Journal of Statistical Planning and Inference 221 (2022) 81–89

Contents lists available at ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Constructions for regular-graph semi-Latin rectangles with block size two

N.P. Uto *, R.A. Bailey

School of Mathematics and Statistics, Mathematical Institute, University of St Andrews, St Andrews, Fife KY16 9SS, UK

ARTICLE INFO

Article history:
Received 18 December 2021
Accepted 25 February 2022
Available online 8 March 2022

MSC:
primary 62K05
62K10
secondary 05B05
05B15

Keywords:
Regular-graph semi-Latin rectangle
Starter
Bi-starter
Balanced tournament design
Trojan square
Undirected terrace

ABSTRACT

Semi-Latin rectangles are generalizations of Latin squares and semi-Latin squares. Although they are called rectangles, the number of rows and the number of columns are not necessarily distinct. There are k treatments in each cell (row–column intersection): these constitute a block. Each treatment of the design appears a definite number of times in each row and also a definite number of times in each column (these parameters also being not necessarily distinct). When $k = 2$, the design is said to have block size two. Regular-graph semi-Latin rectangles have the additional property that the treatment concurrences between any two pairs of distinct treatments differ by at most one. Constructions for semi-Latin rectangles of this class with $k = 2$ which have v treatments, $v/2$ rows and v columns, where v is even, are given in Bailey and Monod (2001). These give the smallest designs when v is even. Here we give constructions for smallest designs with $k = 2$ when v is odd. These are regular-graph semi-Latin rectangles where the numbers of rows, columns and treatments are identical. Then we extend the smallest designs in each case to obtain larger designs.

1. Introduction

1.1. Semi-latin rectangles

The following definition slightly generalizes the one given by Bailey and Monod (2001), because it does not insist that $h < p$.

Definition 1. Let $\mathcal{V}$ denote a set of v treatments. Let Δ be an $h \times p$ rectangular (or in special cases, square) array consisting of k-subsets of $\mathcal{V}$ (blocks), where $k < v$. Suppose that no block contains any treatment more than once. Suppose further that kp and kh are each divisible by v and that, for all $t \in \mathcal{V}$, t occurs n_r times per row and n_c times per column, where n_r and n_c may (or may not) be equal. Then Δ is said to be an $(h \times p)/k$ *semi-Latin rectangle* (SLR) for v treatments.

Row–column designs such as Latin squares and semi-Latin squares can be seen to be special cases of the SLR with $n_r = n_c = 1$: see Uto and Bailey (2020). Discussions on semi-Latin squares can be found in papers such as Preece and Freeman (1983), Bailey (1988, 1992), Bailey and Chigbu (1997), Bailey and Royle (1997), Bedford and Whitaker (2001), Soicher (2012), Soicher (2013) and Bailey and Soicher (2021). SLRs are useful in many experimental settings, including plant disease experiments, consumer testing experiments and food sensory experiments: see Bailey and Monod (2001).

In the remainder of this paper we assume that $k > 1$, which excludes Latin squares, and that $n_r n_c > 1$, which excludes semi-Latin squares.

* Corresponding author.
E-mail addresses: npu@st-andrews.ac.uk (N.P. Uto), rab24@st-andrews.ac.uk (R.A. Bailey).

https://doi.org/10.1016/j.jspi.2022.02.007

Definition 2. Let Λ be a semi-Latin rectangle. Denote by $\Gamma(\Lambda)$ the quotient block design of Λ, which is the block design obtained by ignoring the rows and columns of Λ. If i and i' are distinct treatments, their *concurrence* $\lambda_{ii'}$ is the number of blocks which contain both i and i'. Let ℓ and ℓ' be another pair of distinct treatments. If $|\lambda_{\ell\ell'} - \lambda_{ii'}| \in \{0, 1\}$ for all such treatment pairs then Λ is said to be a *regular-graph semi-Latin rectangle* (RGSLR).

Thus Λ is a RGSLR precisely when $\Gamma(\Lambda)$ is a regular-graph design (RGD) in the sense defined by John and Mitchell (1977).

RGDs are equireplicate binary incomplete-block designs that are close to balanced incomplete-block designs (BIBDs). BIBDs are special cases of RGDs: see Kreher et al. (1996). When BIBDs exist, they are optimal over all incomplete-block designs of their sizes with respect to a range of criteria including the A-, D- and E-criteria: see Shah and Sinha (1989).

However, BIBDs may not exist for certain values of the parameters. In this case, a search is usually restricted to RGDs, as these have been conjectured by John and Mitchell (1977) to contain a D-optimal (or A-optimal or E-optimal) incomplete-block design. This conjecture was confirmed by Cheng (1992), if the number of blocks is sufficiently large.

