A note on m-factorizations of complete multigraphs arising from designs[*]

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Abstract

Some new infinite families of simple, indecomposable m-factorizations of the complete multigraph λK_v are presented. Most of the constructions come from finite geometries.

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1 Introduction

The *complete multigraph* λK_v has v vertices and λ edges joining each pair of vertices. An m-*factor* of the complete multigraph λK_v is a set of pairwise vertex-disjoint m-regular subgraphs, which induce a partition of the vertices. An m-*factorization* of λK_v is a set of pairwise edge-disjoint m-factors such that these m-factors induce a partition of the edges. An m-factorization is called *simple* if the m-factors are pairwise distinct. Furthermore, an m-factorization of λK_v is *decomposable* if there exist positive integers μ_1 and μ_2 such

that $\mu_1 + \mu_2 = \lambda$ and the factorization is the union of the m-factorizations $\mu_1 K_v$ and $\mu_2 K_v$, otherwise it is called *indecomposable*. There is no direct correspondence between simplicity and indecomposability.

Many papers deal with m-factorizations of graphs and multigraphs. This is an interesting problem in its own right, but it is motivated by several applications, too. In particular if $m = 1$, then a one-factorization of K_v corresponds to a schedule of a round robin tournament. For a comprehensive survey on one-factorizations we refer to [29]. A special case of 2-factorizations is the famous *Oberwolfach problem*, see e.g. [2, 8]. Several authors investigated 3-factorizations of λK_v with a certain automorphism group, see e.g. [1, 24]. In general, decompositions of λK_v is also a widely studied problem, see e.g. [12, 13, 18, 27]. As m increases, the structure of an arbitrary m-factor of λK_v can be much more complicated and the existence problem becomes much more difficult. In this paper we restrict ourselves to construct factorizations in which all factors are regular graphs of degree m whose connected components are complete graphs on $(m + 1)$ vertices. In the case $m = 1$ an indecomposable one-factorization of λK_{2n} is denoted by IOF$(2n, \lambda)$. Only a few conditions on the parameters are known: if IOF$(2n, \lambda)$ exists, then $\lambda < 1 \cdot 3 \cdot ... \cdot (2n - 3)$ [4]; each IOF$(2n, \lambda)$ can be embedded in a simple IOF$(2s, \lambda)$, provided that $\lambda < 2n < s$ [16]. Six infinite classes of indecomposable one-factorizations have been constructed so far, namely a simple IOF$(2n, n - 1)$ when $2n - 1$ is a prime [16], IOF$(2(\lambda + p), \lambda)$ where $\lambda > 2$ and p is the smallest prime wich does not divide λ [3] (an improvement of this result can be found in [15]), a simple IOF$(2^h + 2, 2)$ where h is a positive integer [28], IOF$(q^2 + 1, q - 1)$ where q is an odd prime number [26], a simple IOF$(q^2 + 1, q + 1)$ for any odd prime power q [25], and a simple IOF(q^2, q) for any even prime power q [25]. Most of these constructions arise from finite geometry.

The aim of this paper is to construct new simple and indecomposable m-factorizations of λK_v for different values of m, λ and v. In Section 2 we recall the basic combinatorial properties of designs and the geometric properties of finite affine and projective spaces. We also describe a general construction method of m-factorizations which is based on spreads of block designs. In Sections 3 and 4 affine spaces and projective spaces, respectively, are the key objects. We present several new multigraph factorizations using subspaces, subgeometries and other configurations of these structures.

2   Preliminaries

In this section we collect some concepts and results from design theory. For a detailed introduction to block designs we refer to [14].

2.1   Designs

Let v, b, k, r and λ be positive integers with $v > 1$. Let $D = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a triple consisting of a set $\mathcal{P}$ of v distinct objects, called points of D, a set $\mathcal{B}$ of b distinct objects, called blocks of D, and an incidence relation I, a subset of $\mathcal{P} \times \mathcal{B}$. We say that x is incident with y (or y is incident with x) if and only if the ordered pair (x, y) is in I. D is called a $2 - (v, b, k, r, \lambda)$ *design* if it satisfies the following axioms.

   (a) Each block of D is incident with exactly k distinct points of D.

   (b) Each point of D is incident with exactly r distinct blocks of D.

   (c) If x and y are distinct points of D, then there are exactly λ blocks of D incident with

both x and y.

A $2 - (v, b, k, r, \lambda)$ design is called a balanced incomplete block design and is denoted by (v, k, λ)-design, too. The parameters of a $2 - (v, b, k, r, \lambda)$ design are not all independent. The two basic equations connecting them are the following:

$$vr = bk \quad \text{and} \quad r(k - 1) = \lambda(v - 1). \tag{2.1}$$

These necessary conditions are not sufficient, for example no $2 - (43, 43, 7, 7, 1)$ design exists.

