The Spectra of Indefinite Singular Sturm–Liouville Operators

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Abstract

In this thesis we study the spectral properties of singular Sturm–Liouville differential operators of the form

$$Af = \frac{1}{r}\left(-(pf')' + qf\right)$$

with real-valued coefficients p, q and r, where the weight function r is indefinite. We present criteria guaranteeing the stability of the essential spectrum under perturbation with respect to the coefficients. Further, the accumulation of eigenvalues within gaps of the essential spectrum is studied. We show criteria which imply the finiteness or the accumulation of the point spectrum within a gap of the essential spectrum. The results are based on relative oscillation theory and the Floquet theory for periodic Sturm–Liouville problems. Moreover, we focus on the non-real spectra of indefinite Sturm–Liouville operators. We establish bounds on the absolute values and imaginary parts of the non-real eigenvalues. The verification of these bounds bases on a careful analysis of the corresponding eigenfunctions.

Zusammenfassung

In der vorliegenden Arbeit werden die spektralen Eigenschaften singulärer Sturm-Liouville-Differentialoperatoren der Form

$$Af = \frac{1}{r}\left(-(pf')' + qf\right)$$

mit reellwertigen Koeffizienten p, q and r untersucht. Hierbei betrachten wir indefinite Gewichtsfunktionen r. Basierend auf Erkenntnissen der relativen Oszillationstheorie sowie der Floquet-Theorie für periodische Sturm-Liouville-Operatoren werden Kriterien nachgewiesen, welche die Stabilität der essentiellen Spektren unter Störung der Koeffizienten sicherstellen. Außerdem wird die Häufung von Eigenwerten in den Lücken des essentiellen Spektrums untersucht. Wir formulieren Bedingungen, die eine Häufung der Eigenwerte innerhalb einer Lücke implizieren, bzw. eine Häufung ausschließen. Weiterhin werden die nichtreellen Spektren indefiniter Sturm-Liouville-Operatoren untersucht. Hierbei werden Schranken der nichtreellen Eigenwerte hinsichtlich ihres Absolutbetrages and Imaginärteils bestimmt. Der Nachweis der Schranken beruht auf einer gewissenhaften Analyse der zugehörigen Eigenfunktionen.

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Contents

Introduction

Second order differential equations of the form

$$\ell f = \lambda f \qquad \text{with } \ell := \frac{1}{r}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p\frac{\mathrm{d}}{\mathrm{d}x} + q\right) \text{ and } \lambda \in \mathbb{C}, \tag{1}$$

which were initially studied in a series of articles [81, 82, 97] by Sturm and Liouville, have many applications in mathematical physics. For instance, in the case $p = r \equiv 1$ the equation (1) corresponds to the one-dimensional stationary Schrödinger equation, which plays an important role in quantum mechanics. Hence, the problem (1) is intensively researched up to now and there is a quote by Zettl (in [56]) stating that the equation (1) is "the world's most popular differential equation".

The differential equation (1) is considered on an open interval $(a, b) \subset \mathbb{R}$, where suitable boundary conditions are imposed at the endpoints a and b. The coefficients p, q, r are assumed to be real-valued functions on (a, b) such that $1/p$, q, r are locally integrable on (a, b) and $p(x) > 0$ as well as $|r(x)| > 0$ for almost all $x \in (a, b)$.

If the weight function r is positive a. e. on (a, b), then the Sturm–Liouville differential expression ℓ in (1) gives rise to a family of self-adjoint operators, where the underlying Hilbert space is the weighted L^2-space $L^2((a, b), r)$ equipped with the scalar product

$$\langle f, g\rangle_r := \int_a^b f(t)\overline{g(t)}|r(t)|\,\mathrm{d}t, \qquad f,\, g \in L^2((a, b), r). \tag{2}$$

The spectral properties of the associated operators in this so-called *definite* case are comprehensively studied. For an overview we refer the reader to the monographs [11, 78, 86, 98, 100, 101, 102].

The main focus in this thesis is on the *indefinite* case, where the weight function r has sign changes in (a, b). More precisely it is assumed, in addition to $|r(x)| > 0$ for almost all $x \in (a, b)$, that the sets

$$\{x \in (a, b) \,|\, r(x) > 0\} \quad \text{and} \quad \{x \in (a, b) \,|\, r(x) < 0\} \tag{3}$$

have positive Lebesgue measure. Indefinite Sturm–Liouville differential equations arise in various problems in mathematical physics and quantum mechanics. One prominent application is the Camassa–Holm equation

$$u_t - u_{txx} = 2u_x u_{xx} - 3uu_x + uu_{xxx}, \qquad t \in (0, \infty), \quad x \in \mathbb{R}, \tag{4}$$

a non-linear partial differential equation which models the unidirectional wave propagation in shallow water with u representing the height of the water's free surface. An intriguing property of the Camassa–Holm equation is that it allows to describe the phenomenon of wave breaking. The Camassa–Holm equation (4) leads to a one-parameter family of Sturm–Liouville problems

$$\frac{1}{u - u_{xx}}\left(-f'' + \frac{1}{4}f\right) = \lambda f \tag{5}$$

on $\mathbb{R}$ of the form (1), where $p \equiv 1$, $q \equiv 1/4$ and $r = r(\,\cdot\,, t) = u(\,\cdot\,, t) - u_{xx}(\,\cdot\,, t)$ for $t > 0$, which arises as the isospectral problem in the Lax pair associated with the Camassa–Holm equation, see

[6, 29]. For more details see [33, 43, 44]. Other applications of indefinite Sturm–Liouville problems include transport theory and statistical physics, see e. g. [4, 5, 52, 57].

In the early 20th century Haupt [55] and Richardson [92] noticed that indefinite equations of the form (1), where self-adjoint boundary conditions are imposed at the endpoints, may have non-real eigenvalues; for more historical details we refer to the survey paper [84]. In contrast to the definite case, an operator associated with the indefinite differential expression ℓ in (1) subject to self-adjoint boundary conditions is not self-adjoint in the Hilbert space $L^2((a,b),r)$. Therefore, its spectrum may exceed the real line and it is even possible that the non-real spectrum accumulates, see e. g. [12, 13, 62, 63, 79]. In [76, 85] there are examples of Sturm–Liouville operators with empty resolvent set in the case where the coefficient p is allowed to be indefinite as well.

Spectral problems which arise in connection with the indefinite differential expression ℓ can be studied in a natural way in the context of self-adjoint operators in Krein spaces, cf. [36, 39]. Here the space $L^2((a,b),r)$ equipped with the inner product

$$[f,g]_r := \int_a^b f(t)\overline{g(t)}r(t)\,\mathrm{d}t, \qquad f,\, g \in L^2((a,b),r), \tag{6}$$

is a Krein space. The two inner products in (2) and (6) are connected via $[\cdot,\cdot]_r = \langle J\cdot,\cdot\rangle_r$, where J is the multiplication operator by the function $x \mapsto \operatorname{sgn}(r(x))$. Therefore, every self-adjoint realisation A of ℓ in the Krein space $(L^2((a,b),r),\ [\cdot,\cdot]_r)$,

$$Af := \ell f = \frac{1}{r}\big(-(pf')' + qf\big), \tag{7}$$

induces a definite Sturm–Liouville operator $T := JA$,

$$Tf = J(\ell f) = \frac{1}{|r|}\big(-(pf')' + qf\big), \tag{8}$$

which is self-adjoint in the Hilbert space $L^2((a,b),r)$ and vice versa.

The major part of the existing literature concerning indefinite Sturm–Liouville operators focuses on regular problems, i. e. the interval (a,b) is bounded and the coefficients $1/p$, q, r are integrable on (a,b). The qualitative spectral properties of operators associated with ℓ in the regular case are well-understood. We emphasize the contribution of Ćurgus and Langer [36], where it is shown that in the regular case the spectrum of every self-adjoint realisation of ℓ is discrete with at most finitely many non-real eigenvalues, which appear in pairs symmetric with respect to the real line. For a more detailed overview about regular indefinite Sturm–Liouville operators we refer to [36, 84, 102] and the references therein.

For a singular, i. e. non-regular, indefinite Sturm–Liouville operator A the situation is more complicated, as its essential spectrum may be non-empty. In the following we consider an indefinite singular Sturm–Liouville operator A whose corresponding definite operator T is semi-bounded from below, cf. Figure 1. If the lower bound of the spectrum $\sigma(T)$ of T is positive and, hence, $\sigma(T) \subset (0,\infty)$, then the spectrum $\sigma(A)$ of A is real with a gap around 0, see e. g. [70]. These so-called left-definite problems were intensively studied; we refer to [14, 22, 23, 24, 25, 68, 69, 70, 83] and to [102].

If $\sigma(T) \subset [0,\infty)$ and, additionally, the operator A has non-empty resolvent set $\rho(A)$, then $\sigma(A)$ is real, see [77]. In this situation the similarity of the indefinite operator A to a self-adjoint operator in a Hilbert space is of particular interest and is studied in many papers, see e. g. [4, 35, 37, 38, 58, 59, 60, 61, 71, 72] and also the survey [47].

In the case where the essential spectrum $\sigma_{\mathrm{ess}}(T)$ is contained in the interval $(0,\infty)$ it follows that the essential spectrum $\sigma_{\mathrm{ess}}(A)$ of A is real with a gap around 0 and $\sigma(A) \setminus \mathbb{R}$ consists of finitely many eigenvalues, see e. g. [36, 68].

The situation where $\inf \sigma_{\mathrm{ess}}(T) \leq 0$ is more complicated and the non-real spectrum of A may accumulate at the real axis, see e. g. [12, 62, 79]. It is still an open question, under which conditions accumulation occurs, see [7]. Also the non-emptiness of the resolvent set of singular indefinite Sturm–Liouville operators is an open problem and was resolved only under certain restrictions on the weight function r, see [16, 87].

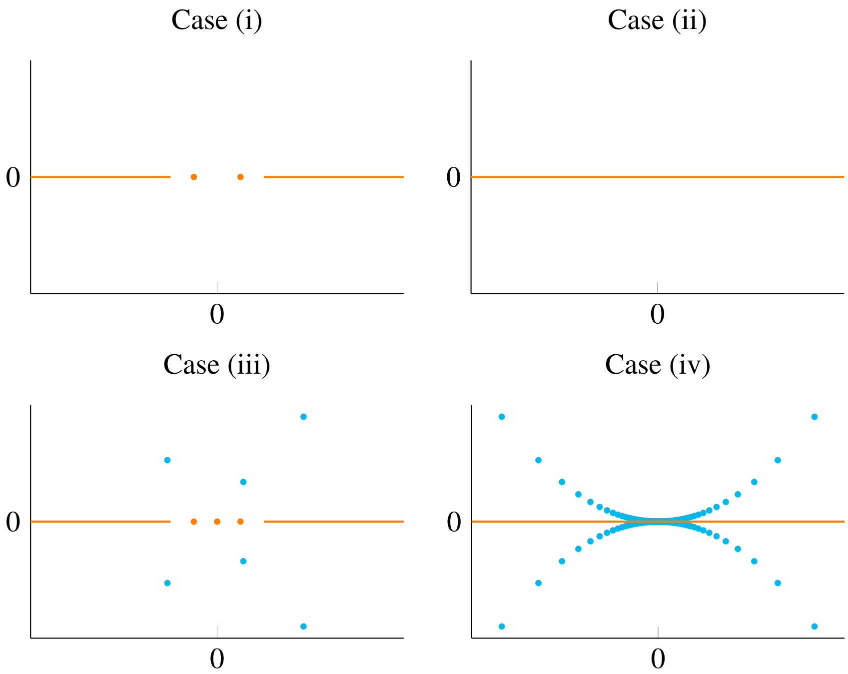

Figure 1: The pictures illustrate the generic structure of $\sigma(A)$ in the case where the corresponding definite operator T is semi-bounded from below. Here, the orange lines indicate the essential spectrum and dots stand for eigenvalues of A. (i) If $\sigma(T) \subset (0,\infty)$, then $\sigma(A) \subset \mathbb{R}$ with a gap around zero. (ii) If $\sigma(T) \subset [0,\infty)$ together with $\rho(A) \neq \emptyset$, then $\sigma(A) \subset \mathbb{R}$. (iii) If $\sigma_{\mathrm{ess}}(T) \subset (0,\infty)$, then $\sigma_{\mathrm{ess}}(A) \subset \mathbb{R}$ with a gap around zero and $\sigma(A) \setminus \mathbb{R}$ consists of at most finitely many eigenvalues. (iv) If $\inf \sigma(T) \leq 0$, then accumulation of $\sigma_{\mathrm{ess}}(A) \setminus \mathbb{R}$ is possible.

The aim of this thesis is to contribute to the quantitative spectral theory of singular indefinite Sturm–Liouville operators. Here we consider certain classes of indefinite operators whose essential spectra are contained in the real line. This includes the complicated case where $\inf \sigma_{\mathrm{ess}}(T) \leq 0$. By means of perturbation theory we study the structure of the essential spectrum and the accumulation of eigenvalues in gaps of the essential spectrum. Furthermore, we establish bounds on the absolute values and imaginary parts of the non-real eigenvalues.

One of the main contributions of this thesis are perturbation results for indefinite Sturm–Liouville operators with respect to the essential spectra. Given two operators A and $\tilde{A}$ of indefinite Sturm–Liouville type we address the following problem:

(P1) *Find criteria in terms of the coefficients of the indefinite Sturm–Liouville operators A and $\tilde{A}$ which guarantee $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(\tilde{A})$.*

The results proved in Section 3.2 allow simultaneous perturbations of all three coefficients p, q, and r. As a direct implication of Theorem 3.3 and Theorem 3.4 presented in Section 3.2 we obtain the following theorem.

Theorem 1. *Consider the coefficients p, q, r and $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ on $\mathbb{R}$ corresponding to the operators A and $\tilde{A}$, respectively, where r and $\tilde{r}$ both have definite sign near each endpoint. Assume that at least one of the following conditions holds:*

(α) *$\tilde{p}$, $\tilde{q}$, $1/\tilde{r}$ are bounded near the endpoints and the differences*

$$\frac{1}{\tilde{p}} - \frac{1}{p}, \quad \tilde{q} - q, \quad \tilde{r} - r \tag{9}$$

decay pointwise at both endpoints;

(β) *$\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ are periodic near the endpoints with the same period at each endpoint and the differences in* (9) *are integrable on $\mathbb{R}$.*

Then $\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(\tilde{A})$.

In general, a modification of the weight function r causes a change of the underlying weighted L^2-space. Therefore, usual stability criteria for the essential spectrum of an operator like compactness of the resolvent difference or compactness in form sense cannot be applied directly. Here we develop a solution to problem (P1) for definite operators in Chapter 2 and apply these results by means of Glazman's decomposition method to indefinite Sturm–Liouville operators. The criteria in the definite case are based on the relative oscillation theory developed by Krüger and Teschl in [73, 74, 75] and the Floquet theory for periodic Sturm–Liouville equations. For definite operators we extend results by Stolz [96] and Brown et al. [28, Chapter 5] to a situation where different weight functions are allowed.

If the essential spectrum of an indefinite Sturm–Liouville operator A has a gap, then it is of particular interest whether the eigenvalues of A within the gap accumulate to the boundary points of the gap. As a second main achievement in Section 3.3 we give a solution to the following problem:

(P2) *Find criteria for an indefinite Sturm–Liouville operator in terms of its coefficients which imply the finiteness or the accumulation of the real eigenvalues in a gap of the essential spectrum.*

Based on finite-rank perturbation results in [15] for self-adjoint operators in Krein spaces, we present in Theorem 3.7 a Kneser type criterion for indefinite Sturm–Liouville operators (see [67] and [98, Theorem 9.42] for Kneser's classical oscillation result). Furthermore, we show in the periodic case (β) of Theorem 1 that each sufficient distant gap of $\sigma_{\text{ess}}(A)$ contains at most finitely many eigenvalues of A, if in addition to the integrability of the differences in (9) a finite first moment condition holds, cf. Theorem 3.8. In that way, we generalize a seminal result by Rofe-Beketov [93] to periodic Sturm–Liouville operators with general coefficients, where perturbations on all three coefficients are allowed.

In Chapter 4 we solve a problem posed in [68, Remark 4.4] (see also [102, Remark 11.4.1]):

(P3) *Find bounds for the the non-real eigenvalues of a singular indefinite Sturm–Liouville operator.*

In the regular case related bounds were obtained in [8, 30, 53, 65, 91] and in [9, 31], where the latter two articles contain the most general results in terms of the assumptions on the coefficients, although, not necessarily the smallest bounds, cf. [65]. We also mention [80], where regular problems with non-locality are considered. In contrast, solutions to the problem (P3) in the more difficult case of singular operators are hardly represented in the literature.

The third main result of this thesis, which is already published in the articles [19, 20], are the bounds for the non-real eigenvalues of singular indefinite Sturm–Liouville operators presented in Chapter 4. The assumptions on the coefficients (cf. Hypothesis 4.1) are rather weak and the results are applicable to a large class of singular indefinite Sturm–Liouville operators. In more detail, besides $(a, b) = \mathbb{R}$ it is assumed that $1/p \in L^{\eta}(\mathbb{R})$, where $1 \le \eta \le \infty$, and q satisfies

$$\|q\|_{\mathrm{u}} := \sup_{n\in\mathbb{Z}} \int_n^{n+1} |q(t)|\,\mathrm{d}t < \infty. \tag{10}$$

The latter condition holds, for instance, if $q \in L^{s}(\mathbb{R})$ with $1 \le s \le \infty$. Further, we assume that $1/r$ is bounded and definite in a neighbourhood of each endpoint ∞ and $-\infty$. In particular, on a compactum r is allowed to change sign infinitely many times. The bounds established in Chapter 4 depend only on the norms of $1/p$, the negative part $q_- := (|q| - q)/2$ of q, and in an implicit way on the weight function r. For weight functions with at most finitely many sign changes explicit bounds are calculated in Section 4.2. For instance, in the case $p \equiv 1$, $r = \operatorname{sgn}$ we find (cf. Corollary 4.16):

Theorem 2. *If $\|q\|_{\mathrm{u}} < \infty$, then the non-real eigenvalues of $A = \operatorname{sgn}\cdot(-\mathrm{d}^2/\mathrm{d}x^2 + q)$ are contained in the set*

$$\Sigma_{\mathrm{u}} := \left\{ \lambda \in \mathbb{C} \;\middle|\; \begin{array}{l} |\operatorname{Im}\lambda| \le 12\cdot\sqrt{3}\left(\|q_-\|_{\mathrm{u}} + 2\|q_-\|_{\mathrm{u}}^2\right), \\ |\lambda| \le (12\cdot\sqrt{3}+9)\left(\|q_-\|_{\mathrm{u}} + 2\|q_-\|_{\mathrm{u}}^2\right) \end{array} \right\}.$$

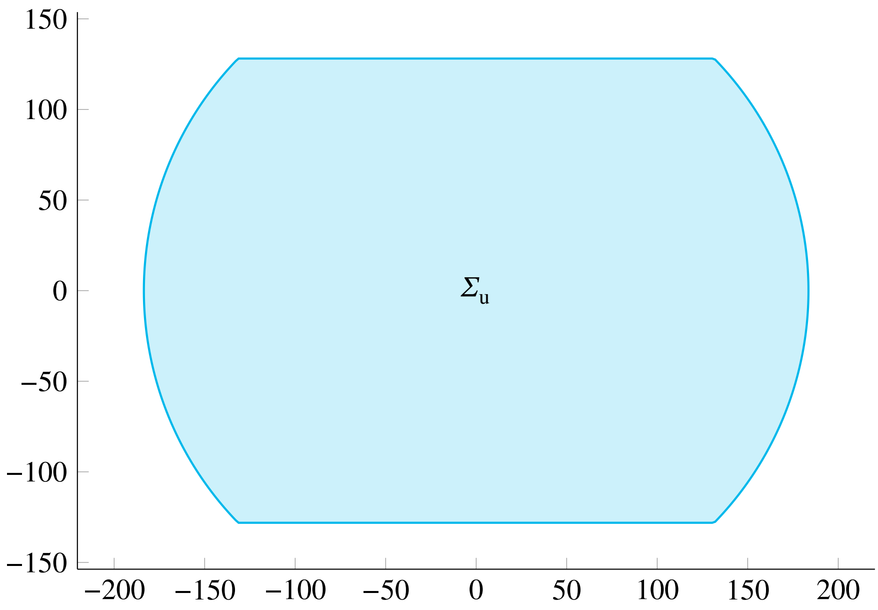

Figure 2: The non-real spectrum of the operator $A = \operatorname{sgn}\cdot(-\mathrm{d}^2/\mathrm{d}x^2 + q)$, where $q = -\varkappa(\varkappa+1)\operatorname{sech}^2$ and $\varkappa \in \mathbb{N}$, consists of $2\varkappa$ non-real eigenvalues, see [13]. The figure illustrates the shape of the set Σ_{u}, which contains the non-real spectrum of A, for the case $\varkappa = 1$.

The results in Chapter 4 are based on a careful analysis of the eigenfunctions corresponding to non-real eigenvalues, where we adapt techniques which were already employed for regular problems, cf. [9, 30].

We emphasize that beyond the results in Chapter 4 (and [19, 20]), bounds for the non-real eigenvalues of indefinite singular Sturm–Liouville operators were achieved only in the case $p \equiv 1$, $r = \operatorname{sgn}$, where q satisfies a certain integrability condition, see [17, 10, 18, 21, 34, 89], and for a special class of operators with two limit-circle endpoints in [90]. In contrast, the results in [17, 34, 89], which are better than our results at least when the potential q is a negative function, the spectral bounds obtained in Chapter 4 for the case $p \equiv 1$, $r = \operatorname{sgn}$ depend only on the negative part q_- of q. Therefore, in the genral case $q \neq q_-$ the findings in Chapter 4 may lead to smaller bounds for the non-real eigenvalues.

Table 1 below summarizes the main results of this thesis and indicates which of them have already been published.

Table 1: Main results in this thesis.

Section	Problem addressed	Main results	published
2.1, 2.2	(P1) for definite operators	Thm. 2.18, 2.26	none
3.2	(P1)	Thm. 3.3, 3.4	none
3.3	(P2)	Thm. 3.7, 3.8	none
4.1	(P3) for general weight functions	Thm. 4.5–4.7	Behrndt, Schmitz, Trunk [20]
4.2	(P3) for weight functions with finitely many sign changes	Thm. 4.13–4.15	Behrndt, Schmitz, Trunk [19, 20]

Chapter 1

Definite Sturm–Liouville operators

In this chapter we recall basic properties of definite Sturm–Liouville operators associated with the differential expression

$$\tau = \frac{1}{r}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p\frac{\mathrm{d}}{\mathrm{d}x} + q\right) \tag{1.1}$$

on an open interval (a, b), where $-\infty \leq a < b \leq \infty$. Throughout this chapter it is imposed that

$$\begin{cases} p,\ q,\ r \text{ are real-valued functions on } (a,b), \\ p(x) > 0,\ r(x) > 0 \text{ for almost all } x \in (a,b), \\ 1/p,\ q,\ r \in L^1_{\mathrm{loc}}(a,b). \end{cases} \tag{1.2}$$

This chapter contains standard material, where we follow parts of [11, 42, 101, 102].

Depending on the integrability of the coefficients, the differential expression τ is classified in the following way. The differential expression τ is said to be *regular* at b if the endpoint b is finite and $1/p$, q, r are integrable on (c, b) for some $c \in (a, b)$. Otherwise τ is *singular* at b. Analogously, τ is called regular at a if the endpoint a is finite and $1/p$, q, r are integrable on (a, c) for some $c \in (a, b)$; otherwise τ is called singular at a. If τ is regular at both endpoints we call τ regular, otherwise singular. For a subinterval $(c, d) \subset (a, b)$ we write $\tau \restriction (c, d)$ for the differential expression restricted to (c, d) corresponding to the coefficients $p \restriction (c, d)$, $q \restriction (c, d)$ and $r \restriction (c, d)$. Observe that $\tau \restriction (a, c)$ and $\tau \restriction (c, b)$ are always regular at c for all $c \in (a, b)$.

For $\lambda \in \mathbb{C}$ and a measurable function $g : (a, b) \to \mathbb{C}$ we call $u : (a, b) \to \mathbb{C}$ solution of the differential equation

$$(\tau - \lambda)u = \frac{1}{r}\big(-(pu')' + (q - \lambda r)u\big) = g \tag{1.3}$$

if u, $pu' \in AC(a, b)$ and u satisfies (1.3) a. e. on (a, b). Here, $AC(a, b)$ denotes the set of locally absolutely continuous functions defined on (a, b) with values in $\mathbb{C}$; if (a, b) comprises the whole real line we write $AC(\mathbb{R})$. If $rg \in L^1_{\mathrm{loc}}(a, b)$, then the differential equation (1.3) subject to the initial condition

$$u(x_0) = c_1, \quad (pu')(x_0) = c_2, \quad \text{where } c_1,\ c_2 \in \mathbb{C},\ x_0 \in (a,b), \tag{1.4}$$

has a unique solution, see e. g. [11, Section 6.1]. Therefore, the solution space of the corresponding homogeneous differential equation $(\tau - \lambda)u = 0$ is two-dimensional. Provided regularity at b and $rg \restriction (c, b) \in L^1(c, b)$ for some $c \in (a, b)$, any solution u of (1.3) and pu' can be continuously extended to the endpoint b. There is a similar statement for the endpoint a.

1.1 Self-adjoint realisations

Let $L^2((a,b),r)$ be the space of all (equivalence classes of) measurable functions $f : (a,b) \to \mathbb{C}$ such that rf^2 is integrable on (a,b). Equipped with the scalar product

$$\langle f, g\rangle_r := \int_a^b f(t)\overline{g(t)}r(t)\,\mathrm{d}t, \qquad f,\, g \in L^2((a,b),r), \tag{1.5}$$

it is a Hilbert space. For a subinterval $(c,d) \subset (a,b)$ we write $L^2((c,d),r)$ instead of $L^2((c,d),r \restriction (c,d))$. The natural domain for τ acting as an operator in $L^2((a,b),r)$ is the linear subspace

$$\mathscr{D}(\tau) = \big\{ f \in L^2((a,b),r) \,\big|\, f,\ pf' \in AC(a,b) \text{ and } \tau f \in L^2((a,b),r) \big\}. \tag{1.6}$$

Let $f, g \in \mathscr{D}(\tau)$ and $\lambda \in \mathbb{C}$. Then for each compact subinterval $[y,x] \subset (a,b)$ integration by parts over $[y,x]$ shows

$$\begin{aligned} \int_y^x \big((\tau-\lambda)f\big)(t)\overline{g(t)}r(t)\,\mathrm{d}t = &\int_y^x \Big(p(t)f'(t)\overline{g'(t)} + \big(q(t)-\lambda r(t)\big)f(t)\overline{g(t)}\Big)\,\mathrm{d}t \\ &+ (pf')(y)\overline{g(y)} - (pf')(x)\overline{g(x)} \end{aligned} \tag{1.7}$$

and the Green's identity

$$\begin{aligned} \int_y^x \Big((\tau f)(t)\overline{g(t)} - f(t)\overline{(\tau g)(t)}\Big)r(t)\,\mathrm{d}t = &\,(pf')(y)\overline{g(y)} - f(y)\overline{(pg')(y)} \\ &- (pf')(x)\overline{g(x)} + f(x)\overline{(pg')(x)}. \end{aligned} \tag{1.8}$$

Provided regularity at an endpoint, each function $f \in \mathscr{D}(\tau)$ together with pf' can be continuously extended to that endpoint. Indeed, assume that τ is regular at b and set $g := \tau f$. By definition of $\mathscr{D}(\tau)$ we have $g \in L^2((a,b),r)$ and the Cauchy–Schwarz inequality yields

$$\int_c^b |r(t)g(t)|\,\mathrm{d}t \le \left(\int_c^b r(t)\,\mathrm{d}t \cdot \int_c^b |g(t)|^2 r(t)\,\mathrm{d}t\right)^{1/2} < \infty$$

for all $c \in (a,b)$. As a solution of the differential equation $\tau f = g$, where rg is integrable near b, the function f as well as pf' can be continuously extended to b. If τ is regular at a a similar argument applies.

The domain, range and kernel of a linear operator T will be denoted by $\mathcal{D}(T)$, $\mathcal{R}(T)$ and $\mathcal{N}(T)$. As usual, for the closure of T and its Hilbert space adjoint we write $\overline{T}$ and T^*. The *maximal operator* $T_{\max}$ associated with τ is defined by

$$T_{\max} f := \tau f = \frac{1}{r}\big(-(pf')' + qf\big), \qquad \mathcal{D}(T_{\max}) = \mathscr{D}(\tau). \tag{1.9}$$

The restriction of the maximal operator

$$T_{\mathrm{pmin}} f := T_{\max} f = \tau f, \qquad \mathcal{D}(T_{\mathrm{pmin}}) = \big\{ f \in \mathscr{D}(\tau) \,\big|\, f \text{ has compact support} \big\}, \tag{1.10}$$

is called the *pre-minimal operator*. The two operators $T_{\max}$ and T_{pmin} are densely defined in $L^2((a,b),r)$, cf. [11, Theorem 6.2.1]. From (1.8) one obtains immediately that the pre-minimal operator is symmetric. Moreover, one can show that $(T_{\text{pmin}})^* = T_{\text{max}}$, cf. [11, Theorem 6.2.1]. The closure of the pre-minimal operator

$$T_{\text{min}} := \overline{T_{\text{pmin}}} = (T_{\text{pmin}})^{**} = (T_{\text{max}})^*, \tag{1.11}$$

is symmetric and is called the *minimal operator*. A self-adjoint extension T of T_{min} (or self-adjoint restriction of T_{max}),

$$T_{\text{min}} \subset T = T^* \subset T_{\text{max}}, \tag{1.12}$$

is called *self-adjoint realisations* of τ. Here, for operators S and T we write $S \subset T$ if $\mathcal{D}(S) \subset \mathcal{D}(T)$ and $Sx = Tx$ for all $x \in \mathcal{D}(S)$. By the extension theory for symmetric operators, see e. g. [98, Section 2.6], the definite Sturm–Liouville differential expression τ admits always self-adjoint realisations. This can be seen by verifying that the defect numbers of the minimal operator T_{min},

$$\begin{aligned} d_+(T_{\text{min}}) &:= \dim \mathcal{R}(T_{\text{min}} - \mathrm{i})^\perp = \dim \mathcal{N}(T_{\text{max}} + \mathrm{i}) \\ d_-(T_{\text{min}}) &:= \dim \mathcal{R}(T_{\text{min}} + \mathrm{i})^\perp = \dim \mathcal{N}(T_{\text{max}} - \mathrm{i}), \end{aligned} \tag{1.13}$$

are equal. Indeed, since the coefficients p, q, r are real-valued, a function f belongs to $\mathcal{N}(T_{\text{max}} - \mathrm{i})$ if and only if $\overline{f}$ is an element of $\mathcal{N}(T_{\text{max}} + \mathrm{i})$, which implies $d_+(T_{\text{min}}) = d_-(T_{\text{min}})$. Moreover, the fact that the number of linearly independent solutions of the differential equation $(\tau - \lambda)u = 0$ for $\lambda \in \mathbb{C}$ is limited by two yields $d_+(T_{\text{min}}) = d_-(T_{\text{min}}) \leq 2$. Therefore, T_{max} and all self-adjoint realisation of τ are finite-dimensional extensions of T_{min}.

Beside regularity there is another classification of the endpoints. We say a solution u of $(\tau - \lambda)u = 0$ for $\lambda \in \mathbb{C}$ *lies right* in $L^2((a, b), r)$ if there exists $c \in (a, b)$ such that $u \restriction (c, b) \in L^2((c, b), r)$. Similarly, u *lies left* in $L^2((a, b), r)$ if $u \restriction (a, c) \in L^2((a, c), r)$ for some $c \in (a, b)$. By Weyl's alternative (see e. g [42, Lemma 4.1]) precisely one of the following possibilities is valid:

(i) For each $\lambda \in \mathbb{C}$ all solutions of $(\tau - \lambda)u = 0$ lie right (resp. left) in $L^2((a, b), r)$.

(ii) For each $\lambda \in \mathbb{C}$ there exists one solution of $(\tau - \lambda)u = 0$ which does not lie right (resp. left) in $L^2((a, b), r)$.

Case (i) is called the *limit-circle case* at b (at a) and (ii) is referred to as *limit-point case* at b (resp. at a). One can show that in the limit-point case at b there exists for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ a unique (up to a constant factor) solution of $(\tau - \lambda)u = 0$ which lies right in $L^2((a, b), r)$; similarly for the limit-point case at a. Since for a regular endpoint the solutions of $(\tau - \lambda)u = 0$ can be continuously extended to this endpoint, each regular endpoint is a limit-circle endpoint. As a direct consequence of Weyl's alternative one has $d_+(T_{\text{min}}) = d_-(T_{\text{min}}) = 2$ if τ is in the limit-circle case at both endpoints, and $d_+(T_{\text{min}}) = d_-(T_{\text{min}}) = 1$ if the limit-circle case prevails at exactly one endpoint. In the case of two limit-point endpoints one has $d_+(T_{\text{min}}) = d_-(T_{\text{min}}) = 0$, cf. [42, Theorem 4.6].

In each of the above cases there are descriptions of all possible self-adjoint realisations of τ in terms of boundary conditions. This is well studied, see e. g. [11, 42, 100]. The self-adjoint realisations of τ can be described as $d_+(T_{\text{min}})$-dimensional restrictions of the maximal operator. At each endpoint, where τ is in the limit-circle case, one additional boundary condition has to be imposed on the functions of $\mathcal{D}(T_{\text{max}}) = \mathscr{D}(\tau)$. For instance, if τ is in the limit-circle case at both endpoints, then two boundary conditions are necessary. In this particular situation the self-adjoint realisations of τ may be divided into realisations, where the boundary conditions at the left and the right endpoint are coupled, and those with separated boundary conditions, cf. [100, Chapter 4].

In the the next proposition which follows from [100, Theorem 5.8][1] we give a description of the self-adjoint realisations of τ. We omit a characterisation of the realisations with coupled boundary conditions, as they are not of particular interest in the following.

Proposition 1.1. *Suppose that the coefficients of τ satisfy* (1.2).

(i) *Let $\lambda \in \mathbb{R}$. If τ is in the limit circle-case at a and in the limit-point case at b then T is a self-adjoint realisation of τ if and only if*

$$Tf = \tau f, \quad \mathcal{D}(T) = \left\{ f \in \mathscr{D}(\tau) \,\middle|\, \lim_{x\to a} \big(f(x)(pu_a')(x) - (pf')(x)u_a(x)\big) = 0 \right\},$$

where u_a is a non-trivial real-valued solution of $(\tau - \lambda)u = 0$. A similar result holds if τ is in the limit-point case at a and in the limit-circle case at b.

(ii) *Let $\lambda \in \mathbb{R}$. If τ is in the limit-circle case at both endpoints then T is a self-adjoint realisation of τ with separated boundary conditions if and only if*

$$Tf = \tau f, \quad \mathcal{D}(T) = \left\{ f \in \mathscr{D}(\tau) \,\middle|\, \begin{array}{l} \lim_{x\to a} \big(f(x)(pu_a')(x) - (pf')(x)u_a(x)\big) = 0, \\ \lim_{x\to b} \big(f(x)(pu_b')(x) - (pf')(x)u_b(x)\big) = 0 \end{array} \right\},$$

where u_a and u_b are non-trivial real-valued solutions of $(\tau - \lambda)u = 0$.

(iii) *If τ is in the limit-point case at both endpoints then $T_{\min} = T_{\max}$. In this case $T = T_{\max}$ is the only self-adjoint realisation of τ.*

If τ is regular at the endpoint a, then the boundary conditions in Proposition 1.1 (i) (similarly, in (ii)) may take the form of point evaluations at a and the self-adjoint realisations of τ can be parametrized by the initial values $((u_a(a), (pu_a')(a))^\top \in \mathbb{R}^2$ of the real-valued solutions u_a. Further, note that replacing a solution u_a with $c \cdot u_a$, where c is a non-zero real constant, does not change the operator domain $\mathcal{D}(T)$. Therefore, it suffices to consider, instead of all $\mathbb{R}^2$-tuples of initial values, only tuples on the semicircle $\{(\sin\alpha, \cos\alpha)^\top \in \mathbb{R}^2 \,|\, \alpha \in [0, \pi)\}$.

Corollary 1.2. *Suppose that the coefficients of τ satisfy* (1.2) *and let τ be regular at a.*

(i) *If τ is in the limit-point case at b then T is a self-adjoint realisation of τ if and only if*

$$Tf = \tau f, \quad \mathcal{D}(T) = \left\{ f \in \mathscr{D}(\tau) \,\middle|\, \cos(\alpha)f(a) - \sin(\alpha)(pf')(a) = 0 \right\},$$

where $\alpha \in [0, \pi)$.

(ii) *Let $\lambda \in \mathbb{R}$. If τ is in the limit-circle case at b then T is a self-adjoint realisation of τ with separated boundary conditions if and only if*

$$Tf = \tau f, \quad \mathcal{D}(T) = \left\{ f \in \mathscr{D}(\tau) \,\middle|\, \begin{array}{r} \cos(\alpha)f(a) - \sin(\alpha)(pf')(a) = 0, \\ \lim_{x\to b} \big(f(x)(pu_b')(x) - (pf')(x)u_b(x)\big) = 0 \end{array} \right\},$$

where $\alpha \in [0, \pi)$ and u_b is a non-trivial real-valued solution of $(\tau - \lambda)u = 0$.

Similar results hold if τ is regular at b or at both endpoints.

1 Note that all scalar products in [100] are defined as anti-linear in the first and linear in the second argument.

1.2 The essential spectrum

For a closed operator T in a Hilbert space $\mathcal{H}$ the resolvent set, the spectrum, the point spectrum and the essential spectrum of T are denoted by $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ess}(T)$. In the literature there are various approaches to define the essential spectrum of a closed operator, cf. [64, 95]. Here, we follow the definition in [50], where the essential spectrum is given as the complement of the Fredholm domain, i. e.

$$\sigma_{ess}(T) = \left\{\lambda \in \mathbb{C} \,\middle|\, \dim \mathcal{N}(T-\lambda) = \infty \text{ or } \dim(\mathcal{H}/\mathcal{R}(T-\lambda)) = \infty\right\}. \tag{1.14}$$

Note that the essential spectrum is invariant under finite-dimensional or compact perturbations; for details we refer to [64]. If T is self-adjoint in $\mathcal{H}$ then, equivalently, $\lambda \in \sigma_{ess}(T)$ if and only if λ is an accumulation point of $\sigma(T)$ or $\dim \mathcal{N}(T-\lambda) = \infty$, cf. [95].

