density (Batchelor). Given this, sound can be defined as a perturbation of both density and pressure. Given these definitions, both the Navier-Stokes equation and the conservation equation of the acoustic field can be updated as (Peyret & Taylor):

$$(\rho_0 + \rho) \cdot \frac{\partial v}{\partial t} + (\rho_0 + \rho) \cdot v \cdot \nabla v + \nabla(p_0 + p) = 0$$

$$\frac{\partial}{\partial t}(\rho_0 + \rho) + (\rho_0 + \rho)(\nabla \cdot v) = 0$$

Where ρ_0 represents the static density, ρ represents the perturbed density, p_0 represents the static pressure, p represents the perturbed pressure, and v represents the velocity vector of the sound. Given that ρ_0 and p_0 both represent constant values, and therefore their derivatives are 0. Given this, the equations can be further updated (Pierce):

$$(\rho_0 + \rho) \cdot \frac{\partial v}{\partial t} + (\rho_0 + \rho) \cdot v \cdot \nabla v + \nabla p = 0$$

$$\frac{\partial\rho}{\partial t} + \rho_0\nabla \cdot v + \nabla \cdot \rho v = 0$$

The first equation represents the Navier-Stokes with the effect of pressure and density perturbation. This equation acts as the motion equation for acoustic theory. The second equation represents conservation in the acoustic field with the effect of density perturbation.

Further advancements can be found when linearization processes are applied to these equations. Under the idea that v, ρ, p are all small quantities and are constrained to being first order, which would imply that ρv approaches 0 in the conservation equation (Epstein & Carhart). When analyzing the equation of motion, terms containing v would also approach zero since v is considered small. These two linearizations result in simpler forms of both the conservation and motion equations (Pierce):

$$(\rho_0 + \rho) \cdot \frac{\partial v}{\partial t} + \nabla p = 0$$

$$\frac{\partial\rho}{\partial t} + \rho_0\nabla \cdot v = 0$$

The equation of motion can be further simplified based on the assumption that the air is considered to have no changing effects from thermal expansion and that all atomic collisions conserve energy. This assumption further demonstrates that perturbations do not occur based on changes in temperature, and are solely based on disturbances of pressure and density (Batchelor). Additionally, the speed of sound based of the medium can be defined via the concept of the bulk modulus (Schobieri).

$$K = \frac{\partial p}{\partial\rho} \cdot \rho_0$$

$$c = \sqrt{\frac{K}{\rho_0}}$$

. All mediums are resistant to changes, and the bulk modulus measures how resistant a fluid is to compression, which is important for sound propagation. Given this, a relation to the speed of sound can be found as the change in pressure with respect to the density, defined as (Peyret & Taylor):

$$c = \sqrt{(\frac{\partial p}{\partial \rho})}$$

With this definition, the speed of sound in relation to density and pressure can be established.

$$p = \rho_0 c^2$$

Given this information, the conservation equation can be updated to consider the changes in pressure of the system, and by extension, the speed of sound within the medium. In order to achieve this, the entire system is multiplied through c^2 and results in a conservation equation accounting for the speed of sound. This does, however, require one additional addendum to be made.

$$\frac{d}{dt}(\rho \cdot \frac{dp}{d\rho}) = \frac{d}{dt}(p) = \frac{dp}{d\rho} \cdot \frac{d\rho}{dt} = \frac{dp}{dt}$$

This allows for a relationship between the values $\frac{\partial \rho}{\partial t}$ and $\frac{\partial p}{\partial t}$ to be made. Now multiplying through by $\frac{\partial p}{\partial \rho}$, the equations of the acoustic field can be established. The equation of motion currently remains unchanged, the density has been divided through so $\frac{\partial v}{\partial t}$ stands on its own (Epstein & Carhart).

$$(\frac{\partial \rho}{\partial t} \cdot \frac{\partial p}{\partial \rho_0}) + (\rho_0 \frac{\partial p}{\partial \rho_0}(\nabla \cdot v)) = 0 \rightarrow \frac{\partial p}{\partial t} + \rho_0 c^2 \cdot \nabla \cdot v = 0$$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho_0}\nabla p = 0$$

While the implications of this change may not seem evident currently, the importance of this state will be explored later.

Thermodynamics

When considering any medium that can change based on heat, simply negating the effects is not sufficient. This is where the importance of analyzing the thermodynamic effect is illustrated. Many fluid properties, such as changes in pressure or density, often have thermodynamic causes that cannot be neglected. General kinetics dictates that thermal energy has effects on the total energy of a system. There are several important concepts from thermodynamics that are crucial to define when looking at their effects. Of note, many of these concepts have definitions in English that apply to everyday life but have physical definitions that are not the same. The first of these are thermal energy and heat. Thermal energy is defined as an internal energy transfer between a set of bodies due to temperature (sometimes referred to as translational kinetic energy), while heat measures this transfer of energies (Chabay & Sherwood). Thermal energy can exist without a source, while heat generally has a source, such as fire or gas, that causes these transfers. More specifically, thermal energy can occur from the collisions of molecules. In this case, the system is the medium that sound passes through which, for the purposes of this thesis, is air. Air, by definition, is a gas composed of molecules that interact with each other, and their movement and interactions contain kinetic energy (Van Wylen & Sonntag). As sound travels through a fluid medium, the dissipation of sound energy generates thermal energy from the collisions of the atoms (Berg & Stork). This results in the fluid being affected by the thermal energy increase that was generated by the sound. Because of these interactions, the effects of thermal energy and temperature on a fluid must not be negated.

Macroscopic thermodynamics plays a key role in sound movement, as pressure and density changes are directly affected by temperature. Further phenomena, like diffraction, happen to the thermodynamic properties of the environment. It is a known fact that temperature has an effect on sound movement and propagation as well(Berg & Stork). While the comprehensive relationship between temperature and sound movement is incredibly complex, some surface-level relationships can be defined. For illustrative purposes, if a room at a constant temperature has a heater that uniformly increases the temperature, the energy transfer of the system will cause all of the individual air molecules to increase in kinetic energy. This, in turn, increases the speed at which the particles interact with each other (Van Wylen & Sonntag). The same concept applies to decreasing temperature, as negative heat transfer decreases the kinetic energy of the particles. This is known as the conservation of energy. This concept is how sound travels faster in higher temperatures. If molecules have higher kinetic energy due to temperature, then the pressure change will be carried faster, assuming different forms of energy aside from kinetic energy are negligible. Mathematically speaking, relationships between the speed of sound in air and temperature are defined through frequency, air temperature, and wavelength (Berg & Stork).

$$v = f \cdot \lambda$$

$$v = 336 +.6 \cdot T$$

$$336 +.6 \cdot T = f \cdot \lambda$$

It is important to remember that certain characteristics of sound, such as frequency, are static. They do not change based on temperature. Wavelength, however, changes as temperature changes. This means that while oscillations do not change, the length of the wave does. This further affects the motion of sound.

Any thermodynamic system has a measure of disorder known as entropy. Temperature is defined as having a direct relationship with the change of energy and inversely related to entropy.

In other words, if an entropic system has an increase in energy, then the temperature increases as well. Mathematically speaking, temperature is inversely proportional to the change of entropy with respect to energy (Chabay & Sherwood).

$$\frac{1}{T} = \frac{\partial S}{\partial E}$$

In standard English, thermal energy and temperature generally tend to be interchangeable due to the defined relationship, as temperature generally increases or decreases when thermal energy changes. This is usually interchangeable with heat since positive or negative heat transfer results in changes in thermal energy. Despite this apparent interchangeability, it is crucial to clarify that these three topics do not define the same thing, and are not interchangeable in the context of physics.

In order to fully comprehend the effects of temperature and dissipative effects on the Navier-Stokes equation, knowledge of tensors is required. Tensors often describe changes within a body, usually in reference to internal forces, stress, or deformation (Schaschke). A simple example of a tensor would be a stress tensor for a cube of length L. The stress tensor would represent the deformation of a cube based on some force exerted on the cube, measured as force per unit area. In order to understand a total deformation of the cube, each face of the cube must be analyzed individually. A cube in $\mathbb{R}^3$ has six faces, but is symmetric along the the $xy,\ xz$, and yz planes, so three faces need to be accounted for instead of six. Deformation of each face can happen in any direction in $\mathbb{R}^3$, implying that a total of 3×3 stress tensors can exist for the cube. The largest caveat about tensors is that the components of a tensor cannot be added. This is because stress in each direction affects the deformation differently, meaning directions for individual deformations can be added, but not the tensors themselves (Dvorkin & Goldschmit). The number of values in this 3×3 stress tensor represents the degrees of freedom, rather than the number of components with direction. In order to accurately depict stress tensors on an object, tensor notation often involves the use of a matrix for describing components of each direction (Schaschke).

$$\sigma_{ij} = \begin{vmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{23} & \sigma_{32} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{vmatrix}$$

Tensors are able to describe what is happening to a medium outside of a coordinate system. If a standard coordinate system like $\mathbb{R}^2$ or $\mathbb{R}^3$ were rotated in an arbitrary direction, the tensor would continue to point in the same direction. Unlike space-dependent constructions like vectors, tensors move independently of space and describe what is happening internally to a system, rather than describing a specific direction in $\mathbb{R}^2$ or $\mathbb{R}^3$ which a particle is moving (Merrill). Tensors are fundamental in describing stress or rates of change within fluids. Concepts like strain-rate tensors and viscous stress tensors appear frequently and are used in the description of internal properties of fluids (Thevenin & Jamiga).