2.2 Resolvability

A *resolution class* (or, a parallel class) of a (v, k, λ)-design is a partition of the point-set of the design into blocks. In general, an *f-resolution class* of a design is a collection of blocks, which together contain every point of the design exactly f times. A *resolution* of a design is a partition of the block-set of the design into r resolution classes. A (v, k, λ)-design with a resolution is called *resolvable*.

Necessary conditions for the existence of a resolvable (v, k, λ)-design are $\lambda(v - 1) \equiv 0$ (mod $(k - 1)$), $v \equiv 0$ (mod k) and $b \geq v + r - 1$, (see [9]).

Let $D = (\mathcal{P}, \mathcal{B}, \mathrm{I})$ be a (v, k, λ)-design, where $\mathcal{P} = \{p_1, p_2, \ldots, p_v\}$ is the set of its points and $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ is the set of its blocks. Identify the points of D with the vertices of the complete multigraph λK_v. Then in the natural way, the set of points of each block of D induces in λK_v a subgraph isomorphic to K_k. For $B_i \in \mathcal{B}$, let G_i be the subgraph of λK_v induced by B_i. Then it follows from the properties of D that a resolution class of D gives a $(k - 1)$-factor of λK_v and a resolution of D gives a $(k - 1)$-factorization of λK_v. Hence we get the following well-known fact.

Lemma 2.1 (Basic Construction). *The existence a resolvable (v, k, λ)-design is equivalent to the existence of a $(k - 1)$-factorization of the complete multigraph λK_v.*

2.3 Projective and affine spaces

Most of our factorizations come from finite geometries. In this subsection we collect the basic properties of these objects. For a more detailed introduction we refer to the book of Hirschfeld [22].

Let V_{n+1} be an $(n + 1)$-dimensional vector space over the finite field of q elements, $\mathrm{GF}(q)$. The *n-dimensional projective space* $\mathrm{PG}(n, q)$ is the geometry whose k-dimensional subspaces for $k = 0, 1, \ldots, n$ are the $(k + 1)$-dimensional subspaces of V_{n+1} with the zero deleted. A k-dimensional subspace of $\mathrm{PG}(n, q)$ is called k-*space*. In particular subspaces of dimension zero, one and two are respectively a *point,* a *line* and a *plane,* while a subspace of dimension $n - 1$ is called a *hyperplane.*

The relation $\sim$

$$\mathbf{x} \sim \mathbf{y} \Leftrightarrow \exists\, 0 \neq \alpha \in \mathrm{GF}(q) : \mathbf{x} = \alpha \mathbf{y}$$

is an equivalence relation on the elements of $V_{n+1} \setminus \mathbf{0}$ whose equivalence classes are the points of $\mathrm{PG}(n, q)$. Let $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ be a vector in $V_{n+1} \setminus \mathbf{0}$. The equivalence class of $\mathbf{v}$ is denoted by $[\mathbf{v}]$. The homogeneous coordinates of the point represented by $[\mathbf{v}]$ are $(v_0 : v_1 : \ldots : v_n)$. Hence two $(n + 1)$-tuples $(x_0 : x_1 : \ldots : x_n)$ and $(y_0 : y_1 : \ldots : y_n)$ represent the same point of $\mathrm{PG}(n, q)$ if and only if there exists $0 \neq \alpha \in \mathrm{GF}(q)$ such that $x_i = \alpha y_i$ holds for $i = 0, 1, \ldots, n$.

A k-space contains those points whose representing vectors $\mathbf{x}$ satisfy the equation $\mathbf{x}A = \mathbf{0}$, where A is an $(n + 1) \times (n - k)$ matrix of rank $n - k$ with entries in $\mathrm{GF}(q)$. In particular a hyperplane contains those points whose homogeneous coordinates $(x_0 : x_1 : \ldots : x_n)$ satisfy a linear equation

$$u_0 x_0 + u_1 x_1 + \cdots + u_n x_n = 0$$

where $u_i \in \mathrm{GF}(q)$ and $(u_0, u_1, \ldots, u_n) \neq \mathbf{0}$.

The basic combinatorial properties of $\mathrm{PG}(n, q)$ can be described by the q-nomial coefficients. $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ equals to the number of k-dimensional subspaces in an n-dimensional vector space over $\mathrm{GF}(q)$, hence it is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q^n - 1)(q^n - q) \ldots (q^n - q^{k-1})}{(q^k - 1)(q^k - q) \ldots (q^k - q^{k-1})}.$$

The proof of the following proposition is straightforward.