1.2. Construction tools

Here we define some combinatorial concepts which we will use to construct RGSLRs with block size two.

If n is any positive integer, we regard $\mathbb{Z}_n$, the set of integers modulo n, as the set $\{1, \ldots, n\}$.

The following definition is given by Bailey and Monod (2001).

Definition 3. Let $\{S_i\}_{i=1}^m$ constitute a partition of $\mathbb{Z}_{2m}$ into subsets of size two. This means that $S_i = \{x_i, y_i\} \subseteq \mathbb{Z}_{2m}$ and $|S_i| = 2$ for $i = 1, \ldots, m$. Moreover, $S_i \cap S_j = \emptyset$ if $i \neq j$. Suppose that

$$\pm(y_i - x_i) = \begin{cases} \pm i & \text{if } i < m, \\ m \text{ twice} & \text{if } i = m. \end{cases}$$

Then $\{S_i\}_{i=1}^m$ is called a *starter* for the cyclic group formed by $\mathbb{Z}_{2m}$ under addition.

Note that $-m = m$ modulo $2m$, so the multiplicity of m in the multiset of differences is 2. Similarly, the multiplicity of each element of the set $\mathbb{Z}_{2m}\backslash\{m, 2m\}$ is 1.

The foregoing definition is a modification of the standard combinatorial definition of *starter* for abelian groups of odd order: see Dinitz (1996). So we need a new name when we modify the foregoing definition to give something suitable for cyclic groups of odd order in our application.

Definition 4. Suppose that v is odd. Let $\{S_i\}_{i=1}^{2v}$ be a set of subsets of $\mathbb{Z}_v$ of size two, with the property that each element of $\mathbb{Z}_v$ occurs in two of the sets. Let $S_i = \{x_i, y_i\}$ for $i = 1, \ldots, 2v$, and let $\delta = (v-1)/2$. Then $\{S_i\}_{i=1}^{2v}$ is called a *bi-starter* for $\mathbb{Z}_v$ if there are precisely three values of i for which $y_i - x_i = \pm\delta$ and, for $1 \leq j \leq \delta - 1$, precisely two values of i for which $y_i - x_i = \pm j$.

Definition 5. Let $V = \{1, \ldots, 2m\}$ denote a set of teams available for a league tournament, which is to consist of $2m-1$ rounds, where each round is to be played on m grounds. The league schedule forms an $m \times (2m-1)$ array whose rows represent the grounds and whose columns represent the rounds. The cells are constituted by the $m(2m-1)$ distinct pairs of teams from V such that each pair of teams plays once overall while each team plays once in each round and at most twice on each ground. Then the league schedule is said to constitute a *balanced tournament design* for the $2m$ teams: see Anderson (1997, Chapter 10).

The following definition was introduced by Darby and Gilbert (1958). See also Edmondson (1998).

Definition 6. Let $\{\Xi_i\}_{i=1}^k$ be a set of k mutually orthogonal Latin squares of order n. For $i = 1, \ldots, k$, let X_i denote the set of symbols in Ξ_i, where $X_i \cap X_{i'} = \emptyset$ whenever $i \neq i'$. If $\Xi_1, \ldots, \Xi_k$ are superimposed and the superimposition is regarded as having nk treatments, rather than k treatment factors with n levels each, then the resulting design is said to be an $(n \times n)/k$ *Trojan square*.

Thus Trojan squares form a sub-class of semi-Latin squares.

1.3. Looking back and forward

Bailey and Monod (2001) gave constructions for efficient SLRs for $2m$ treatments with block size two, where there are m rows and $2m$ columns, for values of m between 2 and 10 inclusive. For those values of m, except for $m = 2$, their constructions, obtained using starters and balanced tournament designs, produce RGSLRs.

SLRs whose quotient block designs are BIBDs are named *balanced semi-Latin rectangles* (BSLRs) by Uto and Bailey (2020). That paper examined the combinatorial properties and necessary conditions for a BSLR to exist, and gave algorithms for constructing these designs when the block size is two.

Table 1
Starters in $\mathbb{Z}_v$ for some even values of v.

v	m	starter								
8	4	{4, 5}	{1, 7}	{3, 8}	{2, 6}					
10	5	{1, 10}	{6, 8}	{2, 5}	{3, 7}	{4, 9}				
16	8	{1, 16}	{8, 10}	{2, 5}	{9, 13}	{4, 15}	{6, 12}	{7, 14}	{3, 11}	
18	9	{3, 4}	{11, 13}	{2, 17}	{1, 15}	{5, 10}	{6, 12}	{7, 14}	{8, 16}	{9, 18}

When a BSLR exists, it is optimal over its class with respect to a range of criteria such as the *A*-, *D*- and *E*-criteria. As discussed in Section 1.1, when a BSLR does not exist, we follow John and Mitchell (1977) in assuming that an optimal design will be found among RGDs. We observe that different RGSLRs may have different values of any given optimality criterion.

