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A Novel Attitude Representation in View of Spacecraft Attitude Reconstruction using Temperature Data ⋆

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Abstract: There are various different attitude representations that describe the orientation of a rigid body in space and allow the transformation between different coordinate systems. Among others, they differ in number of variables, uniqueness of the representation and their continuity. However, in spite of some of them being based on angles, none of them constitutes a simple representation of an angle between two arbitrary vectors. We tackle this issue by proposing a novel attitude representation that directly incorporates the desired angle. The usefulness of this representation is demonstrated in the attitude reconstruction from temperature data, which then leads to an order reduction of a non-linear system.

Keywords: Modeling and Identification of Nonlinear Systems, Aerospace Control Systems, Control of Nonlinear Systems

1. INTRODUCTION

The attitude defines the orientation of a rigid body in space. The three dimensional set of attitudes is the set of all 3×3 orthogonal matrices with determinant equal to one. Since this space is non-euclidean and it uses nine parameter to define a set of dimension three, there exists a number of other attitude representations Shuster et al. (1993); Chaturvedi et al. (2011); Stuelpnagel (1964); Markley and Crassidis (2014); Mortensen (1968); Wie and Barba (1985); Wen and Kreutz-Delgado (1991). Finding the most suitable representation depends on application specific requirements, e.g. the number of variables, uniqueness and the existence of singularities. Attitude representations of dimension three such as Euler angles have the advantage of using the minimal number of variables. However, they suffer from singularities which in this context is a phenomenon called gimbal lock. Furthermore, the same attitude can be described by different Euler angle representations and they do not establish a continuous evolution of these angles if multiple full rotations occur. Quaternions tackle these issues by introducing a fourth variable and a norm constraint. This fixes the continuity and singularity issue. Also, every attitude is represented by (only) exactly two quaternions with opposite sign. There exist many other attitude representations, each with benefits and drawbacks with respect to the points previously discussed. The choice of an attitude representation often is a matter of preference.

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In this paper, we introduce a novel attitude representation which facilitates finding the solution of a non-linear system incorporating quantities that depend on the angle of two vectors given in different frames. The representation is motivated by the temperature evolution of a spacecraft as it depends on the angle between the surface normal and the source of heat. This evolution is used in a variety of works to determine the attitude of the spacecraft Labibian et al. (2017, 2018); Khaniki and Karimian (2016). The proposed transformation simplifies the reconstruction. The motivating system is introduced in Section 3. The new transformation is derived in Section 4. Major properties are discussed in Section 5. The usefulness is shown resorting to the motivating system in Section 6.

2. NOTATION

Throughout this paper we use quaternions and their algebra to introduce the new attitude representation. Most of the notation is borrowed from Markley and Crassidis (2014). Let

$$\mathbb{S}_i := \{x \in \mathbb{R}^{i+1} \mid \|x\| = 1\}$$

denote the i-th unit sphere with $\|\cdot\|$ the Euclidean norm. Let I_n denote the identity matrix of dimension $n \in \mathbb{N}$ and e_i the i-th unity vector of appropriate size. With a slight abuse of notation, we let q denote either a unit quaternion, i.e. $q \in \mathbb{S}_3$, or the function $q : \mathbb{S}_2 \times (-\pi, \pi] \to \mathbb{S}_3$ mapping a rotation vector $r \in \mathbb{S}_2$ and angle $\phi \in (-\pi, \pi]$ onto the corresponding quaternion, i.e.

$$q(r, \phi) := \left[r_1 \sin(\tfrac{\phi}{2}) \;\; r_2 \sin(\tfrac{\phi}{2}) \;\; r_3 \sin(\tfrac{\phi}{2}) \;\; \cos(\tfrac{\phi}{2})\right]^\top .$$

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Let $q_{1:3}$ denote the first three entries of q and q_4 the fourth entry. We extend the definition to arbitrary vectors $r \in \mathbb{R}^3$ with $q(r,\phi) := q\big(\frac{r}{\|r\|},\phi\big)$. The matrices

$$[u\times] := \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}, \quad \Xi(q) := \begin{pmatrix} q_4 I_3 + [q_{1:3}\times] \\ -q_{1:3}^\top \end{pmatrix}$$

represent the cross product matrix and the matrix used later to introduce the quaternion multiplication and the quaternion dynamics. We denote by A the function which maps any attitude representation onto its rotation matrix allowing to transform a vector from one coordinate system into another. In particular, we use symbol A as a function of q with

$$A(q) := (q_4^2 - \|q_{1:3}\|^2) I_3 - 2q_4[q_{1:3}\times] + 2q_{1:3}q_{1:3}^\top \quad (1)$$

and also as a function of $r \in \mathbb{R}^3$ and $\phi \in (-\pi,\pi]$, i.e. $A(r,\phi) := A(q(r,\phi))$. $A^\top(q)$ marks the transpose of $A(q)$. We denote the quaternion multiplication as in Markley and Crassidis (2014) as $\otimes$ with $q, \bar{q} \in \mathbb{S}_3$ and

$$q \otimes \bar{q} := [\Xi(\bar{q})\ \ \bar{q}]\, q \quad (2)$$

$$A^\top(q \otimes \bar{q}) = A^\top(\bar{q}) A^\top(q)\,. \quad (3)$$

For every quaternion we denote its inverse with respect to the quaternion multiplication as q^{-1} with

$$q^{-1} := \begin{bmatrix} -q_{1:3}^\top & q_4 \end{bmatrix}^\top \quad (4)$$

$$A(q)^{-1} = A^\top(q)\,. \quad (5)$$

For two vectors $r, n \in \mathbb{S}_3$ with $e_i \times r \neq 0$ and an angle $\theta \in [0,\pi]$ the following equalities hold

$$r^\top A(e_i \times r, \theta)\, r = \cos\theta \quad (6)$$

$$A(n,\theta)\, n = n\,. \quad (7)$$

Finally, we write arccos to denote the inverse of the cosine function with the image $[0,\pi]$.