Under the conditions in (1.2) the spectrum of a self-adjoint Sturm–Liouville operator T corresponding to the differential expression τ in (1.1) is always unbounded above, see e. g. [102, p. 73]. Observe that for all $\lambda \in \mathbb{R}$ one has $\dim \mathcal{N}(T-\lambda) \le 2$ and, therefore, the essential spectrum of T consists only of the accumulation points of $\sigma(T)$. If τ is in the limit-circle case at both endpoints, in particular if both endpoints are regular, then every self-adjoint realisation T of τ has compact resolvent, cf. [100, Theorem 7.10], and, therefore, $\sigma_{ess}(T) = \emptyset$. The situation is different, however, if the limit-point case prevails at one or both endpoints. Then the essential spectrum may be non-empty. Since the defect numbers of the minimal operator are finite, all self-adjoint realisations of τ are finite-dimensional extensions of the minimal operator and share the same essential spectrum, cf. [98, Theorem 6.20]. Moreover, if one self-adjoint realisation of τ is semi-bounded from below, then all self-adjoint realisations are semi-bounded from below, see e. g. [99, Corollary 2 to Theorem 8.19] or [2, Section 107, Theorem 2].

The essential spectrum is determined only by the behaviour of the coefficients of τ near the (limit-point) endpoints, which can seen as follows by means of Glazman's decomposition method, cf. [49, Section 7]. For an interval $(\alpha, \beta) \subset (a, b)$ let $T_{min}(\alpha, \beta)$, $T_{max}(\alpha, \beta)$ and $T(\alpha, \beta)$ denote the minimal operator, the maximal operator and an arbitrary self-adjoint realisation associated with $\tau \restriction (\alpha, \beta)$ in $L^2((\alpha, \beta), r)$, respectively.

Lemma 1.3. *Suppose that the coefficients of* τ *satisfy* (1.2) *and assume that* T *is any self-adjoint realisation of* τ. *Let* $a < c < d < b$ *and consider*

$$T_0 = \begin{pmatrix} T_{min}(a,c) & 0 & 0 \\ 0 & T_{min}(c,d) & 0 \\ 0 & 0 & T_{min}(d,b) \end{pmatrix}, \qquad \tilde{T} = \begin{pmatrix} T(a,c) & 0 & 0 \\ 0 & T(c,d) & 0 \\ 0 & 0 & T(d,b) \end{pmatrix}, \tag{1.15}$$

where the space $L^2((a,b),r)$ *is identified with the orthogonal sum* $L^2((a,c),r) \oplus L^2((c,d),r) \oplus L^2((d,b),r)$. *Then both operators* T *and* $\tilde{T}$ *are finite-dimensional self-adjoint extensions of* T_0, *and*

$$\sigma_{ess}(T) = \sigma_{ess}(\tilde{T}) = \sigma_{ess}(T(a,c)) \cup \sigma_{ess}(T(d,b)). \tag{1.16}$$

Proof. Since the operators $T_{min}(a,c)$, $T_{min}(c,d)$, $T_{min}(d,b)$ are densely defined, closed and symmetric in the corresponding Hilbert spaces, the operator T_0 is densely defined, closed and symmetric in $L^2((a,b),r)$. It is not difficult to see that $T_0 \subset T_{min}$, where T_{min} is the minimal operator associated with τ on the interval (a,b). The adjoint of T_0 is given by

$$(T_0)^* = \begin{pmatrix} T_{max}(a,c) & 0 & 0 \\ 0 & T_{max}(c,d) & 0 \\ 0 & 0 & T_{max}(d,b) \end{pmatrix}, \tag{1.17}$$

where $T_{\max} = (T_{\min})^* \subset (T_0)^*$. The defect numbers $d_+(T_0)$, $d_-(T_0)$ of T_0 satisfy

$$\begin{aligned} d_+(T_0) &= d_+(T_{\min}(a,c)) + d_+(T_{\min}(c,d)) + d_+(T_{\min}(d,b)) \\ &= d_-(T_{\min}(a,c)) + d_-(T_{\min}(c,d)) + d_-(T_{\min}(d,b)) = d_-(T_0) \le 6. \end{aligned} \tag{1.18}$$

The operator $\tilde{T}$ is a self-adjoint extension of T_0, as easily can be seen from the definition of $\tilde{T}$. By $T_0 \subset T_{\min} \subset T \subset T_{\max} \subset (T_0)^*$ also T is a self-adjoint extension of T_0. From the extension theory for symmetric operators (see e. g. [98, Section 2.6]) and (1.18) we see that both operators are finite-dimensional extensions of T_0 and, therefore, their resolvent difference is a finite-rank operator, i. e.

$$\dim \mathcal{R}\left((T-\lambda)^{-1} - (\tilde{T}-\lambda)^{-1}\right) \le 6 \qquad \text{for all } \lambda \in \rho(T) \cap \rho(\tilde{T}). \tag{1.19}$$

As a consequence the operators T and $\tilde{T}$ share the same essential spectrum, see [98, Theorem 6.19], and we have

$$\sigma_{\mathrm{ess}}(T) = \sigma_{\mathrm{ess}}(\tilde{T}) = \sigma_{\mathrm{ess}}(T(a,c)) \cup \sigma_{\mathrm{ess}}(T(c,d)) \cup \sigma_{\mathrm{ess}}(T(d,b)), \tag{1.20}$$

where $\sigma_{\mathrm{ess}}(T(c,d)) = \emptyset$ as $\tau \upharpoonright (c,d)$ is regular. □

Chapter 2

Perturbations of definite Sturm–Liouville operators

In this chapter we study the spectra of definite Sturm–Liouville operators under perturbations. The main objective is to find criteria for the invariance of the essential spectrum. Moreover, we investigate how perturbations influence the accumulation of eigenvalues at the boundary of the essential spectrum. The key ingredient is relative oscillation theory, where the zeros of the Wronskian determinant corresponding to solutions of two different Sturm–Liouville eigenvalue problems are counted.

As we compare different Sturm–Liouville operators it is convenient to introduce three differential expressions

$$\tau_j = \frac{1}{r_j}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p_j\frac{\mathrm{d}}{\mathrm{d}x} + q_j\right), \quad \text{where } j = 0,\ 1,\ 2, \tag{2.1}$$

on a common open interval (a, b) with finite left endpoint a. Throughout this chapter the conditions

$$\begin{cases} p_j,\ q_j,\ r_j \text{ are real-valued functions on } (a,b), \\ p_j(x) > 0,\ r_j(x) > 0 \text{ for almost all } x \in (a,b), \\ 1/p_j,\ q_j,\ r_j \in L^1_{\mathrm{loc}}(a,b), \\ \tau_j \text{ is regular at } a \end{cases} \tag{2.2}$$

are imposed for $j = 0,\ 1,\ 2$.

2.1 Relative oscillation

We recall well-known results in standard oscillation theory for definite Sturm–Liouville operators, where we follow parts of [100]. Thereafter we introduce the concept of relative oscillation developed by Krüger and Teschl [73, 74, 75].

An important tool in oscillation theory is the Prüfer transformation. Let u be a non-trivial real-valued solution of the differential equation $(\tau_0 - \lambda)u = 0$, where $\lambda \in \mathbb{R}$. Both functions u and pu' do not vanish at the same point $x \in (a, b)$, otherwise uniqueness of the solution would imply that u is the trivial solution. Therefore, u and pu' admit a representation in terms of the Prüfer variables,

$$u = \rho_u \sin\vartheta_u, \qquad pu' = \rho_u \cos\vartheta_u. \tag{2.3}$$

where the *Prüfer radius* ρ_u is given by

$$\rho_u = \sqrt{u^2 + (pu')^2} \tag{2.4}$$

and the *Prüfer angle* ϑ_u satisfies

$$\tan\vartheta_u(x) = \frac{u(x)}{(pu')(x)} \qquad \text{for } (pu')(x) \neq 0 \tag{2.5}$$

or

$$\cot\vartheta_u(x) = \frac{(pu')(x)}{u(x)} \qquad \text{for } u(x) \neq 0, \tag{2.6}$$

cf. [100, Chapter 13]. By (2.4) the Prüfer radius ρ_u is positive and absolutely continuous on (a, b). Observe that $\vartheta_u(x)$ for $x \in (a, b)$ is determined only up to integer multiples of 2π by (2.5) and (2.6). In order to remove this ambiguity we fix a value $\vartheta_u(x_0)$ for an arbitrary $x_0 \in (a, b)$ and require continuity of the Prüfer angle ϑ_u. This leads to a uniquely determined function ϑ_u. A straightforward calculation using (2.5) and (2.6) shows that the Prüfer angle ϑ_u is a locally absolutely continuous function on (a, b) satisfying the differential equation

$$\vartheta_u' = \frac{1}{p}(\cos\vartheta_u)^2 - (q - \lambda r)(\sin\vartheta_u)^2. \tag{2.7}$$

In the same way as u and pu' the Prüfer variables can be continuously extended to the regular endpoint a.

Observe that by (2.3) a point $\xi \in (a, b)$ is a zero of u if and only if $\vartheta_u(\xi)$ is a integer multiple of π. At the zeros of the solution the Prüfer angle is strictly increasing. We present a short proof based on an adaption of the proof of Theorem 13.1 in [100].

Lemma 2.1. *Suppose that* (2.2) *holds for* $j = 0$ *and let* u *be a non-trivial real-valued solution of* $(\tau_0 - \lambda)u = 0$, *where* $\lambda \in \mathbb{R}$. *Consider* $\xi \in [a, b)$ *and* $k \in \mathbb{Z}$. *Then*

(i) $\vartheta_u(\xi) \leq k\pi$ *implies* $\vartheta_u(x) < k\pi$ *for all* $x < \xi$, $x \in (a, b)$, *and*

(ii) $\vartheta_u(\xi) \geq k\pi$ *implies* $\vartheta_u(x) > k\pi$ *for all* $x > \xi$, $x \in (a, b)$.

Proof. Let $\delta := \vartheta_u - k\pi$. Consider

$$f = \frac{1}{p}(\cos\delta)^2 \quad \text{and} \quad h = -(q - \lambda r)\frac{(\sin\delta)^2}{\delta}. \tag{2.8}$$

Then by (2.7) together with the π-periodicity of cos^2 and $\sin^2$ we have $\delta' = f + h\delta$. Here, the functions f, h are integrable on (a, c) for all $c \in (a, b)$ because $f \leq 1/p$ and $|h| \leq |q - \lambda r|$. Consider the positive function g defined by

$$g(x) = \exp\left(-\int_a^x h(t)\,\mathrm{d}t\right). \tag{2.9}$$

Then $(g\delta)' = -h\delta g + (f + h\delta)g = fg \geq 0$, that is $g\delta$ is increasing. By the monotonicity of $g\delta$ we have for $x < \xi$

$$g(x)\big(\vartheta_u(x) - k\pi\big) = (g\delta)(x) \leq (g\delta)(\xi) = g(\xi)\big(\vartheta_u(\xi) - k\pi\big), \tag{2.10}$$

and for $x > \xi$

$$g(\xi)\big(\vartheta_u(\xi) - k\pi\big) = (g\delta)(\xi) \leq (g\delta)(x) = g(x)\big(\vartheta_u(x) - k\pi\big). \tag{2.11}$$

Observe that in the case $\vartheta_u(\xi) = k\pi$ the function $(g\delta)' = fg$ is positive in a neighbourhood of ξ because of $p > 0$ a. e. and $\delta(\xi) = 0$. In this situation the inequalities in (2.10) and (2.11) are strict. Now the positivity of g together with (2.10), (2.11) implies (i) and (ii). □

The growth behaviour of the Prüfer angle allows to count the zeros of the solution u. In the following $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor function and the ceiling function.

Lemma 2.2. *Suppose that* (2.2) *holds for* $j = 0$ *and let* u *be a non-trivial real-valued solution of* $(\tau_0 - \lambda)u = 0$*, where* $\lambda \in \mathbb{R}$*. Then for every* $x \in (a, b)$ *the solution* u *has at most finitely many zeros in* (a, x) *and*

$$N_u(x) := \left\lceil \frac{\vartheta_u(x)}{\pi} \right\rceil - \left\lfloor \frac{\vartheta_u(a)}{\pi} \right\rfloor - 1 \tag{2.12}$$

equals the number of zeros of u *in* (a, x)*. In particular, the function* $N_u : (a, b) \to \mathbb{Z}$ *is non-negative and increasing.*

Proof. Let $x \in (a, b)$ and choose k, $m \in \mathbb{Z}$ such that $\vartheta_u(a) \in [k\pi, (k+1)\pi)$ and $\vartheta_u(x) \in (m\pi, (m+1)\pi]$. Then $N_u(x) = m - k$ by definition. According to Lemma 2.1 one has $k\pi < \vartheta_u(y)$ and $\vartheta_u(y) < (m+1)\pi$ for all $y \in (a, x)$. In particular, $k \le m$. We consider the case $k = m$. Then we have $\vartheta_u(y) \in (k\pi, (k+1)\pi)$ for all $y \in (a, x)$. Hence, u has no zeros in (a, x) and $N_u(x) = 0$. If $k < m$, then $\vartheta_u(y) \in (k\pi, (m+1)\pi)$ for all $y \in (a, x)$. Further, the continuity and the growth behaviour of ϑ_u, see Lemma 2.1, imply that every value $n\pi$, $n \in [k+1, m] \cap \mathbb{Z}$, is attained by ϑ_u exactly once in (a, x). Hence, ϑ_u has $N_u(x) = m - k$ zeros in (a, x). □

In the following we extend the usual definition of the Wronskian determinant. Let u_0 be a solution of $(\tau_0 - \lambda_0)u_0 = 0$ and u_1 a solution of $(\tau_1 - \lambda_1)u_1$, where $\lambda_0, \lambda_1 \in \mathbb{R}$. The *Wronskian determinant* of u_0 and u_1 is defined by

$$W[u_0, u_1] := u_0 p_1 u_1' - u_1 p_0 u_0'. \tag{2.13}$$

The Wronskian $W[u_0, u_1]$ is locally absolutely continuous in (a, b) with

$$W[u_0, u_1]' = \big((q_1 - \lambda_1 r_1) - (q_0 - \lambda_0 r_0)\big)u_0 u_1 + \left(\frac{1}{p_0} - \frac{1}{p_1}\right) p_0 u_0' p_1 u_1'. \tag{2.14}$$

A point $x \in (a, b)$ is a zero of $W[u_0, u_1]$ if and only if the $\mathbb{C}^2$-vectors $(u_0(x), (p_0 u_0')(x))^\top$ and $(u_1(x), (p_1 u_1')(x))^\top$ are linearly dependent. Provided that u_0 and u_1 are real-valued non-trivial solutions, the Wronskian in (2.13) can be represented in terms of Prüfer variables,

$$W[u_0, u_1](x) = -\rho_{u_0}(x)\rho_{u_1}(x)\sin\big(\vartheta_{u_1}(x) - \vartheta_{u_0}(x)\big). \tag{2.15}$$

In this case the zeros of $W[u_0, u_1]$ are exactly those points $\xi \in (a, b)$, where

$$\vartheta_{u_1}(\xi) - \vartheta_{u_0}(\xi) = k\pi \tag{2.16}$$

for some $k \in \mathbb{Z}$. Under certain conditions the difference of two Prüfer angles has at the zeros of the Wronskian a similar growth behaviour as a single Prüfer angle at zeros of the corresponding solution.

Lemma 2.3. *Suppose that* (2.2) *holds for* $j = 0, 1$*. Let* u_0 *and* u_1 *be non-trivial real-valued solutions of* $(\tau_0 - \lambda_0)u_0 = 0$ *and* $(\tau_1 - \lambda_1)u_1 = 0$*, respectively, for* $\lambda_0, \lambda_1 \in \mathbb{R}$*. Consider* $\xi \in [a, b)$ *and* $k \in \mathbb{Z}$*. If*

$$p_0 \ge p_1 \quad \text{and} \quad q_0 - \lambda_0 r_0 \ge q_1 - \lambda_1 r_1 \tag{2.17}$$

holds a. e. on (a, b)*, then*

(i) $\vartheta_{u_1}(\xi) - \vartheta_{u_0}(\xi) < k\pi$ *implies* $\vartheta_{u_1}(x) - \vartheta_{u_0}(x) < k\pi$ *for all* $x < \xi$, $x \in (a, b)$*,*

(ii) $\vartheta_{u_1}(\xi) - \vartheta_{u_0}(\xi) > k\pi$ *implies* $\vartheta_{u_1}(x) - \vartheta_{u_0}(x) > k\pi$ *for all* $x > \xi$, $x \in (a, b)$*, and*

(iii) $\vartheta_{u_1}(\xi) - \vartheta_{u_0}(\xi) = k\pi$ *implies* $\vartheta_{u_1}(x) - \vartheta_{u_0}(x) \le k\pi \le \vartheta_{u_1}(y) - \vartheta_{u_0}(y)$ *for all* $x < \xi$, $x \in (a,b)$ *and for all* $y > \xi$, $y \in (a,b)$.

If

$$p_0 \ge p_1 \quad \textit{and} \quad q_0 - \lambda_0 r_0 > q_1 - \lambda_1 r_1 \tag{2.18}$$

is valid a. e. on (a,b)*, then*

(iv) $\vartheta_{u_1}(\xi) - \vartheta_{u_0}(\xi) \le k\pi$ *implies* $\vartheta_{u_1}(x) - \vartheta_{u_0}(x) < k\pi$ *for all* $x < \xi$, $x \in (a,b)$*, and*

(v) $\vartheta_{u_1}(\xi) - \vartheta_{u_0}(\xi) \ge k\pi$ *implies* $\vartheta_{u_1}(x) - \vartheta_{u_0}(x) > k\pi$ *for all* $x > \xi$, $x \in (a,b)$.

Proof. The proof is similar to the proof of Lemma 2.1 and is an adaption of the proof of Theorem 13.1 in [100] to the case $k \neq 0$. Let $\delta := \vartheta_{u_1} - \vartheta_{u_0} - k\pi$. Then by (2.7) (for $u = u_0$ and $u = u_1$, respectively) we obtain

$$\begin{aligned}\delta' &= \frac{1}{p_1}(\cos\vartheta_{u_1})^2 - (q_1 - \lambda_1 r_1)(\sin\vartheta_{u_1})^2 - \frac{1}{p_0}(\cos\vartheta_{u_0})^2 + (q_0 - \lambda_0 r_0)(\sin\vartheta_{u_0})^2 \\ &= \left(\frac{1}{p_1} - \frac{1}{p_0}\right)(\cos\vartheta_{u_1})^2 + \big((q_0 - \lambda_0 r_0) - (q_1 - \lambda_1 r_1)\big)(\sin\vartheta_{u_0})^2 \\ &\quad - (q_1 - \lambda_1 r_1)\Big((\sin\vartheta_{u_1})^2 - (\sin\vartheta_{u_0})^2\Big) - \frac{1}{p_0}\Big((\cos\vartheta_{u_0})^2 - (\cos\vartheta_{u_1})^2\Big).\end{aligned}$$

The identity $\sin(\vartheta_{u_1} + \vartheta_{u_0})\sin(\vartheta_{u_1} - \vartheta_{u_0}) = (\cos\vartheta_{u_0})^2 - (\cos\vartheta_{u_1})^2 = (\sin\vartheta_{u_1})^2 - (\sin\vartheta_{u_0})^2$ together with $\sin(\vartheta_{u_1} - \vartheta_{u_0}) = (-1)^k \sin\delta$ yields

$$\begin{aligned}\delta' &= \left(\frac{1}{p_1} - \frac{1}{p_0}\right)(\cos\vartheta_{u_1})^2 + \big((q_0 - \lambda_0 r_0) - (q_1 - \lambda_1 r_1)\big)(\sin\vartheta_{u_0})^2 \\ &\quad - (-1)^k \left(\frac{1}{p_0} + q_1 - \lambda_1 r_1\right)\sin(\vartheta_{u_0} + \vartheta_{u_1})\sin\delta.\end{aligned} \tag{2.19}$$

We consider the functions

$$f = \left(\frac{1}{p_1} - \frac{1}{p_0}\right)(\cos\vartheta_{u_1})^2 + \big((q_0 - \lambda_0 r_0) - (q_1 - \lambda_1 r_1)\big)(\sin\vartheta_{u_0})^2 \tag{2.20}$$

and

$$h = -(-1)^k \left(\frac{1}{p_0} + q_1 - \lambda_1 r_1\right)\sin(\vartheta_{u_0} + \vartheta_{u_1})\frac{\sin\delta}{\delta}. \tag{2.21}$$

According to (2.19) we have

$$\delta' = f + h\delta, \tag{2.22}$$

where the functions f, h are integrable on (a,c) for all $c \in (a,b)$ due to

$$|f| \le \left|\frac{1}{p_1} - \frac{1}{p_0}\right| + \big|(q_0 - \lambda_0 r_0) - (q_1 - \lambda_1 r_1)\big|, \qquad |h| \le \left|\frac{1}{p_0} + q_1 - \lambda_1 r_1\right|. \tag{2.23}$$

Consider the positive function g given by

$$g(x) = \exp\left(-\int_a^x h(t)\,\mathrm{d}t\right). \tag{2.24}$$

Then

$$(g\delta)' = -\delta hg + (f + h\delta)g = fg \geq 0 \tag{2.25}$$

by (2.20) and (2.17). Hence, $g\delta$ is an increasing function. For $x < \xi$ one has

$$g(x)\big(\vartheta_{u_1}(x) - \vartheta_{u_0}(x) - k\pi\big) = (g\delta)(x) \leq (g\delta)(\xi) = g(\xi)\big(\vartheta_{u_1}(\xi) - \vartheta_{u_0}(\xi) - k\pi\big) \tag{2.26}$$

and for $x > \xi$ the estimate

$$g(x)\big(\vartheta_{u_1}(x) - \vartheta_{u_0}(x) - k\pi\big) = (g\delta)(x) \geq (g\delta)(\xi) = g(\xi)\big(\vartheta_{u_1}(\xi) - \vartheta_{u_0}(\xi) - k\pi\big) \tag{2.27}$$

holds. Now the positivity of g together with (2.26) and (2.27) implies (i)–(iii).

Suppose that (2.18) holds and let $\vartheta_{u_1}(\xi) - \vartheta_{u_0}(\xi) = k\pi$, that is $\delta(\xi) = 0$. To prove the remaining assertions (iv) and (v) we only need to show that there is no $x \in (a,b)\setminus\{\xi\}$ such that $\vartheta_{u_1}(x) - \vartheta_{u_0}(x) = k\pi$. Assume the existence of $x > \xi$ such that, $\vartheta_{u_1}(x) - \vartheta_{u_0}(x) = \vartheta_{u_1}(\xi) - \vartheta_{u_0}(\xi) = k\pi$, that is $\delta(x) = 0$. Then the monotonicity of δg implies that $(g\delta)(t) = \delta(t) = 0$ for all $t \in [\xi, x]$. With (2.25) we see $f \equiv 0$ on $[\xi, x]$. Since (2.18) holds, by (2.20) there is $m \in \mathbb{Z}$ such that $\vartheta_{u_0}(t) = m\pi$ and, thus, $\vartheta_{u_1} = \vartheta_{u_0} + k\pi = (m+k)\pi$ for all $t \in [\xi, x]$. With (2.7) (for $u = u_1$) we conclude $0 = \vartheta_{u_1}' = 1/p_1$ on $[\xi, x]$; a contradiction. This shows (v). A similar reasoning implies (iv). □

Corollary 2.4 (Sturm's comparison theorem)**.** *Suppose that* (2.2) *holds for* $j = 0, 1$. *Let* u_0 *and* u_1 *be non-trivial real-valued solutions of* $(\tau_0 - \lambda_0)u_0 = 0$ *and* $(\tau_1 - \lambda_1)u_1 = 0$, *respectively, for* $\lambda_0, \lambda_1 \in \mathbb{R}$. *Suppose that* x_0 *and* x_1 *are consecutive zeros of* u_0 *in* (a,b).

(i) *If* (2.18) *holds a. e. on* (a,b), *then there is at least one zero* $y \in (x_0, x_1)$ *of* u_1.

(ii) *If* $p_0 = p_1$, $q_0 = q_1$, $r_0 = r_1$ *a. e. on* (a,b) *and* $\lambda_0 = \lambda_1$, *then there is exactly one zero* $y \in (x_0, x_1)$ *of* u_1 *unless* u_0 *and* u_1 *are linearly dependent.*

Proof. Let $\vartheta_{u_0}(x_0) = k\pi$, $\vartheta_{u_0}(x_1) = (k+1)\pi$ and $\vartheta_{u_1}(x_0) \in [j\pi, (j+1)\pi)$ with $k, j \in \mathbb{Z}$. Then in the case (i) one has

$$(j-k)\pi \leq \vartheta_{u_1}(x_0) - \vartheta_{u_0}(x_0) \tag{2.28}$$

and Lemma 2.3 implies

$$(j-k)\pi < \vartheta_{u_1}(x_1) - \vartheta_{u_0}(x_1) = \vartheta_{u_1}(x_1) - (k+1)\pi. \tag{2.29}$$

Therefore, $\vartheta_{u_1}(x_1) > (j+1)\pi$ which yields the existence of $y \in (x_0, x_1)$ with $\vartheta_{u_1}(y) = (j+1)\pi$, that is $u_1(y) = 0$.

We consider the case (ii). If u_0 and u_1 are linearly independent, the Wronskian $W[u_0, u_1]$ has no zero at x_0. Thus, the inequality in (2.28) is strict, cf. (2.16). By Lemma 2.3 one obtains again (2.29) and the existence of a zero $y \in (x_0, x_1)$ of u_1 follows by the same argument as before. The uniqueness of this zero can be seen by reversing the roles of the solutions and applying the same argument. Here, a second zero $\tilde{y} \in (x_0, x_1)$ of u_1 would lead to another zero x_2 of u_0 between y and $\tilde{y}$, that is $x_2 \in (x_0, x_1)$, which contradicts the assumption that x_0 and x_1 are consecutive zeros of u_0. □

The differential expression $\tau_0 - \lambda$, where $\lambda \in \mathbb{R}$, is called *non-oscillatory* if there is a non-trivial real-valued solution u of $(\tau_0 - \lambda)u = 0$ with at most finitely many zeros in (a,b), that is the limit $\lim_{x\to b} N_u(x)$ (which always exists in $\mathbb{Z} \cup \{\infty\}$, see Lemma 2.2) is finite. Otherwise, $\tau_0 - \lambda$ is called *oscillatory*. Note that this definition does, in fact, not depend on the particular solution, as by the

Sturm's comparison theorem, Corollary 2.4 (ii), the zeros of two linearly independent solutions of $(\tau_0 - \lambda)u = 0$ interlace.

The total number of zeros of a solution of $(\tau_0 - \lambda)u = 0$ is closely related to the spectra of the self-adjoint realisations of τ_0. Proposition 2.5 below is an implication of the results in [100, Chapter 14]. In what follows the spectral projector of a self-adjoint operator T in a Hilbert space corresponding to an interval I is denoted by $P_T(I)$.

Proposition 2.5. *Suppose that* (2.2) *holds for* $j = 0$ *and let* T_0 *be any self-adjoint realisation of* τ_0. *For* $-\infty < \lambda < \mu < \infty$ *we consider non-trivial real-valued solutions* u *and* v *of* $(\tau_0 - \lambda)u = 0$ *and* $(\tau_0 - \mu)v = 0$, *respectively. Then*

(i) $\dim \mathcal{R}\big(P_{T_0}((-\infty, \lambda))\big)$ *is finite if and only if* $\lim_{x\to b} N_u(x)$ *is finite or, equivalently,* $\tau_0 - \lambda$ *is non-oscillatory;*

(ii) $\dim \mathcal{R}\big(P_{T_0}((\lambda, \mu))\big)$ *is finite if and only if* $\liminf_{x\to b}\big(N_v(x) - N_u(x)\big)$ *is finite.*

Moreover, T_0 *is semi-bounded from below if and only if there is* $\lambda \in \mathbb{R}$ *such that* $\tau_0 - \lambda$ *is non-oscillatory. In this case* $\tau_0 - \lambda$ *is non-oscillatory for all* $\lambda < \inf \sigma_{\mathrm{ess}}(T_0)$, *and* $\tau_0 - \lambda$ *is oscillatory at* $\lambda = \inf \sigma_{\mathrm{ess}}(T_0)$ *if and only if the set* $\sigma(T_0) \cap (-\infty, \lambda)$ *consists of an infinite sequence of isolated eigenvalues of* T_0 *which converge to* λ.

Proof. Provided that τ_0 is in the limit-circle case at b, all self-adjoint realisations of τ_0 have empty essential spectrum, see Section 1.2. Further, either every self-adjoint realisation of τ_0 or neither of them is semi-bounded from below, see Section 1.2. Hence, if the limit-circle case prevails at b it suffices to show the assertions (i) and (ii) for one particular self-adjoint realisation. We assume that T_0 is a self-adjoint realisation of τ_0 with separated boundary, that is

$$\cos(\alpha)f(a) - \sin(\alpha)(pf')(a) = 0 \quad \text{for all } f \in \mathcal{D}(T_0), \tag{2.30}$$

for some $\alpha \in [0, \pi)$, cf. Corollary 1.2 (ii). If τ_0 is in the limit-point case at b then (2.30) holds as well for some $\alpha \in [0, \pi)$, see Corollary 1.2 (i).

By Sturm's comparison theorem, Corollary 2.4 (ii), the zeros of two linearly independent solutions u and $\tilde{u}$ of $(\tau_0 - \lambda)u = 0$ interlace and one has $|N_u(x) - N_{\tilde{u}}(x)| \leq 1$ for all $x \in (a, b)$ by Lemma 2.2; similar for the solutions of $(\tau_0 - \mu)v = 0$. Therefore, the convergence of the limits in (i) and (ii) does not depend on the particular solutions. We can choose the solutions u, v of $(\tau_0 - \lambda)u = 0$ and $(\tau_0 - \mu)v = 0$, respectively, such that $\vartheta_u(a) = \vartheta_v(a) = \alpha$, where α is the same as in (2.30). Now (i) and (ii) follow from Theorem 14.1 and Theorem 14.2 in [100].

Let now T_0 be an arbitrary self-adjoint realisation of τ_0. Recall that every eigenvalue λ of T_0 has finite multiplicity, since $\dim \mathcal{N}(T_0 - \lambda) \leq 2$. The remaining assertions follow from the fact that for $\lambda \in \mathbb{R}$ the set $(-\infty, \lambda) \cap \sigma(T_0)$ consists of finitely many isolated eigenvalues if and only if $\dim \mathcal{R}\big(P_{T_0}((-\infty, \lambda))\big) < \infty$, which is by (i) equivalent to property that $\tau_0 - \lambda$ is non-oscillatory. □

In order to compare the spectra of operators corresponding to two different Sturm–Liouville expressions τ_0 and τ_1 we employ techniques developed within the framework of relative oscillation in [73, 74, 75]. Since the results in [73, 74, 75] are formulated only for the case where the corresponding weight functions satisfy $r_0 = r_1$, we adapt these techniques to the general case, where $r_0 \neq r_1$ is allowed. In contrast to the classical oscillation theory, which is connected to the zeros of solutions, the concept of relative oscillation focuses on the zeros of the Wronskian determinant in (2.13). This approach was proposed in [48] to obtain exact eigenvalue counts for Sturm–Liouville operator in cases where classical oscillation theory fails; see the discussion in [48].

For two non-trivial, real-valued solutions u_0 and u_1 of $(\tau_0 - \lambda_0)u_0 = 0$ and $(\tau_1 - \lambda_1)u_1 = 0$, respectively, with $\lambda_0, \lambda_1 \in \mathbb{R}$ we define for $x \in (a,b)$

$$N[u_0,u_1](x) := \left\lceil \frac{\vartheta_{u_1}(x) - \vartheta_{u_0}(x)}{\pi} \right\rceil - \left\lfloor \frac{\vartheta_{u_1}(a) - \vartheta_{u_0}(a)}{\pi} \right\rfloor - 1. \tag{2.31}$$

Under suitable assumptions $N[u_0,u_1]$ counts the zeros of the Wronskian $W[u_0,u_1]$.

Lemma 2.6. *Suppose that* (2.2) *is satisfied for* $j = 0, 1$. *Consider* $\lambda_0, \lambda_1 \in \mathbb{R}$ *and let* u_0, u_1 *be non-trivial real-valued solutions of* $(\tau_0 - \lambda_0)u_0 = 0$ *and* $(\tau_1 - \lambda_1)u_1 = 0$, *respectively.*

(i) *Assume that the conditions in* (2.17) *hold a. e. on* (a,b). *Then the function* $N[u_0,u_1] : (a,b) \to \mathbb{Z}$ *increasing with* $N[u_0,u_1](x) \geq -1$ *for all* $x \in (a,b)$.

(ii) *Assume that the conditions in* (2.18) *hold a. e. on* (a,b). *Then for every* $x \in (a,b)$ *the Wronskian* $W[u_0,u_1]$ *has at most finitely many zeros in* (a,x) *and* $N[u_0,u_1](x)$ *equals the number of zeros of* $W[u_0,u_1]$ *in* (a,x).

Proof. We show part (i). Let $a \leq \xi < x < b$ and assume that $\vartheta_{u_1}(\xi) - \vartheta_{u_0}(\xi) \in (k\pi, (k+1)\pi]$ for some $k \in \mathbb{Z}$. Then by Lemma 2.3 (ii) we have $\vartheta_{u_1}(x) - \vartheta_{u_0}(x) > k\pi$ and, therefore,

$$\left\lfloor \frac{\vartheta_{u_1}(\xi) - \vartheta_{u_0}(\xi)}{\pi} \right\rfloor \leq \left\lceil \frac{\vartheta_{u_1}(\xi) - \vartheta_{u_0}(\xi)}{\pi} \right\rceil = (k+1)\pi \leq \left\lceil \frac{\vartheta_{u_1}(x) - \vartheta_{u_0}(x)}{\pi} \right\rceil. \tag{2.32}$$

This shows that $N[u_0,u_1](\xi) \leq N[u_0,u_1](x)$ and with $\xi = a$ we see $N[u_0,u_1](a) \geq -1$.

Under the condition (2.18) the difference of Prüfer angles $\vartheta_{u_1} - \vartheta_{u_0}$ has a similar growth behaviour compared to a single Prüfer angle, cf. Lemma 2.3 (iv), (v) and Lemma 2.1. Hence, assertion (ii) can be proved in the same way as Lemma 2.2 by replacing in the proof the solution u, the single Prüfer angle ϑ_u and N_u with the Wronskian $W[u_0,u_1]$, the difference $\vartheta_{u_1} - \vartheta_{u_0}$ and $N[u_0,u_1]$, respectively. □

If the condition (2.18) is violated, then $N[u_0,u_1]$ does in general not reflect the number of zeros of the Wronskian. In Lemma 2.7 below there is an example, where $N[u_0,u_1](x)$ and the numbers of zeros of $W[u_0,u_1]$ in (a,x) do not coincide, even though condition (2.17) (but not (2.18)) is satisfied.

Lemma 2.7. *Suppose that* (2.2) *holds for* $j = 0$. *Let* u *and* v *be non-trivial real-valued solutions of* $(\tau_0 - \lambda)u = 0$ *for* $\lambda \in \mathbb{R}$. *If* u *and* v *are linearly dependent solutions then* $N[u,v](x) = -1$ *for all* $x \in (a,b)$. *Otherwise* $N[u,v](x) = 0$ *for all* $x \in (a,b)$.

Proof. Since u and v are solutions of the same differential equation the Wronskian $W[u,v]$ is constant on $[a,b)$, cf. (2.14). If u and v are linearly dependent, then the Wronskian vanishes everywhere and due to the representation by means of Prüfer variables in (2.15) we see $\vartheta_v(x) - \vartheta_u(x) = k\pi$ for all $x \in [a,b)$ and a suitable $k \in \mathbb{Z}$. This implies $N[u,v](x) = -1$ for all $x \in (a,b)$. Otherwise, if both functions are linearly independent then the Wronskian has no zeros in $[a,b)$. Hence, the difference of Prüfer angles $\vartheta_v - \vartheta_u$ does not attain any integer multiple of π. By continuity we have $\vartheta_v(x) - \vartheta_u(x) \in (k\pi, (k+1)\pi)$ for all $x \in [a,b)$ and some $k \in \mathbb{Z}$, which shows $N[u,v](x) = 0$ for all $x \in (a,b)$. □

The difference between the number of zeros of the involved solutions provides an estimate on $N[u_0,u_1]$.

Lemma 2.8. *Suppose that* (2.2) *holds for* $j = 0, 1, 2$ *and let* u_j *be a non-trivial real-valued solution of* $(\tau_j - \lambda_j)u_j = 0$, *where* $\lambda_j \in \mathbb{R}$. *Then*

$$N_{u_1}(x) - N_{u_0}(x) - 3 \le N[u_0, u_1](x) \le N_{u_1}(x) - N_{u_0}(x) + 1 \tag{2.33}$$

for all $x \in (a, b)$. *Further,*

$$-N[u_1, u_0](x) - 2 \le N[u_0, u_1](x) \le -N[u_1, u_0](x) \tag{2.34}$$

and

$$N[u_0, u_1](x) + N[u_1, u_2](x) - 1 \le N[u_0, u_2](x) \le N[u_0, u_1](x) + N[u_1, u_2](x) + 1 \tag{2.35}$$

for all $x \in (a, b)$.