Proposition 2.2.

- *The number of k-dimensional subspaces in $\mathrm{PG}(n, q)$ is $\left[\begin{smallmatrix} n+1 \\ k+1 \end{smallmatrix}\right]_q$.*

- *The number of k-dimensional subspaces of $\mathrm{PG}(n, q)$ through a given d-dimensional $(d \leq k)$ subspace in $PG(n, q)$ is $\left[\begin{smallmatrix} n-d \\ k-d \end{smallmatrix}\right]_q$.*

- *In particular the number of k-dimensional subspaces of $\mathrm{PG}(n, q)$ through two distinct points in $\mathrm{PG}(n, q)$ is $\left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]_q$.*

If $\mathcal{H}_\infty$ is any hyperplane of $\mathrm{PG}(n, q)$, then the n-dimensional affine space over $\mathrm{GF}(q)$ is $\mathrm{AG}(n, q) = \mathrm{PG}(n, q) \setminus \mathcal{H}_\infty$. The subspaces of $\mathrm{AG}(n, q)$ are the subspaces of $\mathrm{PG}(n, q)$ with the points of $\mathcal{H}_\infty$ deleted in each case. The hyperplane $\mathcal{H}_\infty$ is called the *hyperplane at infinity of* $\mathrm{AG}(n, q)$, and for $k = 0, 1, \ldots, n - 2$ the k-dimensional subspaces in $\mathcal{H}_\infty$ are called the *k-spaces at infinity of* $\mathrm{AG}(n, q)$. Let $1 < d < n$ be an integer. Two d-spaces of $\mathrm{AG}(n, q)$ are called *parallel*, if the corresponding d-spaces of $\mathrm{PG}(n, q)$ intersect $\mathcal{H}_\infty$ in the same $(d - 1)$-space. The parallelism is an equivalence relation on the set of d-spaces of $\mathrm{AG}(n, q)$. As a straightforward corollary of Proposition 2.2 we get the following.

Proposition 2.3. *In $\mathrm{AG}(n, q)$ each equivalence class of parallel d-spaces contains q^{n-d} subspaces.*

Projective and affine spaces provide examples of designs.

Example 2.4. Let $i < n$ be positive integers. The projective space $\mathrm{PG}(n, q)$ can be considered as a 2-design $D = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, where $\mathcal{P}$ is the set of points of $\mathrm{PG}(n, q)$, $\mathcal{B}$ is the set of i-spaces of $\mathrm{PG}(n, q)$ and I is the set theoretical inclusion. The parameters of D are $v = \frac{q^{n+1}-1}{q-1}, b = \left[\begin{smallmatrix} n+1 \\ i+1 \end{smallmatrix}\right]_q, k = \frac{q^{i+1}-1}{q-1}, r = \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q$ and $\lambda = \left[\begin{smallmatrix} n-1 \\ i-1 \end{smallmatrix}\right]_q$.

Example 2.5. Let $i < n$ be positive integers. The affine space $\mathrm{AG}(n, q)$ can be considered as a 2-design $D = (\mathcal{P}, \mathcal{B}, \mathrm{I})$, where $\mathcal{P}$ is the set of points of $\mathrm{AG}(n, q)$, $\mathcal{B}$ is the set of i-spaces of $\mathrm{AG}(n, q)$ and I is the set theoretical inclusion. The parameters of D are $v = q^n$, $b = q^{n-i} \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q, k = q^i, r = \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q$ and $\lambda = \left[\begin{smallmatrix} n-1 \\ i-1 \end{smallmatrix}\right]_q$.

In the rest of this paper Examples 2.4 and 2.5 will be denoted by $\mathrm{PG}^{(i)}(n, q)$ and by $\mathrm{AG}^{(i)}(n, q)$, respectively. We will use the terminology from geometry. An i-*spread*, $\mathcal{S}^i$, of $\mathrm{PG}(n, q)$ (or of $\mathrm{AG}(n, q)$) is a set of pairwise disjoint i-dimensional subspaces which gives a partition of the points of the geometry. In general, an f-*fold* i-*spread*, $\mathcal{S}^i_f$, is a set of i-dimensional subspaces such that every point of the geometry is contained in exactly f subspaces of $\mathcal{S}^i_f$. An i-*packing*, $\mathcal{P}^i$, of $\mathrm{PG}(n, q)$ (or of $\mathrm{AG}(n, q)$) is a set of spreads such that each i-dimensional subspace of the geometry is contained in exactly one of the spreads in $\mathcal{P}^i$, i.e., the spreads give a partition of the i-dimensional subspaces of the geometry. The i-spreads, f-fold i-spreads and i-packings induce a resolution class, an f-resolution class and a resolution in $\mathrm{PG}^{(i)}(n, q)$ (or in $\mathrm{AG}^{(i)}(n, q)$), respectively.