3. MOTIVATION: ATTITUDE RECONSTRUCTION USING TEMPERATURE MEASUREMENTS

Determining the position or the attitude of a spacecraft based on temperature data is a vivid subject of research Labibian et al. (2017, 2018); Khaniki and Karimian (2016); Gourabi et al. (2019). We focus on the problem of attitude estimation and illustrate the advantages of the transformation to be proposed by considering an earth orbiting spacecraft in the earth's shadow. We follow the notations and models used in Posielek (2019) and shall consider only infrared irradiation acting on the spacecraft. Then the temperature dynamics are governed by

$$\dot{T} = -\gamma \frac{r_\oplus^2}{\|r(t)\|^3} r^\top(t) A^\top(z)\, n - \delta T^4 \quad (8)$$

where $\gamma, \delta \in \mathbb{R}$ are parameters, $r_\oplus \in \mathbb{R}$ is the earth mean radius, $r(t) \in \mathbb{R}^3$ is the position of the spacecraft in earth-centered inertial frame (ECI), $n \in \mathbb{S}_2$ is the normal of the spacecraft surface in body frame and $T \in \mathbb{R}$ is the temperature of the surface. The variable z denotes any attitude representation of the body frame with respect to the ECI. This might be any of the common representations such as quaternions, Euler angles or rotation matrices. The matrix $A(z) \in \mathcal{SO}_3$ denotes the corresponding rotation matrix for the chosen attitude representation. In the following sections, we often omit the argument of $r(t)$ to allow a more compact notation.

On the one hand, it can be seen that the common attitude representations yield cumbersome expressions for $r^\top A^\top(z) n$ which incorporate in general all variables of z. Indeed, using standard quaternion variables (1) leads to

$$r^\top A^\top(z)\, n = r^\top((q_4^2 - \|q_{1:3}\|^2) I_3 - 2q_4[q_{1:3}\times] + 2q_{1:3}q_{1:3}^\top)^\top n\,. \quad (9)$$

On the other hand, expression $\frac{r^\top}{\|r\|} A^\top(z) n$ is equivalent to the cosine of the angle θ between r and n in ECI coordinates. Naturally, this raises the question if there exists an attitude representation $(\theta, \vartheta_1, \vartheta_2)$ which incorporates θ as one of its variables in order to facilitate the dynamics by using

$$r^\top A^\top(z)\, n = \|r\| \cos(\theta)$$

instead of (9). This representation is introduced in the next section.

4. MAIN CONTRIBUTION: IRRADIATION ANGLE TRANSFORMATION

We derive the new attitude representation via the two mappings $l_1^{r,n} : q \mapsto (\theta, \vartheta_1, \vartheta_2)$ and $l_2^{r,n} : (\theta, \vartheta_1, \vartheta_2) \mapsto q$. The superscripts shall suggest that these mappings change dependent on the vectors r and n. We define the mappings for normalised $r, n \in \mathbb{S}_2$. For not normalised $\bar{r}, \bar{n} \in \mathbb{R}^3$, the mapping $l_1^{\bar{r},\bar{n}}$ is defined by the mapping $l_1^{r,n}$ with $r = \frac{\bar{r}}{\|\bar{r}\|}$ and $n = \frac{\bar{n}}{\|\bar{n}\|}$. As motivated in the previous section, this representation shall incorporate θ as its first variable, which is the angle between two normalised vectors $r, n \in \mathbb{S}_2$ given in different frames that form a linear space of dimension two. Since the attitude space is of dimension three, it is natural to define two additional angles ϑ_1, ϑ_2 for allowing to describe every attitude by the three angles $(\theta, \vartheta_1, \vartheta_2)$. For a reference attitude representation we use quaternions. Thus, in order to show that the three angles $(\theta, \vartheta_1, \vartheta_2)$ are a valid attitude representation, we need to show that for every quaternion q there exists one unique angle triple $(\theta, \vartheta_1, \vartheta_2)$. Then the corresponding mapping $l_1^{r,n}$ is bijective and an inverse mapping $l_2^{r,n}$ exists. Additionally, these mappings need to fulfill certain continuity conditions to render the attitude representation usable.