Proof. Observe that for all $t, s \in \mathbb{R}$ we have $-\lfloor t \rfloor = \lceil -t \rceil$ and

$$\lceil t \rceil + \lceil s \rceil - 1 \le \lceil t + s \rceil \le \lceil t \rceil + \lceil s \rceil, \qquad -\lceil t \rceil \le \lceil -t \rceil \le -\lceil t \rceil + 1. \tag{2.36}$$

Let $t_j = \vartheta_{u_j}(x)/\pi$ and $s_j = \vartheta_{u_j}(a)/\pi$. Then one has

$$\begin{aligned} N_{u_j}(x) &= \lceil t_j \rceil - \lfloor s_j \rfloor - 1 & N[u_j, u_k] &= \lceil t_k - t_j \rceil - \lfloor s_k - s_j \rfloor - 1 \\ &= \lceil t_j \rceil + \lceil -s_j \rceil - 1, & &= \lceil t_k - t_j \rceil + \lceil s_j - s_k \rceil - 1. \end{aligned} \tag{2.37}$$

We show (2.33). According to (2.36) we see

$$\begin{aligned} N[u_0, u_1](x) &= \lceil t_1 - t_0 \rceil + \lceil s_0 - s_1 \rceil - 1 \\ &\le \lceil t_1 \rceil + \lceil -t_0 \rceil + \lceil s_0 \rceil + \lceil -s_1 \rceil - 1 \le \lceil t_1 \rceil - \lceil t_0 \rceil - \lceil -s_0 \rceil + \lceil -s_1 \rceil + 1 \\ &= N_{u_1}(x) - N_{u_0}(x) + 1 \end{aligned}$$

and

$$\begin{aligned} N[u_0, u_1](x) &\ge \lceil t_1 \rceil + \lceil -t_0 \rceil + \lceil s_0 \rceil + \lceil -s_1 \rceil - 3 \ge \lceil t_1 \rceil - \lceil t_0 \rceil - \lceil -s_0 \rceil + \lceil -s_1 \rceil - 3 \\ &= N_{u_1}(x) - N_{u_0}(x) - 3. \end{aligned}$$

Moreover, by (2.36) and (2.37)

$$N[u_0, u_1] \le -\lceil t_0 - t_1 \rceil - \lceil s_1 - s_0 \rceil + 1 = -N[u_1, u_0](x)$$

and

$$N[u_0, u_1] \ge -\lceil t_0 - t_1 \rceil - \lceil s_1 - s_0 \rceil - 1 = -N[u_1, u_0](x) - 2,$$

which shows (2.34). The estimates in (2.35) can be seen in a similar way. Again by (2.36) and (2.37) we find that

$$\begin{aligned} N[u_0, u_2](x) &= \lceil t_2 - t_0 \rceil + \lceil s_0 - s_2 \rceil - 1 = \lceil t_2 - t_1 + t_1 - t_0 \rceil + \lceil s_0 - s_1 + s_1 - s_2 \rceil - 1 \\ &\le \lceil t_2 - t_1 \rceil + \lceil t_1 - t_0 \rceil + \lceil s_0 - s_1 \rceil + \lceil s_1 - s_2 \rceil - 1 \\ &= N[u_0, u_1](x) + N[u_1, u_2](x) + 1 \end{aligned}$$

and

$$\begin{aligned} N[u_0, u_2](x) &\ge \lceil t_2 - t_1 \rceil + \lceil t_1 - t_0 \rceil + \lceil s_0 - s_1 \rceil + \lceil s_1 - s_2 \rceil - 3 \\ &= N[u_0, u_1](x) + N[u_1, u_2](x) - 1. \end{aligned}$$

□

Lemma 2.9. *Suppose that* (2.2) *holds for* $j = 0, 1, 2$ *and let* u_j *be a non-trivial real-valued solution of* $(\tau_j - \lambda_j)u_j = 0$*, where* $\lambda_j \in \mathbb{R}$*. If*

$$q_0 - \lambda_0 r_0 \geq q_1 - \lambda_1 r_1 \geq q_2 - \lambda_2 r_2 \quad \text{and} \quad p_0 \geq p_1 \geq p_2 \tag{2.38}$$

a. e. on (a, b)*, then*

$$-1 \leq N[u_0, u_1](x) \leq N[u_0, u_2](x) + 2, \qquad -1 \leq N[u_1, u_2](x) \leq N[u_0, u_2](x) + 2 \tag{2.39}$$

for all $x \in (a, b)$*.*

Proof. By (2.35) in Lemma 2.8 we see

$$N[u_0, u_1](x) + N[u_1, u_2](x) - 1 \leq N[u_0, u_2](x).$$

Since $N[u_0, u_1](x) \geq -1$ and $N[u_1, u_2](x) \geq -1$ by Lemma 2.6, we obtain (2.39). □

We introduce the concept of relative oscillation. The following definition is due to Krüger and Teschl [73, 74, 75].

Definition 2.10. Suppose that (2.2) is satisfied for $j = 0, 1$ and let u_j be a non-trivial real-valued solution of $(\tau_j - \lambda_j)u_j = 0$, where $\lambda_j \in \mathbb{R}$. We say $\tau_0 - \lambda_0$ is *relatively non-oscillatory* with respect to $\tau_1 - \lambda_1$ if both limits

$$\underline{N}[u_0, u_1] := \liminf_{x \to b} N[u_0, u_1](x), \quad \overline{N}[u_0, u_1] := \limsup_{x \to b} N[u_0, u_1](x), \tag{2.40}$$

are finite. In this case we write

$$(\tau_0 - \lambda_0) \sim (\tau_1 - \lambda_1). \tag{2.41}$$

Otherwise, $\tau_0 - \lambda_0$ is called *relatively oscillatory* with respect to $\tau_1 - \lambda_1$. ◇

Note that this definition is independent of the choice of the solutions. In fact, for another pair of non-trivial real-valued solutions v_0, v_1 of of $(\tau_0 - \lambda_0)u_0 = 0$ and $(\tau_1 - \lambda_1)u_1 = 0$, respectively, the inequality (2.35) in Lemma 2.8 applied twice together with Lemma 2.7 implies

$$\begin{aligned} N[v_0, v_1](x) &\leq N[v_0, u_0](x) + N[u_0, v_1](x) + 1 \\ &\leq N[v_0, u_0](x) + N[u_0, u_1](x) + N[u_1, v_1](x) + 2 \leq N[u_0, u_1](x) + 2 \end{aligned}$$

and

$$\begin{aligned} N[v_0, v_1](x) &\geq N[v_0, u_0](x) + N[u_0, v_1](x) - 1 \\ &\geq N[v_0, u_0](x) + N[u_0, u_1](x) + N[u_1, v_1](x) - 2 \geq N[u_0, u_1](x) - 4 \end{aligned}$$

for all $x \in (a, b)$. Hence, the limits $\overline{N}[u_0, u_1]$ and $\underline{N}[u_0, u_1]$ are finite if and only if $\overline{N}[v_0, v_1]$ and $\underline{N}[v_0, v_1]$ are finite. Further, the relation $\sim$ established in Definition 2.10 is reflexive (cf. Lemma 2.7), symmetric as well as transitive (cf. (2.35), (2.34) in Lemma 2.8), and, therefore, it is an equivalence relation. Moreover, any two non-oscillatory differential expressions (which satisfy (2.2)) are in the same equivalence class, see Lemma 2.11 below.

Lemma 2.11. *Suppose that* (2.2) *is satisfied for* $j = 0, 1$ *and let* $\lambda_0, \lambda_1 \in \mathbb{R}$*. Further, assume that* $\tau_0 - \lambda_0$ *is non-oscillatory. Then* $(\tau_0 - \lambda_0) \sim (\tau_1 - \lambda_1)$ *if and only if* $\tau_1 - \lambda_1$ *is non-oscillatory.*

Proof. Let u_0 and u_1 be non-trivial real-valued solutions of $(\tau_0 - \lambda_0)u_0 = 0$ and $(\tau_1 - \lambda_1)u_1 = 0$, respectively. The functions N_{u_0} and N_{u_1} are non-negative and increasing by Lemma 2.2. Since $\tau_0 - \lambda_0$ is non-oscillatory the solution u_0 has at most finitely many zeros in (a, b) and, thus, there exists $n \in \mathbb{N}$ such that $0 \le N_{u_0}(x) \le n$ for all $x \in (a, b)$. From (2.33) in Lemma 2.8 we obtain

$$N_{u_1}(x) - (n+3) \le N_{u_1}(x) - N_{u_0}(x) - 3 \le N[u_0, u_1](x)$$
$$\le N_{u_1}(x) - N_{u_0}(x) + 1 \le N_{u_1}(x) + 1$$

for all $x \in (a, b)$. This shows that $\lim_{x\to b} N_{u_1}(x)$ is finite if and only if $(\tau_0 - \lambda_0) \sim (\tau_1 - \lambda_1)$. □

Observe that in the case where (2.17) holds, the monotonicity of the function $N[u_0, u_1]$ (see Lemma 2.6) yields

$$-1 \le \underline{N}[u_0, u_1] = \lim_{x\to b} N[u_0, u_1](x) = \overline{N}[u_0, u_1] \le \infty. \tag{2.42}$$

Since the quantity $N[u_0, u_1](x)$ counts the zeros of the Wronskian $W[u_0, u_1]$ if (2.18) holds (see Lemma 2.6), we obtain the the following.

Lemma 2.12. *Suppose that* (2.2) *is satisfied for* $j = 0, 1$ *and let* u_j *be a non-trivial real-valued solution of* $(\tau_j - \lambda_j)u_j = 0$, *where* $\lambda_j \in \mathbb{R}$. *Provided that the conditions in* (2.18) *hold a. e. on* (a, b), *the number of zeros of the Wronskian* $W[u_0, u_1]$ *in* (a, b) *is finite if and only if* $(\tau_0 - \lambda_0) \sim (\tau_1 - \lambda_1)$.

The next lemma is a consequence of Lemma 2.9.

Lemma 2.13. *Suppose that* (2.2) *is satisfied for* $j = 0, 1, 2$ *and let* $\lambda_0, \lambda_1, \lambda_2 \in \mathbb{R}$. *Further, assume that* $(\tau_0 - \lambda_0) \sim (\tau_2 - \lambda_2)$ *and that condition* (2.38) *holds on* (c, b) *for some* $c \in [a, b)$. *Then* $(\tau_0 - \lambda_0) \sim (\tau_1 - \lambda_1)$ *and* $(\tau_1 - \lambda_1) \sim (\tau_2 - \lambda_2)$.

Proof. Let u_j, $j = 0, 1, 2$, be a non-trivial real-valued solution of $(\tau_j - \lambda_j)u_j = 0$, respectively. It suffices to show that the limits $\underline{N}[u_j, u_{j+1}]$ and $\overline{N}[u_j, u_{j+1}]$, $j = 0, 1$ are finite. Observe that due to the regularity of the endpoint a the finiteness of the limits $\underline{N}[u_j, u_k]$ and $\overline{N}[u_j, u_k]$, $j \ne k$, is not affected by the behaviour of the solutions u_j, u_k on the interval $(a, c]$. Therefore, it is no restriction to assume that (2.38) holds in the whole interval (a, b). Passing to the limit $x \to b$ in (2.39) of Lemma 2.9 yields

$$-1 \le \underline{N}[u_j, u_{j+1}] \le \overline{N}[u_j, u_{j+1}] \le \overline{N}[u_0, u_2] + 2$$

for $j = 0, 1$, where $\overline{N}[u_0, u_2]$ is finite by assumption. □

Finally, we establish the relationship between the concept of relative oscillation and the spectra of Sturm–Liouville operators. The following corollary is an immediate consequence of Lemma 2.11 and Proposition 2.5.

Corollary 2.14. *Suppose that* (2.2) *holds for* $j = 0, 1$ *and let* T_j *be a self-adjoint realisation of* τ_j. *If the operator* T_0 *is semi-bounded from below and* $(\tau_0 - \lambda) \sim (\tau_1 - \lambda)$ *for some* $\lambda < \inf \sigma_{\mathrm{ess}}(T_0)$, *then* T_1 *is semi-bounded from below.*

The next theorem can be found in [75, Theorem 3.8] for the case of equal weight functions $r_0 = r_1$.

Theorem 2.15. *Suppose* (2.2) *is satisfied for* $j = 0, 1$ *and let* T_j *be a self-adjoint realisation of* τ_j. *Further, let* $\lambda_0, \lambda_1 \in \mathbb{R}$ *with* $\lambda_0 < \lambda_1$.

(i) *Then* $\dim \mathcal{R}(P_{T_0}((\lambda_0, \lambda_1))) < \infty$ *if and only if* $(\tau_0 - \lambda_0) \sim (\tau_0 - \lambda_1)$.

(ii) *Suppose that* $\dim \mathcal{R} P_{T_0}((\lambda_0,\lambda_1)) < \infty$ *and* $(\tau_0 - \lambda) \sim (\tau_1 - \lambda)$ *for some* $\lambda \in [\lambda_0, \lambda_1]$. *Then* $\dim \mathcal{R}(P_{T_1}((\lambda_0,\lambda_1))) < \infty$ *if and only if* $(\tau_0 - \mu) \sim (\tau_1 - \mu)$ *for all* $\mu \in [\lambda_0, \lambda_1]$.

Proof. We show (i). Let u_0 and u_1 be non-trivial real-valued solutions of $(\tau_0 - \lambda_0)u_0 = 0$ and $(\tau_0 - \lambda_1)u_1 = 0$, respectively. Then with (2.33) in Lemma 2.8 together with (2.42) one sees

$$\liminf_{x\to\infty}\big(N_{u_1}(x) - N_{u_0}(x)\big) - 3 \le \underline{N}[u_0,u_1] = \overline{N}[u_0,u_1] \le \liminf_{x\to\infty}\big(N_{u_1}(x) - N_{u_0}(x)\big) + 1.$$

Thus, Proposition 2.5 (ii) implies (i).

We show (ii). For every $\mu \in (\lambda_0, \lambda_1]$ we have $\dim \mathcal{R} P_{T_0}((\lambda_0,\mu)) < \infty$ and, thus, by part (i) we see $(\tau_0 - \lambda_0) \sim (\tau_0 - \mu)$ for all $\mu \in [\lambda_0,\lambda_1]$. If $\dim \mathcal{R} P_{T_1}((\lambda_0,\lambda_1)) < \infty$, then following the same argument we see $(\tau_1 - \lambda_0) \sim (\tau_1 - \mu)$ for all $\mu \in [\lambda_0,\lambda_1]$. We have

$$(\tau_0 - \mu) \sim (\tau_0 - \lambda_0) \sim (\tau_0 - \lambda) \sim (\tau_1 - \lambda) \sim (\tau_1 - \lambda_0) \sim (\tau_1 - \mu).$$

On the other hand if $(\tau_0 - \mu) \sim (\tau_1 - \mu)$ for all $\mu \in [\lambda_0,\lambda_1]$ we obtain by transitivity and part (i)

$$(\tau_1 - \lambda_0) \sim (\tau_0 - \lambda_0) \sim (\tau_0 - \lambda_1) \sim (\tau_1 - \lambda_1).$$

By applying (i) once again we see $\dim \mathcal{R} P_{T_1}((\lambda_0,\lambda_1)) < \infty$. □

Corollary 2.16. *Suppose that* (2.2) *holds for* $j = 0, 1$ *and let* T_j *be a self-adjoint realisation of* τ_j. *If* $(\tau_0 - \lambda) \sim (\tau_1 - \lambda)$ *for all* $\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(T_0)$*, then* $\sigma_{\mathrm{ess}}(T_1) \subset \sigma_{\mathrm{ess}}(T_0)$.

Proof. Let $\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(T_0)$. Since $\mathbb{R} \setminus \sigma_{\mathrm{ess}}(T_0)$ is open there exists $\varepsilon > 0$ such that $[\lambda - \varepsilon, \lambda + \varepsilon] \subset \mathbb{R} \setminus \sigma_{\mathrm{ess}}(T_0)$, in particular $\dim \mathcal{R} P_{T_0}((\lambda - \varepsilon, \lambda + \varepsilon)) < \infty$. By assumption $(\tau_0 - \mu) \sim (\tau_1 - \mu)$ for all $\mu \in [\lambda - \varepsilon, \lambda + \varepsilon]$. Hence, from Theorem 2.15 follows $\dim \mathcal{R} P_{T_1}((\lambda - \varepsilon, \lambda + \varepsilon)) < \infty$ which shows $\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(T_1)$. □

In what follows we state criteria for relative non-oscillation in terms of the coefficients of Sturm–Liouville expressions.

Lemma 2.17. *Suppose that* (2.2) *holds for* $j = 0, 1$. *Further, assume that* $p_0 = p_1$ *a. e. on* (a,b). *Let* T_0 *be any self-adjoint realisation of* τ_0 *and consider* $\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(T_0)$. *If*

$$\varepsilon := \operatorname*{ess\,sup}_{x\in(c,b)} \left| \frac{q_1(x) - q_0(x)}{r_0(x)} - \lambda \frac{r_1(x) - r_0(x)}{r_0(x)} \right| < \operatorname{dist}(\lambda, \sigma_{\mathrm{ess}}(T_0)) \tag{2.43}$$

for some $c \in [a,b)$*, then* $(\tau_0 - \lambda) \sim (\tau_1 - \lambda)$.

Proof. Condition (2.43) implies $[\lambda - \varepsilon, \lambda + \varepsilon] \subset \mathbb{R} \setminus \sigma_{\mathrm{ess}}(T_0)$ and

$$q_0 - (\lambda + \varepsilon) r_0 \le q_1 - \lambda r_1 \le q_0 - (\lambda - \varepsilon) r_0 \tag{2.44}$$

a. e. on (c,b). Theorem 2.15 (i) implies $(\tau_0 - \lambda) \sim (\tau_0 - (\lambda + \varepsilon))$ and $(\tau_0 - (\lambda - \varepsilon)) \sim (\tau_0 - (\lambda + \varepsilon))$. Since (2.44) holds near b we obtain from Lemma 2.13 that $(\tau_1 - \lambda) \sim (\tau_0 - (\lambda + \varepsilon))$. Hence, by transitivity $(\tau_0 - \lambda) \sim (\tau_1 - \lambda)$. □

The next theorem is the main result of this section and extends Lemma 4.7 in [74] to the case $r_0 \neq r_1$.

Theorem 2.18. *Suppose that* (2.2) *holds for* $j = 0, 1$*, and let* T_j *be a self-adjoint realisation of* τ_j. *Further, assume the following conditions at the endpoint* b:

(α) $\lim_{x\to b} \frac{r_1(x)}{r_0(x)} = 1, \lim_{x\to b} \frac{p_1(x)}{p_0(x)} = 1, \lim_{x\to b} \frac{q_1(x)-q_0(x)}{r_0(x)} = 0$;

(β) *q_0/r_0 is bounded near b, or $p_0 = p_1$ a. e. near b.*

Then the following assertions hold:

(i) $\sigma_{ess}(T_0) = \sigma_{ess}(T_1)$;

(ii) *T_0 is semi-bounded from below if and only if T_1 is semi-bounded from below;*

(iii) $(\tau_0 - \lambda) \sim (\tau_1 - \lambda)$ *for every* $\lambda \in \mathbb{R} \setminus \sigma_{ess}(T_0)$.

Remark 2.19. Observe that the conditions (α) and (β) in Theorem 2.18 are equivalent to the conditions

(α') $\lim_{x\to b} \frac{r_0(x)}{r_1(x)} = 1, \lim_{x\to b} \frac{p_0(x)}{p_1(x)} = 1, \lim_{x\to b} \frac{q_0(x)-q_1(x)}{r_1(x)} = 0$;

(β') q_1/r_1 is bounded near b, or $p_1 = p_0$ a. e. near b.

In fact, this follows immediately from

$$\frac{q_0-q_1}{r_1} = -\frac{q_1-q_0}{r_0}\cdot\left(\frac{r_0}{r_1}-1+1\right), \qquad \frac{q_1}{r_1} = \left(\frac{q_1-q_0}{r_0}+\frac{q_0}{r_0}\right)\cdot\left(\frac{r_0}{r_1}-1+1\right). \qquad \diamond$$

Remark 2.20. If q_0/r_0 is bounded near b then there exists $c \in (a,b)$ such that

$$\lambda := \operatorname*{ess\,inf}_{x\in(c,b)} \frac{q_0(x)}{r_0(x)} > -\infty$$

and, thus, $q_0 - \lambda r_0 \geq 0$ a. e. on (c,b). This already implies that $\tau_0 - \lambda$ is non-oscillatory, see e. g. Lemma 7.4.1 in [102], and T_0 is semi-bounded from below by Proposition 2.5. ⋄

Proof of Theorem 2.18. We show (iii). Let $\lambda \in \mathbb{R}\setminus\sigma_{ess}(T_0)$ and define

$$\varepsilon(y) := \operatorname*{ess\,sup}_{x\in[y,b)} \left|\frac{p_1(x)}{p_0(x)} - 1\right|, \quad y \in (a,b).$$

Choose $y \in (a,b)$ such that $\varepsilon(y) < 1$. This is possible due to (α). Then we have

$$0 < \big(1-\varepsilon(y)\big)p_0 \leq p_1 \leq \big(1+\varepsilon(y)\big)p_0 \tag{2.45}$$

a. e. on (y,b). We consider the differential expressions

$$\tau_+ = \frac{1}{r_1}\left(-\frac{d}{dx}\big(1+\varepsilon(y)\big)p_0\frac{d}{dx}+q_1\right), \quad \tau_- = \frac{1}{r_1}\left(-\frac{d}{dx}\big(1-\varepsilon(y)\big)p_0\frac{d}{dx}+q_1\right)$$

and

$$\tilde\tau_+ = \frac{1}{r_1}\left(-\frac{d}{dx}p_0\frac{d}{dx}+\frac{q_1-\lambda r_1}{1+\varepsilon(y)}+\lambda r_1\right), \quad \tilde\tau_- = \frac{1}{r_1}\left(-\frac{d}{dx}p_0\frac{d}{dx}+\frac{q_1-\lambda r_1}{1-\varepsilon(y)}+\lambda r_1\right).$$

The two differential equations $(\tau_+ - \lambda)u = 0$ and $(\tilde\tau_+ - \lambda)u = 0$ share the same solutions, as well as the two equations $(\tau_- - \lambda)u = 0$ and $(\tilde\tau_- - \lambda)u = 0$. This, of course, implies

$$(\tau_+ - \lambda) \sim (\tilde\tau_+ - \lambda) \quad \text{and} \quad (\tau_- - \lambda) \sim (\tilde\tau_- - \lambda), \tag{2.46}$$

cf. proof of Lemma 2.7. A straightforward calculation shows

$$\frac{1}{r_0}\left(\frac{q_1-\lambda r_1}{1\pm\varepsilon(y)}+\lambda r_1-q_0-\lambda(r_1-r_0)\right)=\frac{q_1-q_0}{r_0(1\pm\varepsilon(y))}-\lambda\frac{r_1-r_0}{r_0(1\pm\varepsilon(y))}$$
$$\mp\frac{\varepsilon(y)}{1\pm\varepsilon(y)}\left(\frac{q_0}{r_0}-\lambda\right).$$

Observe that by conditions (α), (β) and the definition of $\varepsilon(y)$ the L^∞-norm of the term on the right hand side with respect to the interval (y,b) can be made arbitrarily small by increasing y. Hence we obtain (by possible increasing y)

$$\left|\frac{1}{r_0}\left(\frac{q_1-\lambda r_1}{1\pm\varepsilon(y)}+\lambda r_1-q_0-\lambda(r_1-r_0)\right)\right|<\operatorname{dist}(\lambda,\sigma_{\mathrm{ess}}(T_0))$$

a. e. on (y,b). Lemma 2.17 yields $(\tau_0-\lambda)\sim(\tilde\tau_\pm-\lambda)$ and, thus, by transitivity $(\tilde\tau_+-\lambda)\sim(\tilde\tau_--\lambda)$. Moreover, by transitivity we obtain $(\tau_0-\lambda)\sim(\tau_\pm-\lambda)$ and $(\tau_+-\lambda)\sim(\tau_--\lambda)$. Hence, Lemma 2.13 and (2.45) yield $(\tau_+-\lambda)\sim(\tau_1-\lambda)$ and, finally, by transitivity

$$(\tau_0-\lambda)\sim(\tau_1-\lambda). \tag{2.47}$$

With Remark 2.19 we see that (2.47) holds also for all $\lambda\in\mathbb{R}\setminus\sigma_{\mathrm{ess}}(T_1)$. The remaining assertions (i) and (ii) follow from Corollary 2.16 and Corollary 2.14. □

Lemma 2.21. *Suppose that* (2.2) *holds for* $j=0,1$. *If the conditions (α) and (β) in Theorem 2.18 are satisfied, then* τ_0 *is in the limit-point case at* b *if and only if* τ_1 *is in the limit-point case at* b.

Proof. By Corollary 7.4.1 in [102] a sufficient and necessary condition for τ_j, where $j=0,1$, to be in the limit-point case at b is that the differential expression

$$\tilde\tau_j=\frac{1}{r_j}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p_j\frac{\mathrm{d}}{\mathrm{d}x}+q_j+\tilde q_j\right) \tag{2.48}$$

is in the limit-point case at b, provided that $\tilde q_j$ is a real-valued locally integrable function on (a,b) such that $\tilde q_j/r_j$ is bounded near b.

Assume that τ_0 is in the limit-point case at b. We show that the same holds true for τ_1. Fix $\varepsilon\in(0,1)$. By condition (α) in Theorem 2.18 there is $c\in(a,b)$ such that

$$(1-\varepsilon)p_0<p_1<(1+\varepsilon)p_0,\qquad(1-\varepsilon)r_0<r_1<(1+\varepsilon)r_0 \tag{2.49}$$

a. e. on (c,b). As a consequence for every complex-valued measurable function f on (c,b) one has

$$(1-\varepsilon)\int_c^b|f(t)|^2r_0(t)\,\mathrm{d}t\le\int_c^b|f(t)|^2r_1(t)\,\mathrm{d}t\le(1+\varepsilon)\int_c^b|f(t)|^2r_0(t)\,\mathrm{d}t. \tag{2.50}$$

Therefore, f lies right in $L^2((a,b),r_0)$ if and only if f lies right in $L^2((a,b),r_1)$.

Assume that q_0/r_0 is bounded near b. Then by Remark 2.19 the same is true for q_1/r_1. For $j=0$, 1 set $\tilde q_j=-q_j$, where (2.48) reads as

$$\tilde\tau_j=\frac{1}{r_j}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p_j\frac{\mathrm{d}}{\mathrm{d}x}\right). \tag{2.51}$$

By the above observation $\tilde{\tau}_0$ is in the limit-point case at b and it suffices to show that $\tilde{\tau}_1$ is in the limit-point case at b. For $j = 0, 1$ the differential equation $\tilde{\tau}_j u = 0$ is explicitly solvable with a fundamental system given by functions u_j and v_j, where

$$u_j(x) = \int_c^x \frac{1}{p_j(t)}\,\mathrm{d}t, \qquad v_j(x) = 1. \tag{2.52}$$

We have $v_0 = v_1$ and $(1-\varepsilon)u_1 \le u_0 \le (1+\varepsilon)u_1$ a. e. on (c,b) as a consequence of (2.49). Since, the differential expression $\tilde{\tau}_0$ is in the limit-point case at b, at least one of the solutions u_0 and v_0 does not lie right in $L^2((a,b),r_0)$. Therefore, the same holds true for u_1 and v_1 with respect to $L^2((a,b),r_1)$ by (2.50) and $\tilde{\tau}_1$ is in the limit-point case at b.

Suppose that $p_0 = p_1$ a. e. (and q_0/r_0 is unbounded) on (c,b). We set $\tilde{q}_0 = q_1 - q_0$, where $\tilde{q}_0/r_0$ is bounded near b by condition (α) in Theorem 2.18. With the choice of $\tilde{q}_0$ and $p_0 = p_1$ the differential expression $\tilde{\tau}_0$ in (2.48) satisfies

$$\tilde{\tau}_0 = \frac{1}{r_0}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p_1\frac{\mathrm{d}}{\mathrm{d}x} + q_1\right) \tag{2.53}$$

on (c,b). Furthermore, $\tilde{\tau}_0$ as well as τ_0 is in the limit-point case at b. There is a non-trivial solution u of $\tilde{\tau}_0 u = 0$ which does not lie right in $L^2((a,b),r_0)$ and, hence, by (2.50) does not lie right in $L^2((a,b),r_1)$. Observe that $0 = r_0/r_1(\tilde{\tau}_0 u) = \tau_1 u$ a. e. on (c,b). This implies that τ_1 is in the limit-point case at b.

The reverse implication follows by Remark 2.19 and a similar argument. □

Note that the implication $(\tau_0 - \lambda) \sim (\tau_1 - \lambda)$ in Theorem 2.18 does not apply to boundary points λ of the essential spectrum. Therefore, the above result does not help when studying the accumulation of eigenvalues at the boundary of the essential spectrum. The next lemma is a variant of Kneser's classical result [67] (see also [98, Theorem 9.42, Corollary 9.43]) and addresses this question.

Lemma 2.22. *Consider a Sturm–Liouville differential expression τ_1 on (a,∞), where* (2.2) *for $j = 1$ is satisfied. Assume that the limits of the coefficients*

$$q_\infty := \lim_{x\to\infty} q_1(x), \quad p_\infty := \lim_{x\to\infty} p_1(x), \quad r_\infty := \lim_{x\to\infty} r_1(x) \tag{2.54}$$

exist in $\mathbb{R}$ such that $p_\infty > 0$ and $r_\infty > 0$. Then τ_1 is in the limit-point case at ∞ and every self-adjoint realisation T_1 of τ_1 is semi-bounded from below with

$$\sigma_{\mathrm{ess}}(T_1) = [q_\infty/r_\infty, \infty). \tag{2.55}$$

(i) *If*

$$\limsup_{x\to\infty} x^2\left(q_1(x) - \frac{q_\infty}{r_\infty}r_1(x)\right) < -\frac{p_\infty}{4}, \tag{2.56}$$

then $\sigma(T_1) \cap (-\infty, q_\infty/r_\infty)$ consists of an infinite sequence of isolated eigenvalues of T_1 converging to q_∞/r_∞.

(ii) *If*

$$\liminf_{x\to\infty} x^2\left(q_1(x) - \frac{q_\infty}{r_\infty}r_1(x)\right) > -\frac{p_\infty}{4}, \tag{2.57}$$

the set $\sigma(T_1) \cap (-\infty, q_\infty/r_\infty)$ is finite.

Proof. We compare τ_1 with the simple differential expression with constant coefficients

$$\tau_0 = \frac{1}{r_0}\left(-\frac{d}{dx}p_0\frac{d}{dx} + q_0\right), \quad p_0 \equiv p_\infty, \quad q_0 \equiv q_\infty, \quad r_0 \equiv r_\infty, \tag{2.58}$$

on (a, ∞). It is well-known that the differential expression τ_0 is in the limit-point case at ∞ (and regular at a) and every self-adjoint realisation T_0 of τ_0 is semi-bounded from below with essential spectrum $\sigma_{\mathrm{ess}}(T_0) = [q_\infty/r_\infty, \infty)$, cf. [41, Chapter XIII, Section 7, Theorem 16 (b)]. The conditions of Theorem 2.18 are satisfied and we obtain that T_1 is semi-bounded from below with

$$\sigma_{\mathrm{ess}}(T_1) = \sigma_{\mathrm{ess}}(T_0) = [q_\infty/r_\infty, \infty). \tag{2.59}$$

Further, by Lemma 2.21 τ_1 is in the limit-point case at ∞.

Let (2.56) hold. Then there are $\varepsilon > 0$, $\mu \in \mathbb{R}$ and $c \in (a, \infty)$, $c > 0$, such that

$$x^2\left(q_1(x) - \frac{q_\infty}{r_\infty}r_1(x)\right) < \mu < -\frac{p_\infty + \varepsilon}{4}, \qquad p_1(x) < p_\infty + \varepsilon \tag{2.60}$$

for all $x \in (c, \infty)$. For

$$\tau_+ = -\frac{d}{dx}(p_\infty + \varepsilon)\frac{d}{dx} + \tilde{q}, \quad \text{where } \tilde{q}(x) = \frac{\mu}{x^2}, \tag{2.61}$$

consider the differential equation $\tau_+ u = 0$ on (c, ∞). A straight forward calculation shows that the function u given by

$$u(x) = x^{\frac{1}{2}} \cos\left(\sqrt{-\left(\frac{\mu}{p_\infty + \varepsilon} + \frac{1}{4}\right)} \ln(x)\right) \tag{2.62}$$

is a non-trivial solution of $\tau_+ u = 0$. Since $\mu/(p_\infty + \varepsilon) + 1/4 < 0$, the solution u is real-valued and has infinitely many zeros in (c, ∞). We have $\tilde{q} > q_1 - (q_\infty/r_\infty)r_1$ and $p_\infty + \varepsilon > p_1$ on (c, ∞) by (2.60). Therefore, Sturm's comparison theorem, Corollary 2.4 (i), implies that every non-trivial real-valued solution of $(\tau_1 - q_\infty/r_\infty)u = 0$ has infinitely many zeros in (c, ∞). This shows that $\tau_1 - q_\infty/r_\infty$ is oscillatory. Since T_1 is semi-bounded from below with $\inf \sigma_{\mathrm{ess}}(T_1) = q_\infty/r_\infty$, the assertion in (i) follows from Proposition 2.5.

Let (2.57) hold. Then there are $\varepsilon > 0$, $\mu \in \mathbb{R}$ and $c \in (a, \infty)$, $c > 0$, such that

$$x^2\left(q_1(x) - \frac{q_\infty}{r_\infty}r_1(x)\right) > \mu > -\frac{p_\infty - \varepsilon}{4}, \qquad p_1(x) > p_\infty - \varepsilon \tag{2.63}$$

for all $x \in (c, \infty)$. For

$$\tau_- = -\frac{d}{dx}(p_\infty - \varepsilon)\frac{d}{dx} + \tilde{q}, \quad \tilde{q}(x) = \frac{\mu}{x^2}, \tag{2.64}$$

the function u, given by

$$u(x) = x^{\frac{1}{2} + \sqrt{\frac{\mu}{p_\infty - \varepsilon} + \frac{1}{4}}} > 0, \tag{2.65}$$

is a solution of the differential equation $\tau_- u = 0$ on (c, ∞). This shows that $\tau_- - 0$ is non-oscillatory. Assume that $\tau_1 - q_\infty/r_\infty$ is oscillatory. Since $q_1 - (q_\infty/r_\infty)r_1 > \tilde{q}$ and $p_1 > p_\infty - \varepsilon$ on (c, ∞) by (2.63), Sturm's comparison theorem, Corollary 2.4 (i), would imply that $\tau_- - 0$ is oscillatory; a contradiction. Therefore, $\tau_1 - q_\infty/r_\infty$ is non-oscillatory and Proposition 2.5 yields (ii). □

2.2 Perturbations of periodic Sturm–Liouville operators

In this section we consider differential expressions τ_j for $j = 0, 1$ of the form (2.1) on a common open interval (a, ∞), where $a \in \mathbb{R}$. In addition to the conditions in (2.2) we assume that

$$p_0,\ q_0,\ r_0 \text{ are } \omega\text{-periodic on } (a, \infty) \text{ with } \omega > 0 \tag{2.66}$$

and

$$\int_a^\infty \left(\left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + \left| q_1(t) - q_0(t) \right| + \left| r_1(t) - r_0(t) \right| \right) \mathrm{d}t < \infty \tag{2.67}$$

We recall the Floquet theory for periodic Sturm–Liouville equations and collect spectral properties of Sturm–Liouville operators associated with the periodic differential expression τ_0 following [28, 100, 101]. Let $\lambda \in \mathbb{R}$ and denote by $\mathcal{L}$ the two-dimensional complex vector space of solutions of $(\tau_0 - \lambda)u = 0$. By (2.66) for every $f \in \mathcal{L}$ the function $f(\,\cdot\, + \omega)$ is again in $\mathcal{L}$. The map

$$\mathcal{M} : \mathcal{L} \to \mathcal{L}, \quad f \mapsto f(\,\cdot\, + \omega) \tag{2.68}$$

is linear and admits a matrix representation. We consider the two real-valued solutions $\hat{u}$ and $\hat{v}$ of $(\tau_0 - \lambda)u = 0$ which satisfy $\hat{u}(a) = 1$, $(p_0\hat{u}')(a) = 0$ and $\hat{v}(a) = 0$, $(p_0\hat{v}')(a) = 1$. Since $W[\hat{u}, \hat{v}](a) = 1$, these two solutions form a basis of $\mathcal{L}$ and the map $\mathcal{M}$ can be identified with the *monodromy matrix*

$$M = \begin{pmatrix} \hat{u}(a+\omega) & \hat{v}(a+\omega) \\ (p_0\hat{u}')(a+\omega) & (p_0\hat{v}')(a+\omega) \end{pmatrix}.$$

The matrix M is regular with $\det M = W[\hat{u}, \hat{v}](a+\omega) = W[\hat{u}, \hat{v}](a) = 1$. Therefore, all eigenvalues of M are non-zero and, hence, the spectrum of $\mathcal{M}$ and M can be represented in the form

$$\sigma(\mathcal{M}) = \sigma(M) = \left\{ \mathrm{e}^c, \mathrm{e}^{-c} \right\}, \quad \text{where } c \in \mathbb{C}. \tag{2.69}$$

The numbers e^c and e^{-c} are referred to as *Floquet multipliers* and c is called the *Floquet exponent*. Without loss of generality we assume that

$$\operatorname{Re} c \geq 0. \tag{2.70}$$

The eigenvalues e^c and e^{-c} are solutions of the quadratic equation

$$\det(M - z) = z^2 - Dz + 1 = 0, \tag{2.71}$$

where D, the so-called *Hill discriminant*, is given by

$$D = \operatorname{tr}(M) = \hat{u}(a+\omega) + (p_0\hat{v}')(a+\omega) \in \mathbb{R}. \tag{2.72}$$

Hence,

$$\mathrm{e}^{\pm c} = \frac{D}{2} \pm \sqrt{\frac{D^2}{4} - 1} \quad \text{or} \quad \mathrm{e}^{\pm c} = \frac{D}{2} \mp \sqrt{\frac{D^2}{4} - 1}. \tag{2.73}$$

The Hill discriminant and the Floquet exponent are of particular importance for the following analysis. Note that these quantities depend on the spectral parameter λ of the differential equation $(\tau_0 - \lambda)u = 0$. When necessary we emphasize the λ-dependency by writing $D(\lambda)$ for the Hill discriminant and $c(\lambda)$ for the Floquet exponent in order to avoid confusion.

The next lemma is more or less a variant of standard working knowledge in periodic differential operators and is essentially contained in [28, Chapter 1]. For the reader's convenience we provide a short proof.