It is easy to construct spreads and packings in $\mathrm{AG}^{(i)}(n, q)$, because each parallel class of i-spaces is an i-spread. The situation is much more complicated in $\mathrm{PG}^{(i)}(n, q)$. There are only a few constructions of spreads. The following theorem summarizes the known existence conditions.

Theorem 2.6 ([22], Theorems 4.1 and 4.16)**.**

- *There exists an i-spread in $\mathrm{PG}^{(i)}(n, q)$ if and only if $(i + 1)|(n + 1)$.*

- *Suppose that i, l and n are positive integers such that $(l + 1)|\gcd(i + 1, n + 1)$. Then there exists an f-fold i-spread in $\mathrm{PG}^{(i)}(n, q)$, where $f = (q^{i+1} - 1)/(q^{l+1} - 1)$.*

There exist several different 1-spreads (line spreads) in $\mathrm{PG}^{(1)}(3, q)$. We briefly mention two types. Let ℓ_1, ℓ_2 and ℓ_3 be three skew lines in $\mathrm{PG}(3, q)$. The set of the $q + 1$ transversals of ℓ_1, ℓ_2 and ℓ_3 is called *regulus* and it is denoted by $\mathcal{R}(\ell_1, \ell_2, \ell_3)$. The classical construction of a line spread comes from a pencil of hyperbolic quadrics (see e.g. [20], Lemma 17.1.1) and it has the property that if it contains any three lines of a regulus $\mathcal{R}(\ell_1, \ell_2, \ell_3)$, then it contains each of the $q + 1$ lines of $\mathcal{R}(\ell_1, \ell_2, \ell_3)$. This type of spread is called *regular*. A line spread in $\mathrm{PG}(3, q)$ is called *aregular*, if it contains no regulus. An example of an aregular spread can be found in [20], Lemma 17.3.3.

3   Factorizations arising from affine spaces

In this section, we investigate the spreads and packings of $\mathrm{AG}(n, q)$ and the corresponding factorizations of multigraphs. In each case we apply Lemma 2.1, so we identify the points of $\mathrm{AG}(n, q)$ with the vertices of the complete multigraph.

Theorem 3.1. *Let q be a prime power, $i < n$ be positive integers and $\lambda_i = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q$. Then there exists a simple $(q^i - 1)$-factorization $\mathcal{F}^i$ of $\lambda_i K_{q^n}$. $\mathcal{F}^i$ is decomposable if and only if there exists an f-fold $(i - 1)$-spread in $\mathrm{PG}^{(i-1)}(n - 1, q)$ for some $1 \leq f < \lambda_i$.*

Proof. Consider the n-dimensional affine space as $\mathrm{AG}(n, q) = \mathrm{PG}(n, q) \setminus \mathcal{H}_\infty$ where $\mathcal{H}_\infty$ is isomorphic to $\mathrm{PG}(n - 1, q)$. Take the design $D = \mathrm{AG}^{(i)}(n, q)$ and apply Lemma 2.1. If Π^{i-1}_j is an $(i - 1)$-space of $\mathcal{H}_\infty$, then the set of the q^{n-i} parallel affine i-spaces through Π^{i-1}_j is an i-spread of D. This spread induces a $(q^i - 1)$-factor F^i_j for $j \in \{1, \ldots, r\}$. If $\Pi^{i-1}_1, \Pi^{i-1}_2, \ldots, \Pi^{i-1}_g$ are distinct $(i - 1)$-spaces of $\mathcal{H}_\infty$ and they form an f-fold spread, then $f = (g(q^i - 1))/(q^n - 1)$, and the union of the corresponding $(q^i - 1)$-factors F^i_j, for $j = 1, 2, \ldots, g$, gives a $(q^i - 1)$-factorization of fK_{q^n}. Distinct $(i - 1)$-spaces of $\mathcal{H}_\infty$

obviously define distinct $(q^i - 1)$-factors, so this factorization is simple. In particular if we consider all $(i - 1)$-spaces of $\mathcal{H}_\infty$, then

$$g = \begin{bmatrix} n \\ i \end{bmatrix}_q, \quad f = \begin{bmatrix} n \\ i \end{bmatrix}_q \frac{q^i - 1}{q^n - 1} = \begin{bmatrix} n - 1 \\ i - 1 \end{bmatrix}_q = \lambda_i,$$

hence the union of the corresponding factors gives a simple $(q^i - 1)$-factorization $\mathcal{F}^i$ of $\lambda_i K_{q^n}$.