We start by deriving $l_2^{r,n} : (\theta, \vartheta_1, \vartheta_2) \mapsto q$, i.e. the function that maps every angle set $(\theta, \vartheta_1, \vartheta_2)$ onto its quaternion $q \in \mathbb{S}_3$. These steps are illustrated in Figure 1 and define the function $l_2^{r,n}$. First the rotation axis $v := \frac{n \times r}{\|n \times r\|}$ and the angle $\phi := \arccos(n^\top r)$ are used to rotate n on r, i.e.

$$r = A^\top(v, \phi)\, n \quad (10)$$

with A as in (1). Then, a rotation is carried out to achieve the desired angle θ. Define the vector

$$n^{\theta,0} := A^\top\,(e_i \times r, \theta)\, A^\top(v, \phi)\, n \quad (11)$$

where e_i is the first unit vector if $r \notin \text{span}\{e_1\}$, otherwise it is the second unit vector. By definition and using (6), it is clear that $r^\top n^{\theta,0} = \cos(\theta)$. The vector n^{θ,ϑ_1} is defined by a rotation of ϑ_1 around the r axis, hence

$$n^{\theta,\vartheta_1} := A^\top(r, \vartheta_1)\, n^{\theta,0}\,. \quad (12)$$

Finally, rotation of ϑ_2 around n^{θ,ϑ_1} gives the desired attitude q, i.e.

$$q := q(v, \phi) \otimes q\,(e_i \times r, \theta) \otimes q(r, \vartheta_1) \otimes q(n^{\theta,\vartheta_1}, \vartheta_2)\,. \quad (13)$$

Definition 1. For two vectors $r, n \in \mathbb{S}_2$ the attitude defined by the angles θ, ϑ_1 and ϑ_2 is represented as a quaternion defined by the mapping $l_2^{r,n}$ which has the form

$$l_2^{r,n}(\theta,\vartheta_1,\vartheta_2) := q(v,\phi)\otimes q(e_i\times r,\theta)\otimes q(r,\vartheta_1)\otimes q(n^{\theta,\vartheta_1},\vartheta_2)$$

with the rotation axis and rotation angle $v := \frac{n\times r}{\|n\times r\|}$, $\phi := \arccos(n^\top r)$ and axis $n^{\theta,\vartheta_1} := A^\top(r,\vartheta_1)A^\top(e_i\times r,\theta)\,r$.

Note that we have so far deliberately not given the domain and the image of $l_2^{r,n}$. It is clear that the image is a subset of the quaternion space because the definition of $l_2^{r,n}(\theta,\vartheta_1,\vartheta_2)$ is merely a result of a quaternion product which retains the norm condition. The domain is obtained when deriving the inverse transformation. Recall that the original motivation for these three angles is that the first angle θ is the angle between r and n transformed into the same frame as r . This property and a helpful equation for the rotation axis n^{θ,ϑ_1} are shown in the following lemma.

Lemma 2. The quaternion $l_2^{r,n}(\theta,\vartheta_1,\vartheta_2)$ described by the rotation resulting from the three angles $\theta,\vartheta_1,\vartheta_2$ leads to an angle θ between r and transformed n, i.e.

$$r^\top A^\top(l_2^{r,n}(\theta,\vartheta_1,\vartheta_2))\,n = \cos(\theta) \tag{14}$$

where

$$A^\top(l_2^{r,n}(\theta,\vartheta_1,\vartheta_2))\,n = n^{\theta,\vartheta_1}.$$

Proof. We obtain with $q = l_2^{r,n}(\theta,\vartheta_1,\vartheta_2)$ that

$$\begin{aligned} A^\top(q)\,n &\overset{(3)}{=} A^\top(n^{\theta,\vartheta_1},\vartheta_2)A^\top(r,\vartheta_1)A^\top(e_i\times r,\theta)A^\top(v,\phi)\,n \\ &\overset{(10)}{=} A^\top(n^{\theta,\vartheta_1},\vartheta_2)A^\top(r,\vartheta_1)A^\top\,(e_i\times r,\theta)\;r \\ &\overset{(11)}{=} A^\top(n^{\theta,\vartheta_1},\vartheta_2)A^\top(r,\vartheta_1)\,n^{\theta,0} \\ &\overset{(12)}{=} A^\top(n^{\theta,\vartheta_1},\vartheta_2)\,n^{\theta,\vartheta_1} \\ &\overset{(7)}{=} n^{\theta,\vartheta_1}. \end{aligned}$$

By using the definition of n^{θ,ϑ_1} we obtain

$$\begin{aligned} r^\top A^\top(q)n = r^\top n^{\theta,\vartheta_1} &\overset{(12)}{=} r^\top A^\top(r,\vartheta_1)n^{\theta,0} \\ &\overset{(7)}{=} r^\top n^{\theta,0} \overset{(11)}{=} \cos(\theta) \end{aligned}$$

which finishes the proof.

It remains to show that for every attitude q, there is one unique angle set $(\theta,\vartheta_1,\vartheta_2)$. We do so by introducing $l_1^{r,n}$ and show that it is the inverse mapping of $l_2^{r,n}$.