With the concept of tensors defined, the Navier-Stokes equation of energy can begin to be analyzed. Returning back to the conservation of energy for fluid dynamics (Peyret & Taylor):

$$\rho \frac{\partial u}{\partial t} + \rho v \frac{\partial u}{\partial x} + p \frac{du}{dx} - \Psi_{\eta} - \frac{\partial}{\partial x} \cdot \left(k \frac{\partial T}{\partial x}\right) = 0$$

The key points of interest in this equation come from the last two terms. Firstly, Ψ_{η} is defined as the viscous dissipation function, defined as the product of the strain and viscous stress tensors, representing the strain at rate of deformation around a given location (Batchelor).

$$\Psi_{\eta} = \frac{1}{2} e_{ij} p_{ij}$$

With these values defined, they can be substituted back into the conservation of energy equation. In order to make notation simpler for the rest of these calculations, the notation below from Epstein & Carhart will simplify some of these equations.

$$\frac{\partial v}{\partial x} =- \left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)$$

An entropy balance equation can be written for the changes in entropy over some arbitrary volume, and the generation of entropy from different sources. This results in integration over volume (Epstein & Carhart).

$$\frac{\partial}{\partial t}\int_V \rho \cdot s\, dV =- \frac{\partial}{\partial x}\int_V \rho s v\, dV + \frac{\partial}{\partial x}\int_V \frac{k}{T} \cdot \frac{\partial T}{\partial x} dV + \frac{\partial}{\partial t}\int_V S\, dk$$

The first term on the right represents entropy generation through heat flow, where s represents specific entropy. The second term represents entropy generated through heat transfer and thermal effects, while the third term S represents entropy generated from some dissipation process, also referred to as irreversible entropy (Chabay & Sherwood). If V represents any arbitrary material volume, but always equals zero, then the integrated argument must be zero at every point, and the integrals can be removed (Epstein & Carhart; Thevenin & Jamiga).

$$\rho s\frac{dv}{dt} - \frac{\partial}{\partial x}\left(\frac{k}{T} \cdot \frac{\partial T}{\partial x}\right) = \frac{\partial S}{\partial t} \rightarrow \rho\left(\frac{\partial}{\partial t} - v \cdot \frac{\partial}{\partial x}\right)s - \frac{\partial}{\partial x}\left(\frac{k}{T} \cdot \frac{\partial T}{\partial x}\right) = \frac{\partial S}{\partial t}$$

Returning back to the original equation, it is important to look at the effect of s. The first law of thermodynamics states $\partial s = \partial E_{int} + \partial w$ for some infinitely long time period t results in the stress tensor becoming the pressure being exerted ($- p$), as velocity drops out of the tensors due to this time (Matthews & Walker). In addition, $\partial w = p \cdot \partial(\frac{1}{\rho})$, therefore these values can be substituted into the above equation to result in a definition for irreversible entropy (Epstein & Carhart).

$$\frac{\rho}{T}\left(\frac{\partial}{\partial t} - v \cdot \frac{\partial}{\partial x}\right)u + p\left(\frac{\partial}{\partial t} - v \cdot \frac{\partial}{\partial x}\right)\frac{1}{\rho} = \frac{\partial}{\partial x} \cdot \left(k\frac{\partial T}{\partial x}\right))$$

Returning back to the relationship previously established between entropy and temperature, a relation for the energy dissipation function and irreversible entropy can be found (Batchelor; Van Wylen & Sonntag).

$$\frac{\partial S}{\partial E} = \frac{1}{T} \rightarrow T \cdot \partial S = \partial E$$

$$T \cdot \frac{\partial S}{\partial t} = \Psi_{\eta} + \Psi_{T}$$

$$\frac{\partial S}{\partial t} = \frac{1}{T}(\Psi_{\eta} + \Psi_{T})$$

This expression represents the change in entropy generation S is related to some dissipation of energy, represented as Ψ_{η} and Ψ_{T}. To simplify further, the expression defining viscous dissipation function can be eliminated, as values within this expression are irrelevant when evaluating thermal effects in this case. Through this elimination, thermal dissipation can be found (Van Wylen & Sonntag).

$$\frac{1}{T}(\Psi_{\eta} + \frac{\partial}{\partial x} \cdot \left(k\frac{\partial T}{\partial x}\right) - \frac{1}{T}\Psi_{\eta} = E$$

$$\frac{1}{T}\left(\frac{\partial}{\partial x} \cdot \left(k\frac{\partial T}{\partial x}\right)\right) = \Psi_{T}$$

$$\Psi_{T} = \frac{k}{T}(\nabla T)^{2}$$

There is a secondary derivation of this thermal dissipation function worth discussing for conceptual understanding, independent of the Navier-Stokes. The second law of thermodynamics states entropy can never decrease. Entropy production is related to any irreversible process, and can be caused by a variety of actions, including the flow of a fluid. The formula to determine entropy generation involves both the flow of heat and the temperature of the system itself (Carlslaw & Jaeger).

$$\Gamma = J \cdot \nabla(\frac{1}{T}) \rightarrow - J_q \cdot \frac{\nabla T}{T^2}$$

Heat flow (J_q)is defined as the temperature gradient between a system of objects. The flow is not only determined by temperature, but also by the conductivity of the object itself (Brown & Churchill).

$$J_q =- k\nabla T$$

In this case, k is thermal conductivity. Now that heat flow has been defined, this can be put back into the equation to find entropy generation (Carlslaw & Jaeger).

$$\Gamma = k\nabla T \cdot \frac{\nabla T}{T^2}$$

$$\Gamma = \frac{k}{T^2} \cdot (\nabla T)^2$$

Another form of the heat dissipation function is defined as temperature multiplied by entropy production, which describes the energy and flow of heat (Van Wylen & Sonntag).

$$\Psi_T = T \cdot \Gamma \rightarrow \Psi_T = \frac{k}{T}(\nabla T)^2$$

Through both derivations, it proves that the thermal dissipation function is not only an irreversible function, but that it holds true for both the Navier-Stokes and heat flow derivations. This further proves the effect of stress and strain on a fluid, and that temperature does, in fact, have an effect on the fluid.

Diffusion

Many applications of partial differential equations appear when describing complicated systems. One of the most common examples is called the diffusion equation. This equation represents changes in the movement of particles based on the density of the material and mass diffusivity, or the diffusion coefficient. More specifically, it represents systems of particles in Brownian motion based on Fick's laws of diffusion. This diffusion coefficient can be either a function of the density or a constant value, which results in two different equations (Crank).

$$\frac{\partial \varphi}{\partial t}(r, t) = \nabla \cdot (D(\varphi, t) \cdot \nabla\varphi(r, t)), \text{ for } D \text{ is not constant, } r = x, y, z$$

$$\frac{\partial \varphi}{\partial t} = \nabla \cdot D(\nabla\varphi) \rightarrow \frac{\partial \varphi}{\partial t} = D \cdot (\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2}), \text{ for } D \text{ is constant}$$

In these equations, φ represents the density of the material that is diffusing, and D represents the aforementioned diffusion coefficient. When the diffusion equation shows up for constant diffusion coefficients, the form becomes identical to the heat equation. This is due to Fick's laws of diffusion. The first law states that a diffusive flux will travel from high regions of concentration to low regions of concentration (Greiner, Ivrii, & Seco). Physically speaking, for any amount of material passing through a region over some time, high concentrations of the material will travel to low concentrations of the material. This travel happens upon some gradient, also defined as the rate of change of the concentration. Mathematically speaking, Fick's first law is represented by (Crank):

$$J = D \cdot \frac{\partial \varphi}{\partial x} \text{ in one dimension}$$

$$J = D \cdot (\nabla\varphi) \text{ in two or more dimensions}$$

While this works as a solid basis for defining diffusive flux, it does not state any additional information as to how the diffusion changes. However, here is where the similarities to the heat equation appear. Fick's first law states that the flux travels from regions of high concentration to low concentration, which implies that eventually the flux will reach some sort of concentration equilibrium. It can be further implied that regions of very high concentration or very low concentration will reach this form of equilibrium faster than those closer in concentration.

Similar to the heat equation, this is represented through $\frac{\partial^2 \varphi}{\partial x^2}$ in one dimension. Now that the goal of the equation has been stated, the diffusion of some material in one dimension can be described by the equation (Drake):

$$\frac{\partial \varphi}{\partial t} = \nabla \cdot (D(\varphi, t) \cdot \nabla \varphi(r, t)),\ D \text{ is not constant}$$

$$\frac{\partial \varphi}{\partial t} = D \cdot \frac{\partial^2 \varphi}{\partial x^2},\ D \text{ is constant}$$

Besides the trivial difference that heat and diffusion are two different physical processes, one of the most significant differences between the two equations is the dependence on flux density of the diffusion coefficient. In this case, the analogue to the heat equation would be if the thermal conductivity of a material depended on the temperature at given locations. Using this example, a certain material could be defined by conducting heat better at higher temperatures or worse at lower temperatures. Under this constraint, thermal conductivity would be defined as a function $k(u, t)$. If the thermal conductivity depended on temperature, the change in thermal conductivity would also need to be evaluated, since it would not be uniform across the material. Given these changes, the heat equation could now be represented as:

$$\frac{\partial u}{\partial t} = k \cdot \frac{\partial^2 u}{\partial x^2} \rightarrow \frac{\partial u}{\partial t} = \frac{\partial k}{\partial x} \cdot \frac{\partial^2 u}{\partial x^2}, \text{ in one dimension}$$

$$\frac{\partial u}{\partial t} = \nabla(k(u, t) \cdot \nabla u), \text{ in two or more dimensions}$$

Notice that this result is now identical to the diffusion equation.Therefore, in the diffusion equation, the diffusion coefficient can change with respect to the density of the medium.