Suppose that $\mathcal{F}^i$ is decomposable, then there exist two positive integers μ_1 and μ_2 such that $\mu_1 + \mu_2 = \lambda_i$ and $\mathcal{F}^i$ can be written as the union $\mathcal{F}^i = \mathcal{F}_1 \cup \mathcal{F}_2$; $\mathcal{F}_1$ and $\mathcal{F}_2$ are $(q^i - 1)$-factorizations of $\mu_1 K_{q^n}$ and $\mu_2 K_{q^n}$, respectively, having no $(q^i - 1)$-factors in common, since $\mathcal{F}^i$ is simple. For $h = 1, 2$, the relation $\mu_h \binom{q^n}{2} = \binom{q^i}{2} q^{n-i} |\mathcal{F}_h|$ holds, hence $\mu_h(q^n - 1) = (q^i - 1)|\mathcal{F}_h|$. Without loss of generality we can set $\mathcal{F}_1 = \cup_{j=1}^{f_1} F_j^i$ with $f_1 = (\mu_1(q^n - 1))/(q^i - 1)$, and $\mathcal{F}_2 = \mathcal{F}^i \setminus \mathcal{F}_1$, $f_2 = |\mathcal{F}_2|$.

Let u_1 and u_2 be two affine points and let w be the point at infinity of the line $u_1 u_2$. Since $\mathcal{F}_h$ is a factorization of $\mu_h K_{q^n}$, there are exactly μ_h factors of $\mathcal{F}_h$ containing the edge $[u_1, u_2]$, say $F_{j_1}^i, F_{j_2}^i, \ldots, F_{j_{\mu_h}}^i$. The edge $[u_1, u_2]$ belongs the $F_{j_s}^i$ if and only if $w \in \Pi_{j_s}^{i-1}$ for every $1 \le s \le \mu_h$. This happens if and only if $\cup_{j=1}^{f_h} \Pi_j^{i-1}$ contains each point of $\mathcal{H}_\infty$ exactly μ_h times, which means that $\cup_{j=1}^{f_h} \Pi_j^{i-1}$ is a μ_h-fold spread in $\mathcal{H}_\infty$, for every $h = 1, 2$. It is thus proved that if $\mathcal{F}^i$ is decomposable, then $\mathrm{PG}^{(i-1)}(n - 1, q)$ posesses an f-fold spread for some $1 \le f < \lambda_i$.

Vice versa, suppose that there exists a μ_1-fold spread in $\mathrm{PG}^{(i-1)}(n - 1, q)$ for some $1 \le \mu_1 < \lambda_i$. Let $\mathcal{F}_1 = \cup_{j=1}^{f_1} F_j^i$ be a μ_1-fold spread in $\mathcal{H}_\infty$. Then $|\mathcal{F}_1| = f_1 = \mu_1(q^n - 1)/(q^i - 1)$. Let $\mathcal{T}$ be the set of all $(i - 1)$-dimensional subspaces in $\mathcal{H}_\infty$ and let $\mathcal{F}_2 = \mathcal{T} \setminus \mathcal{F}_1$. Then $|\mathcal{T}| = \begin{bmatrix} n \\ i \end{bmatrix}_q$, hence

$$|\mathcal{F}_2| = \begin{bmatrix} n \\ i \end{bmatrix}_q - \mu_1(q^n - 1)/(q^i - 1) = \left(\begin{bmatrix} n - 1 \\ i - 1 \end{bmatrix}_q - \mu_1 \right) \frac{q^n - 1}{q^i - 1},$$

so if $\mu_2 = \begin{bmatrix} n-1 \\ i-1 \end{bmatrix}_q - \mu_1$, then $\mathcal{F}_2$ is a μ_2-fold spread in $\mathcal{H}_\infty$ and $1 \le \mu_2 < \lambda_i$ holds.

As we have already seen, $\mathcal{F}_h$ defines a $(q^i - 1)$-factorization of $\mu_h K_{q^n}$ for $h = 1, 2$. Then $\mathcal{F}^i = \mathcal{F}_1 \cup \mathcal{F}_2$, because $\mu_1 + \mu_2 = \lambda_i$. Hence the $(q^i - 1)$-factorization $\mathcal{F}^i$ of $\lambda_i K_{q^n}$ is decomposable. $\qquad\square$

Corollary 3.2. *If* $\gcd(i, n) > 1$ *then the* $(q^i - 1)$-*factorization* $\mathcal{F}^i$ *of* $\lambda_i K_{q^n}$ *is decomposable.*

Proof. Let $1 < l + 1$ be a divisor of $\gcd(i, n)$. Then it follows from Theorem 2.6 that there exists an $(q^i - 1)/(q^{l+1} - 1)$-fold spread in $\mathcal{H}_\infty$, so $\mathcal{F}^i$ is decomposable. $\qquad\square$

To decide the decomposability of $\mathcal{F}^i$ in the cases $\gcd(i, n) = 1$ is a hard problem in general. We prove its indecomposability in the following important case.