Lemma 2.23. *Suppose that* (2.2) *for* $j = 0$ *and* (2.66) *hold true. Consider* $\lambda \in \mathbb{R}$ *and let* $c = c(\lambda)$ *be the Floquet exponent and* $D = D(\lambda)$ *the Hill discriminant. Then there exists a pair of linearly independent solutions* $u_0 = u_0(\,\cdot\,, \lambda)$ *and* $v_0 = v_0(\,\cdot\,, \lambda)$ *of* $(\tau_0 - \lambda)u = 0$ *such that the functions* $U_0 = U_0(\,\cdot\,, \lambda)$ *and* $V_0 = V_0(\,\cdot\,, \lambda)$ *given by*

$$U_0(x) = \exp\left(c\frac{x-a}{\omega}\right) \cdot \begin{pmatrix} u_0(x) \\ (p_0 u_0')(x) \end{pmatrix}, \quad V_0(x) = \exp\left(-c\frac{x-a}{\omega}\right) \cdot \begin{pmatrix} v_0(x) \\ (p_0 v_0')(x) \end{pmatrix} \tag{2.74}$$

on (a, ∞) *satisfy the following properties:*

(i) *If* $|D| > 2$*, then* U_0 *and* V_0 *are both* ω*-periodic and bounded on* (a, ∞)*, where* $\mathrm{e}^c, \mathrm{e}^{-c} \in \mathbb{R} \setminus \{0\}$ *with* $\operatorname{Re} c > 0$.

(ii) *If* $|D| < 2$*, then* U_0 *and* V_0 *are both* ω*-periodic and bounded on* (a, ∞)*, where* $\mathrm{e}^c, \mathrm{e}^{-c} \in \mathbb{C} \setminus \mathbb{R}$ *with* $\operatorname{Re} c = 0$*. In particular,* $|u_0|$ *and* $|v_0|$ *are* ω*-periodic bounded functions.*

(iii) *If* $|D| = 2$*, then* U_0 *is* ω*-periodic and bounded on* (a, ∞)*, where* $\mathrm{e}^c = \mathrm{e}^{-c} \in \{-1, 1\}$*, that is* $\operatorname{Re} c = 0$*, and, in particular* $|u_0|$ *is an* ω*-periodic and bounded on* (a, ∞)*. Furthermore,* V_0 *satisfies*

$$\|V_0(x)\|_{\mathbb{C}^2} \leq C\left(1 + \frac{x-a}{\omega}\right) \tag{2.75}$$

on (a, ∞) *for some positive constant* C.

In the cases (i) and (iii) the two solutions u_0 *and* v_0 *can be chosen to be real-valued functions.*

Moreover, the differential expression τ_0 *is in the limit-point case at* ∞*. If* $\lambda \in \mathbb{R}$ *such that* $|D(\lambda)| < 2$ *, then for every non-trivial solution* u *of* $(\tau_0 - \lambda)u = 0$ *there is a positive constant* E *such that*

$$\int_{a+n\omega}^{a+(n+1)\omega} |u(t)|^2 r_0(t)\,\mathrm{d}t \geq E \quad \text{for all } n \in \mathbb{N}. \tag{2.76}$$

Proof. If $D = 2$ or $D = -2$ it follows from (2.73) that $\mathrm{e}^c = \mathrm{e}^{-c} = D/2 \in \{-1, 1\}$ and $\operatorname{Re} c = 0$. In the case where $D > 2$ or $D < -2$ we obtain $\mathrm{e}^c, \mathrm{e}^{-c} \in \mathbb{R} \setminus \{0\}$ by (2.73), where $\mathrm{e}^c \neq \mathrm{e}^{-c}$. Therefore, $\mathrm{e}^{2\operatorname{Re} c} = \mathrm{e}^{2c} \neq 1$ and (2.70) yields $\operatorname{Re} c > 0$. In the remaining case $-2 < D < 2$ the Floquet multipliers $\mathrm{e}^c, \mathrm{e}^{-c}$ are non-real and complex conjugates of each other by (2.73). This yields $1 = \mathrm{e}^c \mathrm{e}^{-c} = |\mathrm{e}^c|^2 = \mathrm{e}^{2\operatorname{Re} c}$ and $\operatorname{Re} c = 0$.

For $|D| \neq 2$ the spectrum of $\mathcal{M}$ in (2.68) consists of the two distinct eigenvalues e^c and e^{-c}. Hence, we find corresponding eigenfunctions u_0 and v_0 of $\mathcal{M}$ in $\mathcal{L}$ satisfying

$$u_0(x + \omega) = (\mathcal{M}u_0)(x) = \mathrm{e}^{-c} u_0(x), \qquad v_0(x + \omega) = (\mathcal{M}v_0)(x) = \mathrm{e}^c v_0(x) \tag{2.77}$$

and by the periodicity of p_0

$$(p_0 u_0')(x + \omega) = \mathrm{e}^{-c}(p_0 u_0')(x), \qquad (p_0 v_0')(x + \omega) = \mathrm{e}^c (p_0 v_0')(x) \tag{2.78}$$

on (a, ∞). By (2.77) and (2.78) the functions U_0 and V_0 given in (2.74) are ω-periodic. This proves (i) and (ii).

Assume that $|D| = 2$. Then the Floquet multipliers e^c and e^{-c} coincide. Consequently, the eigenvalue e^{-c} of $\mathcal{M}$ has algebraic multiplicity two. If the geometric multiplicity of $\mathrm{e}^{-c} = \mathrm{e}^c$ is two, then the same argument as in the case $|D| \neq 2$ yields solutions u_0, v_0 such that the functions U_0 and V_0 given by (2.74) are ω-periodic. In this situation (2.75) holds true on (a, ∞) for $C = \sup_{t \in (a, a+\omega]} \|V_0(t)\|_{\mathbb{C}^2}$ which is finite by definition of V_0 in (2.74) and the regularity of τ_0 at a.

Otherwise, if the geometric multiplicity of e^{-c} is one, then there is Jordan chain of length two, i. e. there are non-trivial solutions $u_0, v_0 \in \mathcal{L}$ such that $\mathcal{M}u_0 = e^{-c}u_0$ and $\mathcal{M}v_0 = e^{-c}v_0 + u_0$. Therefore,

$$u_0(x+\omega) = e^{-c}u_0(x), \qquad v_0(x+\omega) = e^{-c}v_0(x) + u_0(x) \tag{2.79}$$

and together with the periodicity of p_0

$$(p_0u_0')(x+\omega) = e^{-c}(p_0u_0')(x), \qquad (p_0v_0')(x+\omega) = e^{-c}(p_0v_0')(x) + (p_0u_0')(x) \tag{2.80}$$

for all $x \in (a,\infty)$. Then again, U_0 given in (2.74) is ω-periodic. Since $\operatorname{Re} c = 0$, by (2.79) and (2.80) we have for V_0 given in (2.74)

$$\|V_0(x+\omega)\|_{\mathbb{C}^2} = \left\| \begin{pmatrix} e^{-c}v_0(x) + u_0(x) \\ e^{-c}(p_0v_0')(x) + (p_0u_0')(x) \end{pmatrix} \right\|_{\mathbb{C}^2} \leq \|V_0(x)\|_{\mathbb{C}^2} + \|U_0(x)\|_{\mathbb{C}^2}. \tag{2.81}$$

Consider an arbitrary $x \in (a,\infty)$ and let $k \leq (x-a)/\omega < k+1$, where $k \in \mathbb{N}$. Inequality (2.81) gives successively

$$\begin{aligned} \|V_0(x)\|_{\mathbb{C}^2} &\leq \|V_0(x-k\omega)\|_{\mathbb{C}^2} + k\|U_0(x-k\omega)\|_{\mathbb{C}^2} \\ &\leq \|V_0(x-k\omega)\|_{\mathbb{C}^2} + \frac{x-a}{\omega}\|U_0(x-k\omega)\|_{\mathbb{C}^2} \\ &\leq \sup_{t\in(a,a+\omega]} \left(\|V_0(t)\|_{\mathbb{C}^2} + \|U_0(t)\|_{\mathbb{C}^2}\right) \cdot \left(1 + \frac{x-a}{\omega}\right), \end{aligned} \tag{2.82}$$

where the supremum on the right-hand side is finite by the definition of U_0, V_0 in (2.74) and the fact that a is a regular endpoint. This shows (iii).

Since in the cases (i) and (iii) the spectrum of $\mathcal{M}$ is real, $\mathcal{M}$ in (2.68) can be regarded as a mapping in the real vector space of real-valued solutions of $(\tau_0 - \lambda)u = 0$ instead of the complex vector space $\mathcal{L}$. Hence, u_0 and v_0 can be chosen as real-valued solutions.

Observe, that in the case $|D| \leq 2$ the solution u_0 does not lie right in $L^2((a,\infty), r_0)$, and in the case $|D| > 2$ the solution v_0 does not lie right in $L^2((a,\infty), r_0)$. Hence, τ_0 is in the limit-point case at ∞. Finally, to show (2.76) consider $\lambda \in \mathbb{R}$ such that $|D| < 2$ and let u_0, v_0 as in (ii). Choose $d \in \mathbb{R}$ such that $w_0 := u_0 + dv_0$ is orthogonal to v_0 in $L^2((a,a+\omega), r_0)$. According to (2.77) we have

$$\begin{aligned} w_0(x+n\omega) &= e^{-cn}u_0(x) + e^{cn}dv_0(x) = e^{-cn}\big(u_0(x) + dv_0(x)\big) + \left(e^{cn} - e^{-cn}\right)dv_0(x) \\ &= e^{-cn}w_0(x) + \left(e^{cn} - e^{-cn}\right)dv_0(x) \end{aligned} \tag{2.83}$$

for every $n \in \mathbb{N}$. Let $u = \alpha w_0 + \beta v_0$ be a non-trivial linear combination, where $\alpha, \beta \in \mathbb{C}$. Recall that $\operatorname{Re} c = 0$. Then (2.77) and (2.83) together with the orthogonality of w_0 and v_0 imply

$$\begin{aligned} &\int_{a+n\omega}^{a+(n+1)\omega} |u(t)|^2 r_0(t)\,dt \\ &= \int_a^{a+\omega} \left|\alpha\Big(e^{-cn}w_0(t) + \left(e^{cn} - e^{-cn}\right)dv_0(t)\Big) + \beta e^{cn}v_0(t)\right|^2 r_0(t)\,dt \\ &= |\alpha|^2 \int_a^{a+\omega} |w_0(t)|^2 r_0(t)\,dt + \left|\alpha\left(e^{cn} - e^{-cn}\right)d + \beta e^{cn}\right|^2 \int_a^{a+\omega} |v(t)|^2 r_0(t)\,dt. \end{aligned}$$

Finally, one sees that (2.76) holds with $E = |\alpha|^2 \int_a^{a+\omega} |w_0(t)|^2 r_0(t)\,dt$ if $\alpha \neq 0$ and otherwise with $E = |\beta|^2 \int_a^{a+\omega} |v(t)|^2 r_0(t)\,dt$. □

By (2.76) in Lemma 2.23 in the case $|D(\lambda)| < 2$ no non-trivial solution of $(\tau_0 - \lambda)u = 0$ lies right in $L^2((a,\infty), r_0)$. Hence,

$$\{\lambda \in \mathbb{R} \,|\, |D(\lambda)| < 2\} \subset \sigma_{\text{ess}}(T_0)$$

for every self-adjoint realisation T_0 of τ_0, cf. [100, Theorem 11.5] and [101, Satz 13.22]. The relationship between the Hill discriminant and the structure of essential spectrum can be summarised as follows, cf. [28, Section 1.6, Section 4.5] and [100, Section 12 and Appendix to Section 12], see also [101, Sektion 16.1].

Proposition 2.24. *Suppose that* (2.2) *for* $j = 0$ *and* (2.66) *hold true. Then the essential spectrum of any self-adjoint realisation* T_0 *of* τ_0 *is given by*

$$\sigma_{\text{ess}}(T_0) = \{\lambda \in \mathbb{R} \,|\, |D(\lambda)| \le 2\} \tag{2.84}$$

with the boundary

$$\partial\, \sigma_{\text{ess}}(T_0) = \{\lambda \in \mathbb{R} \,|\, |D(\lambda)| = 2\}. \tag{2.85}$$

Further, there are $\mu_0 < \mu_1 \le \mu_2 < \mu_3 \le \mu_4 < \ldots$ *with* $\mu_n \to \infty$ *as* $n \to \infty$ *such that*

$$\sigma_{\text{ess}}(T_0) = \bigcup_{n\in\mathbb{N}} [\mu_{2n}, \mu_{2n+1}]. \tag{2.86}$$

In each gap of the essential spectrum $(-\infty, \mu_0)$ *and* (μ_{2n+1}, μ_{2n+2}), *where* $n \in \mathbb{N}$ *and* $\mu_{2n+1} < \mu_{2n+2}$, *there is at most one eigenvalue of* T_0. *In particular,* T_0 *is semi-bounded from below and (the interior of) the essential spectrum* $\sigma_{\text{ess}}(T_0)$ *is non-empty.*

Note that $\sigma_{\text{ess}}(T_0) = [\mu_0, \infty)$ may happen in Proposition 2.24.

We turn to the differential expression τ_1. As before let $u_0 = u_0(\,\cdot\,,\lambda)$ and $v_0 = v_0(\,\cdot\,,\lambda)$ be the linearly independent solutions of $(\tau_0 - \lambda)u = 0$ provided by Lemma 2.23, where $\lambda \in \mathbb{R}$.

Lemma 2.25. *Suppose that* (2.2) *for* $j = 0, 1$ *and* (2.66), (2.67) *hold true. Consider* $\lambda \in \mathbb{R}$ *and let* $c = c(\lambda)$ *be the Floquet exponent and* $D = D(\lambda)$ *the Hill discriminant corresponding to the differential equation* $(\tau_0 - \lambda)u = 0$. *Then there exists a pair of linearly independent solutions* $u_1 = u_1(\,\cdot\,,\lambda)$ *and* $v_1 = v_1(\,\cdot\,,\lambda)$ *of* $(\tau_1 - \lambda)u = 0$ *such that the following properties hold:*

(i) *If* $|D| > 2$, *that is* $\operatorname{Re} c > 0$, *then*

$$\exp\left(\operatorname{Re} c \frac{x-a}{\omega}\right) \cdot \left\| \begin{pmatrix} u_1(x) \\ (p_1 u_1')(x) \end{pmatrix} - \begin{pmatrix} u_0(x) \\ (p_0 u_0')(x) \end{pmatrix} \right\|_{\mathbb{C}^2} \to 0 \quad \text{as } x \to \infty \tag{2.87}$$

and

$$\left\| \begin{pmatrix} u_1(x) \\ (p_1 u_1')(x) \end{pmatrix} \right\|_{\mathbb{C}^2} \le C \exp\left(-\operatorname{Re} c \frac{x-a}{\omega}\right), \tag{2.88}$$

$$\left\| \begin{pmatrix} v_1(x) \\ (p_1 v_1')(x) \end{pmatrix} \right\|_{\mathbb{C}^2} \le C \exp\left(\operatorname{Re} c \frac{x-a}{\omega}\right) \tag{2.89}$$

on (a,∞), *where* $C = C(\lambda)$ *is a positive constant. In particular,* u_1 *is bounded on* (a,∞).

(ii) *If* $|D| < 2$, *that is* $\operatorname{Re} c = 0$, *then* (2.87), (2.88) *and* (2.89) *hold, and*

$$\left\| \begin{pmatrix} v_1(x) \\ (p_1 v_1')(x) \end{pmatrix} - \begin{pmatrix} v_0(x) \\ (p_0 v_0')(x) \end{pmatrix} \right\|_{\mathbb{C}^2} \to 0 \quad \text{as } x \to \infty. \tag{2.90}$$

In particular, u_1 *and* v_1 *are bounded on* (a,∞).

(iii) *If* $|D| = 2$, *that is* $\operatorname{Re} c = 0$, *and*

$$\int_a^\infty \left(\left| \frac{1}{p_1(t)} - \frac{1}{p_0(t)} \right| + \left| q_1(t) - q_0(t) \right| + \left| r_1(t) - r_0(t) \right| \right) |t| \, \mathrm{d}t < \infty \tag{2.91}$$

holds, then u_1 *satisfies* (2.87) *and* (2.88) *on* (a, ∞). *In particular,* u_1 *is bounded on* (a, ∞).

The solutions in (i) and (ii) can be chosen to be real-valued.

Proof. Let $\lambda \in \mathbb{R}$. We consider the differential equations $\varphi' = A\varphi$ and $\xi' = (A+B)\xi$ in $\mathbb{C}^2$, where

$$A = \begin{pmatrix} 0 & \frac{1}{p_0} \\ q_0 - \lambda r_0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & \frac{1}{p_1} - \frac{1}{p_0} \\ (q_1 - q_0) - \lambda(r_1 - r_0) & 0 \end{pmatrix}.$$

Via the identification $\varphi = (u, p_0 u')^\top$ and $\xi = (v, p_1 v')^\top$ we see that the differential equations $(\tau_0 - \lambda)u = 0$ and $(\tau_1 - \lambda)v = 0$ can be expressed equivalently in form of the first-order systems $\varphi' = A\varphi$ and $\xi' = (A+B)\xi$, respectively. With u_0 and v_0 from Lemma 2.23 we consider the fundamental solution Φ of the system $\varphi' = A\varphi$ given by

$$\Phi(x) = \begin{pmatrix} u_0(x) & v_0(x) \\ (p_0 u_0')(x) & (p_0 v_0')(x) \end{pmatrix}, \quad x \in (a, \infty), \tag{2.92}$$

so that

$$\big(\Phi(t)\big)^{-1} = \frac{1}{W[u_0, v_0]} \begin{pmatrix} (p_0 v_0')(t) & -v_0(t) \\ -(p_0 u_0')(t) & u_0(t) \end{pmatrix}, \quad t \in (a, \infty).$$

According to (2.74) in Lemma 2.23 we find a suitable constant $\tilde{E} > 0$ such that for all $x, t \in (a, \infty)$

$$\begin{aligned} \left\| \Phi(x) \big(\Phi(t)\big)^{-1} \right\|_{\mathbb{C}^{2\times 2}} \le{} & \tilde{E} \mathrm{e}^{\operatorname{Re} c \frac{t-x}{\omega}} \|U_0(x)\|_{\mathbb{C}^2} \|V_0(t)\|_{\mathbb{C}^2} \\ & + \tilde{E} \mathrm{e}^{\operatorname{Re} c \frac{x-t}{\omega}} \|U_0(t)\|_{\mathbb{C}^2} \|V_0(x)\|_{\mathbb{C}^2}. \end{aligned} \tag{2.93}$$

We show (i). Let $|D| > 2$. By Lemma 2.23 (i) one has $\operatorname{Re} c > 0$ and the functions U_0, V_0 are bounded on (a, ∞). Together with (2.93) we arrive at the inequality

$$\left\| \Phi(x) \big(\Phi(t)\big)^{-1} \right\|_{\mathbb{C}^{2\times 2}} \le \frac{E}{2} \left(\mathrm{e}^{\operatorname{Re} c \frac{t-x}{\omega}} + \mathrm{e}^{\operatorname{Re} c \frac{x-t}{\omega}} \right) \tag{2.94}$$

for all $x, t \in (a, \infty)$, where E is a suitable positive constant. Observe that for all $x, t \in (a, \infty)$, where $x \le t$, we have

$$\left\| \Phi(x) \big(\Phi(t)\big)^{-1} \right\|_{\mathbb{C}^{2\times 2}} \le E \mathrm{e}^{-\operatorname{Re} c \frac{x-t}{\omega}}. \tag{2.95}$$

We employ Theorem B.1 from Appendix B. Let $C_\beta(a, \infty)$ for $\beta \in \mathbb{R}$ be the Banach space of continuous $\mathbb{C}^2$-valued functions f on (a, ∞) of exponential growth at the rate β, cf. (B.3) in Appendix B, with the corresponding norm

$$\|f\|_{\infty,\beta} = \sup_{x \in (a,\infty)} \mathrm{e}^{-\beta(x-a)} \|f(x)\|_{\mathbb{C}^2}. \tag{2.96}$$

The solution $(u_0, p_0 u_0')^\top$ of $\varphi' = A\varphi$ is contained in $C_{-\operatorname{Re} c/\omega}(a, \infty)$ by Lemma 2.23 (i) and $\|B(\,\cdot\,)\|_{\mathbb{C}^{2\times 2}} \in L^1(a, \infty)$ by assumption. An application of Theorem B.1 (i), where we consider

$\varphi = (u_0, p_0 u_0')^\top$, $\beta = \operatorname{Re} c/\omega$ and $g \equiv E$, provides a solution $\xi \in C_{-\operatorname{Re} c/\omega}(a,\infty)$ of $\xi' = (A+B)\xi$ satisfying

$$e^{\operatorname{Re} c \frac{x-a}{\omega}} \|\xi(x) - \varphi(x)\|_{\mathbb{C}^2} \to 0 \quad \text{as } x \to 0. \tag{2.97}$$

By identifying $(u_1, p_1 u_1')^\top$ with ξ one immediately sees that u_1 solves $(\tau_1 - \lambda)u = 0$ and satisfies the assertions stated in (i).

Inequality (2.94), further, yields

$$\left\|\Phi(x)\big(\Phi(t)\big)^{-1}\right\|_{\mathbb{C}^{2\times 2}} \le E e^{\operatorname{Re} c \frac{x-t}{\omega}} \tag{2.98}$$

for all x, $t \in (a,\infty)$ with $t \le x$. Note that $C_{-\operatorname{Re} c/\omega}(a,\infty) \subset C_{\operatorname{Re} c/\omega}(a,\infty)$. Therefore, both functions $(u_0, p_0 u_0')^\top$, $(v_0, p_0 v_0')^\top$ and, consequently, all solutions of $\varphi' = A\varphi$ are contained in $C_{\operatorname{Re} c/\omega}(a,\infty)$ by Lemma 2.23 (i). With Theorem B.1 (ii), where $\beta = \operatorname{Re} c/\omega$ and $g(\equiv E$, we find a solution $\xi \in C_{\operatorname{Re} c/\omega}(a,\infty)$ of $\xi' = (A+B)\xi$ which is linearly independent of $(u_1, p_1 u_1')^\top$. The identification $\xi = (v_1, p_1 v_1')^\top$ yields the solution v_1 of $(\tau_1 - \lambda)u = 0$ which does not linearly depend on u_1 and satisfies the inequality in (2.89).

We proceed showing (ii) in a similar manner. Suppose that $|D| < 2$. By Lemma 2.23 (ii) one has $\operatorname{Re} c = 0$ and the solutions $(u_0, p_0 u_0')^\top$, $(v_0, p_0 v_0')^\top$ of $\varphi' = A\varphi$ are both contained in $C_0(a,\infty)$. The same reasoning as before yields the estimate in (2.95) for $a < x \le t < \infty$. We apply Theorem B.1 (i), where $\beta = 0$ and $g \equiv E$, to both solutions $(u_0, p_0 u_0')^\top$ and $(v_0, p_0 v_0')^\top$ of $\varphi' = A\varphi$. This yields a suitable pair of linearly independent solutions of $\xi' = (A+B)\xi$ which are contained in $C_0(a,\infty)$ and can be identified with the solutions u_1, v_1 stated in (ii).

Finally, we prove (iii). Suppose that (2.91) holds and let $|D| = 2$. One has $\operatorname{Re} c = 0$ by Lemma 2.23 (iii) and with (2.93) there exists a positive constant E such that for all x, $t \in (a,\infty)$, where $x \le t$,

$$\left\|\Phi(x)\big(\Phi(t)\big)^{-1}\right\|_{\mathbb{C}^{2\times 2}} \le \frac{E}{2}\left(2 + \frac{t-a}{\omega} + \frac{x-a}{\omega}\right) \le E\left(1 + \frac{t-a}{\omega}\right). \tag{2.99}$$

By Lemma 2.23 (iii) the solution $(u_0, p_0 u_0')^\top$ of $\varphi' = A\varphi$ is contained in $C_0(a,\infty)$. We apply Theorem B.1 (i) to $(u_0, p_0 u_0')^\top$, where $\beta = 0$ and $g(t) = E(1 + (t-a)/\omega)$. Note that $g(\cdot)\|B(\cdot)\|_{\mathbb{C}^{2\times 2}} \in L^1(a,\infty)$ by the assumption (2.91). From Theorem B.1 (i) we obtain a solution ξ of $\xi' = (A+B)\xi$ corresponding to $(u_0, p_0 u_0')^\top$ which is contained in $C_0(a,\infty)$ and can be identified with the solution u_1 stated in (iii).

Note that in the cases (i) and (iii) the solutions u_0 and v_0 from Lemma 2.23 can be chosen to be real-valued. Then Φ in (2.92) has values only in $\mathbb{R}^{2\times 2}$ and the solutions u_1, v_1 constructed via Theorem B.1 are also real-valued. □

Note that condition (2.91) implies (2.67).

Our main result in this section is the following theorem which allows to compare the essential spectra of Sturm–Liouville operators associated with τ_0 and τ_1.

Theorem 2.26. *Suppose that* (2.2) *for* $j = 0, 1$ *and* (2.66), (2.67) *hold true. Then* τ_1 *is in the limit-point case at* ∞. *Let* T_0 *and* T_1 *be self-adjoint realisation of* τ_0 *and* τ_1, *respectively. Then the operator* T_1 *is semi-bounded from below and its essential spectrum is given by*

$$\sigma_{\mathrm{ess}}(T_1) = \sigma_{\mathrm{ess}}(T_0) = \bigcup_{n\in\mathbb{N}} [\mu_{2n}, \mu_{2n+1}],$$

where $\mu_0 < \mu_1 \le \mu_2 < \mu_3 \le \mu_4 < \dots$ *and* $\mu_n \to \infty$ *as* $n \to \infty$ *in accordance with Proposition 2.24.*

If (2.91) *holds, then each gap of the essential spectrum* $(-\infty, \mu_0)$ *and* (μ_{2n+1}, μ_{2n+2})*, where* $n \in \mathbb{N}$ *and* $\mu_{2n+1} < \mu_{2n+2}$*, contains at most finitely many eigenvalues of* T_1.

The above theorem extends a seminal result by Rofe-Beketov [93] (see also [66, Theorem 6.13]) for the special case $p_0 = p_1 = r_0 = r_1 = 1$. A similar stability result for the essential spectrum of periodic Sturm–Liouville operators is shown in [96] for $p_0 = p_1 = r_0 = r_1 = 1$, where the assumption (2.67) is replaced by a weaker condition. An extension of this result to more general coefficients, where still $r_0 = r_1$, can be found in [28, Chapter 5].

Proof of Theorem 2.26. For real λ let $D = D(\lambda)$, $c = c(\lambda)$ denote the Hill discriminant and the Floquet exponent. Further, consider the solutions $u_j = u_j(\cdot, \lambda)$ and $v_j(\cdot, \lambda)$, where $j = 0, 1$, provided by Lemma 2.23 and Lemma 2.25. The proof is divided into four steps.

Step 1. By Proposition 2.24 the interior of $\sigma_{\mathrm{ess}}(T_0)$ is non-empty. Let λ be an arbitrary element of the interior of $\sigma_{\mathrm{ess}}(T_0)$, that is $|D| < 2$. We show that no non-trivial solution w_1 of $(\tau_1 - \lambda)u = 0$ lies right in $L^2((a, \infty), r_1)$. Let $w_1 = \alpha u_1 + \beta v_1$ be an arbitrary non-trivial linear combination, where $\alpha, \beta \in \mathbb{C}$. For the same coefficients α and β let $w_0 = \alpha u_0 + \beta v_0$. Then

$$\begin{aligned}
&\left| \int_{a+n\omega}^{a+(n+1)\omega} |w_1(t)|^2 r_1(t) - |w_0(t)|^2 r_0(t)\, \mathrm{d}t \right| \\
&\le \int_{a+n\omega}^{a+(n+1)\omega} |w_1(t)|^2 |r_1(t) - r_0(t)| + \big| |w_1(t)|^2 - |w_0(t)|^2 \big| r_0(t)\, \mathrm{d}t \qquad (2.100) \\
&\le \int_{a+n\omega}^{a+(n+1)\omega} |w_1(t)|^2 |r_1(t) - r_0(t)| + |w_1(t) - w_0(t)| \cdot \big(|w_1(t)| + |w_0(t)|\big) r_0(t)\, \mathrm{d}t
\end{aligned}$$

By Lemma 2.23 (ii) and Lemma 2.25 (ii) the solutions w_0, w_1 are bounded and $|w_0(t) - w_1(t)| \to 0$ as $t \to \infty$. Together with $(r_0 - r_1) \in L^1(a, \infty)$ and the periodicity of r_0 one sees that the integral on the right-hand side of (2.100) tends to zero as $n \to \infty$. By Lemma 2.23 there exists a positive constant E such that (2.76) holds (for $u = w_0$). There is $n_0 \in \mathbb{N}$ such that the integral on the right-hand side of (2.100) is bounded by $E/2$ for all natural numbers $n \ge n_0$. Thus, (2.100) and (2.76) yield

$$\frac{E}{2} \le \int_{a+n\omega}^{a+(n+1)\omega} |w_0(t)|^2 r_0(t)\, \mathrm{d}t - \frac{E}{2} \le \int_{a+n\omega}^{a+(n+1)\omega} |w_1(t)|^2 r_1(t)\, \mathrm{d}t \qquad (2.101)$$

for all $n \in \mathbb{N}$, $n \ge n_0$. This implies that w_1 does not lie right in $L^2((a, \infty), r_1)$.

Consequently, no non-trivial solution of $(\tau_1 - \lambda)u = 0$ lies right in $L^2((a, \infty), r_1)$ and we see $\lambda \in \sigma_{\mathrm{ess}}(T_1)$, cf. [100, Theorem 11.5] and [101, Satz 13.22]. Since the essential spectra are closed sets we obtain

$$\sigma_{\mathrm{ess}}(T_0) \subset \sigma_{\mathrm{ess}}(T_1).$$

Step 2. We prove the converse inclusion $\sigma_{\mathrm{ess}}(T_1) \subset \sigma_{\mathrm{ess}}(T_0)$. Suppose $\lambda \in \mathbb{R} \setminus \sigma_{\mathrm{ess}}(T_0)$, that is $|D| > 2$ and $\operatorname{Re} c > 0$ by Proposition 2.24 and Lemma 2.23 (i). Note that the solutions u_1 and v_1 provided by Lemma 2.25 (i) are real. For $g \in L^2((a, \infty), r_1)$ let

$$(Sg)(x) := \int_a^\infty \frac{G(x,t)}{W[u_1, v_1]} g(t) r_1(t)\, \mathrm{d}t, \quad G(x,t) := \begin{cases} u_1(x) v_1(t) & \text{if } a \le t \le x, \\ u_1(t) v_1(x) & \text{if } a \le x \le t, \end{cases}$$

that is

$$(Sg)(x) = \frac{1}{W[u_1, v_1]} \left(u_1(x) \int_a^x v_1(t) g(t) r_1(t)\, \mathrm{d}t + v_1(x) \int_x^\infty u_1(t) g(t) r_1(t)\, \mathrm{d}t \right). \tag{2.102}$$

Define

$$E := \sup_{n\in\mathbb{N}} \int_{a+n\omega}^{a+(n+1)\omega} r_1(t)\, \mathrm{d}t,$$

which is finite since $(r_0 - r_1) \in L^1(a, \infty)$ and r_0 is periodic and locally integrable. Consider an arbitrary $x \in [a, \infty)$. By (2.88) and (2.89) in Lemma 2.25 (i)

$$\int_a^\infty |G(x,t)| r_1(t)\, \mathrm{d}t \le C^2 \left(\int_a^x \mathrm{e}^{\operatorname{Re} c \frac{(t-x)}{\omega}} r_1(t)\, \mathrm{d}t + \int_x^\infty \mathrm{e}^{\operatorname{Re} c \frac{(x-t)}{\omega}} r_1(t)\, \mathrm{d}t \right).$$

Let $k \in \mathbb{N}$ with $k\omega + a \le x < (k+1)\omega + a$. We further estimate

$$\begin{aligned}\int_a^\infty |G(x,t)| r_1(t)\, \mathrm{d}t &\le C^2 \sum_{n=0}^{k} \mathrm{e}^{\operatorname{Re} c\cdot(1-n)} \int_{a+(k-n)\omega}^{a+(k+1-n)\omega} r_1(t)\, \mathrm{d}t \\ &\quad + C^2 \sum_{n=0}^{\infty} \mathrm{e}^{\operatorname{Re} c\cdot(1-n)} \int_{a+(n+k)\omega}^{a+(n+1+k)\omega} r_1(t)\, \mathrm{d}t \\ &\le 2C^2 E \sum_{n=0}^{\infty} \mathrm{e}^{\operatorname{Re} c\cdot(-n+1)} < \infty.\end{aligned}$$

Due to the symmetry $G(x,t) = G(t,x)$ the same bound holds for $\int_a^\infty |G(x,t)| r_1(x)\, \mathrm{d}x$ evaluated at $t \in [a, \infty)$. As a consequence of the Schur criterion (e. g. [98, Lemma 0.32]) one sees that S is a bounded operator in $L^2((a,\infty), r_1)$. For $g \in L^2((a,\infty), r_1)$ a straight forward calculation using (2.102) and $(\tau_1 - \lambda)u_1 = (\tau_1 - \lambda)v_1 = 0$ shows that Sg, $p_1(Sg)' \in AC(a, \infty)$, and that Sg solves the inhomogeneous differential equation $(\tau_1 - \lambda)u = g$. Thus, $\tau_1(Sg) = \lambda Sg + g \in L^2((a,\infty), r_1)$ and hence Sg is contained in the domain $\mathscr{D}(\tau_1)$ (cf. (1.6)) of the maximal operator associated with τ_1, and S is injective. Moreover, since u_1 and v_1 are real-valued it follows that S is self-adjoint, so that S^{-1} is a self-adjoint restriction of the maximal operator associated with $\tau_1 - \lambda$. In other words, S is the resolvent at λ of some self-adjoint realisation of τ_1 and as all self-adjoint realisations of τ_1 have the same essential spectrum (cf. Section 1.2), we obtain

$$\lambda \notin \sigma_{\mathrm{ess}}(T_1).$$

Thus $\sigma_{\mathrm{ess}}(T_1) \subset \sigma_{\mathrm{ess}}(T_0)$ and together with the first step

$$\sigma_{\mathrm{ess}}(T_1) = \sigma_{\mathrm{ess}}(T_0).$$

Step 3. Recall that the periodic Sturm–Liouville operator T_0 is semi-bounded from below. Let $\lambda < \inf \sigma_{\mathrm{ess}}(T_0)$, that is $|D| > 2$ and $\operatorname{Re} c > 0$ by Proposition 2.24 and Lemma 2.23 (i). The solution u_0 and u_1 provided by Lemma 2.23 (i) and Lemma 2.25 (i) are real-valued. By Proposition 2.5 the solution u_0 has at most finitely many zeros in (a, ∞). Further, Lemma 2.23 (i) implies that the function $\tilde{u}_0$ given by

$$\tilde{u}_0(x) = \mathrm{e}^{c\frac{x-a}{\omega}} u_0(x) \tag{2.103}$$

is ω-periodic. Therefore, the solution u_0 has no zeros and

$$\gamma := \inf_{t\in(a,\infty)} \left|\tilde{u}_0(t)\right| = \min_{t\in[a,a+\omega]} \left|\tilde{u}_0(t)\right| > 0. \tag{2.104}$$

Assume that T_1 is not semi-bounded from below. Then the differential expression $\tau_1 - \lambda$ is oscillatory by Proposition 2.5 and the solution u_1 of $(\tau_1 - \lambda)u = 0$ has infinitely many zeros $x_0 < x_1 < x_2 < \dots$ accumulating at ∞, cf. Lemma 2.2. With (2.87) we obtain

$$0 < \gamma \leq \left|\tilde{u}_0(x_k)\right| = \left|e^{c\frac{x_n-a}{\omega}} u_0(x)\right| = e^{\operatorname{Re} c\frac{x_n-a}{\omega}} \left|u_0(x_n) - u_1(x_n)\right| \to 0 \quad \text{as } n \to \infty; \tag{2.105}$$

a contradiction. This shows the semi-boundedness of T_1.