Diffusion in Acoustics

The term "diffusion" is a word that is often used in the field of acoustic treatment, generally unrelated to the diffusion equation itself. When considering studio treatment or venue architecture, an important consideration is that of the acoustic sound field, specifically with sound dispersion (Everest). The model of the diffusion equation acts, in some sense, as a potential solution to representing the scattering of sound particles. The primary goal when installing sound diffusers is to disperse sound energy in such a way that power post-contact is equal in all directions (Everest). This goal and definition of a sound diffuser are misleading, as diffusers accomplish the act of spreading the sound in random directions rather than creating equal power in all directions. When this model of "sound particles" is involved, the particles become disturbed by changes in pressure and change their motion, but the motion is still affected by the air particles. This combines both perturbations of suspended particles and Brownian motion of the particles themselves (Navarro & Escolano). The fluctuations would act similarly to a force on the particles, but as a guide to determine their interactions rather than their motion.

The acoustic form of the diffusion equation came about from the idea of a "sound particle" diffusing in a fluid medium rather than looking at sound from the wave-based perspective. Current methods for prediction energy levels use sound movement with linear motion to model reflection and diffusion. This is called ray-tracing, and can often fail due to its simplistic nature (Visentin, Prodi, Valeau et al.). If sound followed perfect diffusion models, the acoustic diffusion equation would look identical to the current diffusion equation (Valeau, Picaut & Hodgson):

$$\frac{\partial w}{\partial t} = D \cdot \nabla^2 w(r, t)$$

Two significant issues arise when modeling acoustic diffusion using solely this equation. D is assumed to be constant, which is defined by the dimensions of the enclosure, but what D actually represents is still unanswered. The solution to finding D involves a concept within fluid dynamics, called mean-free path. Mean-free path represents the distance a particle travels in between collisions with other particles before the direction of movement changes. In enclosed spaces where D does not depend on pressure changes, it can be represented by the mean-free path and speed of sound in air (Thevenin & Jamiga). This then raises the question of how the mean-free path is determined. In the modification made for acoustic theory, it represents the distance traveled before reflecting or diffusing against an object, usually the walls, floor, or roof. This modification comes from looking at an acoustic particle moving through a fluid rather than looking at fluid particles colliding with each other (Beranek).

$$D = \frac{1}{3}\Lambda \cdot c$$

$$\Lambda = \frac{4 \cdot V}{S_a}$$

Through both of these definitions, definitive solutions for the mean-free path and the diffusion coefficient can be found. It is important to note that these simplifications work for enclosed spaces and that open spaces result in non-constant diffusion coefficients. What is important to know is that non-constant diffusion coefficients create more complexities by transforming the diffusion equation into a non-homogeneous equation.

The second problem with the “perfect diffusion” model arises in the physical nature of sound versus the general description of what the diffusion equation represents. Standard applications of the diffusion equation can be applied in the form of particles being introduced to a medium that disperses across it, such as when drops of ink are dripped into a cup of water (Greiner, Ivrii & Seco). With applications of sound, it defies this model because sound waves eventually dissipate, meaning the particles dissipate as well. If the current model of the diffusion equation were left unchanged, it would imply that sound pressure would reach an equilibrium that changes the medium itself, rather than dissipate and allow the medium to return to normal. The dichotomy of wave-based perturbations and sound particle models becomes very apparent with this model, as these two different methods of thought create inconsistencies on both sides of the argument.

Continuum Mechanics

At its core, the Navier-Stokes equation is governed by a set of properties that follow from continuum mechanics, while diffusion describes the movement and random nature of material inside of a fluid. On a general level, these are both considered subtopics of fluid dynamics. Navier-Stokes governs the general flow, while diffusion governs how something interacts in the medium. Both of these topics fall under the broader field of continuum mechanics which to understand their link is essential.

Continuum mechanics is a field that looks at models of objects as continuous elements rather than discrete atoms in time. Components of a material such as strain, deformation, and motion can be analyzed through these methods, allowing for more comprehensive descriptions about the material itself (Merrill). When assuming continuity, solutions to differential equations arise to define what physical properties are occurring. Continuum mechanics acts as the foundation for both the diffusion and Navier-Stokes equations. Assumptions of uniformity within a medium are required, as it is assumed that subdivisions of a material fully represent the properties of the entire material (Dvorkin & Goldschmit). Topics like perturbations and deformations are what govern how these mediums change, leading to the foundation of acoustic theory itself. One of the most important concepts within continuum mechanics is that physical and mathematical representations are coordinate-independent. Descriptions tend to be more abstract and reference how something changes over time with respect to the material, rather than physical changes to a coordinate space (Merrill). Concepts like tensors and material derivatives describe what is happening to the material and illustrate these concepts independently of a coordinate system (Schaschke). Understanding their properties is essential to grasping the role in both fluid dynamics and diffusion.

Representation of continuum mechanics is subdivided between Lagrangian and Eulerian representations. These representations create a division between topics that are represented mathematically. Lagrangian representation of deformation follows a particle throughout time, with a frame of reference being the particle (Dvorkin & Goldschmit). Changes within a system do not only happen by a defined function or value in a determined space. Using an example of some object with a set of material coordinates X, the deformation based on a stress tensor over a period of time does not happen solely based on a determination of a distance $\Psi(X)$.The deformation occurs over a period of time, resulting in the mapping of both displacement and time $\Psi(X, t)$ (Dvorkin & Goldschmit).

Eulerian representation of deformation deviates in the form of the coordinates used. Eulerian deformation follows spatial coordinates, where the displacement $\psi(r, t)$ becomes represented by unit vectors describing the frame for the material. This description follows a more generalized concept of deformation, representing multiple particles defined in space but constrained by the material (Smith, Hashemi & Wang). Lagrangian representation, by comparison, lets the material define the space. An important note is that deformation over a time

t is considered to be continuous, therefore deformations happen gradually and the object will deform and have a different configuration at every time t.

The final concept necessary for understanding the basics of continuum mechanics used in this writing is that of material and material derivatives. The term “material” in reference to continuum mechanics can be anything related to the object. Referring back to what changes represent, objects can deform and vary in shape, or flow can change depending on location (Dvorkin & Goldschmit). General terms relating to material functions involve how an aspect of the material can change with respect to time. An example of which relates to fluid flow based on location. The material derivative represents a change in the aspect of a material, whether it be the temperature of a solid or the flow rate of a fluid. This concept is where the two aspects of Lagrangian and Eulerian deformation meet, as material derivatives serve as the bridge between them (Peyret & Taylor).

$$\frac{D\psi}{Dt} = \frac{\partial\psi}{\partial t} + u \cdot \nabla\psi$$

In the above equation, u is defined as a flow field and ψ is defined as a generic field $\psi(x, t)$.The importance of the material derivative is that it uses the Lagrangian representation of following a particle through a material, but accounts for Eulerian flow fields, unifying both perspectives of deformation and change (Dvorkin & Goldschmit).

With the basis of continuum mechanics covered, additional properties within fluid flow can be defined. Starting with the pressure force, it is represented by the pressure multiplied by the identity matrix. This accounts for the regions in which pressure changes within the tensor (Thevenin & Jamiga).

$$f_{pressure} = - p \cdot \delta_{ij} = - p \cdot I$$

The second component is the shear tensor, also called the friction tensor. The shear tensor is generally associated with velocity changes, as kinetic friction is present only when velocity changes are also present. Due to this relationship, the shear tensor has a direct relationship to the deformation tensor (Peyret & Taylor).

$$\tau_{ij} = 2 \cdot \mu \cdot d_{ij} - \frac{2}{3}\mu \cdot (\nabla \cdot u) \cdot I$$

It is crucial to note that this description of the shear tensor does not assume incompressible flow. If fluid flow is incompressible, then $\nabla \cdot u = 0$ and τ_{ij} is only related to the deformation tensor (Dvorkin & Goldwchmit).

$$d_{ij} = \frac{1}{2} \cdot (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

$$\tau_{ij} = \mu \cdot (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

In this case, μ represents the dynamic viscosity of the fluid. The full stress tensor is defined as the sum of the pressure force and the shear stress tensor (Thevenin & Jamiga).

$$\sigma_{ij} =- p \cdot \delta_{ij} + \mu \cdot (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$$

Additional information from the Navier-Stokes equation can be extracted through changing notation. In order to do this, it is important to redefine what the original equation stated (Peyret & Taylor).

$$\frac{\partial v}{\partial t} + v \cdot \nabla v =- \frac{\nabla p}{\rho}$$

With some rearrangement of values, this expression can be written as density multiplied by the material derivative of energy density. This representation of the Navier-Stokes equation represents how an individual particle changes within the acoustic sound field (Batchelor).

$$\rho \cdot \frac{\partial v}{\partial t} + \rho(v \cdot \nabla v) =- \nabla p$$

$$\rho \cdot (\frac{Dv}{Dt}) =- \nabla p$$

$$\nabla p =- \rho \cdot (\frac{Dv}{Dt})$$

One final point is that of vorticity. Vorticity often analyzes the rotation at a point within the field flow, known as a continuum point. Vorticity can be measured as the curl of the velocity, which is noted as $\omega = \nabla \times u$ (Dvorkin & Goldschmit). This can help aid in determining what happens with the Navier-Stokes equations for acoustics when irrotational flow cannot be assumed. It would illustrate what the rotation of the flow would look like, creating potential solutions to rotational flow problems.