Theorem 3.3. *The* $(q^{n-1} - 1)$-*factorization* $\mathcal{F}^{n-1}$ *of* $(q^{n-1} - 1)/(q - 1)K_{q^n}$ *is indecomposable.*

Proof. It is enough to prove that if $\cup_{j=1}^{g} \Pi_j^{n-2}$ is an f-fold $(n-2)$-spread in $\mathcal{H}_\infty$, then $\cup_{j=1}^{g} \Pi_j^{n-2}$ consists of all $(n-2)$-dimensional subspaces of $\mathcal{H}_\infty$, because this implies $f = \lambda_{n-1}$, so the statement follows from Theorem 3.1.

Each Π_j^{n-2} contains exactly $(q^{n-1}-1)/(q-1)$ points, thus the standard double counting of the point-subspace pairs $p \in \Pi_j^{n-2}$ in $\mathcal{H}_\infty$ gives

$$ g\frac{q^{n-1}-1}{q-1} = f\frac{q^n-1}{q-1}, $$

hence

$$ f = \frac{g(q^{n-1}-1)}{q^n-1}. $$

But $\gcd(q^n - 1, q^{n-1} - 1) = q - 1$ and f is an integer, so $g \geq (q^n - 1)/(q-1)$ which implies $g = (q^n - 1)/(q-1)$, hence $f = \lambda_{n-1}$. $\square$

In particular if $n = 2$, we get the following.

Corollary 3.4. *If q is a prime power then there exists a simple and indecomposable $(q-1)$-factorization of K_{q^2}.*

If $q = 2^r$ then each $(q^i - 1)$-factor in $\mathcal{F}^i$ is the vertex-disjoint union of 2^{r-i} complete graphs on 2^i vertices. It is well-known that these graphs can be partitioned into one-factors in many ways (but not in all the ways, it was proved by Hartman and Rosa [19], that there is no cyclic one-factorization of K_{2^i} for $i \geq 3$), hence Theorem 3.1 implies several one-factorizations of $\lambda_i K_{2^r}$.

Each of the one-factorizations arising from $\mathcal{F}^i$ is simple, because distinct $(i-1)$-dimensional subspaces define distinct $(q^i - 1)$-factors of $\mathcal{F}^i$, and the one-factors of $\lambda_i K_{q^n}$ arising from distinct $(q^i - 1)$-factors of $\mathcal{F}^i$ are distinct, because they are the union of q^{n-i} one-factors on q^i vertices of a connected component.

There are both decomposable and indecomposable one-factorizations among these examples. We show it in the smallest case $q = 2$, $n = 3$. Let $\mathcal{F}^2$ be the 3-factorization of $3K_8$ induced by $\mathrm{AG}(3,2)$.

Let $\mathrm{PG}(3,2) = \mathrm{AG}(3,2) \cup \mathcal{H}_\infty$. Then $\mathcal{H}_\infty$ is isomorphic to the Fano plane. Let its points be $0, 1, 2, 3, 4, 5$ and 6 such that for $j = 0, 1, \ldots, 6$, the triples $L_j = (j, j+1, j+3)$ form the lines of the plane, where the addition is taken modulo 7. Now the 3-factors of $\mathcal{F}^2$ can be described in the following way. Let a be a fixed point in $\mathrm{AG}(3,2)$. Then L_j defines a 3-factor F_j^2 whose connected components are complete graphs $K_{2^i} = K_4$. Let $L_{j,a}$ be the complete graph containing a, and let $L_{j,\bar{a}}$ be the other component of F_j^2.

$\mathcal{H}_\infty$ defines one-factors and a one-factorization of K_8 in the following obvious way. The edge joining two points of $\mathrm{AG}(3,2)$, say b and c, belong to the one-factor G_s if and only if b, c and s are collinear points in $\mathrm{PG}(3,2)$. Then $\mathcal{G} = \cup_{s=0}^{6} G_s$ is a one-factorization of K_8.