Step 4. Suppose that (2.91) holds. We show that every gap of the essential spectrum of T_1 contains at most finitely many eigenvalues of T_1. The proof follows a similar scheme as in the third step, but instead of counting the zeros of solutions we count the zeros of Wronskians. Let $\mu, \lambda \in \mathbb{R}$ such that $\mu < \lambda$ with $\sigma_{\mathrm{ess}}(T_0) \cap (\mu,\lambda) = \sigma_{\mathrm{ess}}(T_1) \cap (\mu,\lambda) = \emptyset$. We have

$$\lambda,\ \mu \in \partial\sigma_{\mathrm{ess}}(T_0) \cup \big(\mathbb{R} \setminus \sigma_{\mathrm{ess}}(T_0)\big). \tag{2.106}$$

Let $c(\lambda)$, $c(\mu)$ be the Floquet exponents and $D(\lambda)$, $D(\mu)$ be the Hill discriminants associated with $(\tau_0 - \lambda)u = 0$ and $(\tau_0 - \mu)u = 0$, respectively. Then $|D(\mu)| \geq 2$ and $|D(\lambda)| \geq 2$ by Proposition 2.24. For the real-valued solutions $u_j(\,\cdot\,,\lambda)$ and $u_j(\,\cdot\,,\mu)$, where $j = 0, 1$, provided by Lemma 2.23 (i), (iii) and Lemma 2.25 (i), (iii) we consider the Wronskian

$$W_j(x) := W[u_j(\,\cdot\,,\mu), u_j(\,\cdot\,,\lambda)](x) = \begin{pmatrix} u_j(x,\lambda) \\ p_j(x)u_j'(x,\lambda)\end{pmatrix}^{\top} \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} \begin{pmatrix} u_j(x,\mu) \\ p_j(x)u_j'(x,\mu)\end{pmatrix} \tag{2.107}$$

Observe that

$$\tilde{W}_0(x) := \exp\left(\big(c(\lambda) + c(\mu)\big)\frac{x-a}{\omega}\right) W_0(x) = \big(U_0(x,\lambda)\big)^{\top} \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} U_0(x,\mu), \tag{2.108}$$

where $U_0(\,\cdot\,,\lambda)$ and $U_0(\,\cdot\,,\mu)$ are ω-periodic functions given by (2.74) in Lemma 2.23. Therefore, the function $\tilde{W}_0$ is ω-periodic. Since $\dim \mathcal{R}(P_{T_0}((\mu,\lambda))) \leq 1$ by Proposition 2.24, the differential expression $\tau_0 - \mu$ is relatively non-oscillatory with respect to $\tau_0 - \lambda$ by Theorem 2.15 (i). This implies that the Wronskian W_0 has at most finitely many zeros in (a,∞), cf. Lemma 2.12. According to the periodicity of $\tilde{W}_0$ together with (2.108), the Wronskian W_0 has no zeros and

$$\gamma := \inf_{t\in(a,\infty)} \left|\tilde{W}_0(t)\right| = \min_{t\in[a,a+\omega]} \left|\tilde{W}_0(t)\right| > 0. \tag{2.109}$$

The difference of the Wronskians W_0 and W_1 can written as

$$\begin{aligned} &W_0(x) - W_1(x) \\ &= \left(\begin{pmatrix} u_0(x,\lambda) \\ (p_0(x)u_0'(x,\lambda)\end{pmatrix} - \begin{pmatrix} u_1(x,\lambda) \\ p_1(x)u_1'(x,\lambda)\end{pmatrix}\right)^{\top} \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} \begin{pmatrix} u_0(x,\mu) \\ p_0(x)u_0'(x,\mu)\end{pmatrix} \\ &\quad + \begin{pmatrix} u_1(x,\lambda) \\ p_1(x)u_1'(x,\lambda)\end{pmatrix}^{\top} \begin{pmatrix} 0 & -1 \\ 1 & 0\end{pmatrix} \left(\begin{pmatrix} u_0(x,\mu) \\ p_0(x)u_0'(x,\mu)\end{pmatrix} - \begin{pmatrix} u_1(x,\mu) \\ p_1(x)u_1'(x,\mu)\end{pmatrix}\right). \end{aligned}$$

Combining this with Lemma 2.23 (i), (iii) and Lemma 2.25 (i), (iii) we conclude

$$\exp\Big(\big(c(\lambda)+c(\mu)\big)\frac{x-a}{\omega}\Big)\cdot\big(W_0(x)-W_1(x)\big)\to 0 \quad \text{as } x\to\infty. \tag{2.110}$$

We assume that $\dim\mathcal{R}(P_{T_1}((\mu,\lambda)))=\infty$, that is T_1 has infinitely many eigenvalues in (μ,λ). Then by Theorem 2.15 (i) and Lemma 2.12 the Wronskian W_1 has infinitely many zeros $x_0<x_1<x_2<\dots$ which necessarily accumulate at ∞, cf. Lemma 2.6 (ii). Then (2.109) and (2.110) imply

$$\begin{aligned}
0<\gamma &\le \big|\tilde{W}_0(x_n)\big|\\
&= \Big|\exp\Big(\big(c(\lambda)+c(\mu)\big)\frac{x_n-a}{\omega}\Big)\,W_0(x_n)\Big|\\
&= \Big|\exp\Big(\big(c(\lambda)+c(\mu)\big)\frac{x_n-a}{\omega}\Big)\,\big(W_0(x_n)-W_1(x_n)\big)\Big| \to 0 \quad \text{as } n\to\infty;
\end{aligned}$$

a contradiction. Hence, $\dim\mathcal{R}(P_{T_1}((\mu,\lambda)))<\infty$. □

Chapter 3

Indefinite Sturm–Liouville operators

In this chapter we study the spectral properties of Sturm–Liouville operators associated with the differential expression

$$\ell = \frac{1}{r}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p\frac{\mathrm{d}}{\mathrm{d}x} + q\right) \tag{3.1}$$

on an open interval (a, b), where $-\infty \le a < b \le \infty$. For the coefficients of ℓ we assume that

$$\begin{cases} p,\ q,\ r \text{ are real-valued functions on } (a,b), \\ p(x) > 0,\ r(x) \neq 0 \text{ for almost all } x \in (a,b), \\ 1/p,\ q,\ r \in L^1_{\mathrm{loc}}(a,b) \end{cases} \tag{3.2}$$

with

$$\mu\big(\{x \in (a,b) \,|\, r(x) > 0\}\big) > 0 \quad \text{and} \quad \mu\big(\{x \in (a,b) \,|\, r(x) < 0\}\big) > 0, \tag{3.3}$$

where μ denotes the Lebesgue measure. Sturm–Liouville differential expressions of this type and associated operators are referred to as *indefinite*.

3.1 Self-adjoint realisations in Krein spaces

As in Chapter 1 we denote by $L^2((a,b),r)$ the Hilbert space (of equivalence classes) of measurable functions $f : (a,b) \to \mathbb{C}$ such that rf^2 is integrable on (a,b), with the scalar product $\langle\cdot,\cdot\rangle_r$,

$$\langle f, g\rangle_r := \int_a^b f(t)\overline{g(t)}|r(t)|\,\mathrm{d}t, \qquad f,\ g \in L^2((a,b),r).$$

The weight function r induces a Krein space. Let

$$J : L^2((a,b),r) \to L^2((a,b),r), \qquad (Jf)(x) = \mathrm{sgn}(r(x))f(x). \tag{3.4}$$

Then $J = J^{-1} = J^*$ and the space $L^2((a,b),r)$ equipped with the (indefinite) inner product $[\cdot,\cdot]_r$,

$$[f,g]_r := \langle Jf, g\rangle_r = \int_a^b f(t)\overline{g(t)}r(t)\,\mathrm{d}t, \qquad f,\ g \in L^2((a,b),r), \tag{3.5}$$

is a Krein space with fundamental symmetry J. For the geometrical structure of Krein spaces we refer to the monographs [3, 27]. All topological notations in the Krein space $(L^2((a,b),r),\ [\cdot,\cdot]_r)$ are understood with respect to the topology induced by the norm corresponding to the scalar product $\langle\cdot,\cdot\rangle_r$. Note that any two Banach space norms on $L^2((a,b),r)$ such that the inner product $[\cdot,\cdot]_r$ is continuous (in one, or equivalently, in both arguments) with respect to each of these norms, are equivalent, see [77, Proposition 1.2].

For a densely defined operator A in $L^2((a,b),\infty)$ the adjoint operator with respect to the inner product $[\cdot,\cdot]_r$ is given by

$$A^+ := JA^*J \tag{3.6}$$

and one has

$$[Af,g]_r = [f,A^+g]_r \quad \text{for all } f \in \mathcal{D}(A),\ g \in \mathcal{D}(A^+). \tag{3.7}$$

If $A = A^+$ then A is called self-adjoint in the Krein space $(L^2(a,b),r),\ [\cdot,\cdot]_r)$. The fundamental symmetry J in (3.4) establishes a one-to-one correspondence between the self-adjoint operators in the Krein space $(L^2(a,b),r),\ [\cdot,\cdot]_r)$ and the self-adjoint operators in the Hilbert space $L^2((a,b),r)$. An operator A,

$$Af := \ell f = \frac{1}{r}\big(-(pf')' + qf\big), \qquad \mathcal{D}(A) \subset L^2((a,b),r), \tag{3.8}$$

is a self-adjoint realisation of ℓ in the Krein space $(L^2(a,b),r),\ [\cdot,\cdot]_r)$ if and only if

$$T = JA, \qquad \mathcal{D}(T) = \mathcal{D}(A), \tag{3.9}$$

is a self-adjoint realisation of the definite differential expression

$$\tau = \frac{1}{|r|}\left(-\frac{\mathrm{d}}{\mathrm{d}x}p\frac{\mathrm{d}}{\mathrm{d}x} + q\right) \tag{3.10}$$

in the Hilbert space $L^2((a,b),r)$. If τ is in the limit-point case at both endpoints, then the only self-adjoint realisation A of ℓ in the Krein space $(L^2(a,b),r),\ [\cdot,\cdot]_r)$ is given by $A = JT_{\max}$, where $T_{\max}$ is the maximal operator associated with τ, cf. (1.9) and Proposition 1.1 (iii).

In contrast to definite Sturm–Liouville operators the self-adjoint realisations of ℓ in the Krein space $(L^2(a,b),r),\ [\cdot,\cdot]_r)$ may have non-real spectral points which may accumulate, see e. g. [12, 13, 79, 92]. In general, for a self-adjoint operator in a Krein space the spectrum, the resolvent set and the set of eigenvalues with finite algebraic multiplicity are symmetric with respect to the real axis, see [27, Chapter XI]. If the corresponding definite differential expression τ is regular at both endpoints, then the spectrum of any self-adjoint realisation of ℓ in the Krein space $(L^2(a,b),r),\ [\cdot,\cdot]_r)$ consists only of isolated eigenvalues of finite algebraic multiplicity, where the non-real spectrum is finite, and the real eigenvalues accumulate at ∞ and $-\infty$, cf. [36].

In the case of at least one singular endpoint the situation is more complicated as there may occur non-empty essential spectrum. If the weight function r has constant definite sign near the endpoints, the essential spectrum can be calculated in terms of definite operators. For an interval $(\alpha,\beta) \subset (a,b)$ let $T_{\min}(\alpha,\beta)$ and $T(\alpha,\beta)$ denote the minimal operator (see (1.11)) and an arbitrary self-adjoint realisation associated with definite expression $\tau \upharpoonright (\alpha,\beta)$ in the Hilbert space $L^2((\alpha,\beta),r)$.

Proposition 3.1. *Suppose that the coefficients of ℓ satisfy* (3.2), (3.3) *and let A be a self-adjoint realisation of ℓ in the Krein space $(L^2(a,b),r),\ [\cdot,\cdot]_r)$. Assume that there are c, $d \in (a,b)$, $c < d$, such that $r \upharpoonright (a,c)$ has a. e. constant sign $s_a \in \{-1,1\}$ and $r \upharpoonright (d,b)$ has a. e. constant sign $s_b \in \{-1,1\}$. Then*

$$\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}\big(s_a T(a,c)\big) \cup \sigma_{\mathrm{ess}}\big(s_b T(d,b)\big). \tag{3.11}$$

In particular, the essential spectrum of A is real.

Proof. We identify the space $L^2((a,b),r)$ with the orthogonal sum

$$L^2((a,c),r) \oplus L^2((c,d),r) \oplus L^2((d,b),r). \tag{3.12}$$

Let $J(c,d) = J \restriction L^2((c,d),r)$ and consider

$$\tilde{A} := J\begin{pmatrix} T(a,c) & 0 & 0 \\ 0 & T(c,d) & 0 \\ 0 & 0 & T(d,b) \end{pmatrix} = \begin{pmatrix} s_a T(a,c) & 0 & 0 \\ 0 & J(c,d)T(c,d) & 0 \\ 0 & 0 & s_b T(d,b) \end{pmatrix}. \tag{3.13}$$

Then by Lemma 1.3 the operators $T = JA$ and $\tilde{T} = J\tilde{A}$ are both self-adjoint finite-dimensional extensions of

$$T_0 = \begin{pmatrix} T_{\min}(a,c) & 0 & 0 \\ 0 & T_{\min}(c,d) & 0 \\ 0 & 0 & T_{\min}(d,b) \end{pmatrix} \tag{3.14}$$

in the Hilbert space $L^2((a,b),r)$. Hence, the closed operators A and $\tilde{A}$ are finite-dimensional extensions of JT_0 and share the same essential spectrum, see e. g. [40, Lemma 11.3.2]. Together with (3.13) this shows

$$\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(\tilde{A}) = \sigma_{\text{ess}}(s_a T(a,c)) \cup \sigma_{\text{ess}}(J(c,d)T(c,d)) \cup \sigma_{\text{ess}}(s_b T(d,b)). \tag{3.15}$$

Since the operator $J(c,d)T(c,d)$ is a regular (possibly indefinite) Sturm–Liouville operator it has empty essential spectrum, cf. [36]. □

Remark 3.2. If in addition to $\sigma_{\text{ess}}(A) \subset \mathbb{R}$ the resolvent set of A is non-empty, then the set $\sigma(A) \setminus \sigma_{\text{ess}}(A)$ is countable, with no accumulation points in $\mathbb{C} \setminus \sigma_{\text{ess}}(A)$, and consists of isolated eigenvalues of A with finite algebraic multiplicity, cf. [50, Chapter XVII, Theorem 2.1]. Sufficient conditions implying $\rho(A) \neq \emptyset$ are discussed in [16, 87]. In the situation where τ is, for instance, in the limit-point case at both endpoints, the semi-boundedness of the maximal operator $T_{\max}$, or of at least one of the operators $T(a,c)$ and $T(d,b)$ guarantees $\rho(A) \neq \emptyset$, cf. [16, Theorem 4.5] and [87, Satz 2.14]. ◇

3.2 Stability of the essential spectrum under perturbation

In this section we study the stability of the essential spectra of indefinite Sturm–Liouville operators under perturbations. Let

$$\ell_j = \frac{1}{r_j}\left(-\frac{\mathrm{d}}{\mathrm{d}x} p_j \frac{\mathrm{d}}{\mathrm{d}x} + q_j\right) \tag{3.16}$$

for $j = 0, 1$ be indefinite Sturm–Liouville differential expressions on a joint open interval (a,b), where $-\infty \leq a < b \leq \infty$, such that the coefficients satisfy

$$\begin{cases} p_j,\ q_j,\ r_j \text{ are real-valued functions on } (a,b), \\ p_j(x) > 0,\ r_j(x) \neq 0 \text{ for almost all } x \in (a,b), \\ 1/p_j,\ q_j,\ r_j \in L^1_{\text{loc}}(a,b) \end{cases} \tag{3.17}$$

and

$$\mu\big(\{x \in (a,b) \,|\, r_j(x) > 0\}\big) > 0 \quad \text{and} \quad \mu\big(\{x \in (a,b) \,|\, r_j(x) < 0\}\big) > 0. \tag{3.18}$$

The corresponding definite expressions are denoted by τ_0 and τ_1.

Given a self-adjoint realisation A_0 of ℓ_0 in $(L^2((a,b),r_0),\ [\cdot,\cdot]_{r_0})$ and a self-adjoint realisation A_1 of ℓ_1 in $(L^2((a,b),r_1),\ [\cdot,\cdot]_{r_1})$ we provide conditions in terms of the coefficients of ℓ_0 and ℓ_1 such that

$$\sigma_{\text{ess}}(A_0) = \sigma_{\text{ess}}(A_1).$$

We are particularly interested in the case $r_0 \neq r_1$. The main problem in this situation is, of course, the fact that both operators act in different Hilbert spaces. Therefore, standard perturbation results (see e. g. [64]) are not directly applicable.

In what follows let $T_j(\alpha,\beta)$, $j = 0, 1$, for $(\alpha,\beta) \subset (a,b)$ be an arbitrary self-adjoint realisation of the definite expression $\tau_j \upharpoonright (\alpha,\beta)$ in $L^2((\alpha,\beta), r_j)$.

Theorem 3.3. *Suppose that* (3.17), (3.18) *hold for* $j = 0, 1$ *and let* A_j *be any self-adjoint realisation of* ℓ_j *in the Krein space* $\big(L^2((a,b),r_j),\ [\cdot,\cdot]_{r_j}\big)$. *For each endpoint* $e \in \{a,b\}$ *suppose that the coefficients of* ℓ_0 *and* ℓ_1 *satisfy*

(α) $\lim_{x\to e} \dfrac{r_1(x)}{r_0(x)} = 1,\ \lim_{x\to e} \dfrac{p_1(x)}{p_0(x)} = 1,\ \lim_{x\to e} \dfrac{q_1(x)-q_0(x)}{r_0(x)} = 0;$

(β) q_0/r_0 *is bounded near e, or* $p_0 = p_1$ *in a neighbourhood of e;*

(γ) r_0 *has constant definite sign near e.*

Then

$$\sigma_{\mathrm{ess}}(A_1) = \sigma_{\mathrm{ess}}(A_0) \subset \mathbb{R}. \tag{3.19}$$

Proof. By condition (γ) we find $a < c < d < b$ such that $r_0 \upharpoonright (a,c)$ has definite sign $s_a \in \{-1,1\}$, and $r \upharpoonright (d,b)$ has definite sign $s_b \in \{-1,1\}$. Then by condition (α) we have $|r_1/r_0 - 1| < 1/2$ and, thus,

$$0 < \frac{s_a r_0}{2} < s_a r_1$$

near a. This shows that the sign of r_1 is constant in a neighbourhood of a and coincides there with s_a; without loss of generality we assume that this neighbourhood equals (a,c). A similar argument shows that the sign of r_1 equals s_b (without loss of generality) on (d,b). Theorem 2.18 implies $\sigma_{\mathrm{ess}}(T_0(d,b)) = \sigma_{\mathrm{ess}}(T_1(d,b))$. Similarly, $\sigma_{\mathrm{ess}}(T_0(a,c)) = \sigma_{\mathrm{ess}}(T_1(a,c))$. From Proposition 3.1 we obtain

$$\begin{aligned}\sigma_{\mathrm{ess}}(A_1) &= \sigma_{\mathrm{ess}}(s_a T_1(a,c)) \cup \sigma_{\mathrm{ess}}(s_b T_1(d,b))\\ &= \sigma_{\mathrm{ess}}(s_a T_0(a,c)) \cup \sigma_{\mathrm{ess}}(s_b T_0(d,b)) = \sigma_{\mathrm{ess}}(A_0) \subset \mathbb{R}.\end{aligned}$$ □

We proceed with a perturbation result for indefinite periodic Sturm–Liouville operators. The spectral properties of indefinite Sturm–Liouville operators with periodic coefficients are studied in [88]. In the next theorem the coefficients of the unperturbed operator are required to be periodic at least near the endpoints.

Theorem 3.4. *Suppose that* (3.17), (3.18) *hold for* $j = 0, 1$, *where* $(a,b) = \mathbb{R}$, *and let* A_j *be self-adjoint realisations of* ℓ_j *in the Krein space* $(L^2(\mathbb{R}, r_j),\ [\cdot,\cdot]_{r_j})$. *Assume that the following conditions hold:*

(α) *there exist* $c, d \in \mathbb{R}$, $c < d$, *such that* p_0, q_0, r_0 *are* ω*-periodic,* $\omega > 0$, *on* (d,∞) *and* θ*-periodic,* $\theta > 0$, *on* $(-\infty,c)$*;*

(β) *both functions* r_0 *and* r_1 *have constant definite sign a. e. on* $(-\infty,c)$ *and constant definite sign a. e. on* (d,∞)*;*

(γ) *the coefficients* p_1, q_1, r_1 *satisfy*

$$\int_{\mathbb{R}} \left(\left| \frac{1}{p_0(t)} - \frac{1}{p_1(t)} \right| + \left| q_0(t) - q_1(t) \right| + \left| r_0(t) - r_1(t) \right| \right) \, \mathrm{d}t < \infty. \tag{3.20}$$

Then

$$\sigma_{\mathrm{ess}}(A_1) = \sigma_{\mathrm{ess}}(A_0) \subset \mathbb{R} \quad \textit{and} \quad \rho(A_1) \neq \emptyset. \tag{3.21}$$

Proof. Observe that (3.20) implies that r_0 and r_1 have the same signs near ∞ and near $-\infty$. In fact, if the signs would differ for instance near ∞ then

$$\int_d^\infty |r_1(t) - r_0(t)| \, \mathrm{d}t \geq \int_d^\infty |r_0(t)| \mathrm{d}t, \tag{3.22}$$

where the left integral exists and the right integral diverges because r_0 is periodic on (d, ∞); a contradiction. Let $s_{-\infty}$ and s_∞ denote the joint sign of r_0 and r_1 near $-\infty$ and ∞, respectively. Since $\tau_0 \upharpoonright (d, \infty)$ is periodic, Theorem 2.26 yields $\sigma_{\mathrm{ess}}(T_0(d, \infty)) = \sigma_{\mathrm{ess}}(T_1(d, \infty))$. Similarly one has $\sigma_{\mathrm{ess}}(T_0(-\infty, c)) = \sigma_{\mathrm{ess}} T_1(-\infty, c)$. Finally, Proposition 3.1 implies

$$\begin{aligned} \sigma_{\mathrm{ess}}(A_1) &= \sigma_{\mathrm{ess}}(s_{-\infty} T_1(-\infty, c)) \cup \sigma_{\mathrm{ess}}(s_\infty T_1(-\infty, c)) \\ &= \sigma_{\mathrm{ess}}(s_{-\infty} T_0(-\infty, c)) \cup \sigma_{\mathrm{ess}}(s_\infty T_0(d, \infty)) = \sigma_{\mathrm{ess}}(A_0) \subset \mathbb{R}. \end{aligned} \tag{3.23}$$

A further consequence of Theorem 2.26 is that the corresponding definite differential expression τ_1 is in the limit-point case at the endpoint ∞ and $T_1(d, \infty)$ is semi-bounded from below. Similarly one has that τ_1 is in the limit-point case at $-\infty$. Hence, by Remark 3.2 the resolvent set of A_1 is non-empty. □

Recall that the essential spectra of the periodic definite operators $T_0(-\infty, c)$ and $T_0(d, \infty)$ in the previous proof have a special band structure,

$$\sigma_{\mathrm{ess}}(T_0(-\infty, c)) = \bigcup_{n \in \mathbb{N}} [\mu_{2n}^-, \mu_{2n+1}^-], \qquad \sigma_{\mathrm{ess}}(T_0(d, \infty)) = \bigcup_{n \in \mathbb{N}} [\mu_{2n}^+, \mu_{2n+1}^+], \tag{3.24}$$

where $\mu_0^\pm < \mu_1^\pm \leq \mu_2^\pm < \mu_3^\pm \leq \mu_4^\pm < \ldots$ with $\mu_n^\pm \to \infty$ as $n \to \infty$, cf. Proposition 2.24.

Corollary 3.5. *Let the conditions of Theorem 3.4 hold. If* r_1 *has negative sign near* $-\infty$ *and positive sign near* ∞ *then*

$$\sigma_{\mathrm{ess}}(A_1) = \sigma_{\mathrm{ess}}(A_0) = \bigcup_{n \in \mathbb{N}} \left([\mu_{2n}^+, \mu_{2n+1}^+] \cup [-\mu_{2n+1}^-, -\mu_{2n}^-] \right). \tag{3.25}$$

3.3 Eigenvalues in the gap of the essential spectrum

If an indefinite self-adjoint Stum–Liouville operator A has real essential spectrum and non-empty resolvent set, then for any gap G of the essential spectrum the set $\sigma(A) \cap G$ consists of countably many isolated eigenvalues of A with finite algebraic multiplicity, cf. Remark 3.2. In this section we analyse the accumulation of the point spectrum at the boundaries of gaps.

Let ℓ_0, ℓ_1 be indefinite Sturm–Liouville expressions as in (3.16) on $\mathbb{R}$ with coefficients satisfying (3.17) and (3.18) for $j = 0, 1$. Unless stated otherwise it is assumed that the corresponding definite differential expressions τ_0 and τ_1 are in the limit-point case at both endpoints. Therefore, there are unique self-adjoint realisations A_0 and A_1 of ℓ_0 and ℓ_1, respectively, in the corresponding Krein spaces. As before let $T_0(\alpha,\beta)$, $T_1(\alpha,\beta)$ for $-\infty \le \alpha < \beta \le \infty$ denote arbitrary self-adjoint operators associated with $\tau_0 \upharpoonright (\alpha,\beta)$, $\tau_1 \upharpoonright (\alpha,\beta)$, respectively.

Our analysis is based on the next proposition.

Proposition 3.6 ([15, Theorem 5.1]). *Consider an indefinite Sturm–Liouville differential expression ℓ_1 on $\mathbb{R}$ and suppose that* (3.17), (3.18) *are satisfied for $j = 1$. Further, assume that the following conditions hold:*

(α) *the corresponding definite differential expression τ_1 is in the limit-point case at both endpoints;*

(α) *there are $c, d \in \mathbb{R}$, $c < d$, such that r_1 is negative a. e. on $(-\infty, c)$ and positive a. e. on (d, ∞);*

(γ) *the operators $T_1(-\infty, c)$ and $T_1(d,\infty)$ are semi-bounded from below, where*

$$m_{-\infty} := \inf \sigma_{\mathrm{ess}}(T_1(-\infty, c)) \quad \text{and} \quad m_\infty := \inf \sigma_{\mathrm{ess}}(T_1(d,\infty)). \tag{3.26}$$

Let A_1 be the unique self-adjoint realisation of ℓ_1 in the Krein space $(L^2(\mathbb{R}, r_1),\ [\cdot,\cdot]_{r_1})$.

(i) *Provided that $-m_{-\infty} < m_\infty$, the set $\sigma(A_1) \cap (-m_{-\infty}, m_\infty)$ accumulates at m_∞ if and only if $\sigma(T_1(d,\infty)) \cap (-\infty, m_\infty)$ accumulates at m_∞, and $\sigma(A_1) \cap (-m_{-\infty}, m_\infty)$ accumulates at $-m_{-\infty}$ if and only if $\sigma(T_1(-\infty, c)) \cap (-\infty, m_{-\infty})$ accumulates at $m_{-\infty}$.*

Let G be an open interval with $G \cap \sigma_{\mathrm{ess}}(A_1) = \emptyset$.

(ii) *If $\overline{G} \subset (-m_{-\infty}, \infty)$ and $\sigma(T_1(d,\infty)) \cap G$ is finite, then $\sigma(A_1) \cap G$ is finite.*

(iii) *If $\overline{G} \subset (-\infty, m_\infty)$ and $\sigma(-T_1(-\infty, c)) \cap G$ is finite, then $\sigma(A_1) \cap G$ is finite.*

We prove a variant of Kneser's result [67] for indefinite Sturm–Liouville operators.

Theorem 3.7. *Consider an indefinite Sturm–Liouville differential expression ℓ_1 on $\mathbb{R}$ and suppose that* (3.17), (3.18) *hold for $j = 1$, where the limits*

$$\begin{aligned} r_\infty &:= \lim_{x\to\infty} r_1(x), & p_\infty &:= \lim_{x\to\infty} p_1(x), & q_\infty &:= \lim_{x\to\infty} q_1(x), \\ r_{-\infty} &:= \lim_{x\to-\infty} r_1(x), & p_{-\infty} &:= \lim_{x\to-\infty} p_1(x), & q_{-\infty} &:= \lim_{x\to-\infty} q_1(x) \end{aligned} \tag{3.27}$$

exist in $\mathbb{R}$ such that $r_\infty > 0$, $p_\infty > 0$, $r_{-\infty} < 0$, $p_{-\infty} > 0$. Then the corresponding definite differential expression τ_1 is in the limit-point case at both endpoints and the uniquely determined self-adjoint realisation A_1 of ℓ_1 in $(L^2(\mathbb{R}, r_1),\ [\cdot,\cdot]_{r_1})$ satisfies

$$\sigma_{\mathrm{ess}}(A_1) = (-\infty, q_{-\infty}/r_{-\infty}] \cup [q_\infty/r_\infty, \infty) \quad \text{and} \quad \rho(A_1) \neq \emptyset. \tag{3.28}$$

Provided that $q_{-\infty}/r_{-\infty} < q_\infty/r_\infty$, the essential spectrum has a gap $G = (q_{-\infty}/r_{-\infty}, q_\infty/r_\infty)$ and the set $\sigma(A_1) \cap G$ consists of isolated eigenvalues of A_1 with finite algebraic multiplicity.

(i) *The set $\sigma(A_1) \cap G$ accumulates at q_∞/r_∞ if*

$$\limsup_{x\to\infty} x^2 \left(q_1(x) - \frac{q_\infty}{r_\infty} r_1(x) \right) < -\frac{p_\infty}{4}. \tag{3.29}$$

(ii) *The set* $\sigma(A_1) \cap G$ *does not accumulate at* q_∞/r_∞ *if*

$$\liminf_{x\to\infty} x^2\left(q_1(x) - \frac{q_\infty}{r_\infty} r_1(x)\right) > -\frac{p_\infty}{4}. \tag{3.30}$$

(iii) *The set* $\sigma(A_1) \cap G$ *accumulates at* $q_{-\infty}/r_{-\infty}$ *if*

$$\limsup_{x\to-\infty} x^2\left(q_1(x) - \frac{q_{-\infty}}{r_{-\infty}} r_1(x)\right) < -\frac{p_{-\infty}}{4}. \tag{3.31}$$

(iv) *The set* $\sigma(A_1) \cap G$ *does not accumulate at* $q_{-\infty}/r_{-\infty}$ *if*

$$\liminf_{x\to-\infty} x^2\left(q_1(x) - \frac{q_{-\infty}}{r_{-\infty}} r_1(x)\right) > -\frac{p_{-\infty}}{4}. \tag{3.32}$$

Proof. By (3.27) there are $c, d \in \mathbb{R}$, $c < d$ such that $r \upharpoonright (-\infty, c)$ is negative and $r \upharpoonright (d, \infty)$ is positive. Lemma 2.22 yields that τ_1 is in the limit-point case at ∞ and that the operator $T_1(d,\infty)$ is semi-bounded from below with $\sigma_{\mathrm{ess}}(T_1(d,\infty)) = [q_\infty/r_\infty, \infty)$, where $m_\infty := \inf \sigma_{\mathrm{ess}}(T_1(d,\infty)) = q_\infty/r_\infty$. Similarly one has that τ_1 is in the limit-point case at $-\infty$ and that the operator $T_1(-\infty, c)$ is semi-bounded from below, where $\sigma_{\mathrm{ess}}(T_1(-\infty,c)) = [-q_{-\infty}/r_{-\infty}, \infty)$ with $m_{-\infty} := \inf \sigma_{\mathrm{ess}}(T_1(-\infty,c)) = -q_{-\infty}/r_{-\infty}$. Combining these results with Proposition 3.1 one arrives at

$$\sigma_{\mathrm{ess}}(A_1) = \sigma_{\mathrm{ess}}(-T_1(-\infty,c)) \cup \sigma_{\mathrm{ess}}(T_1(d,\infty)) = (-\infty, q_{-\infty}/r_{-\infty}] \cup [q_\infty/r_\infty, \infty). \tag{3.33}$$

Further, $\rho(A_1) \neq \emptyset$ by Remark 3.2.

Suppose that $q_{-\infty}/r_{-\infty} < q_\infty/r_\infty$ and let $G = (q_{-\infty}/r_{-\infty}, q_\infty/r_\infty) = (-m_{-\infty}, m_\infty)$. We analyse the accumulation of $\sigma(A_1) \cap G$ at q_∞/r_∞. By means of Proposition 3.6 (i) this task reduces the to the investigation of whether $\sigma(T_1(d,\infty)) \cap G$ accumulates at $m_\infty = q_\infty/r_\infty$. By Lemma 2.22 the set $\sigma(T_1(d,\infty)) \cap G$ accumulates at m_∞ if (3.29) holds, and it does not accumulate at m_∞ if (3.30) is valid. This shows (i) and (ii). Similarly, considering $\sigma(-T_1(-\infty,c)) \cap G$ one has (iii) and (iv). □

Recall the band structure of the essential spectra in the case of perturbed periodic operators treated in Theorem 3.4 and Corollary 3.5. We next determine certain gaps of the essential spectrum which contain at most finitely many eigenvalues.

Theorem 3.8. *Suppose that the conditions of Theorem 3.4 and Corollary 3.5 hold, then the essential spectrum of* A_1 *is given by*

$$\sigma_{\mathrm{ess}}(A_1) = \bigcup_{n\in\mathbb{N}} \left([\mu_{2n}^+, \mu_{2n+1}^+] \cup [-\mu_{2n+1}^-, -\mu_{2n}^-]\right).$$

If the condition

$$\int_{\mathbb{R}} \left(\left|\frac{1}{p_0(t)} - \frac{1}{p_1(t)}\right| + |q_0(t) - q_1(t)| + |r_0(t) - r_1(t)|\right) |t|\,\mathrm{d}t < \infty \tag{3.34}$$

is satisfied, then each of the gaps of the essential spectrum

$$\begin{aligned} &(\mu_{2n+1}^+, \mu_{2n+2}^+), \quad \textit{where } -\mu_0^- < \mu_{2n+1}^+ < \mu_{2n+2}^+, \textit{ and} \\ &(-\mu_{2k+2}^-, -\mu_{2k+1}^-), \quad \textit{where } -\mu_{2k+2}^- < -\mu_{2k+1}^- < \mu_0^+, \end{aligned} \tag{3.35}$$

contains each at most finitely many eigenvalues of A_1.

Proof. The corresponding definite differential expression τ_1 is in the limit-point case at both endpoints, cf. Theorem 2.26. We have $m_\infty := \inf \sigma_{\mathrm{ess}}(T_1(d,\infty)) = \mu_0^+$ and $m_{-\infty} := \inf \sigma_{\mathrm{ess}}(T_1(-\infty,c)) = \mu_0^-$, cf. (3.24). Theorem 2.26 implies that each set $\sigma(T_1(d,\infty) \cap (\mu_{2n+1}^+, \mu_{2n+2}^+)$, $n \in \mathbb{N}$, is finite. For $\mu_{2k+1}^+ > -m_{-\infty} = -\mu_0^-$ Proposition 3.6 (ii) implies that the set $\sigma(A_1) \cap (\mu_{2n+1}^+, \mu_{2n+2}^+)$ is finite. In a similar way one can treat the gap $(-\mu_{2k+2}^-, -\mu_{2k+1}^-)$. □

Chapter 4

The non-real spectra of indefinite Sturm–Liouville operators

In this chapter we focus on the non-real spectra of indefinite Sturm–Liouville operators. The main objective is to provide bounds for the non-real eigenvalues of an indefinite Sturm–Liouville operator. These bounds are described in terms of the coefficients of the corresponding differential expression. Parts of this chapter are adapted from the article [20].

In the following the space $L^1_{\mathrm{u}}(\mathbb{R})$ of locally uniformly integrable functions on $\mathbb{R}$, i. e.

$$L^1_{\mathrm{u}}(\mathbb{R}) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}) : \|f\|_{\mathrm{u}} < \infty \right\}, \qquad \|f\|_{\mathrm{u}} := \sup_{n\in\mathbb{Z}} \int_n^{n+1} |f(t)|\,\mathrm{d}t,$$

is of particular importance. Note that $L^s(\mathbb{R}) \subset L^1_{\mathrm{u}}(\mathbb{R})$ for every $s \in [1,\infty] := [1,\infty) \cup \{\infty\}$. This follows from Hölder's inequality which implies $\|f\|_{\mathrm{u}} \leq \|f\|_s$ for $f \in L^s(\mathbb{R})$, $s \in [1,\infty]$, where $\|\cdot\|_s$ denotes the usual norm on $L^s(\mathbb{R})$.

Our standing assumptions on the indefinite differential expression ℓ are collected in the next hypothesis.

Hypothesis 4.1. The differential expression ℓ on $\mathbb{R}$ of the form (3.1) satisfies (3.2), (3.3) and the assumptions

(α) there exist $c, d \in \mathbb{R}$, $c < d$, such that

$$C_r := \operatorname*{ess\,inf}_{x\in\mathbb{R}\setminus[c,d]} |r(x)| > 0$$

and r has constant definite sign a. e. on $(-\infty, c)$ and constant definite sign a. e. on (d,∞);

(β) $q \in L^1_{\mathrm{u}}(\mathbb{R})$;

(γ) $1/p \in L^\eta(\mathbb{R})$ for some $\eta \in [1,\infty]$. ◇

Properties of the corresponding definite differential expression τ and the domain $\mathscr{D}(\tau)$ of the associated maximal operator $T_{\max}$ (cf. (1.6) and (1.9)) are collected in Appendix A.

We now turn to immediate consequences of the conditions in Hypothesis 4.1.

Theorem 4.2. *Under Hypothesis 4.1 the operator* $A = JT_{\max}$ *is the only self-adjoint realisation of* ℓ *in the Krein space* $(L^2(\mathbb{R}, r),\ [\cdot,\cdot]_r)$*. The essential spectrum of* A *is real and* $\rho(A) \neq \emptyset$*. The set* $\sigma(A) \setminus \mathbb{R}$ *is countable, with no accumulation point in* $\mathbb{C} \setminus \mathbb{R}$*, and consists of eigenvalues of* A *with finite algebraic multiplicity.*

Proof. By Corollary A.5 the maximal operator $T_{\max}$ associated with definite differential expression τ is the only self-adjoint realisation of the τ in $L^2(\mathbb{R}, r)$. Therefore, $A = JT_{\max}$ is the only self-adjoint realisation of ℓ in $(L^2(\mathbb{R}, r),\ [\cdot,\cdot]_r)$. Hypothesis 4.1 ($\alpha$) together with Proposition 3.1 yields $\sigma_{\mathrm{ess}}(A) \subset \mathbb{R}$. As another consequence of Corollary A.5 the operator $T_{\max}$ is semi-bounded from below. Thus, $\rho(A) \neq \emptyset$ and the assertion for the non-real spectrum of A follows, cf. Remark 3.2. □

Hereinafter, we always consider the operator $A = JT_{\max}$, where $\mathcal{D}(A) = \mathcal{D}(T_{\max}) = \mathscr{D}(\tau)$.

4.1 Bounds on the non-real eigenvalues

Since the operator A is self-adjoint with respect to the inner product $[\cdot,\cdot]_r$, the sesquilinear form $[A\cdot,\cdot]_r$ on $\mathcal{D}(A)$ is symmetric and $[Af, f]_r \in \mathbb{R}$ for all $f \in \mathcal{D}(A)$. Given $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $f \in \mathcal{D}(A)$ with $Af = \lambda f$ one has $\mathrm{Im}(\lambda[f, f]_r) = \mathrm{Im}([Af, f]_r) = 0$, which yields

$$0 = [f, f]_r = [Af, f]_r = \langle T_{\max} f, f\rangle_r = \langle \tau f, f\rangle_r. \tag{4.1}$$

In particular, one has for the kernel of $A - \lambda$

$$\mathcal{N}(A - \lambda) \subset \mathscr{D}_-(\tau) := \{f \in \mathscr{D}(\tau) \mid \langle \tau f, f\rangle_r \leq 0\} \quad \text{for all } \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{4.2}$$

cf. (A.9) in Appendix A. Based on this observation we derive bounds on the non-real spectrum.

In what follows, we consider the decomposition of the real-valued function q into its positive part q_+ and its negative part q_-, i. e.

$$q = q_+ - q_-, \quad \text{where} \quad q_+ := \frac{|q| + q}{2} \quad \text{and} \quad q_- := \frac{|q| - q}{2}. \tag{4.3}$$

Lemma 4.3. *Suppose that Hypothesis 4.1 holds true and let at least one of the following conditions holds:*

(i) $q_-(x) = 0$ *for almost all* $x \in \mathbb{R}$*;*

(ii) $q(x) \geq c \cdot r(x)$ *for almost all* $x \in \mathbb{R}$ *and some real* $c \neq 0$.