For a quaternion q we define the angle θ along (14) as

$$\theta := \arccos(r^\top A^\top(q)\,n).$$

Consequently, the domain of θ is chosen to be $[0,\pi]$. It is clear that for $n^q := A^\top(q)\,n$ the equations

$$r^\top n^{\theta,0} = r^\top n^q \tag{15a}$$

$$n^{\theta,0\top} n^{\theta,0} = n^{q\top} n^q \tag{15b}$$

hold. Now we calculate the angle ϑ_1 rotating $n^{\theta,0}$ on n^q around the rotation axis r, i.e. $n^q = A^\top(r,\vartheta_1)\,n^{\theta,0}$. Note that this equation is equivalent to (12). Since (15) holds, the angle ϑ_1 can be obtained using Rodrigues' formula

$$\vartheta_1 := \operatorname{atan2}\left((r\times n^{\theta,0})^\top n^q, n^{\theta,0\top} n^q - (r^\top n^{\theta,0})^2\right).$$

Then ϑ_1 is obtained using the atan2 function which maps the two arguments to an angle in $(-\pi,\pi]$ and defines the domain of ϑ_1. Note that for $\theta = 0$ and $\theta = \pi$, both arguments are identical to zero which is where atan2 is undefined. This singularity will be discussed in detail in the next section. For our application, we define atan2 to be identical zero into order to allow a definition on complete $[0,\pi]$. For calculating ϑ_2, we use (13) and define

$$\bar{q} := q^{-1}(r,\vartheta_1)\otimes q^{-1}\,(e_i\times r,\theta)\otimes q^{-1}(v,\phi)\otimes q\,.$$

It can be seen that this quaternion describes a rotation around the $\pm n^q$ axis since

$$\begin{aligned} A^\top(\bar{q})\,n^q &\overset{(3)}{=} A^\top(q)A^\top(q^{-1}(v,\phi))A^\top\left(q^{-1}(e_i\times r,\theta)\right)n^{\theta,0} \\ &\overset{(11)}{=} A^\top(q)A^\top(q^{-1}(v,\phi))\,r \\ &\overset{(10)}{=} A^\top(q)\,n \\ &= n^q\,. \end{aligned}$$

This makes the first three entries of the quaternion describe the n^q axis, i.e. $\frac{\bar{q}_{1:3}}{\|\bar{q}_{1:3}\|} = \pm n^q$ or the unit quaternion and $\bar{q}_{1:3} = 0$. The angle ϑ_2 results from

$$\vartheta_2 := \begin{cases} +2\arccos(\bar{q}_4), & \text{if } \frac{\bar{q}_{1:3}}{\|\bar{q}_{1:3}\|} \in \{0, n^q\} \\ -2\arccos(\bar{q}_4), & \text{if } \frac{\bar{q}_{1:3}}{\|\bar{q}_{1:3}\|} = -n^q \end{cases}.$$

The result of this definition is an angle $\vartheta_2 \in (-2\pi, 2\pi]$ which is chosen to be the domain of ϑ_2. These steps define the mapping $l_1^{r,n}$.

Definition 3. For two vectors $r, n \in \mathbb{S}_2$ the attitude given by q is transformed into the three angles using the mapping $l_1^{r,n} : \mathbb{S}_3 \to [0,\pi]\times(-\pi,\pi]\times(-2\pi,2\pi]$ where $l_1^{r,n}(q) = (\theta,\vartheta_1,\vartheta_2)$ is defined by

$$\begin{aligned} \theta &:= \arccos(r^\top A^\top(q)n) \\ \vartheta_1 &:= \operatorname{atan2}((r\times n^{\theta,0})^\top n^q, n^{\theta,0\top} n^q - (r^\top n^{\theta,0})^2) \\ \vartheta_2 &:= \begin{cases} +2\arccos(\bar{q}_4), & \text{if } \frac{\bar{q}_{1:3}}{\|\bar{q}_{1:3}\|} = n^q \\ -2\arccos(\bar{q}_4), & \text{otherwise} \end{cases} \end{aligned}$$

with the rotation axes $n^q := A(q)^\top n$, $n^{\theta,0} := A^\top\,(e_i\times r,\theta)\,r$ and the quaternion describing the rotation around the n^q axis as $\bar{q} := q^{-1}(r,\vartheta_1)\otimes q^{-1}\,(e_i\times r,\theta)\otimes q^{-1}(v,\phi)\otimes q$ with rotation axis and angle $v := \frac{n\times r}{\|n\times r\|}, \phi := \arccos(n^\top r)$.

This allows to state the main theorem of the section.

Theorem 4. Consider the mappings

$$\begin{aligned} l_1^{r,n} &: \mathbb{S}_3 \to [0,\pi]\times(-\pi,\pi]\times(-2\pi,2\pi] \\ &\quad q \mapsto l_1^{r,n}(q) \end{aligned}$$

and

$$\begin{aligned} l_2^{r,n} &: [0,\pi]\times(-\pi,\pi]\times(-2\pi,2\pi] \to \mathbb{S}_3 \\ &\quad (\theta,\vartheta_1,\vartheta_2) \mapsto l_2^{r,n}(\theta,\vartheta_1,\vartheta_2) \end{aligned}$$

as defined in Definitions 3 and 1. Then $l_1^{r,n}$ is the inverse of $l_2^{r,n}$ for $\theta \notin \{0,\pi\}$ and vice versa. The first argument of $l_2^{r,n}$ describes the cosine of the angle between r and transformed n, i.e.

$$r^\top A^\top(l_2^{r,n}(\theta,\vartheta_1,\vartheta_2))\,n = \cos(\theta)\,. \tag{16}$$

Proof. By the transparent way of defining the functions $l_1^{r,n}$ and $l_2^{r,n}$ it is clear that they are corresponding inverses for θ not identical to zero or π.