Audio Engineering

Spectral Properties

Frequency is a term that has implications in both auditory and non-auditory fields. Frequency is defined as the rate at which an object oscillates, and is inversely proportional to the period of oscillation. The physical implication of frequency in terms of sound is the rate at which molecules are disturbed through a perturbation of pressure, then return back to a “normal” state, where one period is defined through these compressions and rarefactions of pressure (Berg & Stork). Mathematical representations of waves involving frequency often take the form of periodic functions.

$$p = A \cdot sin(2\pi f \cdot x + \phi) \text{or } p = A \cdot cos(2\pi f \cdot x + \phi)$$

General properties of sine and cosine dictate that there exists a phase offset of $\frac{\pi}{2}$ between $sin(x)$and $cos(x)$, leaving these formulas largely interchangeable. The variable f, in this case, represents the frequency. Fluctuations in pressure are not visible, but periodic waves functions work as visual analogues to what is happening in the acoustic field. Peaks represent moments of highest compression, and troughs represent moments of lowest compression. Through this interrelationship, it implies that the amplitude *A* of the wave represents the change in pressure of the medium (Everest). This form works as a visual aid for simple low-dimensional representations of what sound accomplishes; however, becomes trickier to define for higher-dimensional representations.

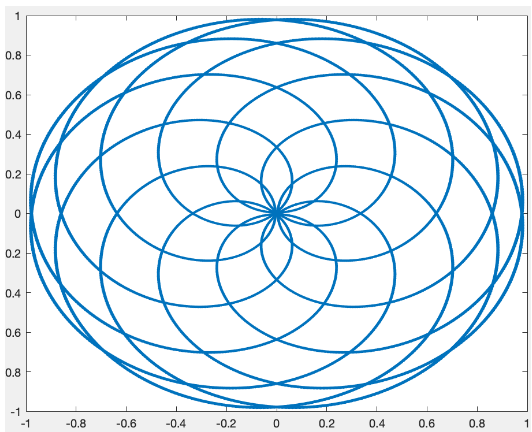
Spectral properties of sound play an influential role in how it interacts with the environment. The interpretation of frequency from a sound engineering perspective is unique in comparison to other characteristics of sound. Reverberation, delay, and amplitude are measured over a period of time, known as time domain. While frequency can be measured over the time domain, it proves much more difficult to analyze frequency in this manner (Case). Modern spectral processors tend to analyze sound waves in terms of intensity per frequency band, using time to illustrate changes in intensity. In this case, the time-dependent definition would depend on the fluctuation of pressure based on a periodic function. A model for pressure fluctuation of a periodic function was previously introduced for a static frequency value (Stewart). Despite this, sounds are rarely pure tones of one frequency, but are instead composed of a variety of frequencies.

Periodic functions, like waves, have incredibly unique properties that relate to functional addition and how waves interact with each other. Given any set of periodic wave functions, the sum of the functions creates a new wave based on the individual values at every point on the waves (Berg & Stork). This is the definition of wave summation, often referred to as Fourier synthesis (Berg & Stork; Case). The difference arises when sound becomes involved. Rates at which pressure changes can vary depending on the frequencies of the waves. Complicated sounds that contain base waves of multiple discrete frequencies can be summed together to create a new wave that oscillates based on the summation of amplitude, frequency, and phase of the base waves that create it (Everest). This is a concept known as wave synthesis and is commonly seen in modern synthesizers to create unique waves, like square and triangle waves. This signal synthesis is great for describing the intensity of the pressure at these points, which is useful from the perturbation perspective, but dictates nothing about frequency intensity. The goal would be to describe at what frequencies the intensity of the pressure is greatest, rather than stating how the pressure changes for a complex wave (Brown & Churchill).

Transform functions commonly appear in several different branches of mathematics, with differential equations being no exception. The goal of a transform function is to transform a function into a different function space that could result in easier operations or less complicated solutions. In the case of frequency analysis, this would be the Fourier transform. The Fourier transform is what transforms periodic acoustic functions from a time-based domain into a domain centered around frequency intensity (Pierce).

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i \xi t} dt \rightarrow \hat{f}(\omega_f) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega_f t} dt,\ \omega_f = 2\pi\xi,\ \xi \in \mathbb{R}$$

Often the substitution of ω_f is used to denote angular frequency in radians per second. There are several complicated components that exist within the Fourier transform, and the methods of the Fourier transform are rather non-intuitive. The Fourier transform uses complex values in order to transfer from periodic motion to frequency. As previously mentioned, $e^{i\theta}$ creates a circle within the complex plane of radius 1 (Brown & Churchill). The inclusion of some time value t represents where the point on the unit circle would lie at some arbitrary time, while ξ represents the rate at which the point moves around the circle. The goal of the Fourier transform is to analyze a function of time and transform it to a function of frequency. The relevance of including the frequency ξ within the expression is to represent the rate at which the point moves on the unit circle over some time t. On its own, the expression $e^{-2\pi i\xi t}$ represents where at some time *t* a point will lie on the unit circle based on some frequency ξ. When the time-dependent pressure function $f(t)$ is added, the expression $e^{-2\pi i\xi t}$ maps the wave function $f(t)$ around the complex unit circle, with higher intensities reaching values close to or above 1 (Pierce).



Pictured above: the graph of $f(t) \cdot e^{-2\pi i\xi t}$, where $f(t) = sin(2\pi * 20t)$ and $\xi = 45\ Hz$

The goal is to find the strength of the frequency ξ among the function $f(x)$, meaning a sampling of points of the function is necessary to find where it is the strongest. This process can be done with either an increasing sample of points discretely or done continuously across the entire function, which is where the derivation for the Fourier transform comes from (Brown & Churchill).

$$\lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} f(t_j) e^{-2\pi i\xi t_j} = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i\xi t} dt$$

An important caveat about the transition from summation to integration is that the number of points N becomes infinity, therefore the expression $\frac{1}{N}$ becomes dt. This results in the final form of the Fourier transform (Pierce).

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i \xi t} dt$$

The goal of the Fourier transform is to analyze the strength of the frequency over a period of time, or in this case, the frequency over an infinite amount of time to see all frequencies. It is reasonable to conclude that the frequencies will be the strongest at the frequencies of the summed waves in $f(x)$. While this is true, suppose the input function was a complex sound wave where the summed waves were unknown. In order to figure out which waves composed the sound, the Fourier transform can be used to determine points where the intensity of a frequency is the strongest. Through this process, strong frequencies can be determined and the simple waves composing the complex wave of various frequencies could be expanded and unscrambled (McCarthy). This equation is how spectral processors function and reasonably determine frequency intensity based on a signal captured by a microphone.

Temporal Properties

Reverb is a simple concept that is unusual to define. An oversimplification of reverb is the amount of time a sound takes to disappear, but that definition skips many key factors in defining reverb. To comprehend reverb, it is important to understand reflections and time of arrival. When sounds are emitted from a source, it emits in multiple directions rather than just a straight line. When sounds reflect off of surfaces and arrive at a given location, they are categorized under reflections. These reflections are further split into early reflections and late reflections (Berg & Stork). Early reflections are defined as reflected sounds that arrive at a location at a time at or before 30ms, while late reflections are those that arrive at the same location at a time greater than 30ms. The intensity of these early and late reflections causes the sound to have a perceived "decay" rather than abruptly ending (Everest). Another concept often explored with acoustics and reverb is RT60, also known as "reverb time 60 (dB)" (Thompson). The concept of RT60 is defined as the amount of time that a sound takes to reduce by 60dB in level, and is often used in current calculations of sound absorption and acoustic treatment. One of the most widely used methods to calculate the RT60 of a room is the Sabine equation, using dimensions and absorptivity of the objects in the room to determine how fast the sound will decay (Beranek).

$$RT_{60} = \frac{.049 \cdot V}{S\bar{a}} \text{ (feet)}$$

$$RT_{60} = \frac{.161 \cdot V}{S\bar{a}} \text{ (meters)}$$

$$S\bar{a} = \sum_{i=1}^{n} s_n a_n$$

In both of these cases, V in both of these cases represents the volumes of the rooms, s_n represents the boundaries for a given object with absorption coefficient a_n, where $n \in \mathbb{Z}^+$ and represents the number of different objects in a given space. Absorption coefficients are static values inherent to materials that represent the effectiveness of a material for absorbing sound, found by the ratio of incident energy to absorbed energy. Absorption coefficients appear frequently in both simple and complicated calculations related to reverb and diffusion, while their values can describe rates and effectiveness of sound energy absorption (Everest). This formula stands on its own as a sufficient measure of RT60 but describes little about what is happening at the particle level.