Then there are no non-real eigenvalues of A *and the spectrum of* A *is real.*

Proof. In the case (i) the assertion follows from (4.2) and Lemma A.9 (vii). If the condition in (ii) holds, then for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $f \in \mathcal{N}(A - \lambda)$

$$\begin{aligned} 0 = c \cdot [f, f]_r = c \cdot \int_{\mathbb{R}} |f(t)|^2 r(t)\,\mathrm{d}t &\leq \int_{\mathbb{R}} q(t)|f(t)|^2\,\mathrm{d}t \\ &= \int_{\mathbb{R}} \big(q_+(t) - q_-(t)\big)|f(t)|^2\,\mathrm{d}t = \|q_+ f^2\|_1 - \|q_- f^2\|_1. \end{aligned} \tag{4.4}$$

The inclusion in (4.2) and Lemma A.6 imply $\|\sqrt{p} f'\|_2 = 0$, which together with Lemma A.10 yields $\|f\|_\infty = 0$. Therefore, $\mathcal{N}(A - \lambda) = \{0\}$. □

The next lemma states bounds on the non-real eigenvalues. These bounds are given in an implicit way and depend on further parameters and a suitable function contained in the set

$$\mathcal{G}_p := \left\{ g \in AC(\mathbb{R}) \cap L^\infty(\mathbb{R}) \;\middle|\; \begin{array}{l} g \text{ is real-valued,} \\ g' \text{ has compact support, } \sqrt{p}g' \in L^2(\mathbb{R}) \end{array} \right\}. \tag{4.5}$$

In the following we use the short hand notations $\{f > c\} := \{x \in \mathbb{R} \,|\, f(x) > c\}$ and $\{f \geq c\}$, $\{f < c\}$, etc. which are defined in a similar fashion. Recall that μ denotes the Lebesgue measure on $\mathbb{R}$.

Lemma 4.4. *Suppose that Hypothesis 4.1 holds. Assume that there are constants $\alpha \geq 0$, $\beta \geq 0$ and a function $g \in \mathcal{G}_p$ such that for every function $f \in \mathcal{D}_-(\tau)$ the estimates*

$$\|q_- f^2\|_1 \leq \alpha \|f\|_2^2 \quad \textit{and} \quad \|f\|_\infty^2 \leq \beta \|f\|_2^2 \tag{4.6}$$

hold and $\omega_g \beta < 1$, where

$$\omega_g := \left(\mu(\{rg < 1\}) + \|g\|_\infty \int_{\{rg<0\}} |r(t)| \,\mathrm{d}t \right). \tag{4.7}$$

If λ is a non-real eigenvalue of A, then

$$|\operatorname{Im} \lambda| \leq \frac{\sqrt{\alpha\beta}\|\sqrt{p}g'\|_2}{\left(1 - \omega_g \beta\right)} \quad \textit{and} \quad |\lambda| \leq \frac{\sqrt{\alpha\beta}\|\sqrt{p}g'\|_2 + 3\|g\|_\infty \alpha}{\left(1 - \omega_g \beta\right)}. \tag{4.8}$$

Proof. Let λ be a non-real eigenvalue of A and f a corresponding eigenfunction. Then $f \in \mathcal{D}_-(\tau)$ by (4.2). Further, f, $\sqrt{p}f' \in L^2(\mathbb{R})$ and $qf^2 \in L^1(\mathbb{R})$ with $\lim_{|x|\to\infty} (pf')(x)\overline{f(x)} = 0$ by Lemma A.4. We consider the real-valued, absolutely continuos functions U and V on $\mathbb{R}$ given by

$$U(x) := \int_x^\infty r(t)|f(t)|^2 \,\mathrm{d}t \quad \text{and} \quad V(x) := \int_x^\infty \left(p(t)|f'(t)|^2 + q(t)|f(t)|^2 \right) \mathrm{d}t. \tag{4.9}$$

Obviously, $\lim_{x\to\infty} U(x) = \lim_{x\to\infty} V(x) = 0$. With the eigenvalue equation $Af = \lambda f$ and integration by parts, cf. (1.7) we arrive at

$$\lambda U(x) = \int_x^\infty (Af)(t)\overline{f(t)} r(t) \,\mathrm{d}t = \int_x^\infty (\tau f)(t)\overline{f(t)} |r(t)| \,\mathrm{d}t = V(x) + (pf')(x)\overline{f(x)}. \tag{4.10}$$

Therefore, $0 = [f,f]_r = \lim_{x\to-\infty} U(x) = \lim_{x\to-\infty} V(x) = 0$. Multiplication of (4.10) by g' and integration lead to

$$\lambda \int_{\mathbb{R}} g'(x)U(x) \mathrm{d}x = \int_{\mathbb{R}} g'(x)(pf')(x)\overline{f(x)} \,\mathrm{d}x + \int_{\mathbb{R}} g'(x)V(x) \,\mathrm{d}x. \tag{4.11}$$

Here, the compact support of g' guarantees the existence of the integrals. With the Cauchy–Schwarz inequality we find for the first integral on the right-hand side of (4.11) that

$$\left| \int_{\mathbb{R}} g'(x)\overline{f(x)}(pf')(x) \,\mathrm{d}x \right| \leq \|f\|_\infty \|\sqrt{p}g'\|_2 \|\sqrt{p}f'\|_2 \leq \sqrt{\alpha\beta} \|\sqrt{p}g'\|_2 \|f\|_2^2, \tag{4.12}$$

where we applied Lemma A.6 and (4.6). Integration by parts of the second term on the right-hand side of (4.11) together with the fact that $\lim_{|x|\to\infty} g(x)V(x) = 0$, Lemma A.6 and (4.6) yields

$$\begin{aligned}
\left|\int_{\mathbb{R}} g'(x)V(x)\,\mathrm{d}x\right| &= \left|-\int_{\mathbb{R}} g(x)V'(x)\,\mathrm{d}x\right| \\
&= \left|\int_{\mathbb{R}} g(x)\left(p(x)|f'(x)|^2 + q(x)|f(x)|^2\right)\,\mathrm{d}x\right| \\
&\leq \|g\|_\infty\left(\|\sqrt{p}f\|_2^2 + \|qf^2\|_1\right) \\
&\leq 3\|g\|_\infty\alpha\|f\|_2^2.
\end{aligned} \tag{4.13}$$

We want to find a lower bound for the left-hand side in (4.11). From integration by parts and $\lim_{|x|\to\infty} g(x)U(x) = 0$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}} g'(x)U(x)\,\mathrm{d}x &= -\int_{\mathbb{R}} g(x)U'(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} g(x)r(x)|f(x)|^2\,\mathrm{d}x \\
&= \int_{\{rg<0\}} g(x)r(x)|f(x)|^2\,\mathrm{d}x + \int_{\{rg\geq 0\}} g(x)r(x)|f(x)|^2\,\mathrm{d}x.
\end{aligned} \tag{4.14}$$

For the first term on the right-hand side we have with the second inequality in (4.6) that

$$\begin{aligned}
\int_{\{rg<0\}} g(x)r(x)|f(x)|^2\,\mathrm{d}x &\geq -\|g\|_\infty \int_{\{rg<0\}} |r(x)||f(x)|^2\,\mathrm{d}x \\
&\geq -\|g\|_\infty\|f\|_\infty^2 \int_{\{rg<0\}} |r(x)|\,\mathrm{d}x \geq -\beta\|g\|_\infty\|f\|_2^2 \int_{\{rg<0\}} |r(x)|\,\mathrm{d}x.
\end{aligned}$$

Further,

$$\begin{aligned}
\int_{\{rg\geq 0\}} g(x)r(x)|f(x)|^2\,\mathrm{d}x &\geq \int_{\{rg\geq 1\}} g(x)r(x)|f(x)|^2\,\mathrm{d}x \geq \int_{\{rg\geq 1\}} |f(x)|^2\,\mathrm{d}x \\
&= \left(\|f\|_2^2 - \int_{\{rg<1\}} |f(x)|^2\,\mathrm{d}x\right) \\
&\geq \left(\|f\|_2^2 - \mu(\{rg<1\})\|f\|_\infty^2\right) \\
&\geq \left(1 - \mu(\{rg<1\})\beta\right)\|f\|_2^2,
\end{aligned} \tag{4.15}$$

where we used again (4.6). From (4.14) and (4.15) follows

$$\int_{\mathbb{R}} g'(x)U(x)\,\mathrm{d}x \geq (1 - \omega_g\beta)\|f\|_2^2 > 0. \tag{4.16}$$

We compare the imaginary parts in (4.11). As a consequence of (4.12), (4.16) together with the fact that g and V are real-valued functions we see

$$|\mathrm{Im}\,\lambda|\big(1-\omega_g\beta\big)\|f\|_2^2 \leq \left|\mathrm{Im}\left(\lambda\int_{\mathbb{R}} g'(x)U(x)\,\mathrm{d}x\right)\right| = \left|\mathrm{Im}\left(\int_{\mathbb{R}} g'(x)(pf')(x)\overline{f(x)}\,\mathrm{d}x\right)\right|$$
$$\leq \sqrt{\alpha\beta}\|\sqrt{p}g'\|_2\|f\|_2^2,$$

which proves the first estimate in (4.8). We compare both sides in (4.11) with respect to the absolute value. Then by (4.13), (4.12), (4.16), and the fact that g, V are real-valued functions we obtain

$$|\lambda|\big(1-\omega\beta\big)\|f\|_2^2 \leq \left|\left(\lambda\int_{\mathbb{R}} g'(x)U(x)\mathrm{d}x\right)\right| = \left|\int_{\mathbb{R}} g'(x)\Big((pf')(x)\overline{f(x)}+V(x)\Big)\,\mathrm{d}x\right|$$
$$\leq \left(\sqrt{\alpha\beta}\|\sqrt{p}g'\|_2 + 3\|g\|_\infty\alpha\right)\|f\|_2^2,$$

which shows the second inequality in (4.8). □

At this point it should be mentioned that an admissible function $g \in \mathcal{G}_p$ always exist, cf. Theorem 4.18. In Section 4.2 we discuss the choice of the function g in more detail. The parameters α and β are specified in Lemma A.9.

As a consequence of Lemma 4.4 and Lemma A.9 we arrive at the main theorems of this section, which are stated in a similar form in [20].

Theorem 4.5. *Assume that Hypothesis 4.1 holds with* $1/p \in L^\infty(\mathbb{R})$ *and define*

$$\alpha := 2\|q_-\|_{\mathrm{u}} + 4\|1/p\|_\infty\|q_-\|_{\mathrm{u}}^2.$$

Choose $g \in \mathcal{G}_p$ *such that*

$$\omega_g\big(4\|1/p\|_\infty\alpha\big)^{\frac{1}{2}} < 1$$

holds for ω_g *as in* (4.7). *Then every non-real eigenvalue* λ *of* A *satisfies*

$$|\mathrm{Im}\,\lambda| \leq \frac{\sqrt{2}\|1/p\|_\infty^{\frac{1}{4}}\|\sqrt{p}g'\|_2\alpha^{\frac{3}{4}}}{\left(1-\omega_g\left(4\|1/p\|_\infty\alpha\right)^{\frac{1}{2}}\right)} \quad \text{and} \quad |\lambda| \leq \frac{\sqrt{2}\|1/p\|_\infty^{\frac{1}{4}}\|\sqrt{p}g'\|_2\alpha^{\frac{3}{4}} + 3\|g\|_\infty\alpha}{\left(1-\omega_g\left(4\|1/p\|_\infty\alpha\right)^{\frac{1}{2}}\right)}.$$

Proof. By Lemma A.9 (i) the estimates in (4.6) hold for all $f \in \mathcal{D}_-(\tau)$ with

$$\alpha = 2\|q_-\|_{\mathrm{u}} + 4\|1/p\|_\infty\|q_-\|_{\mathrm{u}}^2, \quad \beta = \big(4\|1/p\|_\infty\alpha\big)^{\frac{1}{2}}, \quad \text{where } \sqrt{\alpha\beta} = \sqrt{2}\|1/p\|_\infty^{\frac{1}{4}}\alpha^{\frac{3}{4}}.$$

An application of Lemma 4.4 finishes the proof. □

We next state bounds on the non-real spectrum in the cases, where $q_- \in L^s(\mathbb{R})$ and $1/p \in L^\eta(\mathbb{R})$ with $\eta, s \in [1,\infty)$, which can also be found in [20, Theorem 2.4]. In addition, Theorem 4.7 below addresses the case where $s = \infty$ and $\eta \in [1,\infty)$, which was not considered in [20], but can treated in a similar way.

Theorem 4.6. *Assume that Hypothesis 4.1 holds with* $1/p \in L^\eta(\mathbb{R})$ *for some* $\eta \in [1,\infty]$*, and let* $q_- \in L^s(\mathbb{R})$ *for some* $s \in [1,\infty)$*, where* $2 < \eta + s$*. Define*

$$\beta = \begin{cases} \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_s\right)^{\frac{\eta s}{2\eta s-\eta-s}} & \text{if } \eta, s \in [1,\infty), \\ \left(4\|1/p\|_\infty \|q_-\|_s\right)^{\frac{s}{2s-1}} & \text{if } \eta = \infty, s \in [1,\infty). \end{cases} \tag{4.17}$$

Choose $g \in \mathcal{G}_p$ *such that* $\omega_g \beta < 1$ *holds for* ω_g *as in* (4.7)*. Then every non-real eigenvalue* λ *of* A *satisfies*

$$|\operatorname{Im}\lambda| \le \frac{\|q_-\|_s^{\frac12} \beta^{\frac{s+1}{2s}} \|\sqrt{p}g'\|_2}{(1-\omega_g\beta)} \quad \text{and} \quad |\lambda| \le \frac{\|q_-\|_s^{\frac12} \beta^{\frac{s+1}{2s}} \|\sqrt{p}g'\|_2 + 3\|g\|_\infty \|q_-\|_s \beta^{\frac1s}}{(1-\omega_g\beta)}. \tag{4.18}$$

Proof. As a consequence of Lemma A.9 (ii) and (iii) the estimates in (4.6) hold for all $f \in \mathcal{D}_-(\tau)$ with β defined as in (4.17) and

$$\alpha = \|q_-\|_s \beta^{\frac1s}, \quad \text{where } \sqrt{\alpha\beta} = \|q_-\|_s^{\frac12} \beta^{\frac{s+1}{2s}}.$$

An application of Lemma 4.4 shows the assertion. □

Theorem 4.7. *Assume that Hypothesis 4.1 holds with* $1/p \in L^\eta(\mathbb{R})$ *for some* $\eta \in [1,\infty]$ *and let* $q_- \in L^\infty(\mathbb{R})$*. Define*

$$\beta = \begin{cases} \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_\infty\right)^{\frac{\eta}{2\eta-1}} & \text{if } \eta \in [1,\infty), \\ \left(4\|1/p\|_\infty \|q_-\|_\infty\right)^{\frac12} & \text{if } \eta = \infty. \end{cases} \tag{4.19}$$

Choose $g \in \mathcal{G}_p$ *such that* $\omega_g \beta < 1$ *holds for* ω_g *as in* (4.7)*. Then every non-real eigenvalue* λ *of* A *satisfies*

$$|\operatorname{Im}\lambda| \le \frac{(\|q_-\|_\infty \beta)^{\frac12} \|\sqrt{p}g'\|_2}{(1-\omega_g\beta)} \quad \text{and} \quad |\lambda| \le \frac{(\|q_-\|_\infty \beta)^{\frac12} \|\sqrt{p}g'\|_2 + 3\|g\|_\infty \|q_-\|_\infty}{(1-\omega_g\beta)}. \tag{4.20}$$

Proof. With Lemma A.9 (iv) and (v) we see that in (4.6) hold for all $f \in \mathcal{D}_-(\tau)$ with β defined as in (4.19) and

$$\alpha = \|q_-\|_\infty, \quad \text{where } \sqrt{\alpha\beta} = (\|q_-\|_\infty \beta)^{\frac12}.$$

Now Lemma 4.4 finishes the proof. □

Note that the bounds in Theorem 4.5, Theorem 4.6 and Theorem 4.14 do not depend the norm of q but on the norm of the negative part q_-.

The previous two theorems cover all combinations of $1/p \in L^\eta(\mathbb{R})$ and $q_- \in L^s(\mathbb{R})$ for $\eta, s \in [1,\infty]$, except the case $\eta = s = 1$. In this situation, which is slightly different, we give a sufficient condition for the non-real spectrum to be empty.

Theorem 4.8. *Suppose that Hypothesis 4.1 holds with* $1/p$, $q_- \in L^1(\mathbb{R})$*. If in addition* $\|1/p\|_1 \|q_-\|_1 < 1$*, then there are no non-real eigenvalues of* A *and the spectrum of* A *is real.*

Proof. Lemma A.9 (vi) together with (4.2) proves the assertion. □

4.2 Weight functions with finitely many turning points

In this section we discuss the choice of the parameter g appearing in Theorem 4.5, Theorem 4.6 and Theorem 4.7. The aim is to find a function $g \in \mathcal{G}_p$ such that the quantity

$$\omega_g := \mu\big(\{rg < 1\}\big) + \|g\|_\infty \int_{\{rg<0\}} |r(t)|\,\mathrm{d}t \tag{4.21}$$

is sufficiently small.

To formulate the results below we first discuss the notation of sign changes of r or, more precisely, the *turning points* of r. In [36] the turning points of the weigh function r are collected in the set

$$\overline{\{r > 0\}} \cap \overline{\{r < 0\}}. \tag{4.22}$$

As this definition depends on the representative of the equivalence class of r in $L^1_{\mathrm{loc}}(\mathbb{R})$ we use a slightly different approach. Here, the turning points of r are defined as the elements of the set

$$\mathcal{T}_r := \left\{ x \in \mathbb{R} \,\middle|\, \begin{array}{c} \mu(\{r > 0\} \cap I) > 0 \text{ and } \mu(\{r < 0\} \cap I) > 0 \\ \text{for all open intervals } I \text{ containing } x \end{array} \right\}. \tag{4.23}$$

Observe that $\mathcal{T}_r$ is a closed subset of the set in (4.22). In particular, under Hypothesis 4.1 the set $\mathcal{T}_r$ is bounded and, thus, compact. Furthermore, the set $\mathcal{T}_r$ does not depend on the representative of the equivalence class of r in $L^1_{\mathrm{loc}}(\mathbb{R})$.

In preparation for the construction of the function g we need to prove some rather technicals results regarding the weight r.

Lemma 4.9. *Suppose that Hypothesis 4.1 holds and assume that $\mathcal{T}_r$ is finite. Then there exists a function $w \in L^1_{\mathrm{loc}}(\mathbb{R})$ with $w = r$ a. e. such that the disjoint sets $\{w > 0\}$ and $\{w < 0\}$ are finite unions of disjoint open intervals and the boundaries $\partial\{w > 0\}$ and $\partial\{w < 0\}$ satisfy*

$$\partial\{w > 0\} = \partial\{w > 0\} = \overline{\{w > 0\}} \cap \overline{\{w < 0\}} = \{w = 0\} = \mathcal{T}_w = \mathcal{T}_r. \tag{4.24}$$

Proof. Let $\mathcal{F}_+$ be the family of all non-empty open intervals I such that $\mu(I \cap \{r < 0\}) = 0$, and let $\mathcal{F}_-$ be the family of all non-empty open intervals I such that $\mu(I \cap \{r > 0\}) = 0$. Then the sets

$$\Upsilon_+ = \bigcup_{I \in \mathcal{F}_+} I, \qquad \Upsilon_- = \bigcup_{I \in \mathcal{F}_-} I \tag{4.25}$$

are open. For the unions in (4.25) it suffices to consider only open intervals with rational endpoints. Thus, Υ_+ and Υ_- can be represented as unions of countably many intervals. Together with the σ-subadditivity of the Lebesgue measure this implies

$$\mu(\Upsilon_+ \cap \{r < 0\}) = 0, \qquad \mu(\Upsilon_- \cap \{r > 0\}) = 0. \tag{4.26}$$

Let $x \in \mathbb{R} \setminus \mathcal{T}_r$. Then there exists an open interval I containing x with $\mu(I \cap \{r > 0\}) = 0$ or $\mu(I \cap \{r < 0\}) = 0$, and $x \in \Upsilon_+ \cup \Upsilon_-$. This implies

$$\mathbb{R} = \Upsilon_+ \cup \Upsilon_- \cup \mathcal{T}_r. \tag{4.27}$$

By (4.25) it is clear, that $\Upsilon_+ \cap \mathcal{T}_r = \emptyset$ and $\Upsilon_- \cap \mathcal{T}_r = \emptyset$. We want to show $\Upsilon_+ \cap \Upsilon_- = \emptyset$. Assume that $x \in \Upsilon_+ \cap \Upsilon_-$. Then there are open intervals $I_+ \in \mathcal{F}_+$ and $I_- \in \mathcal{F}_-$ both containing x. The intersection

$I_+ \cap I_-$ is a non-empty open interval with $\mu(I_+ \cap I_-) = \mu(I_+ \cap I_- \cap \{r > 0\}) + \mu(I_+ \cap I_- \cap \{r < 0\}) = 0$; a contradiction. Therefore, the union in (4.27) is disjoint.

Since Υ_+ and Υ_- are open and disjoint we have

$$\partial\Upsilon_+ \cap \Upsilon_+ = \emptyset, \quad \partial\Upsilon_- \cap \Upsilon_- = \emptyset, \quad \partial\Upsilon_+ \cap \Upsilon_- = \emptyset, \quad \partial\Upsilon_- \cap \Upsilon_+ = \emptyset. \tag{4.28}$$

Here, (4.27) implies $\partial\Upsilon_+ \subset \mathcal{T}_r$ and $\partial\Upsilon_- \subset \mathcal{T}_r$. Let $x \in \mathcal{T}_r$. Since $\mathcal{T}_r$ is finite there exists an non-empty open interval (a, b) with $(a, b) \cap \mathcal{T}_r = \{x\}$. Then the non-empty open interval (a, x) is a connected set and can be represented as the disjoint union of the two open sets $(a, x) \cap \Upsilon_+$ and $(a, x) \cap \Upsilon_-$. Thus, either $(a, x) \subset \Upsilon_+$ or $(a, x) \subset \Upsilon_-$. A similar argument shows $(x, b) \subset \Upsilon_+$ or $(x, b) \subset \Upsilon_-$. By (4.23) and (4.26) we obtain $(a, x) \subset \Upsilon_+$ and $(x, b) \subset \Upsilon_-$ or, alternatively, $(a, x) \subset \Upsilon_-$ and $(x, b) \subset \Upsilon_+$. This shows $x \in \partial\Upsilon_+ \cap \partial\Upsilon_-$ and, therefore, $\mathcal{T}_r \subset \partial\Upsilon_+ \cap \partial\Upsilon_-$. Together with $\partial\Upsilon_+ \subset \mathcal{T}_r$ and $\partial\Upsilon_- \subset \mathcal{T}_r$ one obtains $\mathcal{T}_r = \partial\Upsilon_+ = \partial\Upsilon_-$. Moreover, (4.28) yields $\overline{\Upsilon_+} \cap \overline{\Upsilon_-} = \mathcal{T}_r$. As $\mathcal{T}_r$ is finite the sets Υ_+ and Υ_- consist of finitely many disjoint open intervals.

We define

$$w(x) = \begin{cases} 1 & \text{if } x \in \Upsilon_+ \setminus \{r > 0\}, \\ -1 & \text{if } x \in \Upsilon_- \setminus \{r < 0\}, \\ r(x) & \text{if } x \in (\Upsilon_+ \cap \{r > 0\}) \cup (\Upsilon_- \cap \{r < 0\}), \\ 0 & \text{if } x \in \mathcal{T}_r. \end{cases}$$

Then $\{w > 0\} = \Upsilon_+$ and $\{w < 0\} = \Upsilon_-$ consist of finitely many open disjoint intervals and we have $\mathcal{T}_r = \{w = 0\}$. Since $\mathcal{T}_r$ has Lebesgue measure zero as well as the sets

$$\Upsilon_+ \setminus \{r > 0\} = \Upsilon_+ \cap (\{r < 0\} \cup \{r = 0\}) \quad \text{and} \quad \Upsilon_- \setminus \{r < 0\} = \Upsilon_- \cap (\{r > 0\} \cup \{r = 0\}),$$

see (4.26), we have $w = r$ a. e. Finally, the properties in (4.24) hold by construction of the sets $\{w > 0\}$ and $\{w < 0\}$. □

Lemma 4.10. *Suppose that Hypothesis 4.1 holds and assume that $\mathcal{T}_r$ is finite. Then for*

$$0 < \delta < \frac{1}{2} \min\{|x - y| \,|\, x, y \in \mathcal{T}_r,\ x \neq y\}$$

there exists $g \in \mathcal{G}_p$ with $rg > 0$ a. e. such that $\|g\|_\infty = 1$,

$$\|\sqrt{p}g\|_2^2 = \sum_{x \in \mathcal{T}_r} \left(\int_{x-\delta}^{x} \frac{1}{p(t)}\, \mathrm{d}t \right)^{-1} + \sum_{x \in \mathcal{T}_r} \left(\int_{x}^{x+\delta} \frac{1}{p(t)}\, \mathrm{d}t \right)^{-1},$$

and

$$\{|g| < 1\} = \bigcup_{x \in \mathcal{T}_r} (x - \delta, x + \delta). \tag{4.29}$$

Proof. By Lemma 4.9 we can assume without loss of generality that the sets $\{r > 0\}$ and $\{r < 0\}$ consist of finitely many disjoint open intervals, where $\partial\{r > 0\} = \partial\{r < 0\} = \{r = 0\} = \mathcal{T}_r$. Then the function $x \mapsto \operatorname{sgn}(r(x))$ is piecewise constant with finitely many discontinuities, namely the points in $\mathcal{T}_r$. Let

$$g(x) := \operatorname{sgn}(r(x)) \begin{cases} \left(\int_y^x \frac{1}{p(t)}\, \mathrm{d}t \right) \left(\int_y^{y+\delta} \frac{1}{p(t)}\, \mathrm{d}t \right)^{-1} & \text{if } x \in [y, y + \delta),\ y \in \mathcal{T}_r, \\ \left(\int_x^y \frac{1}{p(t)}\, \mathrm{d}t \right) \left(\int_{y-\delta}^{y} \frac{1}{p(t)}\, \mathrm{d}t \right)^{-1} & \text{if } x \in (y - \delta, y),\ y \in \mathcal{T}_r, \\ 1 & \text{otherwise.} \end{cases}$$

Then $g \in AC(\mathbb{R})$, $\|g\|_\infty = 1$, and (4.29) holds. Further,

$$\|\sqrt{p}g'\|_2^2 = \sum_{y\in\mathcal{T}_r}\left(\int_{y-\delta}^{y} p(t)|g'(t)|^2\,\mathrm{d}t + \int_{y}^{y+\delta} p(t)|g'(t)|^2\,\mathrm{d}t\right)$$

$$= \sum_{y\in\mathcal{T}_r}\left(\int_{y-\delta}^{y}\frac{1}{p(t)}\,\mathrm{d}t\right)^{-1} + \sum_{y\in\mathcal{T}_r}\left(\int_{y}^{y+\delta}\frac{1}{p(t)}\,\mathrm{d}t\right)^{-1} < \infty,$$

since $1/p \in L^1_{\mathrm{loc}}(\mathbb{R})$ and $p > 0$ a. e. As $\mathcal{T}_r$ is finite the function g is constant near ∞ and $-\infty$. Hence, g' has compact support. Moreover, since $\{g > 0\} = \{r > 0\}$ and $\{g < 0\} = \{r < 0\}$, the product rg is positive. □

Besides the turning points also the points where the weight is close to zero are of special interest for the construction process of the function g. We define the set

$$\mathcal{Z}_r := \left\{x \in \mathbb{R}\,\middle|\, \operatorname*{ess\,inf}_{y\in I} |r(y)| = 0 \text{ for all open intervals } I \text{ with } x \in I\right\}, \tag{4.30}$$

which is again independent of the representative of the equivalence class of r in $L^1_{\mathrm{loc}}(\mathbb{R})$. Note that in general neither $\mathcal{Z}_r \subset \{r = 0\}$, nor $\mathcal{Z}_r \supset \{r = 0\}$.

Lemma 4.11. *If Hypothesis 4.1 holds, then for every $\delta > 0$ and $\Omega = \bigcup_{x\in\mathcal{Z}_r}(x-\delta, x+\delta)$ we have*

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}\setminus\Omega} |r(x)| > 0.$$

Proof. Let $[c, d] \subset \mathbb{R}$ with $C_r = \operatorname{ess\,inf}_{x\in\mathbb{R}\setminus[c,d]} |r(x)| > 0$ as in Hypothesis 4.1 (α), and consider the open set $\Omega = \bigcup_{x\in\mathcal{Z}_r}(x-\delta, x+\delta)$. By the definition of $\mathcal{Z}_r$ in (4.30) there exists for every $x \in \mathbb{R}\setminus\mathcal{Z}_r$ an open interval I_x containing x such that $c_x := \operatorname{ess\,inf}_{y\in I_x}|r(y)| > 0$. Since $[c, d]\setminus\Omega$ is compact and

$$\big([c, d]\setminus\Omega\big) \subset \big(\mathbb{R}\setminus\mathcal{Z}_r\big) \subset \bigcup_{x\notin\mathcal{Z}_r} I_x,$$

we find $x_1, \ldots, x_m \in \mathbb{R}\setminus\mathcal{Z}_r$, $m \in \mathbb{N}$, such that $\big([c, d]\setminus\Omega\big) \subset \bigcup_{k=1}^m I_{x_k}$. Thus, by $(\mathbb{R}\setminus\Omega) \subset (\mathbb{R}\setminus[c, d]) \cup ([c, d]\setminus\Omega)$ we have

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}\setminus\Omega} |r(x)| \geq \min\Big\{C_r, c_{x_1}, \ldots, c_{x_m}\Big\} > 0. \qquad \square$$

The above results allow to construct a suitable function g in the case of finite sets $\mathcal{T}_r$ and $\mathcal{Z}_r$.

Lemma 4.12. *Suppose that Hypothesis 4.1 holds and assume that the set $\mathcal{T}_r \cup \mathcal{Z}_r$ has $n < \infty$ elements. Let $\delta > 0$ such that $\delta < |x-y|/2$ holds for each pair of distinct points $x, y \in \mathcal{T}_r$. Set $\Omega := \bigcup_{x\in\mathcal{Z}_r}(x-\delta, x+\delta)$ and define*

$$\gamma := \operatorname*{ess\,inf}_{x\in\mathbb{R}\setminus\Omega} |r(x)|, \quad P := \left(\sum_{x\in\mathcal{T}_r}\left(\int_{x-\delta}^{x}\frac{1}{p(t)}\,\mathrm{d}t\right)^{-1} + \sum_{x\in\mathcal{T}_r}\left(\int_{x}^{x+\delta}\frac{1}{p(t)}\,\mathrm{d}t\right)^{-1}\right)^{\frac{1}{2}}. \tag{4.31}$$

Then there exists $g \in \mathcal{G}_p$ with $rg > 0$ a. e. such that $\|g\|_\infty = 1/\gamma$, $\|\sqrt{p}g'\|_2 = P/\gamma$ and $\omega_g \leq 2\delta n$ for ω_g as in (4.21).

Proof. By Lemma 4.10 for sufficiently small $\delta > 0$ there is $\tilde{g} \in \mathcal{G}_p$ with $r\tilde{g} > 0$ a. e. such that $\|\tilde{g}\|_\infty = 1$, $\|\sqrt{p}\tilde{g}'\|_2 = P$ and

$$\{|\tilde{g}| < 1\} = \bigcup_{x \in \mathcal{T}_r} (x - \delta, x + \delta).$$

Lemma 4.11 guarantees that γ is positive and $\{|r| < \gamma\} \subset \Omega$. As $r\tilde{g} > 0$ a. e. and $\{r\tilde{g} < \gamma\} \subset \{|r| < \gamma\} \cup \{|\tilde{g}| < 1\} \cup \{r\tilde{g} < 0\}$, we have

$$\begin{aligned}\tilde{\omega} &:= \mu(\{r\tilde{g} < \gamma\}) + \frac{1}{\gamma}\int_{\{r\tilde{g}<0\}} r(t)\,\mathrm{d}t = \mu(\{r\tilde{g} < \gamma\}) \\ &\le \mu\left(\bigcup_{x \in \mathcal{T}_r \cup \mathcal{Z}_r} (x - \delta, x + \delta)\right) = 2\delta n.\end{aligned}$$

Now the assertion follows for $g := \tilde{g}/\gamma \in \mathcal{G}_p$, where $\omega_g = \tilde{\omega}$. □

Observe that the ω_g can by made arbitrarily small by decreasing δ, even though this causes an increase in the norm $\|\sqrt{p}g'\|_2$.

Finally, as a direct consequence of Lemma 4.12 we can reformulate Theorem 4.5, Theorem 4.6 and Theorem 4.7 avoiding the function g. Theorem 4.13, Theorem 4.14, Theorem 4.15 and Corollary 4.16 below are the main results of this section. Except the case where $q_- \in L^\infty(\mathbb{R})$, $1/p \in L^\infty(\mathbb{R})$ in Theorem 4.15, similar statements can be found also in [19, 20].

Theorem 4.13. *Assume that Hypothesis 4.1 holds with $1/p \in L^\infty(\mathbb{R})$ such that $\mathcal{T}_r \cup \mathcal{Z}_r$ has $n < \infty$ elements and let*

$$\alpha := 2\|q_-\|_{\mathrm{u}} + 4\|1/p\|_\infty \|q_-\|_{\mathrm{u}}^2.$$

For $\delta > 0$ with $2\delta n(4\|1/p\|_\infty \alpha)^{1/2} < 1$ and $2\delta \le |x - y|$ for each pair of distinct points $x, y \in \mathcal{T}_r$ define $\Omega := \bigcup_{x \in \mathcal{Z}_r} (x - \delta, x + \delta)$ and let γ, P as in (4.31). *Then every non-real eigenvalue λ of A satisfies*

$$|\operatorname{Im}\lambda| \le \frac{\sqrt{2}\|1/p\|_\infty^{\frac{1}{4}} P \alpha^{\frac{3}{4}}}{\gamma\left(1 - 2\delta n\left(4\|1/p\|_\infty \alpha\right)^{\frac{1}{2}}\right)} \quad \textit{and} \quad |\lambda| \le \frac{\sqrt{2}\|1/p\|_\infty^{\frac{1}{4}} P \alpha^{\frac{3}{4}} + 3\alpha}{\gamma\left(1 - 2\delta n\left(4\|1/p\|_\infty \alpha\right)^{\frac{1}{2}}\right)}.$$

Theorem 4.14. *Suppose that Hypothesis 4.1 holds with $1/p \in L^\eta(\mathbb{R})$ for some $\eta \in [1, \infty]$, $q_- \in L^s(\mathbb{R})$ for some $s \in [1, \infty)$, where $2 < \eta + s$, and assume that $\mathcal{T}_r \cup \mathcal{Z}_r$ has $n < \infty$ elements. Let*

$$\beta = \begin{cases} \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_s\right)^{\frac{\eta s}{2\eta s - \eta - s}} & \text{if } \eta, s \in [1, \infty), \\ \left(4\|1/p\|_\infty \|q_-\|_s\right)^{\frac{s}{2s-1}} & \text{if } \eta = \infty, s \in [1, \infty). \end{cases}$$

For $\delta > 0$ with $2\delta n\beta < 1$ and $2\delta \le |x - y|$ for each pair of distinct points $x, y \in \mathcal{T}_r$ define $\Omega := \bigcup_{x \in \mathcal{Z}_r} (x - \delta, x + \delta)$ and let γ, P as in (4.31). *Then every non-real eigenvalue λ of A satisfies*

$$|\operatorname{Im}\lambda| \le \frac{\|q_-\|_s^{\frac{1}{2}} \beta^{\frac{s+1}{2s}} P}{\gamma(1 - 2\delta n\beta)} \quad \textit{and} \quad |\lambda| \le \frac{\|q_-\|_s^{\frac{1}{2}} \beta^{\frac{s+1}{2s}} P + 3\|q_-\|_s \beta^{\frac{1}{s}}}{\gamma(1 - 2\delta n\beta)}.$$

Theorem 4.15. *Suppose that Hypothesis 4.1 holds with* $1/p \in L^\eta(\mathbb{R})$ *for some* $\eta \in [1,\infty]$, $q_- \in L^\infty(\mathbb{R})$, *and assume that* $\mathcal{T}_r \cup \mathcal{Z}_r$ *has* $n < \infty$ *elements. Let*

$$\beta = \begin{cases} \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_\infty\right)^{\frac{\eta}{2\eta-1}} & \text{if } \eta \in [1,\infty), \\ \left(4\|1/p\|_\infty \|q_-\|_\infty\right)^{\frac{1}{2}} & \text{if } \eta = \infty. \end{cases}$$

For $\delta > 0$ *with* $2\delta n\beta < 1$ *and* $2\delta \le |x-y|$ *for each pair of distinct points* $x, y \in \mathcal{T}_r$ *define* $\Omega := \bigcup_{x \in \mathcal{Z}_r} (x-\delta, x+\delta)$ *and let* γ, P *as in* (4.31). *Then every non-real eigenvalue* λ *of* A *satisfies*

$$|\mathrm{Im}\,\lambda| \le \frac{(\|q_-\|_\infty \beta)^{\frac{1}{2}} P}{\gamma(1-2\delta n\beta)} \quad \text{and} \quad |\lambda| \le \frac{(\|q_-\|_\infty \beta)^{\frac{1}{2}} P + 3\|q_-\|_\infty}{\gamma(1-2\delta n\beta)}.$$

We apply the previous theorems to the case $p \equiv 1$, $r = \mathrm{sgn}$.

Corollary 4.16. *Let* $p \equiv 1$, $r = \mathrm{sgn}$ *and* $q \in L^1_{\mathrm{u}}(\mathbb{R})$, *that is* $A = \mathrm{sgn} \cdot (-\mathrm{d}^2/\mathrm{d}x^2 + q)$.