5. PROPERTIES OF THE MAPPINGS

In this section we discuss some of the properties of the attitude representation and their mappings.

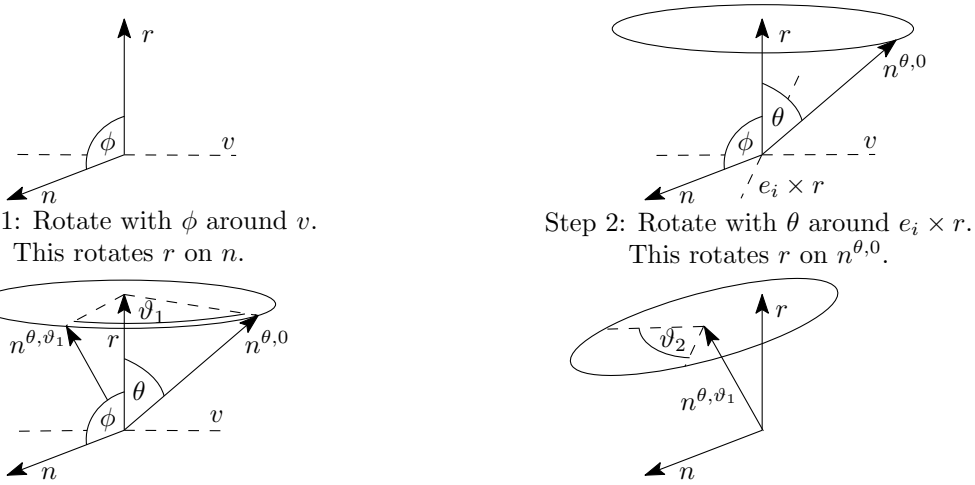

Fig. 1. Illustration of mapping $l_2^{r,n}$ divided into four steps to map the three angles $(\theta, \vartheta_1, \vartheta_2)$ to a quaternion. Each step changes the current attitude using a single rotation. The starting attitude for Step 1 corresponds to the identity quaternion. In all steps the rotations are illustrated showing the resulting normal vector after the rotation. The normal vector in Step 4 remains unchanged after the rotation. The explicit depiction of the current attitude using a coordinate system is omitted to make the illustration clearer.

5.1 Underlying System of Equations

First, we summarise the three equations that are the basis for the two mappings.

Remark 5. The attitude representation is based on the three equations

$$r^\top A^\top(q)\, n = \cos(\theta)$$
$$A^\top(q)\, n = A^\top(r, \vartheta_1) A^\top(e_i \times r, \theta)\, r$$
$$q = q(v, \phi) \otimes q(e_i \times r, \theta) \otimes q(r, \vartheta_1) \otimes q(n^{\theta,\vartheta_1}, \vartheta_2)$$

with the rotation axis and angles as in Definition 3.

5.2 Singularities

It is well known that the attitude space is of dimension three. Attitude representations that use only three variables come with the disadvantage of having singularities while representations with more than three variable have additional constraints. The proposed attitude representation has only three variables, but a singularity occurs when the angle θ is identical to zero or π. In this case, ϑ_1 and ϑ_2 have the same rotation axis, namely r.

Proposition 6. For $\theta = 0$ and $\theta = \pi$ it holds

$$l_2^{r,n}(0, \vartheta_1 - \vartheta_2, \vartheta_2) = l_2^{r,n}(0, \vartheta_1, 0) \tag{17}$$
$$l_2^{r,n}(\pi, \vartheta_1 + \vartheta_2, \vartheta_2) = l_2^{r,n}(\pi, \vartheta_1, 0)\,. \tag{18}$$

For $l_1^{r,n}$, we obtain

$$l_1^{r,n}(l_2^{r,n}(0, \vartheta_1 - \vartheta_2, \vartheta_2)) = (0, 0, \vartheta_1)$$
$$l_1^{r,n}(l_2^{r,n}(\pi, \vartheta_1 + \vartheta_2, \vartheta_2)) = (\pi, 0, -\vartheta_1)\,.$$

Proof. For $\theta = 0$ the rotation axis n^{θ,ϑ_1} is identical to r. Thus, we obtain

$$\begin{aligned} l_2^{r,n}(0, \vartheta_1 - \vartheta_2, \vartheta_2) &\overset{(13)}{=} q(v, \phi) \otimes q(r, \vartheta_1 - \vartheta_2) \otimes q(r, \vartheta_2) \\ &= q(v, \phi) \otimes q(r, \vartheta_1) \otimes q(r, 0) \\ &= l_2^{r,n}(0, \vartheta_1, 0)\,. \end{aligned}$$

The second and third equation follow from the fact that $n^q = n^{\theta,0} = r$ and that we have defined atan2 to be zero for two arguments identical to zero. The same calculations can be made for $\theta = \pi$ and $-r$.

Note that these singularities make $l_1^{r,n}$ not surjective and $l_2^{r,n}$ not injective which is why invertibility was only stated for the sets without these singularities in the previous section.

5.3 Augmentation of the Mappings

The mappings $l_1^{r,n}$ and $l_2^{r,n}$ can be augmented easily for $r \in \operatorname{span}\{n\}$. In this case we define any vector which is perpendicular to r as $v = e_i \times r$ with any unity vector $e_i \notin \operatorname{span}\{n\}$ and proceed as usual.

