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IFAC PapersOnLine 54-14 (2021) 307–312

Dynamic Extension for Adaptive Backstepping Control of Uncertain Pure-Feedback Systems ⋆

Lukas Triska *, **Jhon Portella** **, **and Johann Reger** ***

* *Chair of Automatic Control, Friedrich-Alexander-Universität Erlangen-Nürnberg, Cauerstraße 7, D-91058, Erlangen, Germany (e-mail:lukas.triska@fau.de)*

** *Mechatronics Engineering, Pontifical Catholic University of Peru, 1801 Avenida Universitaria, San Miguel, Lima, Peru (e-mail:jportella@pucp.edu.pe)*

*** *Control Engineering Group, Technische Universität Ilmenau, P.O. Box 10 05 65, D-98684, Ilmenau, Germany (e-mail:johann.reger@tu-ilmenau.de)*

Abstract: An adaptive backstepping algorithm is developed for a class of uncertain systems in pure-feedback form. The control is based on a dynamic state feedback that allows to compensate for parametric uncertainties which enter linearly into the system. As possible in the nominal case, a dynamic extension of just order one is required, in addition to the dynamics of the identifiers for the adaptation. The regularity of the control law only requires standard assumptions.

Keywords: Backstepping Control, Adaptive Control, Dynamic Extensions, Lyapunov Methods

1. INTRODUCTION

Backstepping is a powerful framework to design feedback controllers for triangular systems (Krstić et al., 1995). In the case of systems in strict feedback form the framework has achieved great progress. However, backstepping has only provided limited results for systems in pure feedback form, which is due to the encountered implicit equations. In this light, Zhang and Qian (2017) introduced a dynamic extension to avoid implicit equations. Also Mazenc et al. (2018) have investigated on a dynamic extension with delay in the backstepping framework in order to define an algorithm with bounded outputs. Reger and Triska (2019) extended the procedure by Zhang and Qian (2017) to arbitrary dimension, using a specific dynamic extension of just order one while requiring standard assumptions only. To the best of the authors knowledge, such dynamic extensions have not further been addressed in backstepping.

In a next step towards solving real-world application problems, uncertainties are addressed. To this end, we enhance the dynamic extension approach by adaptive control to tackle linear-parametric uncertainties. Our procedure encompasses a dynamic extension of order one like in (Reger and Triska, 2019) and shapes an adaptive identifier for the parametric uncertainty based on the certainty equivalence principle. The latter involves further dynamics in view of the adaptation law with respect to the parametric uncertainty. We establish asymptotic convergence resorting to standard Lyapunov arguments.

⋆ The second author acknowledges funding from Fondecyt-Peru 2019-4 internship program. The third author has received financial support from the European Union Horizon 2020 research and innovation program, Marie Skłodowska-Curie grant agreement No. 824046.

2. PROBLEM STATEMENT

2.1 System Class

We consider the class of pure-feedback nonlinear systems with linear uncertainties characterized by

$$\left.\left.\left.\begin{aligned} \dot{x}_1 &= f_1(\bar{x}_2) + \psi_1^\top(\bar{x}_2)\Theta \\ \dot{x}_2 &= f_2(\bar{x}_3) + \psi_2^\top(\bar{x}_3)\Theta \end{aligned}\right\}\Sigma_1 \atop \dot{x}_3 = f_3(\bar{x}_4) + \psi_3^\top(\bar{x}_4)\Theta \right\}\Sigma_2 \atop \vdots \atop \dot{x}_k = f_k(\bar{x}_k, u) + \psi_k^\top(\bar{x}_k, u)\Theta \right\}\Sigma_{k-1} \tag{1}$$

where $x_1 \in \mathbb{R}^n$ and $x_2, \dots, x_k \in \mathbb{R}$ together are the states, $\bar{x}_i^\top = (x_1^\top, x_2, \dots, x_i)$, $u \in \mathbb{R}$ is the input, and $\Theta \in \mathbb{R}^r$ a vector of unknown parameters. Let the functions $f_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $f_i : \mathbb{R}^n \times \mathbb{R}^i \to \mathbb{R}$, $\psi_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n\times r}$, $\psi_i : \mathbb{R}^n \times \mathbb{R}^i \to \mathbb{R}^r$, with $i = 2, \dots, k$ be continuously differentiable of sufficient order. For the sake of notation let $x_{k+1} = u$ where appropriate. Further assume

$$f_i(0) = 0, \psi_i^\top(0) = 0, \quad \forall i \in \{1, \dots, k\} \tag{2}$$

such that the origin is an equilibrium point.

2.2 Assumptions

Assumption 1. There is a known function $\alpha_1(x_1, \vartheta^1)$, a radially unbounded Lyapunov function $V(x_1, \vartheta^1)$ and an adaptation law for $\dot{\vartheta}^1$ where ϑ^1 is some first parameter identifier of Θ such that for any ϑ^1 the state $x_1 = 0$ is a stable equilibrium with respect to $\dot{x}_1 = f_1(x_1, \alpha_1(x_1, \vartheta^1)) + \varphi_1^\top(x_1, \alpha_1(x_1, \vartheta^1))\Theta$. Furthermore, the following conditions shall be satisfied:

Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2021.10.371

1) $\alpha_1(0,\vartheta^1)=0$, $\frac{\partial V(x_1,\vartheta^1)}{\partial x_1}\Big|_{x_1=0}=0$, for any ϑ^1,
2) $\dot V=0 \iff x_1=0$.
3) $x_1=0 \Longrightarrow \dot\vartheta^1=0$.

Remark 1. Assumption 1 assures that for $x_1=0$ we have $\dot V=\frac{\partial V(x_1,\vartheta^1)}{\partial x_1}(f_1(x_1,\alpha_1(x_1,\vartheta^1))+\varphi_1^\top(x_1,\alpha_1(x_1,\vartheta^1))\Theta)+\frac{\partial V(x_1,\vartheta^1)}{\partial\vartheta^1}\dot\vartheta^1=0$ and $\dot\vartheta^1=0$.

Assumption 2. For controllability assume that

$$\frac{\partial}{\partial x_{i+1}}(f_i(\bar x_{i+1})+\psi_i^\top(\bar x_{i+1})\Theta)\neq 0 \tag{3}$$

for any $\bar x_{i+1}\in\mathbb{R}^{n+i}$ and $\Theta\in\mathbb{R}^r$, $i=1,\dots,k$ or at least locally in a domain $\mathcal{F}$ including the origin. This extends the assumption in (Reger and Triska, 2019). We also use:

Definition 1. Let the auxiliary functions G_i be given as

$$G_i=\int_0^1\frac{\partial}{\partial v}f_i(\bar x_i,v)\Big|_{v=\alpha_i+\lambda x_{i+1}}\partial\lambda \tag{4}$$

for $i=1,\dots,k$ and for any scalar α_i, such that it holds

$$f_i(\bar x_{i+1})=f_i(\bar x_i,\alpha_i)+G_i(x_{i+1}-\alpha_i). \tag{5}$$

In the case of adaptation laws for ϑ, later we will employ the notation H_i, instead of G_i.

3. MAIN RESULT: ADAPTIVE BACKSTEPPING WITH DYNAMIC EXTENSION

Equipped with these assumptions and definitions we are in the position to state a dynamic state feedback with adaptation law that stabilizes the origin asymptotically against any parametric uncertainty.