Section 3 of the present paper gives constructions for RGSLRs with block size two when the number of treatments v is odd. These designs have the same numbers of rows, columns and treatments. They are the smallest designs for odd v. Section 2 briefly recaps the constructions of Bailey and Monod (2001), which give the smallest RGSLRs for even v. Section 4 gives constructions for larger RGSLRs whose size is an integer multiple of that of the smallest design. For even values of v, Section 5 does this for an odd multiple of 1/2. Finally, Section 6 shows how to obtain a larger RGSLR by extending a smaller RGSLR by a BSLR.

2. Smallest designs for an even number of treatments

2.1. Preliminaries

Some constructions for efficient SLRs with $v = 2m$ treatments in m rows, $2m$ columns and block size two where $2 \leq m \leq 10$ are given by Bailey and Monod (2001). These designs are precisely $(m \times 2m)/2$ SLRs for $2m$ treatments and, in particular, for values of $m \neq 2$, the construction produces RGSLRs. These are the smallest designs for even v with $n_r n_c > 1$.

For situations where $m \equiv 0$ or 1 (mod 4), the construction is obtained using the concept of starter in the cyclic group $\mathbb{Z}_{2m}$. The sets that make the starter constitute the cells (blocks) in the initial column of the design, and the initial block for each row is developed cyclically to generate the remaining $2m - 1$ blocks in that row.

Bailey and Monod (2001) proved that a starter for $\mathbb{Z}_{2m}$ exists if and only if $m \equiv 0$ or 1 (mod 4), and gave a table of starters for small values of $2m$ satisfying this condition. Table 1 shows this, replacing the treatment called 0 by one called $2m$. These starters are needed for some of the constructions in this paper.

For those situations where $m \not\equiv 2$ (mod 3), a different construction uses a balanced tournament design for $2m$ teams. This is obtained by swapping the positions of certain pairs of teams in a cyclic tournament schedule. Then an extra column is added so that each team occurs twice in each row.

We summarize the two constructions in Sections 2.2 and 2.3 since our constructions for larger RGSLRs for even v involve extending these constructions.

2.2. Construction using a starter

Step 1: Label the treatments $1, 2, \ldots, 2m$ and form sets that constitute a starter in the cyclic group $\mathbb{Z}_{2m}$ by partitioning the set of treatments into m subsets of size two, where the differences (modulo $2m$) between the elements of the starter sets are $\pm 1, \pm 2, \ldots, \pm m$. Denote by $\{x_i, y_i\}$ the starter set associated with the differences $\pm i$, where $i = 1, 2, \ldots, m$.
Step 2: Create an $m \times 2m$ array and label its rows $i = 1, 2, \ldots, m$ and its columns $j = 1, 2, \ldots, 2m$.
Step 3: For $i = 1, 2, \ldots, m$, put $\{x_i, y_i\}$, obtained in Step 1, in the cell in position $(i, 1)$ of the array.
Step 4: For $i = 1, 2, \ldots, m$, develop the block in position $(i, 1)$ cyclically, via successive addition of 1 (mod $2m$), thereby generating the block in position (i, j), for all $j = 2, 3, \ldots, 2m$.

2.3. Construction using a balanced tournament design

Step 1: Label the treatments $1, 2, \ldots, w, \infty$, where $w = 2m - 1$. Identify $\{1, \ldots, w\}$ with $\mathbb{Z}_w$.
Step 2: Create an $m \times 2m$ array and label its rows $i = 1, \ldots, m - 1, \infty$ and the columns $j = 1, \ldots, w, \infty$.
Step 3: For $i = 1, \ldots, m - 1$ and $j = 1, \ldots, w$, put $T_{ij} = \{j + i, j - i\}$; and put $T_{\infty j} = \{j, \infty\}$. Here T_{ij} denotes the set of entries in the cell in row i and column j and $T_{\infty j}$ is the set of entries in the cell in row ∞ and column j.
Step 4: For $j = 1, \ldots, w - 1$, let i^* be the unique element in $\{2j, -2j\} \cap \{1, \ldots, m - 1\}$; then exchange $T_{i^* j}$ with $T_{\infty j}$.
Step 5: For $i = 1, \ldots, m - 1$, put $T_{i\infty} = \{3i/2, -3i/2\}$; and put $T_{\infty\infty} = \{w, \infty\}$.