(i) *Then* $\sigma(A) \setminus \mathbb{R}$ *is contained in*

$$\Sigma_{\mathrm{u}} := \left\{ \lambda \in \mathbb{C} \,\middle|\, \begin{array}{l} |\mathrm{Im}\,\lambda| \le 12 \cdot \sqrt{3} \left(\|q_-\|_{\mathrm{u}} + 2\|q_-\|_{\mathrm{u}}^2\right), \\ |\lambda| \le (12 \cdot \sqrt{3} + 9) \left(\|q_-\|_{\mathrm{u}} + 2\|q_-\|_{\mathrm{u}}^2\right) \end{array} \right\}.$$

(ii) *If* $q_- \in L^s(\mathbb{R})$ *for* $s \in [1,\infty)$, *then* $\sigma(A) \setminus \mathbb{R}$ *is contained in*

$$\Sigma_s := \left\{ \lambda \in \mathbb{C} \,\middle|\, \begin{array}{l} |\mathrm{Im}\,\lambda| \le 2^{\frac{2s+1}{2s-1}} \cdot 3 \cdot \sqrt{3} \|q_-\|_s^{\frac{2s}{2s-1}}, \\ |\lambda| \le \left(2^{\frac{2s+1}{2s-1}} \cdot 3 \cdot \sqrt{3} + 2^{\frac{3-2s}{2s-1}} \cdot 9\right) \|q_-\|_s^{\frac{2s}{2s-1}} \end{array} \right\}.$$

(iii) *If* $q_- \in L^\infty(\mathbb{R})$, *then* $\sigma(A) \setminus \mathbb{R}$ *is contained in*

$$\Sigma_\infty := \left\{ \lambda \in \mathbb{C} \,\middle|\, |\mathrm{Im}\,\lambda| \le 6 \cdot \sqrt{3} \|q_-\|_\infty, |\lambda| \le \left(6 \cdot \sqrt{3} + \frac{9}{2}\right) \|q_-\|_\infty \right\}.$$

Proof. Since $\mathcal{T}_r = \{0\}$ and $\mathcal{Z}_r = \emptyset$ we have $n = 1$, $\gamma = 1$ and $P = \sqrt{2/\delta}$ for $\delta > 0$ in the context of Theorem 4.13–Theorem 4.15. Without loss of generality we assume $q_- \neq 0$. Then the estimates in (i) follow from Theorem 4.13, where $\delta = 1/12 \cdot (2\|q_-\|_{\mathrm{u}} + 4\|q_-\|_{\mathrm{u}}^2)^{-1/2}$, (ii) follows from Theorem 4.15 with $\delta = 1/6 \cdot (4\|q_-\|_s)^{-s/(2s-1)}$, and Theorem 4.15 together with the choice $\delta = 1/12 \cdot \|q_-\|_\infty^{-1/2}$ yields (iii). □

Example 4.17. We consider the operator $A = \mathrm{sgn} \cdot (-\mathrm{d}^2/\mathrm{d}x^2 + q)$ with $q = -\varkappa(\varkappa+1)\,\mathrm{sech}^2$, where $\varkappa \in \mathbb{N}$. The operator A has exactly $2\varkappa$ non-real eigenvalues, see [13]. We find

$$\|q_-\|_{\mathrm{u}} = \varkappa(\varkappa+1)\tanh(1), \qquad \|q_-\|_\infty = \varkappa(\varkappa+1)$$

and for $s \in \mathbb{N}$, $s \ge 1$

$$\|q_-\|_s = \varkappa(\varkappa+1) \left(2 \sum_{j=0}^{s-1} \frac{(-1)^j}{2j+1} \binom{s-1}{j} \right)^{\frac{1}{s}},$$

see e. g. [51, p. 115, equation 2.422 1]. Finally, together with Corollary 4.16 one obtains explicit regions which contain the non-real eigenvalues of A, cf. Figure 4.1. ◇

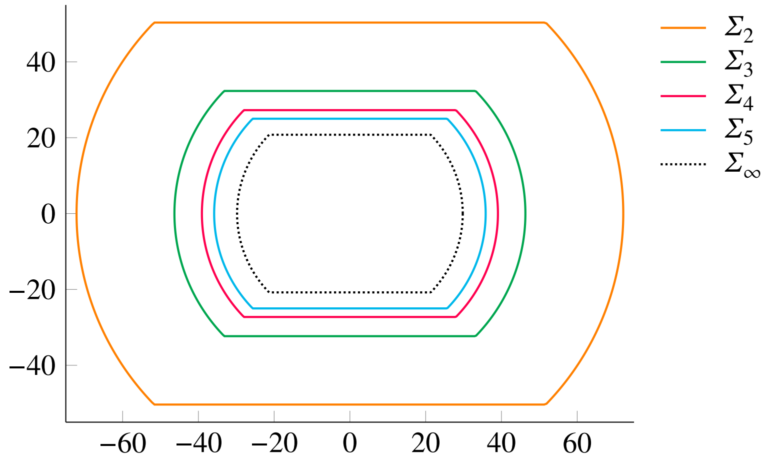

Figure 4.1: The figure depicts the boundaries of sets Σ_s, $s \in [1, \infty]$, for the operator $A = \operatorname{sgn} \cdot (-\mathrm{d}^2/\mathrm{d}x^2 - \varkappa(\varkappa+1)\operatorname{sech}^2)$ considered in Example 4.17, where $\varkappa = 1$.

At the end of this section we prove the existence of an admissible function g in the general case.

Theorem 4.18. *Assume that Hypothesis 4.1 holds. Then for every $\beta > 0$ there exists a function $g \in \mathcal{G}_p$ such that $\omega_g \beta < 1$ holds, where ω_g is defined as in* (4.21).

Proof. Fix $\beta > 0$ and let $[c, d] \subset \mathbb{R}$ be the the compact interval from Hypothesis 4.1 such that $C_r := \operatorname{ess\,inf}_{t \in \mathbb{R}\setminus[c,d]} |r(t)| > 0$. Then $\mu(\{|r| < C_r\}) \leq d - c < \infty$ and with

$$\lim_{n\to\infty} \mu\left(\left\{|r| < \frac{1}{n}\right\}\right) = \mu\left(\bigcap_{n=1}^{\infty}\left\{|r| < \frac{1}{n}\right\}\right) = \mu(\{r = 0\}) = 0$$

there exists $\gamma > 0$ such that

$$\mu(\{|r| < \gamma\}) < \frac{1}{6\beta}. \tag{4.32}$$

Consider the compact set $\mathcal{T}_r \subset [c, d]$ and let $a_0 := \min \mathcal{T}_r$, $b_0 := \max \mathcal{T}_r$. The set

$$\Omega_+ := \{r > 0\} \cap [a_0, b_0] \tag{4.33}$$

has finite Lebesgue measure and, thus, for every $\varepsilon > 0$ there is a finite union Ω_ε of bounded open intervals such that $\mu(\Omega_+ \,\Delta\, \Omega_\varepsilon) < \varepsilon$, where Δ denotes the symmetric difference of two sets; see e. g. [94, Part One, Chapter 3, Proposition 15]. Together with the fact that the measure, which is defined via $\Omega \mapsto \int_\Omega |r(t)|\,\mathrm{d}t$ on the σ-algebra of Lebesgue measurable subsets $\Omega \subset \mathbb{R}$, is absolutely continuous with respect to the Lebesgue measure μ, we find $N \in \mathbb{N}$, $N \geq 1$, and open intervals (a_1, b_1) ,$(a_2, b_2), \ldots, (a_N, b_N)$ such that for $\Omega := \bigcup_{k=1}^{N}(a_k, b_k)$

$$\mu(\Omega_+ \,\Delta\, \Omega) < \frac{1}{6\beta} \quad \text{and} \quad \int_{\Omega_+ \Delta\, \Omega} |r(t)|\,\mathrm{d}t < \frac{\gamma}{6\beta}. \tag{4.34}$$

Since $\Omega_+ \subset [a_0, b_0]$ the estimates in (4.34) are still true if Ω is replaced by the set $\Omega \cap (a_0, b_0)$. Therefore, we may assume that $\Omega \subset (a_0, b_0)$ and that the (possibly empty) intervals (a_k, b_k), $k =$

$1,\dots,N$, are disjoint, contained in (a_0,b_0), and ordered in the way that $b_k < a_{k+1}$, $k=1,\dots,N-1$. Choose $c_0 \in (-\infty, a_0)$ sucht that

$$a_0 - c_0 < \frac{1}{6\beta} \quad \text{and} \quad \int_{c_0}^{a_0} |r(t)|\,\mathrm{d}t < \frac{\gamma}{6\beta}. \tag{4.35}$$

We define

$$w(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ -1 & \text{if } x \in [a_0,b_0] \setminus \Omega, \\ -\operatorname{sgn}(r(x)) & \text{if } x \in [c_0,a_0), \\ \operatorname{sgn}(r(x)) & \text{if } x \in (-\infty,c_0) \cup (b_0,\infty). \end{cases} \tag{4.36}$$

The function w is real-valued, locally integrable on $\mathbb{R}$ and $|w(x)| > 0$ a. e. on $\mathbb{R}$. Recall, that r has constant sign a. e. on $(-\infty, a_0)$ and on (b_0, ∞). The definition of w on $[c_0, a_0)$ and $(-\infty, c_0)$ guarantees that $\mu(\{w < 0\}) > 0$, $\mu(\{w < 0\}) > 0$. Moreover, w has constant definit sign a. e. on $(-\infty, c_0)$ as well as (b_0,∞) and satisfies $\operatorname{ess\,inf}_{t\in\mathbb{R}} |w(t)| \geq 1$. Since $\mathcal{T}_r \subset [a_0,b_0] \subset [c_0,b_0]$ and the signs of r and w coincide outside of $[c_0,b_0]$ we have $\mathcal{T}_w \subset [c_0,b_0]$. More precisely, $\mathcal{T}_w \subset \{c_0,a_0,b_0,a_1,b_1,\dots,a_N,b_N\}$. Further, we have $\{rw<0\} \subset [c_0,b_0]$ and

$$\begin{aligned} \{rw<0\} \cap [a_0,b_0] &\subset \Big(\big(\{r>0\} \cap \{w=-1\}\big) \cup \big(\{r<0\} \cap \{w=1\}\big) \Big) \cap [a_0,b_0] \\ &= \Big(\big(\{r>0\} \cap [a_0,b_0]\big) \setminus \Omega \Big) \cup \Big(\{r<0\} \cap [a_0,b_0] \cap \Omega \Big) \\ &\subset \big(\Omega_+ \setminus \Omega\big) \cup \big(\{r \leq 0\} \cap [a_0,b_0] \cap \Omega\big) = \Omega_+ \,\Delta\, \Omega. \end{aligned} \tag{4.37}$$

Hence, (4.34) together with (4.37) and (4.35) implies

$$\mu\big(\{rw<0\}\big) < \frac{2}{6\beta} \quad \text{and} \quad \int_{\{rw<0\}} |r(t)|\,\mathrm{d}t < \frac{2\gamma}{6\beta}. \tag{4.38}$$

Observe that $\mathcal{T}_w$ is finite and consists of at most $2N+3$ elements. Choose $\delta > 0$ such that

$$\delta < \frac{1}{2}\min\big\{|x-y| : x,y \in \mathcal{T}_w,\ x \neq y\big\}, \quad 2\delta(2N+3) < \frac{1}{6\beta}.$$

We apply Lemma 4.10, where we exchange the weight r with w. This yields $\tilde{g} \in \mathcal{G}_p$ such that $w\tilde{g} > 0$ a. e., $\|\tilde{g}\|_\infty = 1$ and $\mu(\{|\tilde{g}| < 1\}) = 2\delta(2N+3) < 1/(6\beta)$. Since $w\tilde{g} > 0$ a. e., the estimates in (4.38) imply

$$\mu\left(\{r\tilde{g}<0\}\right) < \frac{2}{6\beta} \quad \text{and} \quad \int_{\{r\tilde{g}<0\}} |r(t)|\,\mathrm{d}t < \frac{2\gamma}{6\beta}. \tag{4.39}$$

Observe that $\{r\tilde{g} < \gamma\} \subset \{|r| < \gamma\} \cup \{|\tilde{g}| < 1\} \cup \{r\tilde{g} < 0\}$. Together with (4.32) and (4.39) this yields

$$\tilde{\omega} := \mu(\{r\tilde{g} < \gamma\}) + \frac{1}{\gamma}\int_{\{r\tilde{g}<0\}} |r(t)|\,\mathrm{d}t < \frac{1}{\beta}. \tag{4.40}$$

Finally, the assertion follows for $g := \tilde{g}/\gamma$, where $\omega_g = \tilde{\omega}$. □

4.3 Comparison with other eigenvalue bounds

For the operator

$$A = \operatorname{sgn} \cdot \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + q \right) \tag{4.41}$$

with a real-valued potential $q \in L^s(\mathbb{R})$, where $s \in [1, \infty]$, spectral bounds similar as in Corollary 4.16 have been obtained in [17] and recently in [34, 89]. In [17, 89] perturbation theory for J-non-negative operators in Krein spaces is employed. For $q \in L^\infty(\mathbb{R})$ it is shown in [17] that $\sigma(A) \setminus \mathbb{R}$ is a subset of

$$\Sigma_{\mathrm{BPT},\infty} := \left\{ \lambda \in \mathbb{C} \,\middle|\, \operatorname{dist}(\lambda, (-d, d)) \le 5\|q\|_\infty, |\operatorname{Im} \lambda| \le 2\|q\|_\infty \right\}, \tag{4.42}$$

where $d = 5\|q_-\|_\infty$. Note that, in contrast to the the set Σ_∞ in Corollary 4.16 (iii), which depends only on $\|q_-\|_\infty$, the set $\Sigma_{\mathrm{BPT},\infty}$ depends on $\|q_-\|_\infty$ and $\|q\|_\infty$. Further, in Corollary 4.16 it is assumed that $q \in L^1_{\mathrm{u}}(\mathbb{R})$ and $q_- \in L^\infty(\mathbb{R})$, whereas in [16] the stronger condition $q \in L^\infty(\mathbb{R})$ is required. In the case where $q = -q_-$ one has $\Sigma_{\mathrm{BPT},\infty} \subset \Sigma_\infty$. For a small ratio $\alpha_\infty = \|q_-\|_\infty / \|q\|_\infty$, however, the inclusion $\Sigma_\infty \subset \Sigma_{\mathrm{BPT},\infty}$ holds, cf. Table 4.1. In Figure 4.2 the set $\Sigma_{\mathrm{BPT},\infty}$ and Σ_∞ are compared for different ratios α_∞.

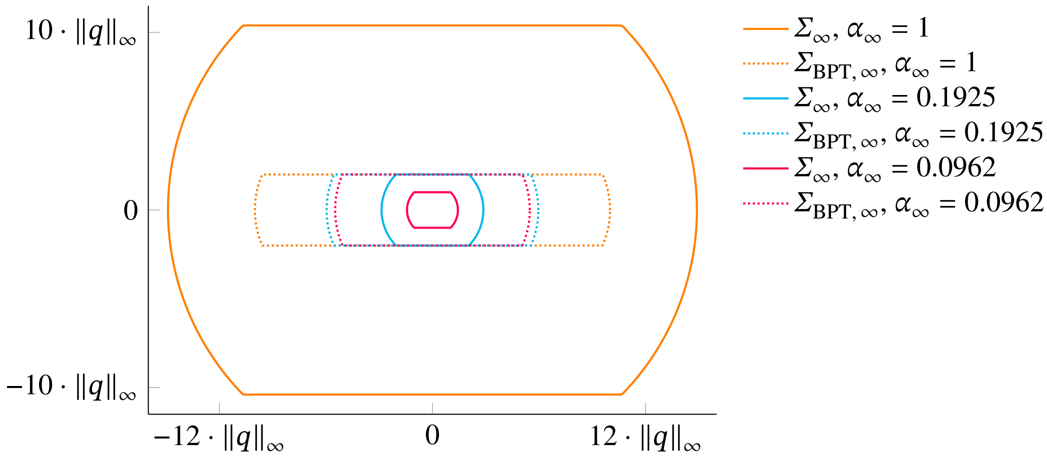

Figure 4.2: For a potential $q \in L^\infty(\mathbb{R})$ the region Σ_∞ is contained in the region $\Sigma_{\mathrm{BPT},\infty}$, provided that the ratio $\alpha_\infty = \|q_-\|_\infty / \|q\|_\infty$ is less or equal than $3^{-3/2} \approx 0.1925$. The figure shows the boundaries of the regions Σ_∞ and $\Sigma_{\mathrm{BPT},\infty}$ for different ratios α_∞.

In [89] it is shown under the condition $q \in L^2(\mathbb{R})$ for some $s \in [2, \infty)$ that the non-real spectrum of A in eq. (4.41) is contained in the rectangle

$$\Sigma_{\mathrm{P},s} := \left\{ \lambda \in \mathbb{C} \,\middle|\, \begin{aligned} &|\operatorname{Im} \lambda| \le c_1(s) c_2(s) \|q\|_s^{\frac{2s}{2s-1}}, \\ &|\operatorname{Re} \lambda| \le c_1(s) \left(\sqrt{6 + 4\sqrt{2}} + c_2(s) \right) \|q\|_s^{\frac{2s}{2s-1}} \end{aligned} \right\}, \tag{4.43}$$

where

$$c_1(s) = (1+\sqrt{2})\sqrt{3-2\sqrt{2}}\sqrt{\frac{2s}{2s-1}}\left(\frac{16\sqrt{2}(3+2\sqrt{2})^2}{3\pi^2 s}\right)^{\frac{1}{4s-2}},$$

$$c_2(s) = \sqrt{2\frac{(17+12\sqrt{2})c_3(s)+4+3\sqrt{2}}{(3+2\sqrt{2})c_3(s)-1-\sqrt{2}}}, \tag{4.44}$$

$$c_3(s) = 4-3\sqrt{2}+(4\sqrt{2}-5)s+\sqrt{44-31\sqrt{2}+(62\sqrt{2}-88)s+(57-40\sqrt{2})s^2}.$$

The rectangle $\Sigma_{\mathrm{P},s}$ depends on $\|q\|_s$. In contrast, the set Σ_s in Corollary 4.16 (ii) depends on $\|q_-\|_s$, where only $q \in L^1_{\mathrm{u}}(\mathbb{R})$ and $q_- \in L^s(\mathbb{R})$ is required. For $q = -q_-$ the inclusion $\Sigma_{\mathrm{P},s} \subset \Sigma_s$ holds. Provided that the ratio $\alpha_s = \|q_-\|_s/\|q\|_s$ is small, one has $\Sigma_s \subset \Sigma_{\mathrm{P},s}$, cf. Table 4.1. In Figure 4.3 the sets $\Sigma_{\mathrm{P},2}$ and Σ_2 are compared for different ratios α_2.

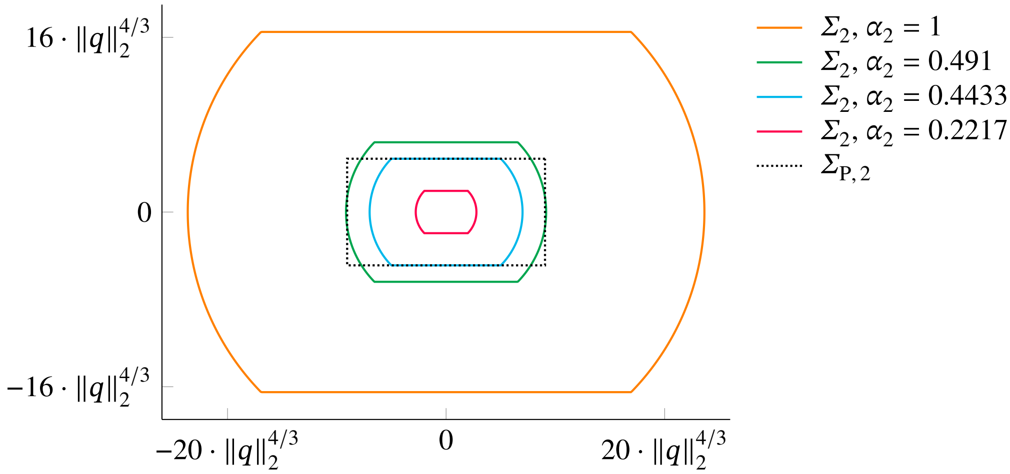

Figure 4.3: For a potential $q \in L^2(\mathbb{R})$ the region Σ_2 is contained in the rectangle $\Sigma_{\mathrm{P},2}$, provided that the ratio $\alpha_2 = \|q_-\|_2/\|q\|_2$ is less or equal than $(c_1(2)c_2(2))^{3/4} \cdot 2^{-5/4} \cdot 3^{-9/8} \approx 0.4433$, where c_1 and c_2 are as in (4.44). The figure shows the boundaries of $\Sigma_{\mathrm{P},2}$ and Σ_2 for different ratios α_2, where $\Sigma_{\mathrm{P},2}$ does not depend on α_2.

In [34] it is shown for $q \in L^s(\mathbb{R})$, where $s \in [1,\infty)$, that every non-real eigenvalue (and every real eigenvalue) of A in (4.41) is contained in the region

$$\Sigma_{\mathrm{CI},s} := \left\{\lambda \in \mathbb{C} \,\middle|\, 2^{\frac{3}{2s}-1}|\lambda|^{\frac{1}{s}}|\mathrm{Im}\,\lambda|^{1-\frac{1}{s}} \le (|\lambda|+|\mathrm{Re}\,\lambda|)^{\frac{1}{2s}}\|q\|_s\right\}. \tag{4.45}$$

The results in [34] are based on an approach similar to the the Birman–Schwinger principle which already lead to spectral bounds for Schrödinger operators with complex-valied potentials, see [1], and eigenvalue estimates for the operator A in the case where $q \in L^1(\mathbb{R})$, see [19]. The result in [34] for $s = 1$ improves the eigenvalue estimate obtained in [19]. Unlike the set Σ_s in Corollary 4.16 (ii) the set $\Sigma_{\mathrm{CI},s}$ depends on $\|q\|_s$. For $q = -q_-$ one has $\Sigma_{\mathrm{CI},s} \subset \Sigma_s$. If the ratio $\alpha_s = \|q_-\|_s/\|q\|_s$ is small, then the inclusion $\Sigma_s \subset \Sigma_{\mathrm{CI},s}$ holds, cf. Table 4.1. In Figure 4.4 and Figure 4.5 we compare the sets $\Sigma_{\mathrm{CI},s}$ and Σ_s for different ratios α_s in the cases $s = 1$ and $s = 2$. Note that the region $\Sigma_{\mathrm{CI},s}$ for $s > 1$ is unbounded and comprises the whole real axis, cf. Figure 4.5.

Table 4.1: A small ratio of $\|q_-\|_s$ to $\|q\|_s$.

$q \in L^s(\mathbb{R})$	Ratio $\alpha_s = \|q_-\|_s / \|q\|_s$ less or equal than	Inclusion
$s \in [1, \infty)$	$2^{-4/2s} \cdot 3^{-3(2s-1)/4s}$	$\Sigma_s \subset \Sigma_{\mathrm{CI},s}$
$s \in [2, \infty)$	$(c_1(s)c_2(s))^{(2s-1)/2s} \cdot 2^{-(2s+1)/2s} \cdot 3^{-3(2s-1)/4s}$	$\Sigma_s \subset \Sigma_{\mathrm{P},s}$
$s = \infty$	$3^{-3/2}$	$\Sigma_\infty \subset \Sigma_{\mathrm{BPT},\infty}$

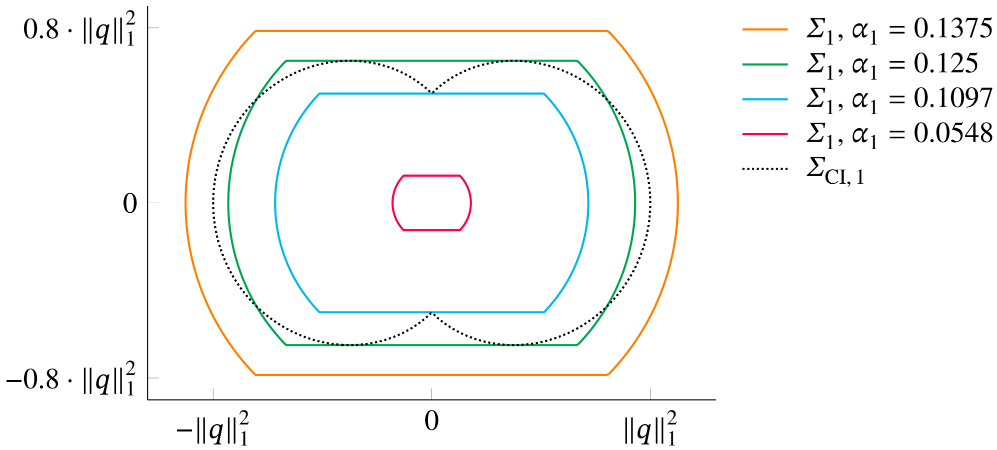

Figure 4.4: For a potential $q \in L^1(\mathbb{R})$ the region Σ_1 is contained in the region $\Sigma_{\mathrm{CI},1}$, provided that the ratio $\alpha_1 = \|q_-\|_1 / \|q\|_1$ is less or equal than $3^{-3/4}/4 \approx 0.1097$. The figure shows the boundaries of the regions $\Sigma_{\mathrm{CI},1}$ and Σ_1 for different ratios α_1, where $\Sigma_{\mathrm{CI},1}$ does not depend on α_1

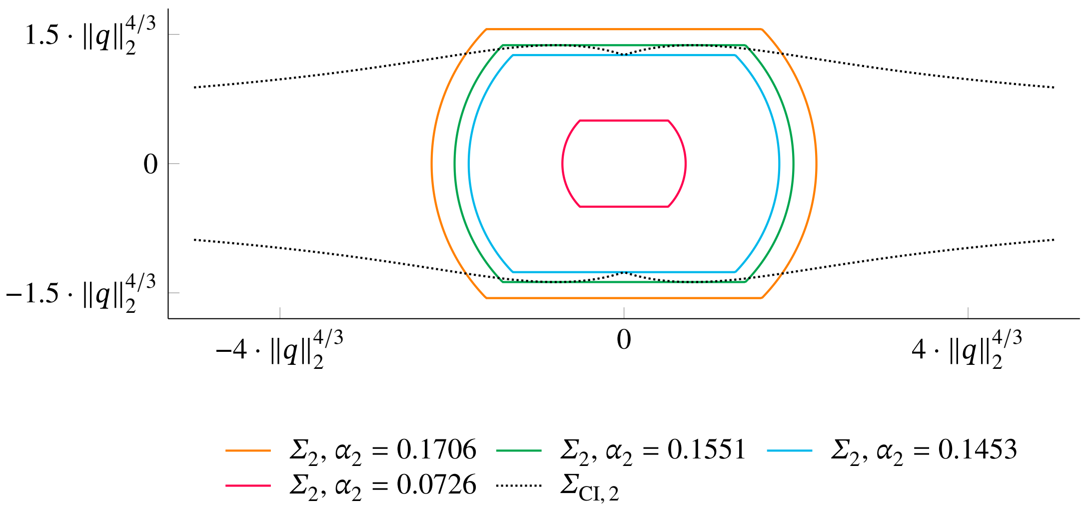

Figure 4.5: For a potential $q \in L^2(\mathbb{R})$ the region Σ_2 is contained in the region $\Sigma_{\mathrm{CI},2}$, provided that the ratio $\alpha_2 = \|q_-\|_2 / \|q\|_2$ is less or equal than $3^{-9/8}/2 \approx 0.1453$. The figure shows the boundaries of the regions $\Sigma_{\mathrm{CI},2}$ and Σ_2 for different ratios α_2, where $\Sigma_{\mathrm{CI},2}$ does not depend on α_2.

Appendix A

Sturm–Liouville operators with uniformly locally integrable potentials

In this section we investigate the properties of a certain class of definite Sturm–Liouville expressions on $\mathbb{R}$ and their associated operators. As in Chapter 4 let $L^1_u(\mathbb{R})$ denote the normed space of uniformly locally integrable functions from $\mathbb{R}$ to $\mathbb{C}$, i. e.

$$L^1_u(\mathbb{R}) = \left\{f \in L^1_{\text{loc}}(\mathbb{R}) : \|f\|_u < \infty\right\}, \qquad \|f\|_u = \sup_{n\in\mathbb{Z}} \int_n^{n+1} |f(t)|\,\mathrm{d}t.$$

The Sturm–Liouville differential expressions we are interested in are characterized by the following properties.

Hypothesis A.1. The differential expression τ on $\mathbb{R}$ of the form (1.1) satisfies (1.2) and the assumptions

(α) there exist $c, d \in \mathbb{R}$ with $c < d$ such that $c_r := \operatorname{ess\,inf}_{t\in\mathbb{R}\setminus[c,d]} r(t) > 0$;

(β) $q \in L^1_u(\mathbb{R})$;

(γ) $1/p \in L^\eta(\mathbb{R})$ for some $\eta \in [1, \infty]$. ◇

Particularly, condition (γ) in Hypothesis A.1 implies the following.

Lemma A.2. *If Hypothesis A.1 holds, then $P : \mathbb{R} \to \mathbb{R}$, $x \mapsto \int_0^x 1/p(t)\,\mathrm{d}t$ is uniformly continuous.*

Proof. We show that P is Hölder continuous in the cases where $\eta \in (1, \infty]$ which implies the uniform continuity. Let $x, y \in \mathbb{R}$ with $x \geq y$. For $\eta \in (1, \infty)$ by Hölder's inequality we obtain

$$P(x) - P(y) = \int_y^x \frac{1}{p(t)}\,\mathrm{d}t \leq (x-y)^{\frac{\eta-1}{\eta}} \left(\int_y^x \left(\frac{1}{p(t)}\right)^\eta \mathrm{d}t\right)^{\frac{1}{\eta}} \leq (x-y)^{\frac{\eta-1}{\eta}} \|1/p\|_\eta$$

and in the case $\eta = \infty$ we have

$$P(x) - P(y) = \int_y^x \frac{1}{p(t)}\,\mathrm{d}t \leq (x-y)\|1/p\|_\infty.$$

In the case $\eta = 1$ we have

$$I_n := \int_{\mathbb{R}\setminus(-n,n)} \frac{1}{p(t)}\,\mathrm{d}t \to 0 \quad \text{as } n \to \infty. \tag{A.1}$$

Let $\varepsilon > 0$ and choose $n \in \mathbb{N}$, $n \geq 1$, with $I_n < \varepsilon$. Since P is continuous, it is uniformly continuous on the compact intervall $[-2n, 2n]$ and there is $\delta \in (0, 1)$ such that $|P(x) - P(y)| < \varepsilon$ for all

$x, y \in [-2n, 2n]$ with $|x-y| < \delta$. Let $x, y \in \mathbb{R}$ with $y < x$ and $x - y < \delta$. Then the interval $[y, x]$ is contained in at least one of the intervals $[-2n, 2n]$, $(-\infty, -n]$ and $[n, \infty)$. Provided that $[y, x] \subset (-\infty, n]$ or $[y, x] \subset [n, \infty)$, the choice of n yields

$$P(x) - P(y) = \int_y^x \frac{1}{p(t)}\,\mathrm{d}t \le I_n < \varepsilon. \tag{A.2}$$

This shows the uniformly continuity of P. □

In the propositions below we collect properties of the domain $\mathscr{D}(\tau)$ and the maximal operator $T_{\max}$ associated with τ, cf. (1.6) and (1.9). We employ standard techniques in Sturm–Liouville theory, see [11, Chapter 6], [45, 46], [98, §9.7] and [100, Appendix to section 6].

Lemma A.3. *Suppose that Hypothesis A.1 holds. Then for every* $\varepsilon > 0$ *there exists* $\gamma > 0$ *such that for all* $f \in \mathscr{D}(\tau)$ *and every* $\xi \in \mathbb{R}$

$$\sup_{t\in[\xi,\xi+1]} |f(t)|^2 \le \varepsilon \int_\xi^{\xi+1} p(t)|f'(t)|^2\,\mathrm{d}t + \gamma \int_\xi^{\xi+1} |f(t)|^2 r(t)\,\mathrm{d}t. \tag{A.3}$$

Proof. Fix $\varepsilon > 0$ and consider an arbitrary $f \in \mathscr{D}(\tau)$. Since $f \in AC(\mathbb{R})$ we have $f' \in L^1_{\mathrm{loc}}(\mathbb{R})$. Together with $pf' \in AC(\mathbb{R})$ we see that $p(f')^2 \in L^1_{\mathrm{loc}}(\mathbb{R})$. For all $x, y \in \mathbb{R}$ we obtain by $2\alpha\beta \le \alpha^2 + \beta^2$ for $\alpha, \beta \in \mathbb{R}$ and the Cauchy–Schwarz inequality

$$\begin{aligned} |f(x)|^2 &= \left| f(y) + \int_y^x f'(t)\,\mathrm{d}t \right|^2 \le 2|f(y)|^2 + 2\left(\int_y^x |f'(t)|\,\mathrm{d}t\right)^2 \\ &\le 2|f(y)|^2 + 2\int_y^x \frac{1}{p(t)}\,\mathrm{d}t \int_y^x p(t)|f'(t)|^2\,\mathrm{d}t. \end{aligned} \tag{A.4}$$

Due to the uniform continuity of P in Lemma A.2 there exists $\delta > 0$ such that

$$\int_{x-\delta}^{x+\delta} \frac{1}{p(t)}\,\mathrm{d}t = P(x+\delta) - P(x-\delta) < \frac{\varepsilon}{2}$$

for all $x \in \mathbb{R}$. It is no restriction to assume that $\delta < \frac{1}{2}$. Further, by Hypothesis A.1 (α) we find $\tilde{\gamma} > 0$ such that

$$\int_{x-\frac{\delta}{2}}^{x+\frac{\delta}{2}} r(t)\,\mathrm{d}t > \tilde{\gamma}$$

for all $x \in \mathbb{R}$. Let $I(x, \xi) := (x - \delta, x + \delta) \cap [\xi, \xi + 1]$ for $\xi \in \mathbb{R}$ and $x \in [\xi, \xi + 1]$. Then the length of the interval $I(x, \xi)$ is bounded from below by δ but does not exceed 2δ. Thus, with (A.4)

$$\begin{aligned} |f(x)|^2 \int\limits_{I(x,\xi)} r(y)\,\mathrm{d}y &\le 2 \int\limits_{I(x,\xi)} |f(y)|^2 r(y)\,\mathrm{d}y \\ &\quad + 2 \int\limits_{I(x,\xi)} r(y)\,\mathrm{d}y \int\limits_{I(x,\xi)} \frac{1}{p(t)}\,\mathrm{d}t \int\limits_{I(x,\xi)} p(t)|f'(t)|^2\,\mathrm{d}t \\ &\le 2\int_\xi^{\xi+1} |f(t)|^2 r(t)\,\mathrm{d}t + \varepsilon \int\limits_{I(x,\xi)} r(y)\,\mathrm{d}t \int_\xi^{\xi+1} p(t)|f'(t)|^2\,\mathrm{d}t. \end{aligned}$$

We divide by $\int_{I(x,\xi)} r(y)\,\mathrm{d}y$ and define $\gamma := 2/\tilde{\gamma}$. This finishes the proof. □

Lemma A.4. *Suppose that Hypothesis A.1 holds. Then τ is in the limit point case at both endpoints and for all $f, g \in \mathscr{D}(\tau)$*

(i) $f, \sqrt{p}f' \in L^2(\mathbb{R})$ *and* $qf^2 \in L^1(\mathbb{R})$,

(ii) *there exists a sequence* $(x_n)_{n\in\mathbb{Z}}$ *in* $\mathbb{R}$ *with* $\lim_{n\to\infty} x_n = \infty$ *and* $\lim_{n\to-\infty} x_n = -\infty$ *such that* $\lim_{|n|\to\infty} f(x_n) = 0$,

(iii) $\lim_{|x|\to\infty}(pf')(x)\overline{g(x)} = 0$.

Moreover, there exists $\lambda > 0$ such that $\langle(\tau - \lambda)f, f\rangle_r \geq 0$ for all $f \in \mathscr{D}(\tau)$.

Proof. If suffices to prove (i)–(iii) only for real-valued functions. Assume that $f, g \in \mathscr{D}(\tau)$ are real-valued. Observe that for all $\xi \in \mathbb{R}$

$$\begin{aligned}\int_\xi^{\xi+1} |q(t)||f(t)|^2\,\mathrm{d}t &\leq \left(\int_{\lfloor\xi\rfloor}^{\lceil\xi\rceil+1} |q(t)|\,\mathrm{d}t\right)\left(\sup_{t\in[\xi,\xi+1]} |f(t)|^2\right) \\ &\leq 2\|q\|_{\mathrm{u}}\left(\sup_{t\in[\xi,\xi+1]} |f(t)|^2\right).\end{aligned} \tag{A.5}$$

Let $x, y \in \mathbb{R}$ with $1 < x - y$ and set $n = \lfloor x - y\rfloor$. Then by (A.5)

$$\begin{aligned}\int_y^x |q(t)||f(t)|^2\,\mathrm{d}t &\leq \int_y^{y+n} |q(t)||f(t)|^2\,\mathrm{d}t + \int_{x-n}^x |q(t)||f(t)|^2\,\mathrm{d}t \\ &\leq 2\|q\|_{\mathrm{u}}\sum_{k=0}^{n-1}\left(\sup_{t\in[y+k,y+k+1]} |f(t)|^2 + \sup_{t\in[x-k-1,x-k]} |f(t)|^2\right).\end{aligned}$$

By Lemma A.3 for $\varepsilon = 1/(8\|q\|_{\mathrm{u}} + 1)$ we find $\gamma > 0$ such that (A.3) holds. This implies

$$\begin{aligned}\int_y^x |q(t)||f(t)|^2\,\mathrm{d}t &\leq 4\|q\|_{\mathrm{u}}\left(\varepsilon\int_y^x p(t)|f'(t)|^2\,\mathrm{d}t + \gamma\int_y^x |f(t)|^2 r(t)\,\mathrm{d}t\right) \\ &\leq \frac{1}{2}\int_y^x p(t)|f'(t)|^2\,\mathrm{d}t + 4\gamma\|q\|_{\mathrm{u}}\int_y^x |f(t)|^2 r(t)\,\mathrm{d}t.\end{aligned} \tag{A.6}$$

Let $\lambda = -4\gamma\|q\|_{\mathrm{u}}$. Then

$$\int_y^x \left(p(t)|f'(t)|^2 + \big(q(t) - \lambda r(t)\big)|f(t)|^2\right)\mathrm{d}t \geq \frac{1}{2}\int_y^x p(t)|f'(t)|^2\,\mathrm{d}t$$

and integration by parts, cf. (1.7), yields

$$\int_y^x \big((\tau - \lambda)f\big)(t)f(t)r(t)\,\mathrm{d}t \geq \frac{1}{2}\int_y^x p(t)|f'(t)|^2\,\mathrm{d}t + (pf')(y)f(y) - (pf')(x)f(x). \tag{A.7}$$

Fix $y = 0$ and let $x > 1$. If $p|f'|^2$ is not integrable near ∞ then the integral on the right hand side of (A.7) diverges monotone to infinity while the integral an the left hand side converges as x tends ∞.