Theorem 1. Let system (1) satisfy Assumptions 1-2 in a domain $\mathcal{F}$ containing the origin. Let (x,u,ϑ) be initialized inside a compact, positively invariant set $\Omega\subset\mathcal{F}$. Then the following dynamic state feedback stabilizes the extended system such that $\lim_{t\to\infty}(x,u)=0$ with any element of the series of identifiers $\bar\vartheta^{k+1}=(\vartheta^1,\dots,\vartheta^{k+1})$ bounded:

$$\begin{aligned}\dot u=&\Big(\tfrac{\partial f_k(\bar x_k,u)}{\partial u}+\tfrac{\partial\varphi_k^\top(\bar x_k,u)\vartheta^k}{\partial u}-\tfrac{\partial\dot{\hat\alpha}_{k-1}(\bar x_k,u,\bar\vartheta^k)}{\partial u}\Big)^{-1}\\&\Big[-K_k(u-\alpha_k(\bar x_k,u,\bar\vartheta^k))-(x_k-\alpha_{k-1}(\bar x_k,\bar\vartheta^{k-1}))\\&+\textstyle\sum_{i=1}^k\frac{\partial\alpha_k(\bar x_k,u,\bar\vartheta^k)}{\partial x_i}(f_i(\bar x_{i+1})+\varphi_i^\top(\bar x_{i+1})\vartheta^{k+1})\\&+\textstyle\sum_{i=1}^k\frac{\partial\alpha_k(\bar x_k,u,\bar\vartheta^k)}{\partial\vartheta^i}\dot\vartheta^i\Big].\end{aligned}\tag{6}$$

In this dynamic feedback, $\alpha_1(x_1,\vartheta^1)$ is an initial function that satisfies Assumption 1 and

$$\begin{aligned}\alpha_2(\bar x_3,\bar\vartheta^2)=&x_3-\tfrac{\partial V(x_1,\vartheta^1)}{\partial x_1}\Big[G_1(x_1,x_2-\alpha_1(x_1,\vartheta^1))\\&+H_1^\top(x_1,x_2-\alpha_1(x_1,\vartheta^1))\vartheta^2\Big]-(f_2(\bar x_3)+\varphi_2^\top(\bar x_3)\vartheta^2)\\&+\dot{\hat\alpha}_1(\bar x_2,\bar\vartheta^2)-K_1(x_2-\alpha_1(x_1,\vartheta^1)).\end{aligned}\tag{7}$$

Furthermore, we have the adaptation law

$$\begin{aligned}\dot\vartheta^{2\top}=&(x_2-\alpha_1(\bar x_1,\bar\vartheta^1))\Big[\tfrac{\partial V(x_1,\vartheta^1)}{\partial x_1}H_1^\top(x_1,x_2-\alpha_1(x_1,\vartheta^1))\\&+\varphi_2^\top(\bar x_3)-\tfrac{\partial\alpha_1(x_1,\vartheta^1)}{\partial x_1}\varphi_1^\top(\bar x_2)\Big]\Gamma_2\end{aligned}\tag{8}$$

and update of an estimate described by

$$\dot{\hat\alpha}_1(\bar x_2,\bar\vartheta^2)=\tfrac{\partial\alpha_1(x_1,\vartheta^1)}{\partial x_1}(f_1(\bar x_2)+\varphi_1^\top(\bar x_2)\vartheta^2)+\tfrac{\partial\alpha_1(x_1,\vartheta^1)}{\partial x_1}\dot\vartheta^1.\tag{9}$$

For the remaining $i=3,\dots,k$, we have

$$\begin{aligned}\alpha_i(\bar x_{i+1},\bar\vartheta^i)=&x_{i+1}-(x_{i-1}-\alpha_{i-2}(\bar x_{i-1},\bar\vartheta_{i-2}))\\&-(f_i(\bar x_{i+1})+\varphi_i^\top(\bar x_{i+1})\vartheta^i)+\dot{\hat\alpha}_{i-1}(\bar x_{i+1},\bar\vartheta^i),\end{aligned}\tag{10}$$

and for the further adaptation laws firstly

$$\begin{aligned}\dot\vartheta^{i\top}=&(x_i-\alpha_{i-1}(\bar x_i,\bar\vartheta^{i-1}))\\&\big[\varphi_i^\top(\bar x_{i+1})-\textstyle\sum_{l=1}^i\frac{\partial\alpha_{i-1}(\bar x_i,\bar\vartheta^{i-1})}{\partial x_l}\varphi_l^\top(\bar x_{l+1})\big]\Gamma_{i-1}\end{aligned}\tag{11}$$

and finally

$$\dot\vartheta^{k+1\top}=-(u-\alpha_k(\bar x_k,u,\bar\vartheta^k))\textstyle\sum_{l=1}^k\frac{\partial\alpha_k(\bar x_k,u,\bar\vartheta^k)}{\partial x_l}\varphi_l^\top(\bar x_{l+1})\Gamma_k.\tag{12}$$

For $i=2,\dots,k$ we have

$$\begin{aligned}\dot{\hat\alpha}_i(\bar x_{i+2},\bar\vartheta^{i+1})=&\textstyle\sum_{l=1}^i\frac{\partial\alpha_i(\bar x_{i+1},\bar\vartheta^i)}{\partial\vartheta^l}\dot\vartheta^l\\&+\textstyle\sum_{l=1}^{i+1}\frac{\partial\alpha_i(\bar x_{i+1},\bar\vartheta^i)}{\partial x_l}(f_l(\bar x_{l+1})+\varphi_l^\top(\bar x_{l+1})\vartheta^{i+1}).\end{aligned}\tag{13}$$

The gains may be freely chosen as $K_j>0$ and adaptations gain matrices as $\Gamma_j=\Gamma_j^\top>0\in\mathbb{R}^{r\times r}$ for $j=1,\dots,k$. □

Remark 2. Dynamic feedback (6) works for any initialization. However, if $x_i(0)$ and some initial parameter estimates $\vartheta^i(0)$ are available, convergence can be improved by choosing $u_0=u(0)$ as solution of the implicit equation

$$u_0-\alpha_k(\bar x_k(0),u_0,\bar\vartheta^k(0))=0.\tag{14}$$

4. PROOF OF THE MAIN RESULT

Proof. *Step 1*: The first step will focus on Σ_1 of (1). Note that the indices in the following $V_{a,i,j}(\bar x_{i+1},\bar\vartheta^j)$ shall spot the dependency on $\bar x_{i+1}$ and $\bar\vartheta^j$. The objective at this stage is to obtain the stabilizing term $\alpha_2(\bar x_3,\vartheta^1,\vartheta^2)$ and the adaptation law for the right-hand side of $\dot\vartheta^2$.

We take the starting Lyapunov function candidate

$$V_{a,1,1}(\bar x_2,\vartheta^1)=V(x_1,\vartheta^1)+\frac12(x_2-\alpha_1(x_1,\vartheta^1))^2.\tag{15}$$

Its time derivative reads

$$\begin{aligned}\dot V_{a,1,1}(\bar x_3,\vartheta^1)=&\tfrac{\partial V(x_1,\vartheta^1)}{\partial x_1}(f_1(\bar x_2)+\varphi_1^\top(\bar x_2)\Theta)\\&+\tfrac{\partial V(x_1,\vartheta^1)}{\partial\vartheta^1}\dot\vartheta_1+(x_2-\alpha_1(x_1,\vartheta^1))\big(f_2(\bar x_3)\\&+\varphi_2^\top(\bar x_3)\Theta-\dot\alpha_1(\bar x_2,\vartheta^1)\big).\end{aligned}\tag{16}$$

Assumption 1 may be employed using Definition 1 as per

$$\begin{aligned}f_1(\bar x_2)=&f_1(x_1,\alpha_1(x_1,\vartheta^1))+G_1(x_1,x_2-\alpha_1(x_1,\vartheta^1))\\&(x_2-\alpha_1(x_1,\vartheta^1))\end{aligned}\tag{17}$$

$$\begin{aligned}\varphi_1^\top(\bar x_2)=&\varphi_1^\top(x_1,\alpha_1(x_1,\vartheta^1))+H_1^\top(x_1,x_2-\alpha_1(x_1,\vartheta^1))\\&(x_2-\alpha_1(x_1,\vartheta^1)).\end{aligned}\tag{18}$$