As previously mentioned, sound is defined as a pressure wave. A common measurement of sound comes in the form of Sound Pressure Level, also shortened to SPL. SPL is considered to be an objective measurement of sound intensity by measuring the deviations of air pressure caused by sound waves to a reference pressure, defined to be $2 \cdot 10^{-5}$ Pa. The ratio of the pressure is calculated then converted to a logarithmic scale. The general SPL equation takes the form (Berg & Stork):

$$dB_{SPL} = 20 \cdot log_{10}(\frac{P_m}{P_R})$$

Where P_m is the measured pressure and P_R is the reference pressure of $2 \cdot 10^{-5}$Pa. This ratio is used in a multitude of different areas for acoustics and functions great for measurement purposes, but only measures deviations or SPL level at specific discrete points rather than across a region.

Results

Derivation of the Wave Equation

The implications of the linearized Navier-Stokes equation that serves as the foundation of acoustic theory is the basis for further exploration for the acoustic field. Despite this foundation, more information can be found if the propagation of energy is considered to be uniform. By doing this, it is assumed that the field has irrotational flow. By definition of irrotational flow, $\nabla \times v = 0$ and there exists a potential function V such that $v = \nabla V$, or in this case, $v = - \nabla V$.

When substituting these values into the two acoustic field equations, it results in two systems of partial differential equations (Rival):

$$\frac{\partial p}{\partial t} - \rho_0 c^2 \cdot \nabla^2 V = 0$$

$$-\nabla \frac{\partial V}{\partial t} + \frac{1}{\rho_0} \nabla p = 0$$

The bottom equation can be solved in terms of p by performing operations on each side to yield an equation in terms of p. The importance of this allows for substitution back into the first equation and subsequent solutions to this second-order space and time partial differential equation (Fetter & Walecka):

$$-\nabla \frac{\partial V}{\partial t} + \frac{1}{\rho_0} \nabla p = 0 \rightarrow \frac{1}{\rho_0} \nabla p = \nabla \frac{\partial V}{\partial t}$$

$$\nabla p = \rho_0 \nabla \frac{\partial V}{\partial t}$$

$$p = \rho_0 \frac{\partial V}{\partial t}$$

This finding states that the pressure perturbation is proportional to the change in velocity potential over time and the density of the initial medium. As previously stated, this result can be utilized for $\frac{\partial p}{\partial t}$, now having an illustration of how the pressure behaves in relation to other values.

$$\rho_0 \frac{\partial^2 V}{\partial t^2} - \rho_0 c^2 \cdot \nabla^2 V = 0 \rightarrow \frac{1}{c^2} \cdot \frac{\partial^2 V}{\partial t^2} - \nabla^2 V = 0$$

$$\frac{1}{c^2} \cdot \frac{\partial^2 V}{\partial t^2} = \nabla^2 V$$

This outcome has important implications in the form of the wave equation. The wave equation is a standardized model of acoustic sound movement and governs acoustic wave movement through a medium. From this finding, it illustrates that the energy potential function obeys the laws of the wave equation under linearizations and assumptions of the fluid medium.

Vorticity Diffusion

Having seen these cases where irrotational flow has been assumed, it is important to see what would happen when irrotational flow cannot be assumed. Due to viscosity being a measure of deformation, the value is represented as a stress tensor of a fluid. In order to analyze these effects, an equation for changes fluid momentum over some arbitrary volume V can be written. The change must be proportional to the dissipative effects of pressure forces and deformation of the fluid itself. The result defines this momentum within a material volume (Thevenin & Jamiga).

$$\iiint_V \rho \cdot \left(\frac{\partial u}{\partial t} + (\nabla \cdot u) \cdot u\right) dV = \iiint_V \rho \cdot g \, dV + \iint_A \sigma_{ij} \, dA$$

$$\iiint_V \rho \cdot \left(\frac{Du}{Dt}\right) dV = \iiint_V [\rho \cdot g + \nabla(p \cdot \delta_{ij})] \, dV + \iiint_V \mu \cdot \nabla\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)] \, dV$$

$$\iiint_V [\rho \cdot \left(\frac{Du}{Dt}\right) - \rho \cdot g + \nabla(p \cdot \delta_{ij}) - \mu \cdot \nabla\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)] \, dV = 0$$

The significance of having the entire integral equate to zero has implications about the state of what is being integrated. If V represents any arbitrary material volume, but always equals zero, then the integrated volume must be zero at every point within the defined space.

$$\rho \cdot \left(\frac{Du}{Dt}\right) - \rho \cdot g + \nabla(p \cdot \delta_{ij}) - \mu \cdot \nabla\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = 0$$

Rearranging the equation results in a partial differential equation that represents momentum conservation of viscous flow. Further simplifications and mathematical operators can be performed to alleviate the notation and transform the stress tensors (Merrill).

$$\rho \cdot \left(\frac{Du}{Dt}\right) = \rho \cdot g - \nabla(p \cdot \delta_{ij}) + \mu \cdot \nabla\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\rho \cdot \left(\frac{Du}{Dt}\right) = \rho \cdot g - \nabla p + \mu \cdot \left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_i^2}\right)$$

$$\rho \cdot \left(\frac{Du}{Dt}\right) = \rho \cdot g - \nabla p + \mu \nabla^2 u$$

At this point, the momentum conservation equation within a given material has been established with the current simplification. This equation can be taken further when the material derivative is expanded and solved for the change in velocity over time.

$$\rho \cdot \left(\frac{\partial u}{\partial t} + (\nabla \cdot u) \cdot u\right) = (\rho \cdot g - \nabla p) + \mu \nabla^2 u$$

$$\rho \cdot \frac{\partial u}{\partial t} = - \nabla p + (\rho \cdot g) - \rho(\nabla \cdot u) \cdot u + \mu \nabla^2 u$$

Having established the momentum conservation with the above simplification, considering incompressible flow allows for more simplifications. In the case of incompressible flow, continuity is written as $\nabla \cdot u = 0$, stating that nothing is leaving or entering the system. The previous equation can then be further simplified as (Pierce):

$$\rho \cdot \frac{\partial u}{\partial t} = - \nabla p + (\rho \cdot g) + \mu \nabla^2 u$$

This equation is often seen as the simplest form of the Navier-Stokes equation. However, one more additional modification can be made. It comes in the form of vorticity. If the flow is no longer considered irrotational, then ω would have to exist and abide by some form of motion. In order to achieve this, the curl of both sides can be taken. The value ρ has been accounted for, isolating $\frac{\partial u}{\partial t}$ to the left hand side.

$$\nabla \times \frac{\partial u}{\partial t} = \nabla \times [-\ \nabla p \cdot \frac{1}{\rho} + \frac{1}{\rho} \cdot (\rho \cdot g) + \frac{1}{\rho} \cdot \ \mu \nabla^2 u]$$

$$\nabla \times \frac{\partial u}{\partial t} = \nabla \times [-\ \nabla p \cdot \frac{1}{\rho} + \frac{1}{\rho} \cdot (\rho \cdot g)] + \frac{1}{\rho} \cdot \nabla \times \ \mu \nabla^2 u$$

The effect that curl has on each set of values must be considered. Operators involving derivatives and curl can be rearranged in such a way that the curl of a time-dependent partial derivative is the same as taking the partial derivative with respect to time of the curl of a vector. A similar operation can be performed with the curl of a Laplacian operator.

$$\nabla \times \frac{\partial u}{\partial t} \rightarrow \frac{\partial}{\partial t}(\nabla \times u) \rightarrow \frac{\partial \omega}{\partial t}$$

$$\nabla \times \nabla^2 u \rightarrow \nabla^2(\nabla \times u) \rightarrow \nabla^2 \omega$$

The final values to consider are those of the pressure change and density. These values all represent constant values; therefore, their curl is 0. Pressure is not a function of density change provided that the density is uniform.

$$\frac{\partial \omega}{\partial t} = (\frac{\mu}{\rho}) \cdot \nabla^2 \omega$$

After all of the operations are finished, the result is an equation with identical form to the Diffusion equation in one dimension, with $\frac{\mu}{\rho}$ acting as the analogue to the diffusion coefficient.