Table 2
Bi-starters in $\mathbb{Z}_v$ for some odd values of v.

v	bi-starter										
5	{1, 5}	{5, 2}	{2, 4}	{4, 3}	{3, 1}						
7	{1, 7}	{7, 2}	{2, 6}	{6, 3}	{3, 5}	{5, 4}	{4, 1}				
9	{1, 9}	{9, 2}	{2, 8}	{8, 3}	{3, 7}	{7, 4}	{4, 6}	{6, 5}	{5, 1}		
11	{1, 11}	{11, 2}	{2, 10}	{10, 3}	{3, 9}	{9, 4}	{4, 8}	{8, 5}	{5, 7}	{7, 6}	{6, 1}

3. Smallest designs for an odd number of treatments

3.1. Description of the designs

Let v be odd. Since v must divide $2h$ and $2p$, the smallest designs have v rows and v columns. Each treatment in these designs appears twice per row and twice per column. Hence it appears $2v$ times overall, and therefore the sum of concurrences with any given treatment is $2v$.

To get quotient block designs that are RGDs, the only pattern of concurrences is for each treatment to concur with two other treatments three times each and concur twice with each of the remaining $v-3$ treatments. Thus, to have a RGD, every pair of distinct treatments needs to appear in either two or three blocks. The designs are precisely $(v \times v)/2$ RGSLRs for v treatments. When $v = 3$, this construction gives a $(3 \times 3)/2$ BSLR for three treatments. This can be obtained more directly by starting with any 3×3 Latin square and replacing each letter by one pair of distinct numbers from $\{1, 2, 3\}$.

We give constructions for these designs using bi-starters, obtained via undirected terraces for $\mathbb{Z}_v$.

3.2. Undirected terrace and bi-starter sets

For a given odd integer v, the sequence $(1, v, 2, v-1, 3, v-2, \ldots, (v+1)/2)$ constitutes an *undirected terrace* for $\mathbb{Z}_v$: see, for example, Bailey (1984), Durier et al. (1997), Ollis and Willmott (2015) and Anderson et al. (2017). This means that the undirected differences between successive pairs in the sequence give every non-zero element twice. In fact, they are $\pm 1, \pm 2, \ldots, \pm(v-1)/2, \pm(v-1)/2, \pm(v-3)/2, \ldots, \pm 1$. If the sequence is regarded as a circle that joins up the two ends then the extra undirected difference is $\pm(v-1)/2$.

For example, the sequence $(1, 5, 2, 4, 3)$ constitutes an undirected terrace for $\mathbb{Z}_5$. The successive undirected differences are $\pm 1, \pm 2, \pm 2, \pm 1, \pm 2$. Similarly, $(1, 7, 2, 6, 3, 5, 4)$ constitutes an undirected terrace for $\mathbb{Z}_7$. Its successive undirected differences are $\pm 1, \pm 2, \pm 3, \pm 3, \pm 2, \pm 1, \pm 3$.

To obtain the subsets that constitute the bi-starter, we take the v pairs of consecutive elements of the circular sequence. For instance, we obtain, for the bi-starter in $\mathbb{Z}_5$, the sets $\{1, 5\}, \{5, 2\}, \{2, 4\}, \{4, 3\}, \{3, 1\}$.

Table 2 shows these bi-starters in $\mathbb{Z}_v$ for some odd values of v. An algorithm for the construction of the design is presented in Section 3.3.

3.3. An algorithm for constructing the design

Step 1: Label the treatments $1, 2, \ldots, v$.

Step 2: Obtain v sets that constitute a bi-starter in $\mathbb{Z}_v$ by forming a sequence of the elements of $\mathbb{Z}_v$ that constitutes an undirected terrace for $\mathbb{Z}_v$. One possibility is $(1, v, 2, v-1, 3, v-2, \ldots, (v+1)/2)$. Considering the sequence as a circle, combine successive pairs of elements to obtain sets $S_1, \ldots, S_v$ which make a bi-starter.

Step 3: Create a $v \times v$ array and label its rows $i = 1, 2, \ldots, v$ and its columns $j = 1, 2, \ldots, v$.

Step 4: For $j = 1, 2, \ldots, v$, put S_j in the cell in position $(1, j)$ of the array.

Step 5: For $j = 1, 2, \ldots, v$, develop the block in position $(1, j)$, cyclically, via successive addition of 1 modulo v, thereby generating the v blocks in each column.

Example 1. Let $v = 5$. Table 2 shows that the sets $\{1, 5\}, \{5, 2\}, \{2, 4\}, \{4, 3\}$ and $\{3, 1\}$ constitute a bi-starter in $\mathbb{Z}_5$. Using the algorithm, we obtain the $(5 \times 5)/2$ RGSLR for five treatments shown in Fig. 1.

4. Designs whose size is an integer multiple of that of the smallest designs

4.1. Strategy

Our strategy for building larger designs is based on the following result.