The use of these expressions in (16) yields

$$\begin{aligned}\dot V_{a,1,1}(\bar x_3,\vartheta^1)=&\tfrac{\partial V(x_1,\vartheta^1)}{\partial x_1}\big[f_1(x_1,\alpha_1(x_1,\vartheta^1))\\&+\varphi_1^\top(x_1,\alpha_1(x_1,\vartheta^1))\Theta\big]+\tfrac{\partial V(x_1,\vartheta^1)}{\partial\vartheta^1}\dot\vartheta^1\\&+(x_2-\alpha_1(x_1,\vartheta^1))\Big(\tfrac{\partial V(x_1,\vartheta^1)}{\partial x_1}\big[G_1(x_1,x_2-\alpha_1(x_1,\vartheta^1))\\&+H_1^\top(x_1,x_2-\alpha_1(x_1,\vartheta^1))\Theta\big]\\&+f_2(\bar x_3)+\varphi_2^\top(\bar x_3)\Theta-\dot\alpha_1(\bar x_2,\vartheta^1)\Big).\end{aligned}\tag{19}$$

The terms between the big parentheses in (19) shall be stabilized by choice of $\alpha_2(\bar x_3,\bar\vartheta^2)$. This may be done letting

$$x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2) = \frac{\partial V(x_1,\vartheta^1)}{\partial x_1}\big[G_1(x_1, x_2 - \alpha_1(x_1,\vartheta^1)) + H_1^\top(x_1, x_2 - \alpha_1(x_1,\vartheta^1))\vartheta^2\big] + f_2(\bar{x}_3) + \varphi_2^\top(\bar{x}_3)\vartheta^2 - \hat{\dot{\alpha}}_1(\bar{x}_2, \bar{\vartheta}^2) + K_1(x_2 - \alpha_1(x_1,\vartheta^1)) \quad (20)$$

where $\hat{\dot{\alpha}}_1(\bar{x}_2, \bar{\vartheta}^2)$ is an estimate of $\dot{\alpha}_1$ defined by

$$\hat{\dot{\alpha}}_1(\bar{x}_2, \bar{\vartheta}^2) = \frac{\partial \alpha_1(x_1,\vartheta^1)}{\partial x_1}(f_1(\bar{x}_2) + \varphi_1^\top(\bar{x}_2)\vartheta^2) + \frac{\partial \alpha_1(x_1,\vartheta^1)}{\partial \vartheta^1}\dot{\vartheta}^1. \quad (21)$$

The following notation is introduced for simplicity:

$$\dot{\hat{V}}_1 := \frac{\partial V(x_1,\vartheta^1)}{\partial x_1}\big[f_1(x_1, \alpha_1(x_1,\vartheta^1)) + \varphi_1^\top(x_1, \alpha_1(x_1,\vartheta^1))\Theta\big] + \frac{\partial V(x_1,\vartheta^1)}{\partial \vartheta^1}\dot{\vartheta}^1 \quad (22)$$

With the expressions (20) and (22) in (19), we obtain

$$\dot{V}_{a,1,1}(\bar{x}_3, \bar{\vartheta}^2) = \dot{\hat{V}}_1 - K_1(x_2 - \alpha_1(x_1,\vartheta^1))^2 + (x_2 - \alpha_1(x_1,\vartheta^1))(x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2)) + (x_2 - \alpha_1(x_1,\vartheta^1))\Big[\frac{\partial V(x_1,\vartheta^1)}{\partial x_1}H_1^\top(x_1, x_2 - \alpha_1(x_1,\vartheta^1)) + \varphi_2^\top(\bar{x}_3) - \frac{\partial \alpha_1(x_1,\vartheta^1)}{\partial x_1}\varphi_1^\top(\bar{x}_2)\Big](\Theta - \vartheta^2). \quad (23)$$

To attenuate the influence of the estimation error $\Theta - \vartheta^2$, we further augment the Lyapunov function candidate, i.e.

$$V_{a,1,2}(\bar{x}_2, \bar{\vartheta}^2) = V_{a,1,1}(\bar{x}_2, \vartheta^1) + \frac{1}{2}(\Theta - \vartheta^2)^\top\Gamma_1^{-1}(\Theta - \vartheta^2), \quad (24)$$

which first yields

$$\dot{V}_{a,1,2}(\bar{x}_2, \bar{\vartheta}^2) = \dot{V}_{a,1,1}(\bar{x}_2, \vartheta^1) - \dot{\vartheta}^{2\top}\Gamma_1^{-1}(\Theta - \vartheta^2) \quad (25)$$

and can be rewritten with (23) as

$$\dot{V}_{a,1,2}(\bar{x}_2, \bar{\vartheta}_2) = \dot{\hat{V}}_1 - K_1(x_2 - \alpha_1(x_1,\vartheta^1))^2 + (x_2 - \alpha_1(x_1,\vartheta^1))(x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2)) + \Big((x_2 - \alpha_1(x_1,\vartheta^1))\Big[\frac{\partial V(x_1,\vartheta^1)}{\partial x_1}H_1^\top(x_1, x_2 - \alpha_1(x_1,\vartheta^1)) + \varphi_2^\top(\bar{x}_3) - \frac{\partial \alpha_1(x_1,\vartheta^1)}{\partial x_1}\varphi_1^\top(\bar{x}_2)\Big] - \dot{\vartheta}^{2\top}\Gamma_1^{-1}\Big)(\Theta - \vartheta^2). \quad (26)$$

The adaptation law for $\dot{\vartheta}^2$ is chosen to cancel the influence of the estimation error $\Theta - \vartheta^2$, thus we use

$$\dot{\vartheta}^{2\top} = (x_2 - \alpha_1(x_1,\vartheta^1))\Big[\frac{\partial V(x_1,\vartheta^1)}{\partial x_1}H_1^\top(x_1, x_2 - \alpha_1(x_1,\vartheta^1)) + \varphi_2^\top(\bar{x}_3) - \frac{\partial \alpha_1(x_1,\vartheta^1)}{\partial x_1}\varphi_1^\top(\bar{x}_2)\Big]\Gamma_1. \quad (27)$$

With the notation $\dot{\hat{V}}_2 := \dot{\hat{V}}_1 - K_1(x_2 - \alpha_1(x_1,\vartheta^1))^2$ the time derivative of $V_{a,1,2}(\bar{x}_2, \bar{\vartheta}^2)$ can be expressed as

$$\dot{V}_{a,1,2}(\bar{x}_2, \bar{\vartheta}^2) = \dot{\hat{V}}_2 + (x_2 - \alpha_1(x_1,\vartheta^1))(x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2)). \quad (28)$$

The term $(x_2 - \alpha_1(x_1,\vartheta^1))(x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2))$ will be handled in the following step.