Thermodynamics

Identifying the full effects of temperature on acoustics requires the equation of energy to be written in terms of the acoustic field and linearized in similar fashion to the original Navier-Stokes equations (Peyret & Taylor).

$$\rho_0 \cdot \frac{\partial u}{\partial t} + p_0(\nabla \cdot v) - k \cdot \nabla^2 T = 0$$

When dealing with concepts of thermodynamics, it is important to remember that energy density and pressure are both functions of density and temperature. When analyzing how these functions change over time, the change in both density and temperature must also be analyzed. Both of these derivations are written below, both of which will be analyzed later.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \rho} \cdot \frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial T} \cdot \frac{\partial T}{\partial t}$$

$$\frac{dp}{dt} = \frac{\partial p}{\partial \rho} \cdot \frac{\partial \rho}{\partial t} + \frac{\partial p}{\partial T} \cdot \frac{\partial T}{\partial t}$$

Two important terms are crucial for the next set of operations. In thermodynamics, a process is termed “adiabatic” if a process occurs and heat is not exchanged during the process. Thermal expansion is the tendency at which an object or fluid changes shape, volume, and density in response to a change in temperature. Thermal expansion plays a critical role in determining the compressibility of a fluid. The thermal expansion coefficient is defined as the change in density due to temperature multiplied by the negative reciprocal of the density unaffected by temperature (Thevenin & Jamiga).

$$\vartheta = - \frac{1}{\rho_0} \cdot \frac{\partial \rho}{\partial T}$$

$$\frac{\partial \rho}{\partial T} = - \rho_0 \vartheta$$

$$\frac{\partial p}{\partial \rho} = \frac{v_q^2}{\gamma}, \gamma = \frac{C_p}{C_v}$$

In this formula, γ is defined as the ratio of specific heat. The term C_p is defined as the specific heat at a constant pressure, while C_v is defined as the specific heat at a constant volume. An important fact is that within any fluid, the ratio of specific heat defines the speed of sound in a system. This is evident based on the definition of $\frac{\partial p}{\partial \rho}$ (Thevenin & Jamiga; Van Wylen & Sonntag).

$$\frac{\partial u}{\partial T} = C_v$$

$$\frac{\partial p}{\partial T} = \frac{\partial p}{\partial \rho} \cdot \left(-\frac{\partial \rho}{\partial T}\right) = \frac{v_q^2}{\gamma} \cdot \rho_0 \vartheta = \frac{v_q^2 \rho_0 \vartheta}{\gamma}$$

In order to solve the equation involving $\frac{\partial u}{\partial t}$, the corresponding values that compose $\frac{\partial u}{\partial t}$ must be solved first, giving the following relation (Epstein & Carhart).

$$\rho_0^2 \cdot \frac{\partial u}{\partial \rho} = p_0 - T_0 \cdot \frac{\partial p}{\partial T} \rightarrow \rho_0^2 \cdot \frac{\partial u}{\partial \rho} = p_0 - T_0\left(\frac{v_q^2 \rho_0 \vartheta}{\gamma}\right)$$

Returning to the previously mentioned equations, the known values can now be substituted for $\frac{\partial u}{\partial t}$ values and for the energy conservation equation.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \rho} \cdot \frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial T} \cdot \frac{\partial T}{\partial t}$$

$$\rho_0 \cdot \frac{\partial u}{\partial t} + p_0(\nabla \cdot v) - k \cdot \nabla^2 T = 0 \rightarrow \rho_0 \cdot \frac{\partial u}{\partial t} = - p_0(\nabla \cdot v) + k \cdot \nabla^2 T$$

Proper substitutions can be made based on what was determined to result in a linearized form of the energy equation. Since $\frac{\partial u}{\partial t}$ has been defined, the substituted values can be added into the equation for energy. The proper substitution values are listed below, as well as the resulting equations that follow from them (Van Wylen & Sonntag).

$$\frac{\partial u}{\partial T} = C_v, \frac{\partial \rho}{\partial T} = \rho_0 \vartheta$$

Now that the proper values have been substituted for $\frac{\partial u}{\partial t}$, the energy conservation equation can be updated by the values represented. Before this occurs, the value for $\frac{\partial \rho}{\partial t}$ must be accounted for. Returning to the conservation equation for the acoustic field, $\frac{\partial \rho}{\partial t}$ can be solved for proper substitution.

$$\frac{\partial \rho}{\partial t} + \rho_0 \nabla \cdot v = 0$$

$$\frac{\partial\rho}{\partial t} =- \rho_0 \nabla \cdot v$$

Now that all of these values have been appropriately covered, $\frac{\partial u}{\partial t}$ can properly be substituted. The first equation represents the fully substituted values, while the second and third equations are there to give proper form.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \rho} \cdot (-\rho_0 \nabla \cdot v) + p_0 \cdot (\nabla v) + C_v \cdot \frac{\partial T}{\partial t}$$

$$\frac{\partial u}{\partial t} =- \rho_0 (\frac{\partial u}{\partial \rho} \cdot \nabla \cdot v) + p_0 \cdot (\nabla v) + C_v \frac{\partial T}{\partial t}$$

In order to properly finish this equation, $\frac{\partial u}{\partial \rho}$ must be represented in the form of a non-partial derivative. Using the substitution that was previously explored and multiplying by density, the derivation can be solved.

$$\rho_0 \cdot (-\rho_0 \frac{\partial u}{\partial \rho} \nabla \cdot v) + C_v \frac{\partial T}{\partial t} + p_0 \cdot (\nabla v) - k \cdot \nabla^2 T = 0$$

$$-(\rho_0^{\ 2} \cdot \frac{\partial u}{\partial \rho}) \nabla \cdot v + C_v \frac{\partial T}{\partial t} + p_0 \cdot (\nabla v) - k \cdot \nabla^2 T = 0$$

$$(-(p_0 - T_0 (\frac{v_q^{\ 2} \rho_0 \vartheta}{\gamma})) \nabla \cdot v + C_v \frac{\partial T}{\partial t}) + p_0 \cdot (\nabla v) - k \cdot \nabla^2 T = 0$$

$$(\nabla \cdot v)(-p_0 + T_0 (\frac{v_q^{\ 2} \rho_0 \vartheta}{\gamma}) + p_0) + C_v \frac{\partial T}{\partial t} - k \cdot \nabla^2 T = 0$$

$$(\nabla \cdot v)(T_0 \cdot \frac{v_q^{\ 2} \rho_0 \vartheta}{\gamma}) + C_v \frac{\partial T}{\partial t} - k \cdot \nabla^2 T = 0$$

$$(\nabla \cdot v)(T_0 \cdot \frac{v_q^{\ 2} \rho_0 \vartheta}{\gamma}) = k \cdot \nabla^2 T - C_v \frac{\partial T}{\partial t}$$

$$\nabla \cdot v = \frac{\gamma k}{T_0 v_q^{\ 2} \rho_0 \vartheta} (\nabla^2 T - \frac{\partial u}{\partial T} \cdot \frac{\partial T}{\partial t})$$

$$\nabla \cdot v = \frac{k}{T_0} \cdot \frac{\partial T}{\partial \rho} ((\nabla^2 T) - \frac{\partial u}{\partial t})$$

In the equation above, it states that the divergence of the sound velocity is proportional the change in temperature with respect to density, multiplied by how the temperature changes with respect to space, while removing how the temperature changes over time.

Spectral Properties

The Fourier Transform can be applied to the pressure waves that propagate from a given source to determine at which frequencies the pressure wave is the strongest (Pierce).

$$\widehat{p}(\xi) = \int_{-\infty}^{\infty} p(t) \cdot e^{-2\pi i\xi t} dt$$

In the case of $p(x)$ being a summation of any arbitrary number of sinusoidal functions. Any complex value consists of two components: the real and imaginary components. When analyzing the results of the Fourier Transform it is important to look at the real-valued solutions in order to extract information about frequency intensity. Imaginary solutions have no importance here, as frequency intensity is concerned with only the real-valued solutions of the Fourier Transform (Kierkegaard; Pierce).

$$p = Re\{\widehat{p} \cdot e^{-2\pi i\xi t}\}$$

Where the term Re represents the real valued parts, and $\widehat{p}$ represents the intensity of acoustic pressure in terms of frequency. In addition, it is possible to take the Fourier Transform of a velocity function as well, resulting in a value $\widehat{v}(\xi)$. This describes frequency intensity based on the velocity. Similar to the acoustic pressure, the interest in what the Fourier Transform of velocity describes is represented only by the real-valued solutions of $\widehat{v}$ (Pierce).

$$v = Re\{\widehat{v} \cdot e^{-2\pi i\xi t}\}$$

The idea of a velocity function having a frequency seems unusual since velocity usually is not a periodic function, the definition of $\hat{v}$ serves as a function for v in the complex-valued function space. The representations of both v and p act as real-valued time-dependent analogues to the real-valued frequency-dependent values that are produced by the Fourier Transform. Returning back at the linearized acoustic forms of the Navier-Stokes equations, both velocity and pressure are now defined with values (Pierce). Despite this, the time-dependent partial derivatives have not yet been defined. To define the complex-valued analogue, the operation $\frac{\partial}{\partial t}$ will be treated as the input argument for $f(t)$.Through this process, the partial derivative can be taken inside the Fourier Transform, and an expression to define $\frac{\partial}{\partial t}$ can be found (Brown & Churchill; Kierkegaard).

$$\int_{-\infty}^{\infty} f(t) \cdot e^{-2\pi i\xi t} dt = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \cdot e^{-2\pi i\xi t} dt$$

$$\frac{\partial}{\partial t} \cdot e^{-2\pi i\xi t} =- 2\pi i\xi \cdot e^{-2\pi i\xi t}$$

$$\frac{\partial}{\partial t} =- 2\pi i\xi$$

Given what has been found, starting from the Fourier Transforms of both pressure and velocity, the linearized Navier-Stokes equations is as follows:

$$\hat{p}(\xi) = \int_{-\infty}^{\infty} p(t) \cdot e^{-2\pi i\xi t} dt$$

$$\hat{v}(\xi) = \int_{-\infty}^{\infty} v(t) \cdot e^{-2\pi i\xi t} dt$$

Converting the respected values for both pressure, velocity, and $\frac{\partial}{\partial t}$, updated forms of the Navier-Stokes equations for frequency are produced.

$$\frac{\partial p}{\partial t} + \rho_0 c^2 \cdot \nabla \cdot v = 0 \rightarrow -\ 2\pi i\xi \cdot \widehat{p} + \rho_0 c^2 \cdot \nabla \cdot \widehat{v} = 0$$

$$\frac{\partial v}{\partial t} + \frac{1}{\rho_0}\nabla p = 0 \rightarrow -\ 2\pi i\xi \cdot \widehat{v} + \frac{1}{\rho_0}\nabla\widehat{p} = 0$$

These two forms are significant in that the equations are no longer time-dependent, and instead are frequency dependent representations of acoustic movement.