Theorem 1. *Suppose that Λ_1 is an $(h \times p_1)/k$ semi-Latin rectangle and Λ_2 is an $(h \times p_2)/k$ semi-Latin rectangle, both with treatment set $\mathcal{V} = \{1, 2, \ldots, v\}$. Let $p = p_1 + p_2$. Then the design Δ obtained by putting Λ_1 and Λ_2 side by side in an $h \times p$ array is an $(h \times p)/k$ semi-Latin rectangle for v treatments.*

1 5	5 2	2 4	4 3	3 1
2 1	1 3	3 5	5 4	4 2
3 2	2 4	4 1	1 5	5 3
4 3	3 5	5 2	2 1	1 4
5 4	4 1	1 3	3 2	2 5

Fig. 1. A (5 × 5)/2 RGSLR for five treatments.

2 5	2 ∞	3 ∞	5 3	1 4	4 1	5 ∞	5 3	1 3	∞ 1	4 2	2 4
1 ∞	4 5	5 1	4 ∞	2 3	3 2	4 3	2 ∞	∞ 4	2 3	5 1	1 5
3 4	3 1	4 2	1 2	5 ∞	5 ∞	1 2	1 4	2 5	4 5	∞ 3	∞ 3

Fig. 2. A (3 × 12)/2 RGSLR for six treatments.

Proof. Each cell of Δ contains k distinct treatments. Let i be any treatment in $\mathcal{V}$. Then i occurs hk/v times in each column of Λ_1, and also hk/v times in each column of Λ_2: hence it occurs hk/v times in each column of Δ. Moreover, i occurs p_1k/v times in each row of Λ_1 and p_2k/v times in each row of Λ_2: hence it occurs pk/v times in each row of Δ. Therefore Δ is a semi-Latin rectangle. □

The main idea in this section shows how to double the size of the designs in Sections 2–3, using the strategy in Theorem 1.

Suppose that Λ_1 is one of those designs. Let P_1 be the set (not multi-set) of pairs of distinct treatments with the higher concurrence. It is always possible to find a permutation α of the set $\mathcal{V}$ of treatments which, when applied to all pairs in P_1, gives a set P_2 of pairs which is disjoint from P_1. Thus the blocks in $P_1 \cup P_2$ form a RGD. Applying α to the whole of Λ_1 gives a second SLR Λ_2 of the same size. Placing Λ_1 and Λ_2 side by side gives a RGSLR which is twice the size of Λ_1.

Later, we extend this method to larger sizes.

4.2. Doubling the size of the smallest designs for an even number of treatments

These designs are precisely $(m \times 4m)/2$ RGSLRs for $2m$ treatments. To obtain these, we extend the constructions in Section 2 by permuting the treatments in the smallest design, where the permutation is chosen so that $P_1 \cup P_2$ forms a polygon. The original design and the permuted design are placed side by side.

Here is the algorithm.

Step 1: Construct an $(m \times 2m)/2$ SLR Λ_1 as in Section 2.

Step 2: Denote by P_1 the parallel class formed by the m pairs with concurrence two. If we are using the algorithm in Section 2.2, P_1 consists of the distinct pairs in row m. If we are using the algorithm in Section 2.3, P_1 consists of the pairs in column ∞. Choose a permutation α of the treatments and apply it to each set in P_1 to obtain another parallel class P_2. Do this in such a way that P_2 contains no pair in common with P_1 and $P_1 \cup P_2$ gives the edges of a connected design, that is, a single polygon on $2m$ vertices.

Step 3: Apply α to every treatment in the first design Λ_1, to obtain a second $(m \times 2m)/2$ SLR Λ_2.

Step 4: Create an $(m \times 4m)/2$ SLR by placing Λ_1 in columns 1 to $2m$ and Λ_2 in columns $2m + 1$ to $4m$.

Example 2. Let $v = 6$. Then $m = 3$. Step 1 of the algorithm gives the $(3 \times 6)/2$ SLR Λ_1 on the left-hand side of Fig. 2. In this, $P_1 = \{\{1, 4\}, \{2, 3\}, \{5, \infty\}\}$. Choosing the permutation

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \infty \\ 4 & 5 & 1 & 2 & \infty & 3 \end{pmatrix},$$

gives $P_2 = \{\{4, 2\}, \{5, 1\}, \{\infty, 3\}\}$, so that the six pairs in $P_1 \cup P_2$ do indeed form a single hexagon. Applying α to Λ_1 gives the $(3 \times 6)/2$ SLR Λ_2 on the right-hand side of Fig. 2.