Step 2: The procedure in this step is similar to Step 1. However, the objective will be to calculate a stabilizing function $\alpha_3(\bar{x}_4, \bar{\vartheta}_3)$ and the adaptation law for $\dot{\vartheta}_3$. An augmentation for the Lyapunov function candidate is

$$V_{a,2,1}(\bar{x}_3, \bar{\vartheta}^2) = V_{a,1,2}(\bar{x}_2, \bar{\vartheta}^2) + \frac{1}{2}(x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2))^2 \quad (29)$$

leading to the time derivative

$$\dot{V}_{a,2,1}(\bar{x}_3, \bar{\vartheta}^2) = \dot{V}_{a,1,2}(\bar{x}_2, \bar{\vartheta}^2) + (x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2))\big(f_3(\bar{x}_4) + \varphi_3^\top(\bar{x}_4)\Theta - \dot{\alpha}_2(\bar{x}_4, \bar{\vartheta}^2)\big). \quad (30)$$

Using the expression for $\dot{V}_{a,1,2}$ from (28), (30) reads

$$\dot{V}_{a,2,1}(\bar{x}_3, \bar{\vartheta}^2) = \dot{\hat{V}}_2 + (x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2))\big(x_2 - \alpha_1(x_1,\vartheta^1) + f_3(\bar{x}_4) + \varphi_3^\top(\bar{x}_4)\Theta - \dot{\alpha}_2(\bar{x}_4, \bar{\vartheta}^2)\big). \quad (31)$$

The terms between the big parentheses in (31) shall be stabilized by a choice of $\alpha_3(\bar{x}_4, \bar{\vartheta}^3)$, given by

$$x_4 - \alpha_3(\bar{x}_4, \bar{\vartheta}^3) = x_2 - \alpha_1(x_1,\vartheta^1) + f_3(\bar{x}_4) + \varphi_3^\top(\bar{x}_4)\vartheta^3 - \hat{\dot{\alpha}}_2(\bar{x}_4, \bar{\vartheta}^3) + K_2(x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2)) \quad (32)$$

where $\hat{\dot{\alpha}}_2(\bar{x}_4, \bar{\vartheta}^3)$ is an estimate of $\dot{\alpha}_2$ defined by

$$\hat{\dot{\alpha}}_2(\bar{x}_4, \bar{\vartheta}^3) = \sum_{i=1}^{3}\frac{\partial \alpha_2(\bar{x}_3,\bar{\vartheta}^2)}{\partial x_i}(f_i(\bar{x}_{i+1}) + \varphi_i^\top(\bar{x}_{i+1})\vartheta^3) + \sum_{i=1}^{2}\frac{\partial \alpha_2(\bar{x}_3,\bar{\vartheta}^2)}{\partial \vartheta^i}\dot{\vartheta}^i. \quad (33)$$

With expression (33) and (32), (31) can be expressed as

$$\dot{V}_{a,2,1}(\bar{x}_3, \bar{\vartheta}^2) = \dot{\hat{V}}_2 - K_2(x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2))^2 + (x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2))(x_4 - \alpha_3(\bar{x}_4, \bar{\vartheta}^3)) + (x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2))\Big[\varphi_3^\top(\bar{x}_4) - \sum_{i=1}^{3}\frac{\partial \alpha_2(\bar{x}_3,\bar{\vartheta}^2)}{\partial x_i}\varphi_i^\top(\bar{x}_{i+1})\Big](\Theta - \vartheta^3) \quad (34)$$

To attenuate the influence of the estimation error $\Theta - \vartheta^3$, we further augment the Lyapunov function candidate, i.e.

$$V_{a,2,2}(\bar{x}_3, \bar{\vartheta}^3) = V_{a,2,1}(\bar{x}_3, \vartheta^2) + \frac{1}{2}(\Theta - \vartheta^3)^\top\Gamma_2^{-1}(\Theta - \vartheta^3), \quad (35)$$

which yields

$$\dot{V}_{a,2,2}(\bar{x}_3, \bar{\vartheta}^3) = \dot{V}_{a,2,1}(\bar{x}_4, \bar{\vartheta}^3) - \dot{\vartheta}^{3\top}\Gamma_2^{-1}(\Theta - \vartheta^3) \quad (36)$$

and with (34) then can be rewritten as

$$\dot{V}_{a,2,2}(\bar{x}_3, \bar{\vartheta}^3) = \dot{\hat{V}}_2 - K_2(x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2))^2 + (x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2))(x_4 - \alpha_3(\bar{x}_4, \bar{\vartheta}^3)) + \Big((x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2))\Big[\varphi_3^\top(\bar{x}_4) - \sum_{i=1}^{3}\frac{\partial \alpha_2(\bar{x}_3,\bar{\vartheta}^2)}{\partial x_i}\varphi_i^\top(\bar{x}_{i+1})\Big] - \dot{\vartheta}^{3\top}\Gamma_2^{-1}\Big)(\Theta - \vartheta^3). \quad (37)$$

The adaptation law for $\dot{\vartheta}^3$ is chosen to cancel the influence of the estimation error $\Theta - \vartheta^3$, hence

$$\dot{\vartheta}^{3\top} = (x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2))\Big[\varphi_3^\top(\bar{x}_4) - \sum_{i=1}^{3}\frac{\partial \alpha_2(\bar{x}_3,\bar{\vartheta}^2)}{\partial x_i}\varphi_i^\top(\bar{x}_{i+1})\Big]\Gamma_2 \quad (38)$$

and introducing notation $\dot{\hat{V}}_3 := \dot{\hat{V}}_2 - K_2(x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2))^2$, the time derivative of $V_{a,2,2}(\bar{x}_4, \bar{\vartheta}^3)$ results in

$$\dot{V}_{a,2,2}(\bar{x}_4, \bar{\vartheta}^3) = \dot{\hat{V}}_3 + (x_3 - \alpha_2(\bar{x}_3, \bar{\vartheta}^2))(x_4 - \alpha_3(\bar{x}_4, \bar{\vartheta}^3)). \quad (39)$$

Step k-1: The procedures from Step 1 to Step k-1 are all similar. The objective in this stage will be to calculate a stabilizing function $\alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)$ and the adaptation law for $\dot{\vartheta}^k$. This leads to a new augmentation for the Lyapunov function candidate, that is

$$V_{a,k-1,1}(\bar{x}_k, \bar{\vartheta}^{k-1}) = V_{a,k-2,2}(\bar{x}_{k-1}, \bar{\vartheta}^{k-2}) + \frac{1}{2}(x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1}))^2 \quad (40)$$

whose time derivative is

$$\dot{V}_{a,k-1,1}(\bar{x}_k, \bar{\vartheta}^{k-1}) = \dot{V}_{a,k-2,2}(\bar{x}_k, \bar{\vartheta}^{k-2}) + (x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1}))\Big[f_k(\bar{x}_k, u) + \varphi_k^\top(\bar{x}_k, u)\Theta - \dot{\alpha}_{k-1}(\bar{x}_k, u, \bar{\vartheta}^{k-1})\Big]. \quad (41)$$

The expression for $\dot{V}_{a,k-2,2}(\bar{x}_k, \bar{\vartheta}^{k-2})$ from Step k-2 is

$$\dot{V}_{a,k-2,2}(\bar{x}_k, \bar{\vartheta}^{k-2}) = \dot{\hat{V}}_{k-2} - K_{k-2}(x_{k-1} - \alpha_{k-2}(\bar{x}_{k-1}), \bar{\vartheta}^{k-2}))^2 + (x_{k-1} - \alpha_{k-2}(\bar{x}_{k-1}, \bar{\vartheta}^{k-2}))(x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1})). \quad (42)$$

Using $\dot{\hat{V}}_{k-1} := \dot{\hat{V}}_{k-2} - K_{k-2}(x_{k-1} - \alpha_{k-2}(\bar{x}_{k-1})\bar{\vartheta}^{k-2}))^2$ we may rewrite (41) in the following way:

$$\dot{V}_{a,k-1,1}(\bar{x}_k, \bar{\vartheta}^{k-1}) = \dot{\hat{V}}_{k-1} + (x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1})) \Big[x_{k-1} - \alpha_{k-2}(\bar{x}_{k-1}, \bar{\vartheta}^{k-2}) + f_k(\bar{x}_k, u) + \varphi_k^\top(\bar{x}_k, u)\Theta - \dot{\alpha}_{k-1}(\bar{x}_k, u, \bar{\vartheta}^{k-1})\Big]. \quad (43)$$