Diffusion Models

To account for the problems found within the diffusion models for sound dissipation, values must be subtracted from the changes in sound energy density. The reasoning is that sound energy is not only changing, it is also dissipating. In regards to absorption in the environment not caused by air, a mean absorption coefficient is taken by the sum of all absorption coefficients divided by the mean free path. Taking all of this into consideration, a potential new model for the diffusion of sound in a simple closed system can be modeled (Navarro & Escolano).

$$\frac{\partial w}{\partial t} = D \cdot \nabla^2 w(r,t) - \alpha c \cdot w(r,t) - C_a c \cdot w(r,t)$$

$$\alpha = \frac{\bar{a}}{\Lambda} \rightarrow \alpha = \frac{\bar{a} \cdot S}{4V}$$

In this equation, C_a represents the absorption of air and c is the speed of sound. This equation illustrates loss over a unit volume, something initially unaccounted for. In order to resolve this, a function of the sound energy per unit volume must be added, acting as an "initial value" from which sound can begin dissipating.

$$\frac{\partial w}{\partial t} = D \cdot \nabla^2 w(r,t) - \alpha c \cdot w(r,t) - C_a c \cdot w(r,t) + E_0(r,t)$$

With the addition of energy per unit volume $E_0(r,t)$, the acoustic diffusion equation is properly modified. The concept behind this modified partial differential equation is that the change in sound energy density over time does not simply reach equal energy dispersion, but loses energy as time continues (Jing & Xiang). This is via absorption of energy from air and boundaries in a closed system, such as a room, closed venue, or studio. It further states that this absorption is dependent on the properties of the closed system, like volume of and materials present in the room.

It is important to address the case in which D, as the diffusion coefficient is not a constant value. Possible solutions to this problem have been theorized and presented, but have resulted in additional questions or inconsistencies (Muñoz, Navarro-Ruiz, & Hornikx). One possible solution is a wave-addition-based method in which the values for D can be taken at discrete points. The values would result in a function $D(\bar{r})$ that predicted the diffusion coefficient at different regions in several dimensions (Navarro, Escolano, & López). Additional assumptions would have to be made, such as if changes in D is minimal and sound decay is smooth.

$$\frac{\partial w}{\partial t} = \nabla(D(w,r)\cdot\nabla w(r,t)) - \alpha c\cdot w(r,t) - C_a c\cdot w(r,t) + E_0(r,t)$$

Analyzing the gradient of the diffusion function in this context is valid since it describes how diffusion of energy density would change with respect to space. However, it must be noted that this is based on the original diffusion equation, and is simply a conjecture as to what the modification for a diffusion function would look like.

Based on potential solutions to this diffusion equation model, a new measurement for looking at a sound pressure levels based on this diffusion model arises:

$$SPL(r,t) = 20\cdot log_{10}\left(\frac{w(r,t)\cdot\rho c^2}{P^2}\right)$$

Under the assumption that real valued solutions exist, ρc^2 represents the bulk modulus of the medium, and P^2 represent the static sound pressure, acting as a reference pressure level in which no disturbances occur.

Discussion

The mathematics that describes Acoustic Theory has proven to be incredibly complex, but it is important to discuss how it relates to other properties of sound as well. Starting from the linearized Navier-Stokes equation, the mathematics that governs fluid flow eventually leads to something that governs sound, which in this case, is the wave equation. In the derivation of the wave equation, the potential function of some velocity function has been shown to follow behaviors similar to those of standing waves. This illustrates a link to wave behavior that comes from the equations of the acoustic field under constraints of irrotational flow. The derivation of the wave equation from the linearized Navier-Stokes equation shows the mathematical backing of the acoustic theory itself.

The theory-based links between the Navier-Stokes equations and vorticity of a sound field resulting in a diffusion-like model show interest in how this model of vorticity can affect the sound field. Through this process, the diffusion equation was able to be related to the Navier-Stokes equation under the constraint that rotational flow occurs. What resulted is an idea that any kind of rotation within the fluid field would diffuse throughout the field, traveling from high-density levels to lower density levels. Implications of what this could mean for acoustics is a topic in which further research could be considered. What is significant, however, is that diffusion happens under constraints of the Navier-Stokes, linking the two equations together.

This link can act as a reasonable implication that acoustic movement could obey models of the diffusion equation, something that would provide a crucial bridge between these two models.

The effect of temperature on sound movement is incredibly complex, but an attempt toward defined descriptions of the original equations can be made. The Navier-Stokes equation of energy conservation through a fluid field represents how energy is preserved through the acoustic field with respect to thermal expansion and temperature. The derivation of finding the divergence of sound with respect to temperature and density helps illustrate how sound is affected by thermodynamic properties but remains a question in terms of its significance. Comprehensively, this model would work well for closed-system areas, such as studios or inside venues. Open systems, including open-air venues, are more complicated to handle, as there is no definitive equation or method to determine how the flow will move due to an indeterminable number of external factors. Of biggest importance is that temperature is not something that can simply be factored out when evaluating acoustic movement, as thermal dissipation and the effects of temperature on sound contribute to changes in sound movement.

Applications on how temperature affects sound tend to appear heavily in system placement, tuning, and mixing. Much of these applications tend to be highly focused on predictive and preventative measures since the temperature is constantly changing, especially in large or open venues. Live sound engineers often use the basic principles of how sound changes due to temperature, rather than looking at properties of thermal expansion or specific heat. This is generally acceptable, as the specificities of topics like thermal expansion and dissipation become infinitely more complex depending on the openness of an environment. These assumptions further exist within current prediction software for live sound engineers as well. Most softwares utilize the assumption that the air temperature does not change based on location, and that uniform temperature exists throughout a given volume. Even though the software tends to result in decently accurate predictions, this oversimplification of an environment is incredibly unlikely to occur in the real world. However, it would not necessarily be impossible to consider simpler closed environments for these types of thermodynamic analyses. Closed systems with relatively small numbers of people could help determine both sound direction, speed, and travel with predictive software to higher degrees of accuracy. This could further extend into finding and accounting for anticipated changes in this process. As environments become more open, like large outdoor venues that require multiple sets of speaker arrays, these predictions become much, much more difficult and less accurate. Atmospheric temperature and pressure changes happen constantly, and it is known that engineers may have to make changes to system properties to account for these changes. Possible solutions to open venues would be evaluating the divergence of sound, given all of the accountable factors that exist.

A more simplified potential solution to this problem could be to treat an open environment like a closed environment and evaluate the characteristics as an individual point in time. Evaluating multiple of these individual points in time can result in a set of these points that illustrate the changes in the environment. Based on this, realistic changes that can be made could be implemented as time continues. This method would result in an average of what changes over time or a more accurate “best fit” for the environment. This may be the best option given the current knowledge base but is more heavily dependent on more uncontrollable factors. It is important to recognize that as the field of acoustic theory advances, the effects of thermodynamics may be more accurately accounted for in new theories and models.

The Fourier Transforms in relation to the linearized Navier-Stokes equation have illustrated connections to the frequency domain and the propagation of sound. What results from this are relationships between frequency strength, pressure change, and spatial dependence that results in models of frequency strength from acoustic movement. Current testing and simulations are being made; but most of these simulations and computations have been restricted to two dimensions due to limitations of computational ability. In addition, the current focus on using and testing these equations involves looking into ducts and industrial applications, rather than a production setting. Despite this, as more research continues, the ability to extrapolate from ducts to full-sized venues is not unreasonable, and would allow additional strides with acoustic tuning to be made. These would include frequency strength within boundary conditions, leading to predictions in the form of acoustic dampeners within modeled rooms. Furthermore, by modeling defined signals in enclosed spaces or venues, frequency strength at given locations based on an input signal would be possible to map. The result would be more accurate predictions for acoustic tuning. The previously stated limitations of the Navier-Stokes equation carry over into the time-independent frequency domain. Gaps in the current model, therefore, cause additional gaps in the frequency domain in the form of transformations of the time-dependent gaps having unknown implications in the frequency-dependent domain. This implies that what the gaps represent will not be known until they are filled, thereby only being able to evaluate these implications in retrospect. Despite this, research is still being conducted on how well the model fits (Kierkegaard). Currently, this representation of a frequency-dependent Navier-Stokes equation is not commonly found since frequency generally does not occur in fluid flow as it does for sound. Despite this, it could lead to the representation of frequency change by the equations serving as the models for acoustic theory.

Research on the use of the diffusion equation has illustrated definitive links between sound propagation and diffusion in relatively-controlled environments. Current research focuses on urban acoustics and industrial applications, including soundproofing in architectural applications and loudness control between buildings and streets. As previously mentioned, the current diffusion model fails for outdoor solutions and incredibly large rooms due to non-constant diffusion coefficients as the sound continues to move. Current research focuses on this aspect of modifying and creating a diffusion equation that works for both of these applications. In terms of the proposed equation, having a non-homogeneous diffusion coefficient would result in changes also present in the original diffusion equations. Similar to the situation with the Navier-Stokes equation, research on acoustics and the diffusion equation include gaps in current knowledge, specifically regarding the previously mentioned non-constant diffusion factors. Current estimations on calculations focus on statistical methods for predicting spatial-dependent diffusion coefficients. While this has worked with relative degrees of success, testing is limited and inconssistencies in calculations have currently rendered these methods inconclusive.