4.3. Doubling the size of the smallest designs for an odd number of treatments

These designs are precisely $(v \times 2v)/2$ RGSLRs for v treatments, where v is odd. For $v = 3$, we simply place two $(3 \times 3)/2$ BSLRs for three treatments side by side. For larger odd values of v, we extend the construction in Section 3

1 7	7 2	2 6	6 3	3 5	5 4	4 1	2 7	7 4	4 5	5 6	6 3	3 1	1 2
2 1	1 3	3 7	7 4	4 6	6 5	5 2	4 2	2 6	6 7	7 1	1 5	5 3	3 4
3 2	2 4	4 1	1 5	5 7	7 6	6 3	6 4	4 1	1 2	2 3	3 7	7 5	5 6
4 3	3 5	5 2	2 6	6 1	1 7	7 4	1 6	6 3	3 4	4 5	5 2	2 7	7 1
5 4	4 6	6 3	3 7	7 2	2 1	1 5	3 1	1 5	5 6	6 7	7 4	4 2	2 3
6 5	5 7	7 4	4 1	1 3	3 2	2 6	5 3	3 7	7 1	1 2	2 6	6 4	4 5
7 6	6 1	1 5	5 2	2 4	4 3	3 7	7 5	5 2	2 3	3 4	4 1	1 6	6 7

Fig. 3. A $(7 \times 14)/2$ RGSLR for seven treatments.

by permuting the treatments in the smallest design, where the permutation involves multiplying every treatment in the smallest design by an element of $\mathbb{Z}_v \setminus \{1, v-1, v\}$ that is coprime to v. This gives a second constituent design, which is placed beside the original design.

Here is the algorithm.

Step 1: Construct an $(v \times v)/2$ SLR Λ_1 as in Section 3.
Step 2: Choose an element of $\mathbb{Z}_v \setminus \{1, v-1, v\}$ that is coprime to v. Multiply every treatment in Λ_1 by this to obtain a second $(v \times v)/2$ SLR Λ_2.
Step 3: Create a $(v \times 2v)/2$ SLR by placing Λ_1 in columns 1 to v and Λ_2 in columns $v+1$ to $2v$.

Example 3. Let $v = 7$. Step 1 of the algorithm gives the $(7 \times 7)/2$ SLR Λ_1 on the left-hand side of Fig. 3. Multiplying every treatment by 2 (which is coprime to 7) gives the $(7 \times 7)/2$ SLR Λ_2 on the right-hand side of Fig. 3. Together, these give a $(7 \times 14)/2$ RGSLR for 7 treatments.

4.4. Comments

The pair of double vertical lines shown in Figs. 2 and 3 simply shows the method of construction. This should not be included in the design that is given to the experimenter. In particular, randomization ignores these lines. For example, to randomize the design in Fig. 2, first re-order the three rows by applying a random permutation of three objects. Then re-order the twelve columns by applying a random permutation of twelve objects. Finally, within each of the 36 blocks independently, randomize the order of the two treatments.

In the doubling construction, it is also possible to place the array Λ_2 below the array Λ_1. This gives a $(2m \times 2m)/2$ RGSLR for $2m$ treatments, or a $(2v \times v)/2$ RGSLR for v treatments when v is odd. Transposition also gives a $(4m \times m)/2$ RGSLR for $2m$ treatments.

These remarks also apply to the subsequent constructions in this paper.

4.5. Even larger designs

The methods in Sections 4.2 and 4.3 easily extend to larger designs.

For $v = 2m$, create Λ_1 and P_1 as before. For $2ms$ columns, where s is an integer with $s > 1$, choose sets of pairs P_2, …, P_s such that each set contains each treatment once and the ms pairs in $P_1, \ldots, P_s$ (accounting for multiplicity) form a regular-graph block design. For $u = 2, \ldots, s$, choose a permutation α_u of the treatments such that $\alpha_u(P_1) = P_u$; then apply α_u to Λ_1 to obtain the SLR Λ_u. Finally, place $\Lambda_1, \ldots, \Lambda_s$ side by side.

Example 4. Suppose that we want a $(3 \times 18)/2$ RGSLR for six treatments. If we take $P_1 = \{\{1, 4\}, \{2, 3\}, \{5, \infty\}\}$ and $P_2 = \{\{4, 2\}, \{5, 1\}, \{\infty, 3\}\}$, as in Example 2, then we may choose $P_3 = \{\{1, 3\}, \{2, 5\}, \{4, \infty\}\}$, so that the pairs in $P_1 \cup P_2 \cup P_3$ consist of all edges between the disjoint triples $\{1, 2, \infty\}$ and $\{3, 4, 5\}$. A result of Cheng and Bailey (1991) shows that the quotient block design of the resulting RGSLR is A-optimal.