The terms between the brackets in (43) shall be stabilized by choice of $\alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)$. Note that here we choose

$$u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k) = x_{k-1} - \alpha_{k-2}(\bar{x}_{k-1}, \vartheta^{k-2}) + f_k(\bar{x}_k, u) + \varphi_k^\top(\bar{x}_k, u)\vartheta^k - \dot{\hat{\alpha}}_{k-1}(\bar{x}_k, u, \bar{\vartheta}^k) + K_{k-1}(x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1})) \quad (44)$$

where $\dot{\hat{\alpha}}_{k-1}(\bar{x}_k, u, \bar{\vartheta}^k)$ is an estimate of $\dot{\alpha}_{k-1}$ defined by

$$\dot{\hat{\alpha}}_{k-1}(\bar{x}_k, u, \bar{\vartheta}^k) = \sum_{i=1}^k \frac{\partial \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1})}{\partial x_i}(f_i(\bar{x}_{i+1}) + \varphi_i^\top(\bar{x}_{i+1})\vartheta^k) + \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1})}{\partial \vartheta^i}\dot{\vartheta}^i. \quad (45)$$

Hence, with (45) and (44), we may express (43) as

$$\dot{V}_{a,k-1,1}(\bar{x}_k, u, \bar{\vartheta}^k) = \dot{\hat{V}}_{k-1} - K_{k-1}(x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1}))^2 + (x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1})(u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)) + (x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1}))\Big[\varphi_k^\top(\bar{x}_k, u) - \sum_{i=1}^k \frac{\partial \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1})}{\partial x_i}\varphi_i^\top(\bar{x}_{i+1})\Big](\Theta - \vartheta^k). \quad (46)$$

In order to attenuate the influence of the estimation error $\Theta - \vartheta^k$, a new augmentation is introduced as

$$V_{a,k-1,2}(\bar{x}_k, \bar{\vartheta}^k) = V_{a,k-1,1}(\bar{x}_k, \vartheta^{k-1}) + \frac{1}{2}(\Theta - \vartheta^k)^\top \Gamma_{k-1}^{-1}(\Theta - \vartheta^k) \quad (47)$$

which leads to

$$\dot{V}_{a,k-1,2}(\bar{x}_k, u, \bar{\vartheta}^k) = \dot{V}_{a,k-1,1}(\bar{x}_k, u, \bar{\vartheta}^k) - \dot{\vartheta}^{k\top}\Gamma_{k-1}^{-1}(\Theta - \vartheta^k). \quad (48)$$

With (46) may now rewrite (48), that is

$$\dot{V}_{a,k-1,2}(\bar{x}_k, u, \bar{\vartheta}^k) = \dot{\hat{V}}_{k-1} - K_{k-1}(x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1}))^2 + (x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1}))(u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)) + \Big((x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1}))\Big[\varphi_k^\top(\bar{x}_k, u) - \sum_{i=1}^k \frac{\partial \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1})}{\partial x_i}\varphi_i^\top(\bar{x}_{i+1})\Big] - \dot{\vartheta}^{k\top}\Gamma_{k-1}^{-1}\Big)(\Theta - \vartheta^k). \quad (49)$$

The adaptation law for $\dot{\vartheta}^k$ is chosen to cancel the estimation error $\Theta - \vartheta^k$, i.e.

$$\dot{\vartheta}^{k\top} = (x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1}))\Big[\varphi_k^\top(\bar{x}_k, u) - \sum_{i=1}^k \frac{\partial \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1})}{\partial x_i}\varphi_i^\top(\bar{x}_{i+1})\Big]\Gamma_{k-1} \quad (50)$$

Using notation $\dot{\hat{V}}_k := \dot{\hat{V}}_{k-1} - K_{k-1}(x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1}))^2$, the time derivative of $V_{a,k-1,2}(\bar{x}_k, u, \bar{\vartheta}^k)$ reads

$$\dot{V}_{a,k-1,2}(\bar{x}_k, u, \bar{\vartheta}^k) = \dot{\hat{V}}_k + (x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1}))(u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)). \quad (51)$$

Finally, product $(x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1}))(u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k))$ is compensated by the dynamic extension $\dot{u}$ in the last step.

Step k: The explicit solution that ensures the equality $u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k) = 0$ in (51) cannot be assumed available. Thus, the Lyapunov function candidate is augmented again and introduces a single dynamic state feedback in order to obtain $\dot{V}_{a,k,2}(\bar{x}_k, u, \bar{\vartheta}^k) \le 0$, and to attenuate the estimation error $\Theta - \vartheta^{k+1}$. With this in mind, we use

$$V_{a,k,1}(\bar{x}_k, u, \bar{\vartheta}^k) = V_{a,k-1,2}(\bar{x}_k, \bar{\vartheta}^k) + \frac{1}{2}(u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k))^2 \quad (52)$$

and its time derivative is

$$\dot{V}_{a,k,1}(\bar{x}_k, u, \bar{\vartheta}^k) = \dot{V}_{a,k-1,2}(\bar{x}_k, u, \bar{\vartheta}^k) + (u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k))(\dot{u} - \dot{\alpha}_k(\bar{x}_k, u, \dot{u}, \bar{\vartheta}^k)). \quad (53)$$

Using the expression for $\dot{V}_{a,k-1,2}(\bar{x}_k, u, \bar{\vartheta}^k)$ from (51), we may rephrase (53) as

$$\dot{V}_{a,k,1}(\bar{x}_k, u, \bar{\vartheta}^k) = \dot{\hat{V}}_k + (u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)) \Big[x_k - \alpha_{k-1}(\bar{x}_k, \vartheta^{k-1}) + \dot{u} - \dot{\alpha}_k(\bar{x}_k, u, \dot{u}, \bar{\vartheta}^k)\Big]. \quad (54)$$

The terms between the brackets are completing a square when choosing

$$\dot{u} - \dot{\hat{\alpha}}_k(\bar{x}_k, u, \dot{u}, \bar{\vartheta}^{k+1}) = -x_k + \alpha_{k-1}(\bar{x}_k, \vartheta^{k-1}) - K_k(u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)). \quad (55)$$

where $\dot{\hat{\alpha}}_k(\bar{x}_k, u, \dot{u}, \bar{\vartheta}^{k+1})$ is an estimate of $\dot{\alpha}_k$ defined by

$$\dot{\hat{\alpha}}_k(\bar{x}_k, u, \dot{u}, \bar{\vartheta}^{k+1}) = \sum_{i=1}^k \frac{\partial \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)}{\partial x_i}\big(f_i(\bar{x}_{i+1}) + \varphi_i^\top(\bar{x}_{i+1})\vartheta^{k+1}\big) + \frac{\partial \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)}{\partial u}\dot{u} + \sum_{i=1}^k \frac{\partial \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)}{\partial \vartheta^i}\dot{\vartheta}^i. \quad (56)$$

We have the difference $\dot{\alpha}_k - \dot{\hat{\alpha}}_k$ given by

$$\dot{\alpha}_k(\bar{x}_k, u, \dot{u}, \bar{\vartheta}^k) - \dot{\hat{\alpha}}_k(\bar{x}_k, u, \dot{u}, \bar{\vartheta}^{k+1}) = \sum_{i=1}^k \frac{\partial \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)}{\partial x_i}\varphi_i^\top(\bar{x}_{i+1})(\Theta - \vartheta^{k+1}) \quad (57)$$

which with (56) yields

$$\dot{u}\Big(1 - \frac{\partial \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)}{\partial u}\Big) = -K_k(u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)) - (x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1})) + \sum_{i=1}^k \frac{\partial \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)}{\partial x_i}\big(f_i(\bar{x}_{i+1}) + \varphi_i^\top(\bar{x}_{i+1})\vartheta^{k+1}\big) + \sum_{i=1}^k \frac{\partial \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)}{\partial \vartheta^i}\dot{\vartheta}_i. \quad (58)$$