For reconstruction purposes it might be beneficial to expand the domain of $l_2^{r,n}$. This can avoid discontinuities in the attitude representation. In this case, the periodicity of the functions allows to distinguish equivalent attitude representations.

Proposition 7. If we extend the domain of l_2 to $[0,\pi] \times \mathbb{R} \times \mathbb{R}$, we obtain the negative quaternions as

$$l_2^{r,n}(\theta, \vartheta_1, \vartheta_2) = -l_2^{r,n}(\theta, \vartheta_1 - 2\pi, \vartheta_2) \tag{19a}$$
$$l_2^{r,n}(\theta, \vartheta_1, \vartheta_2) = -l_2^{r,n}(\theta, \vartheta_1, \vartheta_2 - 2\pi) \tag{19b}$$

while for a shift in ϑ_1 and ϑ_2 the identities are

$$l_2^{r,n}(\theta, \vartheta_1, \vartheta_2) = l_2^{r,n}(\theta, \vartheta_1 - 2\pi, \vartheta_2 - 2\pi) \tag{20a}$$
$$l_2^{r,n}(\theta, \vartheta_1, \vartheta_2) = l_2^{r,n}(\theta, \vartheta_1, \vartheta_2 - 4\pi). \tag{20b}$$

Proof. With the identities $q(n^{\theta,\vartheta_1}, \vartheta_2) = -q(n^{\theta,\vartheta_1}, \vartheta_2 - 2\pi)$, $q(r, \vartheta_1) = -q(r, \vartheta_1 - 2\pi)$ and (13) we obtain (19a) and (19b). Then, by using the identities

$$-l_2^{r,n}(\theta, \vartheta_1, \vartheta_2 - 2\pi) = l_2^{r,n}(\theta, \vartheta_1 - 2\pi, \vartheta_2 - 2\pi)$$
$$-l_2^{r,n}(\theta, \vartheta_1, \vartheta_2 - 2\pi) = l_2^{r,n}(\theta, \vartheta_1, \vartheta_2 - 4\pi)$$

we obtain (20a) and (20b).

On the other hand, the domain of $l_2^{r,n}$ for ϑ_2 is chosen in the definition of the mappings to have the form $(-2\pi, 2\pi]$. This is done to achieve the bijectivity on the quaternion space. However, restricting this domain to $(-\pi, \pi]$ is sufficient to represent all orientations because $\pm q$ describe the same orientation and (19b) allows the transformation into the reduced domain.

5.4 Smoothness

The smoothness properties of $l_1^{r,n}$ and $l_2^{r,n}$ are inherited from their defining functions. The function $l_1^{r,n}$ is not continuous on its complete domain. The points of discontinuity are defined by the following subset

$$\mathcal{Q}^{dc} = \Big\{ q \in \mathbb{S}_3 \mid \exists\, \theta \in [0,\pi],\, \vartheta_1 \in (-\pi,\pi],\, \vartheta_2 \in (-2\pi, 2\pi] : \\ q \in \{ l_1^{r,n}(\bar{\theta}, \vartheta_1, \vartheta_2), l_1^{r,n}(\theta, \pi, \vartheta_2), l_1^{r,n}(\theta, \vartheta_1, 2\pi) \} \\ \text{for } \bar{\theta} \in \{0, \pi\} \Big\}.$$

We summarise the smoothness properties as follows.

Proposition 8. The function $l_2^{r,n}$ is smooth everywhere while $l_1^{r,n}$ is only smooth on $\mathbb{S}_3 \setminus \mathcal{Q}^{\mathrm{dc}}$.

Proof. The definition of $l_2^{r,n}$ is the result of a composition of smooth functions. Thus $l_2^{r,n}$ is smooth Amann and Escher (2005). Also the smoothness of $l_1^{r,n}$ is the result of the individually composed functions. Possible discontinuities are induced by atan2 and the piecewise definition of ϑ_2. Indeed the discontinuities due to atan2 at $\theta \in \{0, 2\pi\}$ can be shown using two simple sequences and the property (18). The same can be done for the singularities at $\vartheta_1 = \pi$ and $\vartheta_2 = 2\pi$ and the properties (20).

5.5 Dynamics

Considering the three angles as functions of time we may define their dynamics by using either of the functions $l_1^{r,n}$, $l_2^{r,n}$ and the quaternion dynamics from Markley and Crassidis (2014), i.e.

$$\dot{q} = \frac{1}{2} \Xi^\top(q)\, \omega \,. \tag{21}$$

By differentiating $(\theta, \vartheta_1, \vartheta_2) = l_1^{r,n}(q)$ with respect to time we obtain

$$\begin{bmatrix} \dot{\theta} \\ \dot{\vartheta}_1 \\ \dot{\vartheta}_2 \end{bmatrix} = \frac{\partial l_1^{r,n}(l_2^{r,n}(\theta,\vartheta_1,\vartheta_2))}{\partial q} \frac{1}{2} \Xi^\top(l_2^{r,n}(\theta,\vartheta_1,\vartheta_2))\, \omega \\ + \frac{\partial l_1^{r,n}(l_2^{r,n}(\theta,\vartheta_1,\vartheta_2))}{\partial r}\, \dot{r} \,,$$

defined for all $(\theta, \vartheta_1, \vartheta_2)$ with $l_2^{r,n}(\theta, \vartheta_1, \vartheta_2) \in \mathbb{S}_3 \setminus \mathcal{Q}^{\mathrm{dc}}$. Note that we also assume r to be time-varying in view of the motivating system. These dynamics have the disadvantage that they are not defined for ϑ_1 or ϑ_2 at their boundaries. This can be avoided using $l_2^{r,n}$ instead.