If v is odd, the treatment pairs which have the higher concurrence in Λ_1 form a single polygon with v vertices. Call this set of pairs Q_1. For vs columns, where $s > 1$, choose polygons $Q_2, \ldots, Q_s$, each containing all treatments, such that the vs pairs in $Q_1, \ldots, Q_s$ (accounting for multiplicity) form a regular-graph block design. Then proceed with permutations as in the even case.

1 2	2 3	3 4	4 5	5 6	6 7	7 8	8 1	1 2	3 4	5 6	7 8
5 7	6 8	7 1	8 2	1 3	2 4	3 5	4 6	3 6	1 8	7 2	5 4
3 6	4 7	5 8	6 1	7 2	8 3	1 4	2 5	5 8	7 6	1 4	3 2
4 8	5 1	6 2	7 3	8 4	1 5	2 6	3 7	7 4	5 2	3 8	1 6

Fig. 4. A $(4 \times 12)/2$ RGSLR for eight treatments.

5. Designs of intermediate size for a number of treatments divisible by four

5.1. Overview

Suppose that $v = 2m$. So long as $m \notin \{1, 2, 6\}$, there is a pair of mutually orthogonal Latin squares of order m, and hence an $(m \times m)/2$ Trojan square. In this, the set $\mathcal{V}$ of treatments is partitioned into two subsets $\mathcal{V}_1$ and $\mathcal{V}_2$, both of size m. Let x and y be two treatments. If x and y are both in $\mathcal{V}_1$, or both in $\mathcal{V}_2$, then their concurrence in the Trojan square is zero; otherwise, it is one.

Our strategy in this section is to adjoin a Trojan square to one of the designs Λ constructed in Section 2, 4.2 or 4.5. In order for this to produce a RGSLR, if $\{x, y\}$ is any pair of treatments with the higher concurrence in Λ, then either $\{x, y\} \subset \mathcal{V}_1$ or $\{x, y\} \subset \mathcal{V}_2$. In the design Λ_1 in Section 4.2, the pairs with the higher concurrence are precisely those in the parallel class P_1. Therefore, we require half of these pairs to be contained in $\mathcal{V}_1$, and half to be contained in $\mathcal{V}_2$. For this to be possible, each of $\mathcal{V}_1$ and $\mathcal{V}_2$ must be composed of $m/2$ disjoint pairs from P_1, and so m must be even.

5.2. Designs with three times as many columns as rows

Suppose that $v = 2m$, where m is even, $m \neq 2$ and $m \neq 6$. Here is our algorithm for constructing an $(m \times 3m)/2$ RGSLR.

Step 1: Construct an $(m \times 2m)/2$ SLR Λ_1 as in Section 2.
Step 2: Denote by P_1 the parallel class formed by the m pairs with concurrence two, as in Section 4.2.
Step 3: Make a pair of mutually orthogonal Latin squares of order m, where the treatment set $\mathcal{V}_1$ of one consists of those treatments in any $m/2$ pairs in P_1 while the remaining treatments form the treatment set $\mathcal{V}_2$ of the other Latin square. Superpose these Latin squares to make an $(m \times m)/2$ Trojan square Λ_2.
Step 4: Create an $(m \times 3m)/2$ SLR by placing Λ_1 in columns 1 to $2m$ and Λ_2 in columns $2m + 1$ to $3m$.

Example 5. Let $v = 8$, so that $m = 4$. Then Step 1 of the algorithm gives the $(4 \times 8)/2$ SLR on the left-hand side of Fig. 4. In this, $P_1 = \{\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}\}$, so we may put $\mathcal{V}_1 = \{1, 3, 5, 7\}$ and $\mathcal{V}_2 = \{2, 4, 6, 8\}$ and obtain the Trojan square on the right of Fig. 4.

5.3. Larger designs

Trojan squares can also be adjoined to $(m \times 2um)/2$ SLRs with $u > 1$. However, the best choice for the parallel classes $P_1, P_2, \ldots, P_u$ given in Section 4.5 is not compatible with the need to split each parallel class equally between $\mathcal{V}_1$ and $\mathcal{V}_2$. Therefore, the intermediate $(m \times 2um)/2$ SLRs are not the designs given in Section 4.5.