We may analytically express the derivative $\frac{\partial \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)}{\partial u}$ via (44) and substitute the result in (58) to obtain

$$\dot{u} = \Big(\frac{\partial}{\partial u}(f_k(\bar{x}_k, u) + \varphi_k^\top(\bar{x}_k, u)\vartheta^k) - \frac{\partial \dot{\hat{\alpha}}_{k-1}(\bar{x}_k, u, \bar{\vartheta}^k)}{\partial u}\Big)^{-1} \Big[-K_k(u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)) - (x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1})) + \sum_{i=1}^k \frac{\partial \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)}{\partial x_i}(f_i(\bar{x}_{i+1}) + \varphi_i^\top(\bar{x}_{i+1})\vartheta^{k+1}) + \sum_{i=1}^k \frac{\partial \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)}{\partial \vartheta^i}\dot{\vartheta}^i\Big]. \quad (59)$$

Now with (57), equation (54) results in

$$\dot{V}_{a,k,1}(\bar{x}_k, u, \bar{\vartheta}^{k+1}) = \dot{\hat{V}}_k - K_k(u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k))^2 - (u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k))\sum_{i=1}^k \frac{\partial \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)}{\partial x_i}\varphi_i^\top(\bar{x}_{i+1})(\Theta - \vartheta^{k+1}) \quad (60)$$

whose influence of the estimation error is canceled with the last augmentation of the Lyapunov function candidate

$$V_{a,k,2}(\bar{x}_k,u,\bar{\vartheta}^{k+1}) = V_{a,k,1}(\bar{x}_k,u,\vartheta^k) + \frac{1}{2}(\Theta-\vartheta^{k+1})^\top\Gamma_k^{-1}(\Theta-\vartheta^{k+1}) \quad (61)$$

with time derivative

$$\dot{V}_{a,k,2}(\bar{x}_k,u,\bar{\vartheta}^{k+1}) = \dot{V}_{a,k,1}(\bar{x}_k,u,\bar{\vartheta}^{k+1}) - \dot{\vartheta}^{k+1^\top}\Gamma_k^{-1}(\Theta-\vartheta^{k+1}) \quad (62)$$

leading with (60) to

$$\dot{V}_{a,k,2}(\bar{x}_k,u,\bar{\vartheta}^{k+1}) = \dot{\tilde{V}}_k - K_k(u-\alpha_k(\bar{x}_k,u,\bar{\vartheta}^k))^2 + \Big((u-\alpha_k(\bar{x}_k,u,\bar{\vartheta}^k))\Big[-\sum_{i=1}^k \tfrac{\partial\alpha_k(\bar{x}_k,u,\bar{\vartheta}^k)}{\partial x_i}\varphi_i^\top(\bar{x}_{i+1})\Big] - \dot{\vartheta}^{k+1^\top}\Gamma_k^{-1}\Big)(\Theta-\vartheta^{k+1}). \quad (63)$$

The final adaptation law for $\dot{\vartheta}^{k+1}$ is chosen to cancel the influence of the estimation error $\Theta-\vartheta^{k+1}$, i.e.

$$\dot{\vartheta}^{k+1^\top} = (u-\alpha_k(\bar{x}_k,u,\bar{\vartheta}^k))\Big[-\sum_{i=1}^k \tfrac{\partial\alpha_k(\bar{x}_k,\bar{\vartheta}^k)}{\partial x_i}\varphi_i^\top(\bar{x}_{i+1})\Big]\Gamma_k \quad (64)$$

which substituted in (63) establishes

$$\dot{V}_{a,k,2}(\bar{x}_k,u,\bar{\vartheta}^{k+1}) = \dot{\tilde{V}}_k - K_k(u-\alpha_k(\bar{x}_k,u,\bar{\vartheta}^k))^2, \quad (65)$$

or to highlight the negative definiteness, equivalently

$$\dot{V}_{a,k,2}(\bar{x}_k,u,\bar{\vartheta}^{k+1}) = \dot{\tilde{V}}_1 - \sum_{i=2}^k K_{i-1}(x_i-\alpha_{i-1}(\bar{x}_i,u,\bar{\vartheta}^{i-1}))^2 - K_k(u-\alpha_k(\bar{x}_k,u,\bar{\vartheta}^k))^2. \quad (66)$$

Remark 3. In order to avoid singularities in (59), condition $\frac{\partial}{\partial u}(f_k(\bar{x}_k,u)+\varphi_k^\top(\bar{x}_k,u)\vartheta^k) - \frac{\partial\dot{\hat{\alpha}}_{k-1}(\bar{x}_k,u,\bar{\vartheta}^k)}{\partial u} \neq 0$ must be fulfilled on $\mathcal{F}$. For this purpose, consider

$$\hat{\tau}_i = \tfrac{\partial\dot{\hat{\alpha}}_i(\bar{x}_{i+2},\bar{\vartheta}^{i+1})}{\partial x_{i+2}} \quad (67)$$

which for $i=2,\ldots,k-1$ in explicit terms reads

$$\hat{\tau}_i = \tfrac{\partial}{\partial x_{i+2}}\big[\big(-\tfrac{\partial}{\partial x_{i+1}}(f_i(\bar{x}_{i+1})+\varphi_i^\top(\bar{x}_{i+1})\vartheta^i)+1+\hat{\tau}_{i-1}\big)\big(f_{i+1}(\bar{x}_{i+2})+\varphi_{i+1}^\top(\bar{x}_{i+2})\vartheta^{i+1}\big)\big] \quad (68)$$

and vanishes for $i=1$, i.e. $\hat{\tau}_1=0$. With (67) and (68), the required condition can be rewritten as

$$\tfrac{\partial}{\partial u}(f_k(\bar{x}_k,u)+\varphi_k^\top(\bar{x}_k,u)\vartheta^k) - \hat{\tau}_{k-1} \neq 0. \quad (69)$$

For clarity in the analysis we drop the arguments. We have

$$\begin{aligned}
&\tfrac{\partial}{\partial u}(f_k+\varphi_k^\top\vartheta^k) - \hat{\tau}_{k-1} = \tfrac{\partial}{\partial u}(f_k+\varphi_k^\top\vartheta^k) \\
&-\tfrac{\partial}{\partial u}\big[\big(-\tfrac{\partial}{\partial x_k}(f_{k-1}+\varphi_{k-1}^\top\vartheta^{k-1})+1+\hat{\tau}_{k-2}\big)(f_k+\varphi_k^\top\vartheta^k)\big] \\
&=\tfrac{\partial}{\partial u}(f_k+\varphi_k^\top\vartheta^k)\big(\tfrac{\partial}{\partial x_k}(f_{k-1}+\varphi_{k-1}^\top\vartheta^{k-1})-\hat{\tau}_{k-2}\big) \\
&=\tfrac{\partial}{\partial u}(f_k+\varphi_k^\top\vartheta^k)\Big(\tfrac{\partial}{\partial x_k}(f_{k-1}+\varphi_{k-1}^\top\vartheta^{k-1}) \\
&\quad -\tfrac{\partial}{\partial x_k}\big[\big(-\tfrac{\partial}{\partial x_{k-1}}(f_{k-2}+\varphi_{k-2}^\top\vartheta^{k-2})+1+\hat{\tau}_{k-3}\big) \\
&\quad (f_{k-1}+\varphi_{k-1}^\top\vartheta^{k-1})\big]\Big) \\
&=\tfrac{\partial}{\partial u}(f_k+\varphi_k^\top\vartheta^k)\tfrac{\partial}{\partial x_k}(f_{k-1}+\varphi_{k-1}^\top\vartheta^{k-1}) \\
&\quad \big(\tfrac{\partial}{\partial x_{k-1}}(f_{k-2}+\varphi_{k-2}^\top\vartheta^{k-2})-\hat{\tau}_{k-3}\big) \\
&\;\vdots \\
&=\tfrac{\partial}{\partial u}(f_k+\varphi_k^\top\vartheta^k)\tfrac{\partial}{\partial x_k}(f_{k-1}+\varphi_{k-1}^\top\vartheta^{k-1})\cdots \\
&\quad \cdots\tfrac{\partial}{\partial x_4}(f_3+\varphi_3^\top\vartheta^3)\big(\tfrac{\partial}{\partial x_3}(f_2+\varphi_2^\top\vartheta^2)-\hat{\tau}_1\big) \\
&=\tfrac{\partial}{\partial u}(f_k+\varphi_k^\top\vartheta^k)\tfrac{\partial}{\partial x_k}(f_{k-1}+\varphi_{k-1}^\top\vartheta^{k-1})\cdots \\
&\quad \cdots\tfrac{\partial}{\partial x_3}(f_2+\varphi_2^\top\vartheta^2)\neq 0. \qquad (70)
\end{aligned}$$

Hence by Assumption 2 we conclude that controller (59) has no singularity.