Thus, there exists $b \in (0,\infty)$ such that $(pf')(x)f(x) > 0$ for all $x \in [b,\infty)$ and we obtain

$$\int_b^\infty |f(t)|^2 r(t)\,\mathrm{d}t = \int_b^\infty \left(|f(b)|^2 + 2\int_b^t \frac{(pf')(s)f(s)}{p(s)}\,\mathrm{d}s\right) r(t)\,\mathrm{d}t$$
$$\geq \int_b^\infty |f(b)|^2 r(t)\,\mathrm{d}t = \infty,$$

since r is bounded from below near ∞ by Hypothesis A.1 (α). But this contradicts $f \in L^2(\mathbb{R}, r)$ and, therefore, $p|f'|^2$ is integrable near ∞. In a similar way one shows that $p|f'|^2$ is integrable near $-\infty$ and $\sqrt{p}f' \in L^2(\mathbb{R})$ follows. Passing to the limits $x \to \infty$ and $y \to -\infty$ in (A.6) yields $qf^2 \in L^1(\mathbb{R})$. Further, by Hypothesis A.1 (α) we obtain

$$\int_{\mathbb{R}} |f(t)|^2\,\mathrm{d}t \leq (d-c)\cdot \sup_{t\in[c,d]} |f(t)|^2 + \frac{1}{c_r}\int_{\mathbb{R}\setminus[c,d]} |f(t)|^2 r(t)\,\mathrm{d}t < \infty,$$

which implies $f \in L^2(\mathbb{R})$ and finishes the proof of assertion (i).

We construct the sequence in (ii). Since f is continuous on $\mathbb{R}$, by the mean value theorem we find for every $n \in \mathbb{Z}$ a point $x_n \in [n, n+1]$ such that

$$\int_n^{n+1} |f(t)|^2\,\mathrm{d}t = |f(x_n)|^2.$$

As f is square integrable on $\mathbb{R}$ the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to zero for $n \to \infty$ and $n \to -\infty$.

We show (iii). By formula (1.7) together with (i) we see that $\lim_{x\to\infty}(pf')(x)g(x)$ exists and is finite. Assume

$$\lim_{x\to\infty} |(pf'(x)g(x)| =: \alpha > 0.$$

Then there exists $b \in \mathbb{R}$ such that

$$|g(x)| > 0 \quad \text{and} \quad p(x)|f'(x)| \geq \frac{\alpha}{2|g(x)|}$$

for all $x \in [b,\infty)$. Multiplication by $|g'(x)|$ and integration yield

$$\int_b^x p(t)|f'(t)g'(t)|\,\mathrm{d}t \geq \frac{\alpha}{2}\int_b^x \frac{|g'(t)|}{|g(t)|}\,\mathrm{d}t \geq \frac{\alpha}{2}\left|\int_b^x \frac{g'(t)}{g(t)}\,\mathrm{d}t\right| = \frac{\alpha}{2}\left|\ln\left(\frac{|g(x)|}{|g(b)|}\right)\right|. \qquad \text{(A.8)}$$

Let x run through the sequence provided by (ii) for g. Then the right hand side in (A.8) diverges to ∞ while the left hand side is bounded as $\sqrt{p}f', \sqrt{p}g' \in L^2(\mathbb{R})$ by (i); a contradiction. Therefore, $\alpha = 0$ and $\lim_{x\to\infty}(pf')(x)g(x) = 0$. Similarly one obtains $\lim_{x\to-\infty}(pf')(x)g(x) = 0$ which finishes the proof of (iii). As a consequence of (iii) the differential expression τ is in the limit-point case at both endpoints, cf. [46].

Let λ be defined as before. Then from (A.7) together with (iii) we know that $\langle(\tau-\lambda)f, f\rangle_r \geq 0$ for all real-valued $f \in \mathscr{D}(\tau)$. An arbitrary $f \in \mathscr{D}(\tau)$ can be decomposed as $f = g + \mathrm{i}h$, where $g, h \in \mathscr{D}(\tau)$ are real-valued, and a simple calculation employing (1.8) and (iii) shows

$$\langle(\tau-\lambda)f, f\rangle_r = \langle(\tau-\lambda)g, g\rangle_r + \langle(\tau-\lambda)h, h\rangle_r + \mathrm{i}\langle(\tau-\lambda)h, g\rangle_r - \mathrm{i}\langle(\tau-\lambda)g, h\rangle_r$$
$$= \langle(\tau-\lambda)g, g\rangle_r + \langle(\tau-\lambda)h, h\rangle_r \geq 0. \qquad \square$$

The following corollary is an immediate consequence of Lemma A.4 and Proposition 1.1 (iii).

Corollary A.5. *If Hypothesis A.1 holds, then the maximal operator $T_{\max}$ is the only self-adjoint realisation of τ in $L^2(\mathbb{R}, r)$ and $T_{\max}$ is semi-bounded from below.*

We next investigate lower bounds for the spectrum of the maximal operator. In this context we consider the set

$$\mathscr{D}_-(\tau) := \{ f \in \mathscr{D}(\tau) \,|\, \langle \tau f, f\rangle_r \leq 0 \}. \tag{A.9}$$

Further, the decomposition of the real-valued function q into its positive part q_+ and its negative part q_-, i. e.

$$q = q_+ - q_-, \quad \text{where} \quad q_+ := \frac{|q| + q}{2} \quad \text{and} \quad q_- := \frac{|q| - q}{2}, \tag{A.10}$$

is of particular importance.

Lemma A.6. *Under Hypothesis A.1 every function $f \in \mathscr{D}_-(\tau)$ satisfies*

$$\|\sqrt{p} f'\|_2^2 \leq \|q_- f^2\|_1 \quad \textit{and} \quad \|q f^2\|_1 \leq 2\|q_- f^2\|_1. \tag{A.11}$$

Moreover, for $f \in \mathscr{D}_-(\tau)$ the inequality $\|q_- f^2\|_1 \leq \|q_+ f^2\|_1$ implies $\|\sqrt{p} f'\|_2 = 0$.

Proof. For $f \in \mathscr{D}_-(\tau)$ integration by parts, cf. (1.7), together with the decomposition $q = q_+ - q_-$ and Lemma A.4 (i), (iii) yields

$$0 \geq \langle \tau f, f\rangle_r = \int_{\mathbb{R}} p(t)|f'(t)|^2 \,\mathrm{d}t + \int_{\mathbb{R}} q(t)|f(t)|^2 \,\mathrm{d}t = \|\sqrt{p} f'\|_2^2 + \|q_+ f^2\|_1 - \|q_- f^2\|_1. \tag{A.12}$$

This implies $\|\sqrt{p} f'\|_2^2 \leq \|q_- f^2\|_1$ and $\|q_+ f^2\|_1 \leq \|q_- f^2\|_1$. Therefore, with $|q| = q_+ + q_-$ we have

$$\|q f^2\|_1 = \|q_+ f^2\|_1 + \|q_- f^2\|_1 \leq 2\|q_- f^2\|_1.$$

If $\|q_- f^2\|_1 \leq \|q_+ f^2\|_1$ holds, then (A.12) implies $\|\sqrt{p} f'\|_2 = 0$. □

Lemma A.7. *Suppose that Hypothesis A.1 holds and assume that there are constants $\alpha \geq 0$, $\beta \geq 0$ and a non-negative function $g \in L^\infty(\mathbb{R})$ such that*

(i) *for all $f \in \mathscr{D}_-(\tau)$ the estimates*

$$\|q_- f^2\|_1 \leq \alpha \|f\|_2^2 \quad \textit{and} \quad \|f\|_\infty^2 \leq \beta \|f\|_2^2 \tag{A.13}$$

hold, and

(ii) $\mu(\Omega)\beta < 1$, *where $\Omega := \{x \in \mathbb{R} \,|\, r(x)g(x) < 1\}$ and μ denotes the Lebesgue measure.*

Then the spectrum of the maximal operator $T_{\max}$ is bounded from below by

$$\inf \sigma(T_{\max}) \geq \frac{-\alpha \|g\|_\infty}{1 - \mu(\Omega)\beta}. \tag{A.14}$$

Proof. Let $f \in \mathscr{D}_-(\tau)$. Then one has

$$\begin{aligned} \|g\|_\infty \langle f, f\rangle_r &= \|g\|_\infty \int_{\mathbb{R}} |f(t)|^2 r(t) \,\mathrm{d}t \geq \int_{\mathbb{R}} |f(t)|^2 |r(t)g(t)| \,\mathrm{d}t \\ &\geq \int_{\mathbb{R}\setminus\Omega} |f(t)|^2 |g(t)r(t)| \,\mathrm{d}t \geq \|f\|_2^2 - \int_\Omega |f(t)|^2 \,\mathrm{d}t \\ &\geq \|f\|_2^2 - \mu(\Omega)\|f\|_\infty^2 \geq \big(1 - \mu(\Omega)\beta\big)\|f\|_2^2. \end{aligned} \tag{A.15}$$

Further, we have by (A.12)

$$\langle \tau f, f\rangle_r = \|\sqrt{p} f'\|_2^2 + \|q_+ f^2\|_1 - \|q_- f^2\|_1 \geq -\|q_- f^2\|_1 \geq -\alpha \|f\|_2^2.$$

This together with (A.15) yields

$$\langle T_{\max} f, f\rangle_r = \langle \tau f, f\rangle_r \geq -\frac{\alpha \|g\|_\infty}{1 - \mu(\Omega)\beta} \langle f, f\rangle_r. \tag{A.16}$$

Obviously, the inequality in (A.16) holds also for $f \in \mathscr{D}(\tau) \setminus \mathscr{D}_-(\tau)$ and, thus, for all $f \in \mathcal{D}(T_{\max}) = \mathscr{D}(\tau)$. This implies (A.14). □

Remark A.8. By Hypothesis A.1 one has $\mu(\{x \in \mathbb{R} \mid r(x) < c_r\}) \leq d - c < \infty$ and

$$\begin{aligned}\lim_{n\to\infty} \mu(\{x \in \mathbb{R} \mid r(x) < 1/n\}) &= \mu\left(\bigcap_{n=1}^{\infty} \{x \in \mathbb{R} \mid r(x) < 1/n\}\right) \\ &= \mu(\{x \in \mathbb{R} \mid r(x) = 0\}) = 0.\end{aligned}$$

Therefore, given $\beta \geq 0$ there is always a constant function g on $\mathbb{R}$ satisfying the conditions in Lemma A.7. In the particular case where $1/r \in L^\infty(\mathbb{R})$ one can choose $g = 1/r$. Then the set Ω in Lemma A.7 is a Lebesgue null set and (A.14) reads as $\inf \sigma(T_{\max}) \geq -\alpha\|1/r\|_\infty$. ◇

Suitable constants α and β are collected in the lemma below.

Lemma A.9. *Suppose that Hypothesis A.1 holds.*

(i) *If $1/p \in L^\infty(\mathbb{R})$, then the estimates in* (A.13) *hold for all $f \in \mathscr{D}_-(\tau)$ with*

$$\alpha = 2\|q_-\|_{\mathrm{u}} + 4\|1/p\|_\infty \|q_-\|_{\mathrm{u}}^2, \quad \beta = \left(4\|1/p\|_\infty \alpha\right)^{\frac{1}{2}}. \tag{A.17}$$

(ii) *If $1/p \in L^\eta(\mathbb{R})$ and $q_- \in L^s(\mathbb{R})$, where η, $s \in [1,\infty)$ with $\eta + s > 2$, then the estimates in* (A.13) *hold for all $f \in \mathscr{D}_-(\tau)$ with*

$$\alpha = \|q_-\|_s \beta^{\frac{1}{s}}, \quad \beta = \left(\left(\frac{2\eta - 1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_s\right)^{\frac{\eta s}{2\eta s - \eta - s}}. \tag{A.18}$$

(iii) *If $1/p \in L^\infty(\mathbb{R})$ and $q_- \in L^s(\mathbb{R})$, where $s \in [1,\infty)$, then the estimates in* (A.13) *hold for all $f \in \mathscr{D}_-(\tau)$ with*

$$\alpha = \|q_-\|_s \beta^{\frac{1}{s}}, \quad \beta = \left(4\|1/p\|_\infty \|q_-\|_s\right)^{\frac{s}{2s-1}}. \tag{A.19}$$

(iv) *If $1/p \in L^\eta(\mathbb{R})$, where $\eta \in [1,\infty)$, and $q_- \in L^\infty(\mathbb{R})$, then the estimates in* (A.13) *hold for all $f \in \mathscr{D}_-(\tau)$ with*

$$\alpha = \|q_-\|_\infty, \quad \beta = \left(\left(\frac{2\eta - 1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_\infty\right)^{\frac{\eta}{2\eta - 1}}. \tag{A.20}$$

(v) *If $1/p \in L^\infty(\mathbb{R})$ and $q_- \in L^\infty(\mathbb{R})$, then the estimates in* (A.13) *hold for all $f \in \mathscr{D}_-(\tau)$ with*

$$\alpha = \|q_-\|_\infty, \quad \beta = 2\sqrt{\|1/p\|_\infty \|q_-\|_\infty}. \tag{A.21}$$

(vi) *If* $1/p \in L^1(\mathbb{R})$ *and* $q_- \in L^1(\mathbb{R})$ *such that* $\|1/p\|_1 \|q_-\|_1 < 1$, *then* $\mathscr{D}_-(\tau) = \{0\}$ *and, in particular,* $\inf \sigma(T_{\max}) \geq 0$.

(vii) *If* $q_-(x) = 0$ *a. e. on* $\mathbb{R}$, *then* $\mathscr{D}_-(\tau) = \{0\}$ *and, in particular,* $\inf \sigma(T_{\max}) \geq 0$.

Before we prove Lemma A.9 we establish estimates on the L^∞-norm of functions contained in $\mathscr{D}(\tau)$.

Lemma A.10. *Suppose that Hypothesis A.1 holds.*

(i) *If* $1/p \in L^\eta(R)$, *where* $\eta \in [1, \infty)$, *then*

$$\|f\|_\infty \leq \left(\frac{2\eta - 1}{\eta} \sqrt{\|1/p\|_\eta} \|\sqrt{p} f'\|_2\right)^{\frac{\eta}{2\eta-1}} \|f\|_2^{\frac{\eta-1}{2\eta-1}} \tag{A.22}$$

for all $f \in \mathscr{D}(\tau)$.

(ii) *If* $1/p \in L^\infty(\mathbb{R})$ *then*

$$\|f\|_\infty \leq \left(2\sqrt{\|1/p\|_\infty} \|\sqrt{p} f'\|_2 \|f\|_2\right)^{\frac{1}{2}} \tag{A.23}$$

for all $f \in \mathscr{D}(\tau)$. *Moreover, for every* $\varepsilon > 0$ *and all* $n \in \mathbb{Z}$ *the estimate*

$$\sup_{t \in [n, n+1]} |f(t)|^2 \leq \varepsilon \|1/p\|_\infty \int_n^{n+1} p(t) |f'(t)|^2 \,\mathrm{d}t + \left(1 + \frac{1}{\varepsilon}\right) \int_n^{n+1} |f(t)|^2 \,\mathrm{d}t \tag{A.24}$$

holds for all $f \in \mathscr{D}(\tau)$.

Proof. Let $f \in \mathscr{D}(\tau)$ and $(x_n)_{n \in \mathbb{Z}}$ the sequence in $\mathbb{R}$ provided by Lemma A.4 with $f(x_n) \to 0$ as $|n| \to \infty$. We consider the case $1/p \in L^\eta(\mathbb{R})$ with $\eta \in [1, \infty)$. Define $\theta := \frac{2\eta-1}{\eta}$. For arbitrary $x \in \mathbb{R}$ we obtain with $(f^\theta)' = \theta f^{\theta-1} f'$

$$|f(x)|^\theta \leq |f(x_n)|^\theta + \theta \int_{x_n}^x |f(t)|^{\theta-1} |f'(t)| \,\mathrm{d}t$$

and, thus,

$$\|f\|_\infty^\theta \leq \theta \int_{\mathbb{R}} |f(t)|^{\theta-1} |f'(t)| \,\mathrm{d}t. \tag{A.25}$$

The integral in (A.25) can be further estimated by means of the Hölder's inequality,

$$\begin{aligned}
\int_{\mathbb{R}} |f(t)|^{\theta-1} |f'(t)| \,\mathrm{d}t &\leq \|\sqrt{p} f'\|_2 \left(\int_{\mathbb{R}} \frac{|f(t)|^{2(\theta-1)}}{p(t)} \,\mathrm{d}t\right)^{\frac{1}{2}} \\
&\leq \|\sqrt{p} f'\|_2 \sqrt{\|1/p\|_\eta} \left(\int_{\mathbb{R}} |f(t)|^{\frac{2(\theta-1)\eta}{\eta-1}} \,\mathrm{d}t\right)^{\frac{\eta-1}{2\eta}} \\
&\leq \|\sqrt{p} f'\|_2 \sqrt{\|1/p\|_\eta} \|f\|_2^{\frac{\eta-1}{\eta}}.
\end{aligned} \tag{A.26}$$

Combining (A.25) and (A.26) leads to (A.22).

If $1/p \in L^\infty(\mathbb{R})$, we obtain for arbitrary $x \in \mathbb{R}$

$$|f(x)|^2 = |f(x_n)|^2 + 2 \operatorname{Re} \int_{x_n}^x \overline{f(t)} f'(t) \,\mathrm{d}t$$

and, therefore,

$$\|f\|_\infty^2 \le 2\left(\int_{\mathbb{R}} p(t)|f'(t)|^2\,\mathrm{d}t \int_{\mathbb{R}} \frac{|f(t)|^2}{p(t)}\,\mathrm{d}t\right)^{\frac{1}{2}} \le 2\sqrt{\|1/p\|_\infty}\|\sqrt{p}f'\|_2\|f\|_2.$$

This shows (A.23). Let $\varepsilon > 0$ and $n \in \mathbb{Z}$. Then for $x, y \in [n, n+1]$

$$|f(x)|^2 = |f(y)|^2 + 2\,\mathrm{Re}\int_y^x f'(t)\overline{f(t)}\,\mathrm{d}t.$$

By the mean value theorem we can choose y in such a way that $|f(y)|^2 = \int_n^{n+1}|f(t)|^2\,\mathrm{d}t$. Thus, by the Cauchy–Schwarz inequality and $2\alpha\beta \le \alpha^2 + \beta^2$ for $\alpha, \beta \in \mathbb{R}$ we obtain

$$\begin{aligned}|f(x)|^2 &\le \int_n^{n+1}|f(t)|^2\,\mathrm{d}t + 2\left(\frac{1}{\varepsilon}\int_n^{n+1}|f(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}}\cdot\left(\|1/p\|_\infty\varepsilon\int_n^{n+1}p(t)|f'(t)|^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\le \varepsilon\|1/p\|_\infty\int_n^{n+1}p(t)|f'(t)|^2\,\mathrm{d}t + \left(1+\frac{1}{\varepsilon}\right)\int_n^{n+1}|f(t)|^2\,\mathrm{d}t\end{aligned}$$

which shows (A.24). □

Proof of Lemma A.9. Let $f \in \mathscr{D}_-(\tau)$. If $q_-(x) = 0$ a. e. on $\mathbb{R}$, then by Lemma A.6 we have $\|\sqrt{p}f'\|_2 = 0$ and Lemma A.10 implies $\|f\|_\infty = 0$. This shows (vii). In the case where $1/p, q_- \in L^1(\mathbb{R})$ Lemma A.6 and Lemma A.10 (i) yield

$$\|f\|_\infty^2 \le \|1/p\|_1\|\sqrt{p}f'\|_2^2 \le \|1/p\|_1\|q_-f^2\|_1 \le \|1/p\|_1\|q_-\|_1\|f\|_\infty^2.$$

If the condition $\|1/p\|_1\|q_-\|_1 < 1$ holds, then we have $\|f\|_\infty = 0$. This shows the assertion (vi).

For the proofs of the remaining cases it is no restriction to assume that $f \in \mathscr{D}_-(\tau)\setminus\{0\}$ and that q_- is positive on a set of positive Lebesgue measure. We show (i). Let $1/p \in L^\infty$ and consider α, β as in (A.17). Choose $\varepsilon = (2\|q_-\|_\mathrm{u}\|1/p\|_\infty)^{-1} > 0$. The estimate in (A.24) of Lemma A.10 yields

$$\begin{aligned}\|q_-f^2\|_1 = \int_{\mathbb{R}} q_-(t)|f(t)|^2\,\mathrm{d}t &\le \|q_-\|_\mathrm{u}\sum_{n\in\mathbb{Z}}\sup_{t\in[n,n+1]}|f(t)|^2 \\ &\le \|q_-\|_\mathrm{u}\left(\varepsilon\|1/p\|_\infty\|\sqrt{p}f'\|_2^2 + \left(1+\frac{1}{\varepsilon}\right)\|f\|_2^2\right) \\ &= \frac{1}{2}\|\sqrt{p}f'\|_2^2 + \left(\|q_-\|_\mathrm{u} + 2\|1/p\|_\infty\|q_-\|_\mathrm{u}^2\right)\|f\|_2^2 \\ &= \frac{1}{2}\|\sqrt{p}f'\|_2^2 + \frac{\alpha}{2}\|f\|_2^2.\end{aligned} \tag{A.27}$$

Together with Lemma A.6 we obtain

$$\|\sqrt{p}f'\|_2^2 = 2\|\sqrt{p}f'\|_2^2 - \|\sqrt{p}f'\|_2^2 \le 2\|q_-f^2\|_1 - \|\sqrt{p}f'\|_2^2 \le \alpha\|f\|_2^2.$$

With (A.23) in Lemma A.10 and (A.27) we see

$$\|f\|_\infty^2 \le 2\sqrt{\|1/p\|_\infty\alpha}\|f\|_2^2 = \beta\|f\|_2^2 \quad \text{and} \quad \|q_-f^2\|_1 \le \alpha\|f\|_2^2.$$

We show (ii). Suppose that $1/p \in L^\eta(\mathbb{R})$ and $q_- \in L^s(\mathbb{R})$, where $\eta, s \in [1,\infty)$ with $\eta + s > 2$. Since $\eta + s > 2$ we obtain

$$2\eta s - \eta - s = \eta(s-1) + s(\eta-1) \geq s - 1 + \eta - 1 > 0.$$

Let α and β as in (A.18). From Hölder's inequality we obtain

$$\begin{aligned}\|q_- f^2\|_1 &\leq \|f\|_\infty^{\frac{2}{s}} \int_{\mathbb{R}} |q_-(t)||f(t)|^{\frac{2(s-1)}{s}} \, \mathrm{d}t \leq \|f\|_\infty^{\frac{2}{s}} \left(\int_{\mathbb{R}} |q_-(t)|^s \, \mathrm{d}t\right)^{\frac{1}{s}} \left(\int_{\mathbb{R}} |f(t)|^2 \, \mathrm{d}t\right)^{\frac{s-1}{s}} \\ &= \|q_-\|_s \|f\|_\infty^{\frac{2}{s}} \|f\|_2^{\frac{2(s-1)}{s}}.\end{aligned} \tag{A.28}$$

Thus, together with Lemma A.10 (i) and Lemma A.6 we obtain

$$\begin{aligned}\|f\|_\infty^2 &= \left(\frac{\|f\|_\infty^{\frac{2(2\eta-1)}{\eta}}}{\|f\|_\infty^{\frac{2}{s}}}\right)^{\frac{\eta s}{2\eta s - \eta - s}} \leq \left(\frac{\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|\sqrt{p} f'\|_2^2 \|f\|_2^{\frac{2(\eta-1)}{\eta}}}{\|f\|_\infty^{\frac{2}{s}}}\right)^{\frac{\eta s}{2\eta s - \eta - s}} \\ &\leq \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_s\right)^{\frac{\eta s}{2\eta s - \eta - s}} \|f\|_2^2 = \beta \|f\|_2^2.\end{aligned}$$

The estimate from (A.28) yields

$$\|q_- f^2\|_1 \leq \|q_-\|_s \beta^{\frac{1}{s}} \|f\|_2^2 = \alpha \|f\|_2^2.$$

We show (iii). Suppose that $1/p \in L^\infty(\mathbb{R})$ and $q_- \in L^s(\mathbb{R})$, where $s \in [1,\infty)$. Let α and β as in (A.19). Again Hölder's inequality yields (A.28). Lemma A.10 (ii), (A.28) and Lemma A.6 imply

$$\begin{aligned}\|f\|_\infty^2 &= \left(\frac{\|f\|_\infty^4}{\|f\|_\infty^{\frac{2}{s}}}\right)^{\frac{s}{2s-1}} \leq \left(\frac{4\|1/p\|_\infty \|\sqrt{p} f'\|_2^2 \|f\|_2^2}{\|f\|_\infty^{\frac{2}{s}}}\right)^{\frac{s}{2s-1}} \\ &\leq \left(4\|1/p\|_\infty \|q_-\|_s\right)^{\frac{s}{2s-1}} \|f\|_2^2 = \beta \|f\|_2^2.\end{aligned}$$

By applying this to the estimate in (A.28) we arrive at

$$\|q_- f^2\|_1 \leq \|q_-\|_s \beta^{\frac{1}{s}} \|f\|_2^2 = \alpha \|f\|_2^2.$$

We show (iv). Let $1/p \in L^\eta(\mathbb{R})$, where $\eta \in [1,\infty)$, and $q_- \in L^\infty(\mathbb{R})$. Choose α and β as in (A.20). Observe that

$$\|q_- f^2\|_1 \leq \|q_-\|_\infty \|f\|_2^2 = \alpha \|f\|_2^2. \tag{A.29}$$

Lemma A.10 (i) in combination with Lemma A.6 and (A.29) leads to

$$\begin{aligned}\|f\|_\infty^2 &\leq \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|\sqrt{p} f'\|_2^2\right)^{\frac{\eta}{2\eta-1}} \|f\|_2^{\frac{2(\eta-1)}{2\eta-1}} \\ &\leq \left(\left(\frac{2\eta-1}{\eta}\right)^2 \|1/p\|_\eta \|q_-\|_\infty\right)^{\frac{\eta}{2\eta-1}} \|f\|_2^2 = \beta \|f\|_2^2.\end{aligned}$$

The assertion in (v) follows in a similar way. Let $1/p, q_- \in L^\eta(\mathbb{R})$ and consider α, β in (A.21). As before (A.29) holds. Lemma A.10 (ii) in combination with Lemma A.6 and (A.29) implies

$$\|f\|_\infty^2 \leq 2\sqrt{\|1/p\|_\infty}\|\sqrt{p}f'\|_2\|f\|_2 \leq 2\sqrt{\|1/p\|_\infty\|q_-\|_\infty}\|f\|_2^2 = \beta\|f\|_2^2. \qquad \square$$

Appendix B

Asymptotic integration for differential systems

We consider the linear system

$$\varphi' = A\varphi \tag{B.1}$$

in $\mathbb{C}^2$ on an open interval (a,∞), where $a \in \mathbb{R}$, with a measurable function $A : (a,\infty) \to \mathbb{C}^{2\times 2}$ such that $\|A(\cdot)\|_{\mathbb{C}^{2\times 2}} \in L^1_{\mathrm{loc}}(a,\infty)$. We call $\varphi : (a,\infty) \to \mathbb{C}^2$ a solution of (B.1) if each component of φ is locally absolutely continuous and φ satisfies the equation (B.1) a. e. on (a,∞). From the theory of ordinary differential equations, see e. g. [32, 54], it is well-known that there is a unique solution of the differential equation (B.1) subject to the initial condition $\varphi(x_0) = y_0$, where $x_0 \in (a,\infty)$ and $y_0 \in \mathbb{C}^2$. In this section we compare the asymptotic behaviour of the solutions of (B.1) and those of the perturbed system

$$\xi' = (A + B)\xi, \tag{B.2}$$

where $B : (a,\infty) \to \mathbb{C}^{2\times 2}$ is measurable such that $\|B(\cdot)\|_{\mathbb{C}^{2\times 2}} \in L^1_{\mathrm{loc}}(a,\infty)$. The type of asymptotic analysis we employ is based on a technique which is referred to as asymptotic integration, see [26].

For $\beta \in \mathbb{R}$ let $C_\beta(a,\infty)$ denote the Banach space of continuous $\mathbb{C}^2$-valued functions on (a,∞) of exponential growth at the rate β, that is

$$C_\beta(a,\infty) := \left\{ f : (a,\infty) \to \mathbb{C}^2 \text{ continuous} \;\middle|\; \begin{array}{l} \|f(x)\|_{\mathbb{C}^2} \le \gamma \mathrm{e}^{\beta(x-a)} \\ \text{for some } \gamma \ge 0 \text{ and all } x \in (a,\infty) \end{array} \right\}, \tag{B.3}$$

with the corresponding norm

$$\|f\|_{\infty,\beta} := \sup_{x\in(a,\infty)} \mathrm{e}^{-\beta(x-a)} \|f(x)\|_{\mathbb{C}^2}. \tag{B.4}$$

Theorem B.1. *Let $\beta \in \mathbb{R}$ and A, $B : (a,\infty) \to \mathbb{C}^{2\times 2}$ be measurable such that $\|A(\cdot)\|_{\mathbb{C}^{2\times 2}}$ and $\|B(\cdot)\|_{\mathbb{C}^{2\times 2}}$ are locally integrable on (a,∞). Consider a fundamental solution $\Phi : (a,\infty) \to \mathbb{C}^{2\times 2}$ of the system* (B.1)*.*

(i) *If there is a measurable non-negative function g defined on (a,∞) such that $g(\cdot)\|B(\cdot)\|_{\mathbb{C}^{2\times 2}} \in L^1(a,\infty)$ and*

$$\left\| \Phi(x)\big(\Phi(t)\big)^{-1} \right\|_{\mathbb{C}^{2\times 2}} \le \mathrm{e}^{-\beta(x-t)} g(t) \tag{B.5}$$

holds for all $x, t \in (a,\infty)$ with $x \le t$, then there is a bijective linear operator T from $C_{-\beta}(a,\infty)$ onto $C_{-\beta}(a,\infty)$ such that for every solution $\varphi \in C_{-\beta}(a,\infty)$ of (B.1) *the function $\xi := T\varphi \in C_{-\beta}(a,\infty)$ solves* (B.2) *and*

$$\mathrm{e}^{\beta(x-a)} \|\xi(x) - \varphi(x)\|_{\mathbb{C}^2} \to 0 \quad \text{as } x \to \infty. \tag{B.6}$$

(ii) *If there is a measurable non-negative function g defined on (a,∞) such that $g(\cdot)\|B(\cdot)\|_{\mathbb{C}^{2\times 2}} \in L^1(a,\infty)$ and*

$$\left\|\Phi(x)\big(\Phi(t)\big)^{-1}\right\|_{\mathbb{C}^{2\times 2}} \le \mathrm{e}^{\beta(x-t)} g(t) \tag{B.7}$$

holds for all $x,t \in (a,\infty)$ with $t \le x$, then there is a bijective linear operator S from $C_\beta(a,\infty)$ onto $C_\beta(a,\infty)$ such that for every solution $\varphi \in C_\beta(a,\infty)$ of (B.1) *the function $\xi := S\varphi \in C_\beta(a,\infty)$ solves* (B.2).

If A, B, Φ are $\mathbb{R}^{2\times 2}$-valued and φ is $\mathbb{R}^2$-valued then ξ is $\mathbb{R}^2$-valued.

Proof. We show (i). For $\xi \in C_{-\beta}(a,\infty)$ and $x \in (a,\infty)$ we define

$$(T_0\xi)(x) = -\Phi(x)\int_x^\infty \big(\Phi(t)\big)^{-1} B(t)\xi(t)\,\mathrm{d}t. \tag{B.8}$$

Since (B.5) holds for all $x,t \in (a,\infty)$ with $x \le t$, we have

$$\begin{aligned} \|(T_0\xi)(x)\|_{\mathbb{C}^2} &\le \int_x^\infty \left\|\Phi(x)\big(\Phi(t)\big)^{-1}\right\|_{\mathbb{C}^{2\times 2}} \cdot \|B(t)\|_{\mathbb{C}^{2\times 2}} \cdot \|\xi(t)\|_{\mathbb{C}^2}\,\mathrm{d}t \\ &\le \mathrm{e}^{-\beta(x-a)}\|\xi\|_{\infty,-\beta}\int_x^\infty g(t)\|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t \end{aligned} \tag{B.9}$$

for all $x \in (a,\infty)$. Multiplying the inequality (B.9) by $\mathrm{e}^{\beta(x-a)}$ and taking the supremum we arrive at

$$\left\|T_0\xi\right\|_{\infty,-\beta} \le \|\xi\|_{\infty,-\beta}\int_a^\infty g(t)\|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t. \tag{B.10}$$

Therefore, (B.8) defines a bounded operator T_0 in $C_{-\beta}(a,\infty)$. We show inductively that for the kth power of T_0, where $k \in \mathbb{N}$ with $k \ge 1$, one has

$$\left\|(T_0^k\xi)(x)\right\|_{\mathbb{C}^2} \le \mathrm{e}^{-\beta(x-a)}\frac{\|\xi\|_{\infty,-\beta}}{k!}\left(\int_x^\infty g(t)\|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t\right)^k \tag{B.11}$$

for all $x \in (a,\infty)$. This is true for $k = 1$ by (B.9). By (B.5) we can estimate for $k > 1$

$$\begin{aligned} \left\|(T_0^k\xi)(x)\right\|_{\mathbb{C}^2} &\le \int_x^\infty \left\|\Phi(x)\big(\Phi(t)\big)^{-1}\right\|_{\mathbb{C}^{2\times 2}} \cdot \|B(t)\|_{\mathbb{C}^{2\times 2}} \cdot \|(T_0^{k-1}\xi)(t)\|_{\mathbb{C}^2}\,\mathrm{d}t \\ &\le \mathrm{e}^{-\beta(x-a)}\frac{\|\xi\|_{\infty,-\beta}}{(k-1)!}\int_x^\infty g(t)\|B(t)\|_{\mathbb{C}^{2\times 2}}\left(\int_t^\infty g(s)\|B(s)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}s\right)^{k-1}\mathrm{d}t. \end{aligned} \tag{B.12}$$

Consider the function G_k defined by

$$G_k(x) := \frac{1}{k}\left(\int_x^\infty g(t)\|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t\right)^k. \tag{B.13}$$

The function G_k is locally absolutely continuos, where

$$G_k'(x) = -g(x)\|B(x)\|_{\mathbb{C}^{2\times 2}}\left(\int_x^\infty g(t)\|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t\right)^{k-1}, \tag{B.14}$$

and we have by (B.12)

$$\left\|\left(T_0^k\xi\right)(x)\right\|_{\mathbb{C}^2} \le \mathrm{e}^{-\beta(x-a)}\frac{\|\xi\|_{\infty,-\beta}}{(k-1)!}\int_x^\infty \left(-G_k'(t)\right)\mathrm{d}t = \mathrm{e}^{-\beta(x-a)}\frac{\|\xi\|_{\infty,-\beta}}{(k-1)!}G_k(x) \tag{B.15}$$

which shows (B.11).

By multiplying the inequality in (B.11) by $\mathrm{e}^{-\beta(x-a)}$ and taking the supremum we obtain

$$\left\|T_0^k\xi\right\|_{\infty,-\beta} \le \frac{1}{k!}\left(\int_a^\infty g(t)\|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t\right)^k \|\xi\|_{\infty,-\beta}. \tag{B.16}$$

Hence, the Neumann series $(I-T_0)^{-1} = \sum_{k\in\mathbb{N}} T_0^k$ converges absolutely with respect to the operator norm induced by $\|\cdot\|_{\infty,-\beta}$. We set

$$T := (I-T_0)^{-1}. \tag{B.17}$$

For a solution φ of (B.1) in $C_{-\beta}(a,\infty)$ the function $\xi := T\varphi = (I-T_0)^{-1}\varphi \in C_{-\beta}(a,\infty)$ satisfies

$$\xi = \varphi + T_0\xi. \tag{B.18}$$

Differentiation on both sides yields

$$\xi' = (\varphi + T_0\xi)' = A\varphi + AT_0\xi + B\xi = A(\varphi + T_0\xi) + B\xi = (A+B)\xi,$$

which shows that ξ is a solution of (B.2). The asymptotic behaviour in (B.6) follows from (B.18) and (B.9).

The assertion in (ii) can be shown analogously. One considers for $\xi \in C_\beta(a,\infty)$

$$(S_0\xi)(x) := \Phi(x)\int_a^x \left(\Phi(t)\right)^{-1}B(t)\xi(t)\,\mathrm{d}t. \tag{B.19}$$

In a similar way as in (i), using (B.7) one shows that (B.19) defines a bounded operator S_0 in $C_\beta(a,\infty)$, where the kth power of S_0 for $k \in \mathbb{N}$ with $k \ge 1$ satisfies

$$\left\|\left(S_0^k\xi\right)(x)\right\|_{\mathbb{C}^2} \le \mathrm{e}^{\beta(x-a)}\frac{\|\xi\|_{\infty,\beta}}{k!}\left(\int_a^x g(t)\|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t\right)^k \tag{B.20}$$

on (a,∞) and

$$\left\|S_0^k\xi\right\|_{\infty,\beta} \le \frac{\|\xi\|_{\infty,\beta}}{k!}\left(\int_a^\infty g(t)\|B(t)\|_{\mathbb{C}^{2\times 2}}\,\mathrm{d}t\right)^k. \tag{B.21}$$

Hence, the Neumann series $(I-S_0)^{-1} = \sum_{k\in\mathbb{N}} S_0^k$ is absolutely convergent with respect to the operator norm induced by $\|\cdot\|_{\beta,\infty}$. A straightforward calculation shows that the operator $S := (I-S_0)^{-1}$ establishes a one-to-one correspondence between the solutions contained in $C_\beta(a,\infty)$ of (B.1) and of (B.2). □

Symbols

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