Example 6. Let $v = 8$. The $(4 \times 8)/2$ SLR Λ_1 on the left-hand side of Fig. 4 has $P_1 = \{\{1, 5\}, \{2, 6\}, \{3, 7\}, \{4, 8\}\}$. To construct a $(4 \times 16)/2$ RGSLR as in Section 4.2, we could choose $P_2 = \{\{1, 2\}, \{6, 8\}, \{4, 3\}, \{7, 5\}\}$, so that $P_1 \cup P_2$ forms a single octagon. On the other hand, to construct a $(4 \times 20)/2$ RGSLR with $\mathcal{V}_1 = \{1, 3, 5, 7\}$ and $\mathcal{V}_2 = \{2, 4, 6, 8\}$, the only possibilities for P_2 are

$$\begin{array}{lcl} \{\{1,3\},\{5,7\},\{2,4\},\{6,8\}\}, & & \{\{1,3\},\{5,7\},\{2,8\},\{4,6\}\}, \\ \{\{1,7\},\{3,5\},\{2,4\},\{6,8\}\} & \text{and} & \{\{1,7\},\{3,5\},\{2,8\},\{4,6\}\}. \end{array}$$

If we put $P_2 = \{\{1, 3\}, \{5, 7\}, \{2, 4\}, \{6, 8\}\}$ then we may choose

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 5 & 6 & 3 & 4 & 7 & 8 \end{pmatrix}.$$

Then one choice of the Trojan square gives the RGSLR in Fig. 5, shown with the treatments in each block one below the other.

1	2	3	4	5	6	7	8	1	2	5	6	3	4	7	8	1	3	5	7
2	3	4	5	6	7	8	1	2	5	6	3	4	7	8	1	2	4	6	8
5	6	7	8	1	2	3	4	3	4	7	8	1	2	5	6	3	1	7	5
7	8	1	2	3	4	5	6	7	8	1	2	5	6	3	4	6	8	2	4
3	4	5	6	7	8	1	2	5	6	3	4	7	8	1	2	5	7	1	3
6	7	8	1	2	3	4	5	4	7	8	1	2	5	6	3	8	6	4	2
4	5	6	7	8	1	2	3	6	3	4	7	8	1	2	5	7	5	3	1
8	1	2	3	4	5	6	7	8	1	2	5	6	3	4	7	4	2	8	6

Fig. 5. A $(4 \times 20)/2$ RGSLR for eight treatments.

2	2	3	5	1	4	1	2	3	2	3	4	3	4	5	4	5	1	5	1	2
5	∞	∞	3	4	1	∞	5	4	∞	1	5	∞	2	1	∞	3	2	∞	4	3
1	4	5	4	2	3	3	1	2	4	2	3	5	3	4	1	4	5	2	5	1
∞	5	1	∞	3	2	4	∞	5	5	∞	1	1	∞	2	2	∞	3	3	∞	4
3	3	4	1	5	5	2	3	1	3	4	2	4	5	3	5	1	4	1	2	5
4	1	2	2	∞	∞	5	4	∞	1	5	∞	2	1	∞	3	2	∞	4	3	∞

Fig. 6. A $(3 \times 21)/2$ RGSLR for six treatments.

6. Extending a RGSLR by adding a BSLR

The following theorem shows that another way to obtain a larger RGSLR from a smaller one is to adjoin a BSLR with the same set of treatments and the same number of rows. In practice, this is most likely to be useful for a small number of treatments.

Theorem 2. *Let Λ_1, Λ_2, Δ, h, p_1, p_2, p, k, $\mathcal{V}$ and v be as in Theorem 1. If Λ_1 is a regular-graph semi-Latin rectangle and Λ_2 is a balanced semi-Latin rectangle then Δ is a regular-graph semi-Latin rectangle.*

Proof. Let $\{x, x'\}$ and $\{y, y'\}$ be two pairs of distinct treatments. Suppose that their concurrences in Λ_1 are λ_1 and λ_2 respectively. Then $|\lambda_1 - \lambda_2| \leq 1$, because Λ_1 is a RGSLR.

Since Λ_2 is a BSLR, there is a constant λ such that all concurrences in Λ_2 are λ. Hence the concurrence of $\{x, x'\}$ in Δ is $\lambda_1 + \lambda$, while that of $\{y, y'\}$ in Δ is $\lambda_2 + \lambda$. Then $|(\lambda_1 + \lambda) - (\lambda_2 + \lambda)| = |\lambda_1 - \lambda_2| \leq 1$.

This is true for all distinct pairs of treatments. Hence Δ is a RGSLR. □

For the case that $k = 2$, Uto and Bailey (2020) give BSLRs. Theorem 2 shows that these can be adjoined to RGSLRs of suitable size constructed in Sections 2–5 to obtain larger RGSLRs. This method also applies if Λ_1 is a semi-Latin square whose quotient block design is a regular-graph design.

Example 7. Figure 4 of Uto and Bailey (2020) shows a $(3 \times 15)/2$ BSLR for six treatments. This can be adjoined to the $(3 \times 12)/2$ RGSLR in Fig. 2 to obtain a $(3 \times 27)/2$ RGSLR. Alternatively, it can be adjoined to the $(3 \times 6)/2$ RGSLR on the left-hand side of Fig. 2 to obtain a $(3 \times 21)/2$ RGSLR. This is shown in Fig. 6, with the treatments in each block shown one below the other.

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