We can define a decomposable one-factorization of $3K_8$ in the following way. Take $L_{j,a}$ and $L_{j,\bar{a}}$ and let $s \in L_j$ be any point. Then G_s gives a one-factor of $L_{j,a}$ and a one-factor of $L_{j,\bar{a}}$. Hence $\mathcal{G}_j = \cup_{s \in L_j} G_s$ is the union of three one-factors of $3K_8$, and $\mathcal{G}' = \cup_{j=0}^{6} \mathcal{G}_j$ is a one-factorization of $3K_8$.

In $\mathcal{H}_\infty$ there are three lines through the point s, hence $\mathcal{G}'$ contains each one-factor G_s three times. Thus $\mathcal{G}'$ is decomposable, because it is obviously the union of three copies of $\mathcal{G}$.

But we can define an indecomposable one-factorization, too. Let L_j be a line in $\mathcal{H}_\infty$, take $L_{j,a}$ and $L_{j,\overline{a}}$ and let M_j^1 be the one-factor which contains the following pairs of points in $AG(3,2)$:

– (b,c) if $b,c \in L_{j,a}$ and b,c,j are collinear in $PG(3,2)$.

– (b,c) if $b,c \in L_{j,\overline{a}}$ and $b,c,j+1$ are collinear in $PG(3,2)$.

Let M_j^2 be the one-factor which contains the following pairs of points in $AG(3,2)$:

– (b,c) if $b,c \in L_{j,a}$ and $b,c,j+1$ are collinear in $PG(3,2)$.

– (b,c) if $b,c \in L_{j,\overline{a}}$ and $b,c,j+3$ are collinear in $PG(3,2)$.

Finally let M_j^3 be the one-factor which contains the following pairs of points in $AG(3,2)$:

– (b,c) if $b,c \in L_{j,a}$ and $b,c,j+3$ are collinear in $PG(3,2)$.

– (b,c) if $b,c \in L_{j,\overline{a}}$ and b,c,j are collinear in $PG(3,2)$.

Then $\mathcal{M}_j = \cup_{t=1}^3 M_j^t$ is a union of three one-factors of $3K_8$, and $\mathcal{M} = \cup_{j=0}^6 \mathcal{M}_j$ is a one-factorization of $3K_8$.

Suppose that this one-factorization is decomposable. Then it contains a one-factorization $\mathcal{E}$ of K_8. $\mathcal{E}$ is the union of seven one-factors. We may assume without loss of generality, that M_0^1 belongs to $\mathcal{E}$. It contains an edge through a, let it be (a,b), and a pair (c,d) for which the lines ab and cd are parallel lines in $AG(3,2)$. There are two more lines in the parallel class of ab, say ef and gh. It follows from the definition of the one-factors that exactly one of them contains the pairs (e,f) and (a,b), another one contains the pairs (e,f) and (c,d), and a third one contains the pairs (e,f) and (g,h). But $\mathcal{E}$ contains each pair exactly once, hence it must contain the one-factor containing the pairs (e,f) and (g,h). But this is a one-factor of type M_0^t, where $t \neq 1$. Hence $\mathcal{E}$ contains M_0^t where $t = 2$ or 3. If we repeat the previous argument, we get that $\mathcal{E}$ must contain M_0^l for $1 \neq l \neq t$, too. Thus $\mathcal{E}$ is the union of triples of type M_j^t, $t = 1, 2, 3$, but this is a contradiction, because $\mathcal{E}$ consists of seven one-factors.

4 Factorizations arising from projective spaces

There are two basic types of partitioning the point-set of finite projective spaces. Both types give factorizations of some multigraphs. In this section we discuss these constructions.

4.1 Spreads consisting of subspaces

It is easy to construct spreads in $PG^{(i)}(n,q)$, Theorem 2.6 gives a necessary and sufficient existence condition. Packings are much more complicated objects. Only a few packings in $PG^{(1)}(n,q)$ have been constructed so far. In each case of the known packings either n or q satisfies some conditions.

Theorem 4.1 (Beutelspacher, [6])**.** *Let $1 < k$ be an integer and let $n = 2^k - 1$. Then there exists a packing in $PG^{(1)}(n,q)$.*

Theorem 4.2 (Baker, [5])**.** *Let $1 < k$ be an integer. Then there exists a packing in $PG^{(1)}(2k-1,2)$.*

Applying the Basic Construction Lemma, we get the following existence theorems.

Corollary 4.3. *Let q be a prime power, $1 < k$ be an integer and $v = \frac{q^{2^k}-1}{q-1}$. Then there exists a q-factorization of K_v induced by a line-packing in $PG(2^k-1,q)$.*

Corollary 4.4. *Let $1 < k$ be an integer and $v = \frac{q^{2k}-1}{q-1}$. There exists a 2-factorization K_v induced by a line-packing in $\mathrm{PG}(2k-1, 2)$.*

If $k = 2$ then Corollary 4.4 gives a solution of *Kirkman's fifteen schoolgirls problem*, which was first posed in 1850 (for the history of the problem we refer to [7]), while Corollary 4.3 gives a solution of the generalised problem in the case of $(q^2 + 1)(q + 1)$ schoolgirls.