In terms of smaller environments like smaller venues, the acoustic diffusion equation could work as an accurate model for acoustic treatment, given known absorption coefficients in the room. Based on these factors, sound pressure level estimations could be made within the environment based on these findings. Treatment elements could be placed in rooms based on these predictions. Returning back to the shortcomings of prediction software, they often do not take into account reflections and interference patterns of sound. This results in sharp cutoffs where surfances should be, acting as a boundary rather than a place for reflection and absorption. These results fail considerably in real-world applications, as sound does not simply disappear once a boundary is reached, while further interferes in the form of reflections. Using this diffusion model, the issues with boundary cutoffs would not be an issue, as the diffusion method would illustrate how sound diffuses within the room itself while accounting for the surfaces themselves. In the case of outdoor venues, the focus may need to be less on acoustical treatment with more focus on system design and optimization. Audio equipment characteristics could be more carefully selected based on the constraints of the venue. Optimizing proper dispersion and adequate coverage while constrained to a budget helps not only for engineers but also for venues and concertgoers.

The link between the microscopic and macroscopic perspectives of using the diffusion equation could exist based on the sound pressure level equation. Solutions to this modified diffusion equation would come with the creation of boundary conditions for the sound particles, and result in a function $w(r, t)$ to describe the scattering of these particles. The result of this function would model the energy levels from a sound source as a function of time given a set of constraints. Furthermore, this can be converted to a $SPL(r, t)$ function, and systems within rectangular rooms could be calculated, as well as finding sound decay values.

The most important topic to evaluate is the background of the Navier-Stokes equation itself. The Navier-Stokes equation is indeed an unsolvable problem. Despite this reputation, the Navier-Stokes equation is capable of producing limited time-dependent results of laminar flow in two and three dimensions. Turbulent flow is one of the biggest questions that physicists are trying to answer today about the Navier-Stokes equation, as nonlinear flow is unaccounted for in the equation. The significance of this question is how reflecting sound could create some form of turbulence within the fluid medium. If this were true, it would severely complicate acoustic theory and would rely on solutions of the turbulent flow for general fluid movement first before additional progress can be made. In the future, solutions to the Navier-Stokes equation may be found, which could give greater insight into the complexity of sound movement. Given both of these outcomes, solutions that either ratify or disprove elements of acoustic theory still serve to further the understanding of acoustics. Positive solutions can serve to further concepts within the fluid-based sound model; elements that are reasonably disproved expand knowledge of what does not work, allowing for revisions and further conjectures to be made. This is not to say that the basis for acoustic theory is flawed, but rather that the foundation is unfinished. Answers to current questions can be revealed by what is currently unknown with future research.

Several questions continue unresolved: how do all of these concepts come together into one cohesive theory? The goal of acoustic theory is to unify topics across all aspects of sound on both a microscopic and macroscopic level. The acoustic field equations represent how the flow of the sound moves, which relates down to the particle level. The diffusion equation also represents particles of sound diffusing within the sound field, but with more of a focus on the dispersion of the particles rather than their motion. Both of these relate to the particle level. The Fourier transform of the Navier-Stokes equation represents both the oscillation of pressure and flow of the frequencies, relating to both particle states and what people can hear within the frequency spectrum. The disconnect comes in the form of describing the singular particle theory and extrapolating upwards from single particles to multiple particles, then from multiple particles to all particles. The diffusion equation model relating to sound pressure level attempts to combine these, but has not been fully successful. The more research that is performed on these aspects, the closer a more accurate and inclusive acoustic theory becomes. Conversely, it is also plausible that the term "acoustic theory" might shift from describing the current macro-aspects of acoustics, to becoming a term that describes a branch of acoustical physics. Another transformation could be to have math and models that govern the motion of the particles resulting in equations and models of interaction with other objects within the parameters of motion. This process could also lead to a set of "Navier-Stokes-Diffusion" equations for the acoustic process of diffusion within a medium. Not only could this result in the advancement of acoustic theory, but it would also have implications within the broader field of fluid dynamics. The new model could potentially be able to describe motion while also describing interference and "turbulence" of the particle motion within the fluid. Other possibilities could result in essentially splitting acoustic theory into two different aspects: acoustic movement and acoustic interaction. Acoustic movement would be governed by the previously found Navier-Stokes equations of acoustic movement, while the interactions of sound would be governed by the diffusion equations.

From a theoretical perspective, the continued research of acoustic theory in math and physics is a necessity, but the underlying question of how acoustic theory can be applied by audio engineers and sound engineers remains. Aspects of live sound have been addressed, but mostly in a predictive sense. Given the current knowledge of acoustic theory, it would be difficult to describe every facet of what this kind of software could hold. Of key importance would be possible different methods of describing acoustic movement. Additionally, applications of temperature and pressure could be used to generate predictions of flow with reflections and diffusion. Other elements that this software could include might be frequency strength based on given parameters of a room, which would help immensely for adding absorption elements to help create a more even frequency distribution.

Moving away from solely the applications within predictive software, there are further applications of acoustic theory within system optimization. Tuning a sound system is to improve responsiveness within a room or venue. If frequencies are determined to be too strong in certain locations within the venue through the presented methods, and absorption panels are not viable options, then using spectral processors to eliminate those strong frequencies is a method of system tuning using this model, illustrating how these theories can be used in practice. Additional applications within system design would take the form of system creation itself. Directionality and placement of loudspeakers can affect the energy density of sound in a room. The concepts of beamforming and beam-steering are synonymous with line arrays, as the directionality of sound movement becomes more narrow with the strength of the beam that forms from the speaker arrays. As stronger beams are formed, sound energy density increases towards the center of the beam. Using the equations of motion can help with beam-steering and speaker placement to optimize coverage within a room. Taking into account everything from the thermal effects of an audience to how sound would diffuse within a complex environment, proper placement, angling, and tuning will yield better coverage and better listening experiences for the audience. This allows for better equipment selection, as having an understanding of optimal coverage results in proper equipment selection. By using equipment that properly fits these parameters, it can result in more cost-effective equipment selection, optimizing coverage under potential financial constraints.

To summarize, understanding what has been presented through the seemingly disjoint topics helps illustrate the current state of acoustic theory. The current models show significant promise in furthering the advancement of software and techniques used in production settings, but still remain inconclusive from a lack of answers. Ultimately, this all comes down to testing rooted in new mathematical equations and physics. If solutions can be found and connected through research, then significant changes to the world of production seem highly likely. Despite the lack of definitive connection, as research is ongoing, the hope for the development of a robust acoustic theory seems likely for the future.

Conclusion

Acoustic theory has shown to be a very complicated topic with considerable potential applications. The foundation of acoustic theory itself, however, has a dearth of information in general representation and special cases that create a substantial knowledge void. Its interdisciplinary nature undoubtedly serves as part of the reason acoustic theory appears so disjoint, as elements from several fields must interact and be represented cohesively. Theories that sit at the intersection of multiple fields are not only difficult from a physical standpoint but vary in the application within these distinctive fields. Multiple sections of acoustic theory focus on finding linear analogues to wave-based properties. The current theory is successful in this application. Additional aspects of acoustics, such as frequency, are time-independent whereas equations of motion and energy are not. Due to this, the transformation from a time-dependent to a time-independent representation requires an additional set of equations to be analyzed. This further complicates acoustic theory. Particle diffusion models have been studied, suggesting links between diffusion and sound movement, requiring yet another layer of calculations and equations to be included within the current theory. If links can be discovered providing support for both, it could potentially create additional support for model testing and ultimately prove or disprove acoustic theory in its current state. It also could conceivably increase the scope of the current theory to include both macro and micro acoustic influences. The current challenge with acoustic theory is linking particle movement to the scale at which sound is experienced. Current extrapolations within the diffusion model have been found for sound intensity but have yet gone untested. Frequency intensity based on acoustic movement has been hypothesized and tested, but ultimately was inconclusive and requires more scrutiny. Ultimately, the theory sits in a place of uncertainty due to current cracks and inconsistencies but leaves hope for the future once these become resolved.

The applications of acoustic theory stretch far beyond just general representations of theoretical physics. Acoustic theory would serve as the underlying physics for new types of predictive software that can be applied for both concert venues and general studio treatment. Through this software, proper system placement and tuning can be determined for concerts, as well as potential problems that could arise from these choices. Furthermore, techniques for combating issues within environments can be advanced, changing the methodology and current techniques of sound engineers in the industry in the process. Acoustic treatment would benefit from illustrations of problems and inconsistencies of sound propagation, determining placement for acoustical panels and diffusers to ensure uniform energy spread or decreasing frequency strength at certain locations. These applications also produce more cost-effective methods for both system equipment and treatment equipment. As these methods are further studied, more solutions and techniques for optimization in both equipment and action may appear for audio engineers. Currently, they serve as the framework for what would be the intersection of complex physics and practice for audio engineers. While currently there is no straightforward equation or rule for acoustic theory, the hope is that as time continues, establishing these foundations can hopefully be achieved.

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