Stability considerations: $V_{a,k,2}(\bar{x}_k,u,\bar{\vartheta}^{k+1})$ in view of (66) is only a weak Lyapunov function since its derivative is only negative semi-definite in $x_1,\cdots,x_k,u,\vartheta^1,\ldots,\vartheta^{k+1}$. We invoke Krasovskii-LaSalle's invariance principle for analyzing the asymptotic convergence of the states (Khalil, 1996). To formalize the analysis, let a compact positively invariant set $\Omega\subset\mathcal{F}$ with respect to the controlled system be such that for some $l>0$

$$\Omega := \{\bar{x}_k\in\mathbb{R}^{n+k-1},\ u\in\mathbb{R},\ \bar{\vartheta}^{k+1}\in\mathbb{R}^{r\times(k+1)} : V_{a,k,2}(\bar{x}_k,u,\bar{\vartheta}^{k+1})\leq l\}. \quad (71)$$

Furthermore, let E be the set of all points in Ω where

$$E := \{(\bar{x}_k,u,\bar{\vartheta}^{k+1})\in\Omega : \dot{V}_{a,k,2}(\bar{x}_k,u,\bar{\vartheta}^{k+1})=0\} \quad (72)$$

and M be the largest invariant set in E, inspecting (66).

From Assumption 1 we have that $x_1=0$ if and only if

$$\dot{\tilde{V}}_1 = \tfrac{\partial V(x_1,\vartheta^1)}{\partial x_1}\big[f_1(x_1,\alpha_1(x_1,\vartheta^1)) + \varphi_1^\top(x_1,\alpha_1(x_1,\vartheta^1))\Theta\big] + \tfrac{\partial V(x_1,\vartheta^1)}{\partial\vartheta^1}\dot{\vartheta}^1 = 0.$$

From $\dot{\tilde{V}}_2$ defined before (28), that is

$$\dot{\tilde{V}}_2 = \dot{\tilde{V}}_1 - K_1(x_2-\alpha_1(x_1,\vartheta^1))^2,$$

and $\alpha_1(0,\vartheta^1)=0$ from Assumption 1 we conclude that $\dot{\tilde{V}}_2=0$ implies $\bar{x}_2=0$.

Analyzing $\dot{\tilde{V}}_3=0$, we shall inspect the last expression of

$$\dot{\tilde{V}}_3 = \dot{\tilde{V}}_2 - K_2(x_3-\alpha_2(\bar{x}_3,\bar{\vartheta}^2))^2, \quad (73)$$

recalled from (20):

$$x_3-\alpha_2(\bar{x}_3,\bar{\vartheta}^2) = \tfrac{\partial V(x_1,\vartheta^1)}{\partial x_1}\big[G_1(x_1,x_2-\alpha_1(x_1,\vartheta^1)) + H_1^\top(x_1,x_2-\alpha_1(x_1,\vartheta^1))\vartheta^2\big] + f_2(\bar{x}_3) + \varphi_2^\top(\bar{x}_3)\vartheta^2 - \dot{\hat{\alpha}}_1(\bar{x}_2,\bar{\vartheta}^2) + K_1(x_2-\alpha_1(x_1,\vartheta^1)).$$

Using again Assumption 1 and that $\bar{x}_2=0$, we see that

$$x_3-\alpha_2(0,x_3,\bar{\vartheta}^2) = f_2(0,x_3)+\varphi_2^\top(0,x_3)\vartheta^2. \quad (74)$$

By Assumption 2, $\frac{\partial}{\partial x_3}(f_2(0,x_3)+\psi_2^\top(0,x_3)\Theta)\neq 0$. Then, with the implicit function theorem and (2) we have that $f_2(0,x_3)+\varphi_2^\top(0,x_3)\vartheta^2=0$ implies $x_3=0$, achieving that $\dot{\tilde{V}}_3=0 \iff \bar{x}_3=0$.

In a similar way, but with somewhat different expressions, we examine

$$\dot{\tilde{V}}_4 = \dot{\tilde{V}}_3 - K_3(x_4-\alpha_3(\bar{x}_4,\bar{\vartheta}^3))^2. \quad (75)$$

From (32) recall that

$$x_4-\alpha_3(\bar{x}_4,\bar{\vartheta}^3) = x_2-\alpha_1(x_1,\vartheta^1)+f_3(\bar{x}_4)+\varphi_3^\top(\bar{x}_4)\vartheta^3 - \dot{\hat{\alpha}}_2(\bar{x}_4,\bar{\vartheta}^3)+K_2(x_3-\alpha_2(\bar{x}_3,\bar{\vartheta}^2)).$$

The analysis so far and Assumption 1 results in

$$x_4-\alpha_3(\bar{x}_4,\bar{\vartheta}^3) = f_3(0,x_4)+\varphi_3^\top(0,x_4)\vartheta^3-\dot{\hat{\alpha}}_2(0,x_4,\bar{\vartheta}^3) \quad (76)$$

in which via (13) we draw

$$\dot{\hat{\alpha}}_2(0,x_4,\bar{\vartheta}^3) = \sum_{l=1}^3 \tfrac{\partial\alpha_2(\bar{x}_3=0,\bar{\vartheta}^2)}{\partial x_l}(f_l(\bar{x}_{l+1})+\varphi_l^\top(\bar{x}_{l+1})\vartheta^3) + \sum_{l=1}^2 \tfrac{\partial\alpha_2(\bar{x}_3=0,\bar{\vartheta}^2)}{\partial\vartheta^l}\dot{\vartheta}^l. \quad (77)$$

Then from $\bar{x}_1 \implies \dot{\vartheta}^1=0$, $\bar{x}_3=0 \implies \dot{\vartheta}^2=0$, and (74), we may simplify (77) to get

$$\dot{\hat{\alpha}}_2(0,x_4,\bar{\vartheta}^3) = \Big(1-\tfrac{\partial}{\partial x_3}(f_2(\bar{x}_3=0)+\varphi_2^\top(\bar{x}_3=0)\vartheta^2)\Big)\big(f_3(\bar{x}_4)+\varphi_3^\top(\bar{x}_4)\vartheta^3\big) \quad (78)$$

which inserted in (76) leads to

$$x_4 - \alpha_3(\bar{x}_4, \bar{\vartheta}^3) = \frac{\partial}{\partial x_3}(f_2(\bar{x}_3 = 0) + \varphi_2^\top(\bar{x}_3 = 0)\vartheta^2) \big(f_3(\bar{x}_4) + \varphi_3^\top(\bar{x}_4)\vartheta^3\big). \quad (79)$$

From Assumption 2, $\frac{\partial}{\partial x_3}(f_2(\bar{x}_3 = 0) + \varphi_2^\top(\bar{x}_3 = 0)\vartheta^2) \neq 0$. Then, $\dot{\tilde{V}}_4 = 0$ requires that $f_3(\bar{x}_4) + \varphi_3^\top(\bar{x}_4)\vartheta^3 = 0$. With $\frac{\partial}{\partial x_4}(f_3(0, x_4) + \varphi_3^\top(0, x_4)\vartheta^3) \neq 0$ and the implicit function theorem we conclude that $\dot{\tilde{V}}_4 = 0 \iff \bar{x}_4 = 0$.