Proposition 9. For the functions of time $(r, \theta, \vartheta_1, \vartheta_2)$ with $q = l_2^{r,n}(\theta, \vartheta_1, \vartheta_2)$ and q obeying the quaternion dynamics (21), the dynamics of $(\theta, \vartheta_1, \vartheta_2)$ have the form

$$\begin{bmatrix} \dot{\theta} \\ \dot{\vartheta}_1 \\ \dot{\vartheta}_2 \end{bmatrix} = \frac{\partial l_2^{r,n}(\theta,\vartheta_1,\vartheta_2)}{\partial(\theta,\vartheta_1,\vartheta_2)}^{+} \frac{1}{2} \Xi^\top(l_2^{r,n}(\theta,\vartheta_1,\vartheta_2))\, \omega \\ - \frac{\partial l_2^{r,n}(\theta,\vartheta_1,\vartheta_2)}{\partial(\theta,\vartheta_1,\vartheta_2)}^{+} \frac{l_2^{r,n}(\theta,\vartheta_1,\vartheta_2)}{\partial r}\, \dot{r} \tag{22}$$

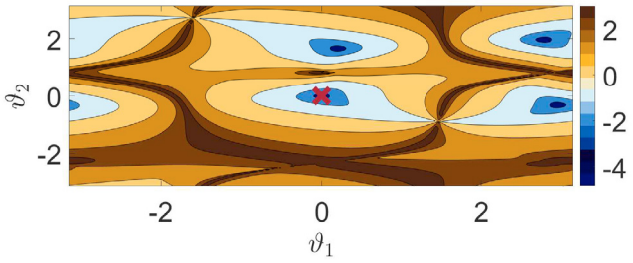

Fig. 2. Contour map showing the image of the cost function from the optimisation problem (29).

where $A^+ := (A^\top A)^{-1} A^\top$ denotes the Moore–Penrose inverse of a matrix A with full rank. The domain of the dynamics is $(0, \pi) \times (-\pi, \pi] \times (-2\pi, 2\pi]$.

Proof. Differentiating $q = l_2^{r,n}(\theta, \vartheta_1, \vartheta_2)$ yields

$$\dot{q} = \frac{\partial l_2^{r,n}(\theta,\vartheta_1,\vartheta_2)}{\partial(\theta,\vartheta_1,\vartheta_2)} \begin{bmatrix} \dot{\theta} \\ \dot{\vartheta}_1 \\ \dot{\vartheta}_2 \end{bmatrix} + \frac{l_2^{r,n}(\theta,\vartheta_1,\vartheta_2)}{\partial r}\, \dot{r} \,. \tag{23}$$

With $\partial l_2^{r,n} := \frac{\partial l_2^{r,n}(\theta,\vartheta_1,\vartheta_2)}{\partial(\theta,\vartheta_1,\vartheta_2)}$ the product of the jacobians is

$$\partial l_2^{r,n\top} \partial l_2^{r,n} = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.25 & 0.25\cos\theta \\ 0 & 0.25\cos\theta & 0.25 \end{pmatrix} . \tag{24}$$

This was validated in a numerical fashion. The expression for $\partial l_2^{r,n}$ itself is rather complicated and computed using computer algebra. The matrix (24) has full rank for $\theta \in (0, \pi)$. It does not have full rank at the singularities $\theta \in \{0, \pi\}$. Consequently, the Moore–Penrose inverse $\frac{\partial l_2^{r,n}(\theta,\vartheta_1,\vartheta_2)}{\partial(\theta,\vartheta_1,\vartheta_2)}^{+}$ exists on $(0, \pi) \times (-\pi, \pi] \times (-2\pi, 2\pi]$. Multiplying $\frac{\partial l_2^{r,n}(\theta,\vartheta_1,\vartheta_2)}{\partial(\theta,\vartheta_1,\vartheta_2)}^{+}$ from the right to (23) and inserting the quaternion dynamics (21) leads to (22).

6. APPLICATION: ATTITUDE RECONSTRUCTION USING TEMPERATURE MEASUREMENTS

Augmenting the motivating example with the attitude dynamics we obtain

$$\dot{T} = -\gamma \frac{r_\oplus^2}{\|r(t)\|^3} r^\top(t) A^\top(q)\, n - \delta T^4 \tag{25a}$$

$$\dot{q} = \frac{1}{2} \Xi^\top(q)\, \omega \tag{25b}$$

$$\dot{\omega} = J^{-1}(-\omega \times J\omega) \tag{25c}$$

where $J \in \mathbb{R}^{3\times 3}$ denotes the inertia matrix, $q \in \mathbb{S}_3$ the attitude described by quaternions and $\omega \in \mathbb{R}^3$ the angular velocity. Note that we start with the quaternion dynamics and use the proposed attitude representation later in the analysis.

An estimation of the attitude shall be achieved using only the temperature measurement and its derivatives under the assumption that the angular velocity ω and time t are known via e.g. other measurements or estimations. We denote by $\mathcal{L}_f^{(i)} h(x)$ the i-th Lie derivative of h with respect to f and write $x = [T, q^\top, \omega^\top, t]^\top$, $h(x) = T$ and $f(x)$ for the right hand side of (25). This allows to define the derivatives of y as $L_2(q) := \ddot{y} = \mathcal{L}_f^{(2)} h(x)$ and

$L_3(q) := \dddot{y} = \mathcal{L}_f^{(3)} h(x)$. Note that we have defined the derivatives of y as functions of only the attitude q, as all the other states T, ω and t are already assumed to be available and do not need to be estimated from these derivatives. Then the attitude can be reconstructed solving the non-linear system of equations

$$\dot{T} = -\gamma \frac{r_\oplus^2}{\|r(t)\|^3} r^\top(t) A^\top(q) n - \delta T^4 \tag{26a}$$
$$\ddot{T} = L_2(q) \tag{26b}$$
$$\dddot{T} = L_3(q) \tag{26c}$$
$$q^\top q = 0 \tag{26d}$$

for q. Note that the quaternion constraint must be added to obtain attitudes as solutions of the system. It is not clear whether the solution of the system is unique and if an analytical expression for it exists. For the attitude representations in quaternions all four equations contain all four variables and no straightforward simplification can be made to change that. Thus, if the solution is not unique, all solutions solving the system are generally described by a non-trivial function of all variables of q. This can be changed by using the proposed attitude representation. For $q = l_2^{-r,n}(\theta, \vartheta_1, \vartheta_2)$ we see that (26) simplifies to

$$\dot{T} = \gamma \frac{r_\oplus^2}{\|r(t)\|^2} \cos(\theta) - \delta T^4 \tag{27a}$$
$$\ddot{T} = L_2(l_2^{r,n}(\theta, \vartheta_1, \vartheta_2)) \tag{27b}$$
$$\dddot{T} = L_3(l_2^{r,n}(\theta, \vartheta_1, \vartheta_2)) \tag{27c}$$

which allows the calculation of θ via

$$\theta = \arccos\left(\frac{1}{\gamma} \frac{\|r\|^2}{r_\oplus^2} (\dot{T} + \delta T^4) \right). \tag{28}$$

Thus, the angle θ can be uniquely calculated and we have reduced the problem by two dimensions. The remaining two angles ϑ_1, ϑ_2 to describe the attitude may now be obtained solving the remaining two non-linear equations (27b)-(27c). Note that a benefit of a two dimensional problem is that it is possible to visualise the state space. We formulate (27b)-(27c) into the optimisation problem

$$\min_{\vartheta_1, \vartheta_2} \quad \left\| \begin{bmatrix} \frac{\partial H_{3:4}(\vartheta_1, \vartheta_2)}{\partial(\vartheta_1, \vartheta_2)} \end{bmatrix}^{-1} \begin{bmatrix} H_3(\vartheta_1, \vartheta_2) - \ddot{T} \\ H_4(\vartheta_1, \vartheta_2) - \dddot{T} \end{bmatrix} \right\|_2 \tag{29a}$$
$$\text{s.t.} \quad [\vartheta_1, \vartheta_2] \in (-\pi, \pi] \times (-\pi, \pi]. \tag{29b}$$

with the definitions $H_3(\vartheta_1, \vartheta_2) := L_2(l_2^{r,n}(\theta, \vartheta_1, \vartheta_2))$ and $H_4(\vartheta_1, \vartheta_2) := L_3(l_2^{r,n}(\theta, \vartheta_1, \vartheta_2))$. Figure 2 show the image of the cost function for $\ddot{T}$ and $\dddot{T}$ resulting from the sample parameter configuration as seen in Table 1. The red cross depicts the real attitude. We can see that the system has four solutions for this parameter configuration. Which of the solutions is found depends on the algorithm and the chosen initial state and is discussed in another work. The proposed attitude representation allows to reduce the order of the system and the number of variables that need to be numerically reconstructed.

7. CONCLUSIONS

We have proposed an attitude representation incorporating the angle between two vectors as its first variable. We have given a detailed derivation and have derived a bijective mapping to the quaternion space. Like every three dimensional attitude representation, the proposed one has singularities. Further, we have given the properties necessary to augment the domain to allow continuous angle estimations. Finally, the advantages of the proposed attitude representation were presented for the application of attitude reconstruction using solely a single temperature measurement. The introduced representation yields an order reduction of the non-linear system which reduces the numerical effort for the reconstruction considerably and also generates a better understanding of the solutions.

Table 1. Parameters and Constants

Parameter	Value	Parameter	Value
γ	0.0673	δ	1.6×10^{-11}
J	diag[5.4 5.4 0.9]	r_1	3.1×10^6
r_2	-1.1×10^6	r_3	-6.1×10^6
$r_\oplus$	6371000	n	e_3
T	292	θ	$\frac{\pi}{2}$
ϑ_1	0	ϑ_2	0
ω_1	0.0058	ω_2	0.0058
ω_3	0.0058		

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