A similar procedure can be performed for the other augmented Lyapunov functions up to

$$\dot{V}_{a,k,2}(\bar{x}_k, u, \bar{\vartheta}^{k+1}) = \dot{\tilde{V}}_k - K_k(u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k))^2. \quad (80)$$

From the recursive analysis we have $\dot{\tilde{V}}_k = 0$. Hence, we focus on $u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k)$ which from (44) has the form

$$\begin{aligned} u - \alpha_k(\bar{x}_k, u, \bar{\vartheta}^k) = {} & x_{k-1} - \alpha_{k-2}(\bar{x}_{k-1}, \vartheta^{k-2}) + f_k(\bar{x}_k, u) \\ & + \varphi_k^\top(\bar{x}_k, u)\vartheta^k - \dot{\hat{\alpha}}_{k-1}(\bar{x}_k, u, \bar{\vartheta}^k) \\ & + K_{k-1}(x_k - \alpha_{k-1}(\bar{x}_k, \bar{\vartheta}^{k-1})). \end{aligned}$$

With the results of the analysis before, we may rewrite

$$u - \alpha_k(0, u, \bar{\vartheta}^k) = f_k(0, u) + \varphi_k^\top(0, u)\vartheta^k - \dot{\hat{\alpha}}_{k-1}(0, u.\bar{\vartheta}^k) \quad (81)$$

in which resorting to (13) we have

$$\begin{aligned} \dot{\hat{\alpha}}_{k-1}(0, u, \bar{\vartheta}^k) = {} & \textstyle\sum_{l=1}^{k-1} \frac{\partial \alpha_{k-1}(\bar{x}_k = 0, \bar{\vartheta}^{k-1})}{\partial \vartheta^l} \dot{\vartheta}^l \\ & + \textstyle\sum_{l=1}^{k} \frac{\partial \alpha_{k-1}(\bar{x}_k = 0, \bar{\vartheta}^{k-1})}{\partial x_l} (f_l(\bar{x}_{l+1}) + \varphi_l^\top(\bar{x}_{l+1})\vartheta^k). \quad (82) \end{aligned}$$

Then $\bar{x}_1 = 0 \implies \dot{\vartheta}^1 = 0, \ldots, \bar{x}_k = 0 \implies \dot{\vartheta}^{k-1} = 0$, and $\alpha_{k-1}(\bar{x}_k, \bar{\vartheta}_{k-1})$ derived from (10) and (67), we have

$$\dot{\hat{\alpha}}_{k-1}(0, u, \bar{\vartheta}^k) = \big(f_k(0, u) + \varphi_k^\top(0, u)\vartheta^k\big) \Big(1 - \frac{\partial}{\partial x_k}\big(f_{k-1}(\bar{x}_k = 0) + \varphi_{k-1}^\top(\bar{x}_k = 0)\vartheta^{k-1} + \hat{\tau}_{k-2}\big)\Big). \quad (83)$$

Substitution in (81) yields

$$u - \alpha_k(0, u, \bar{\vartheta}^k) = \frac{\partial}{\partial x_k}\big(f_{k-1}(\bar{x}_k = 0) + \varphi_{k-1}^\top(\bar{x}_k = 0)\vartheta^{k-1} - \hat{\tau}_{k-2}\big)\big(f_k(0, u) + \varphi_k^\top(0, u)\vartheta^k\big). \quad (84)$$

In view of (70), term $\hat{\tau}_{k-2}$ can be expanded, leading to

$$u - \alpha_k(0, u, \bar{\vartheta}^k) = \Big(\frac{\partial}{\partial x_k}(f_{k-1} + \varphi_{k-1}^\top\vartheta^{k-1}) \frac{\partial}{\partial x_{k-1}}(f_{k-2} + \varphi_{k-2}^\top\vartheta^{k-2}) \cdots \frac{\partial}{\partial x_3}(f_2 + \varphi_2^\top\vartheta^2)\Big)\Big|_{\bar{x}_k = 0} \big(f_k(0, u) + \varphi_k^\top(0, u)\vartheta^k\big). \quad (85)$$

From Assumption 2, the expression in the big parentheses taken at $\bar{x}_k = 0$ is non-zero. So $f_k(0, u) + \varphi_k^\top(0, u)\vartheta^k = 0$. Moreover, $\frac{\partial}{\partial x_{k+1}}(f_k(\bar{x}_k, u) + \varphi_k^\top(\bar{x}_k, u)\vartheta^k)\big|_{\bar{x}_k = 0} \neq 0$. Then inspecting $u - \alpha_k(0, u, \bar{\vartheta}^k) = 0$ with the implicit function theorem we draw that $\dot{V}_{a,k,2}(\bar{x}_k, u, \bar{\vartheta}^{k+1}) = 0$ if and only if both $\bar{x}_k = 0$ and $u = 0$, for which also $\dot{x}_i = 0$ and $\dot{u} = 0$. In other words,

$$E = \{(\bar{x}_k, u, \bar{\vartheta}^{k+1}) \in \Omega : \bar{x}_k = 0,\ u = 0\} \quad (86)$$

itself is the largest positively invariant subset. Therefore, according to Krasovskii-LaSalle's invariance principle all solutions starting in Ω eventually converge to E. Furthermore from Assumption 1 we inherit that (61) is also radially unbounded. Consequently $\bar{x}_k$ and u will converge to E irrespective of their initial conditions.

However, we cannot conclude that the ϑ^i converge to Θ. We only know that ϑ^i is bounded and takes a limit. This can be seen as follows. From $\dot{V}_{a,k,2}(\bar{x}_k, u, \bar{\vartheta}^{k+1}) \leq 0$ and the lower bound of $V_{a,k,2}(\bar{x}_k, u, \bar{\vartheta}^{k+1})$ we have that the Lyapunov function in (47) is bounded at any time. Further we have that the Lyapunov function $V_{a,k,2}(\bar{x}_k, u, \bar{\vartheta}^{k+1})$ is radially unbounded, hence, its level sets are compact. Consequently, all variables in the respective squares are bounded. This also includes that ϑ^i for any $i = 2, 3, \cdots, k$ is bounded. Since $V_{a,k,2}(\bar{x}_k, u, \bar{\vartheta}^{k+1})$ is bounded from below, does not increase and is continuous, it takes a limit. As shown above, also $\bar{x}_k$ and u take a limit. Therefore we also conclude that ϑ^i for any $i = 2, 3, \cdots, k$ takes a limit.

Finally we comment on the existence and uniqueness of the solution of the system in closed loop. Let the states $\bar{x}_k$, u, $\bar{\vartheta}^{k+1}$ be initialized on the compact set Ω. As shown above, the states will remain there, consequently prohibiting any finite escape. Then Lipschitz continuity on Ω with respect to the functions f_i, ψ_i, for all $i = 1, \ldots, k$, the function α_1 and the right-hand side of $\dot{\vartheta}^1$ implies a unique solution.

5. CONCLUSIONS

This paper has broadened the applicability of the dynamic state extension approach for backstepping to systems in pure-feedback systems that show a linear-parametric uncertainty. Resorting to Lyapunov arguments we have shown the asymptotic stability of the states and control action in the closed-loop system. The dynamic control law devised does not show singularities. Future work will focus on taking control saturation into account.

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