The complete classification of packings in $\mathrm{PG}^{(i)}(n, q)$ is known only in the case $i = 1$, $n = 3$ and $q = 2$. There are 240 projectively distinct packings of lines in $\mathrm{PG}(3, 2)$ (see [20], Subsection 17.5).

If $\gcd(q + 1, 3) = 3$, then there is a construction of aregular spreads in $\mathrm{PG}^{(1)}(3, q)$ due to Bruen and Hirschfeld [11] which is completly different from the constructions of Theorems 4.1 and 4.2. It is based on the geometric properties of twisted cubics.

A normal rational curve of order 3 in $\mathrm{PG}(3, q)$ is called *twisted cubic*. It is known that a twisted cubic is projectively equivalent to the set of points $\{(t^3 : t^2 : t : 1) : t \in \mathrm{GF}(q)\} \cup \{(1 : 0 : 0 : 0)\}$. In [20] it was shown that there exist aregular spreads given by a twisted cubic. For a detailed description of twisted cubics and the proofs of the following theorems we refer to [20], Section 21.

Theorem 4.5. *Let G_q be the group of projectivities in $\mathrm{PG}(3, q)$ fixing a twisted cubic $\mathcal{C}$. Then*

- $G_q \cong \mathrm{PGL}(2, q)$ *and it acts triply transitively on the points of $\mathcal{C}$.*
- *If $q \geq 5$ then the number of twisted cubics in $\mathrm{PG}(3, q)$ is $q^5(q^4 - 1)(q^3 - 1)$.*

Theorem 4.6. *Let $\mathcal{C}$ be a twisted cubic in $\mathrm{PG}(3, q)$. If $\gcd(q + 1, 3) = 3$, then there exists a spread in $\mathrm{PG}^{(1)}(3, q)$ induced by $\mathcal{C}$.*

Using the spreads associated to twisted cubics and the Basic Construction Lemma, we get the following multigraph factorization.

Theorem 4.7. *Let $q \geq 5$ be a prime power, $\lambda = q^5(q^4-1)(q-1)$ and $v = q^3 + q^2 + q + 1$. If $\gcd(q + 1, 3) = 3$, then there exists a simple q-factorization of λK_v induced by the set of twisted cubics in $\mathrm{PG}(3, q)$.*

Proof. Let C be the set of twisted cubics in $\mathrm{PG}(3, q)$. For $\mathcal{C} \in \mathsf{C}$ let $\mathcal{L}_{\mathcal{C}}$ be the spread in $\mathrm{PG}^{(1)}(3, q)$ induced by $\mathcal{C}$. If ℓ is a line and c_ℓ denotes the number of twisted cubics $\mathcal{C}$ with the property that ℓ belongs to $\mathcal{L}_{\mathcal{C}}$, then it follows from Theorem 4.5 that c_ℓ does not depend on ℓ. Hence

$$c_\ell = \frac{|\{\text{twisted cubics in } \mathrm{PG}(3, q)\}| \times |\{\text{lines in a spread of } \mathrm{PG}(3, q)\}|}{|\{\text{lines in } \mathrm{PG}(3, q)\}|}$$
$$= \frac{q^5(q^4 - 1)(q^3 - 1) \times (q^2 + 1)}{(q^2 + 1)(q^2 + q + 1)} = q^5(q^4 - 1)(q - 1).$$

Thus C induces a $|\mathsf{C}|$-fold spread in $\mathrm{PG}^{(1)}(3, q)$. Each spread $\mathcal{L}_{\mathcal{C}}$ induces a q-factor in K_v, therefore the Basic Construction Lemma gives that $\bigcup_{\mathcal{C} \in \mathsf{C}} \mathcal{L}_{\mathcal{C}}$ is a q-factorization of λK_v. Any two distinct twisted cubics define different spreads, hence the factorization is simple by definition. $\square$

4.2   Constructions from subgeometries

If the order of the base field is not prime, then projective spaces can be partitioned by subgeometries. Let $1 < k$ be an integer. Since $\mathrm{GF}(q)$ is a subfield of $\mathrm{GF}(q^k)$, so $\mathrm{PG}(n, q)$ is naturally embedded into $\mathrm{PG}(n, q^k)$ if the coordinate system is fixed. Any $\mathrm{PG}(n, q)$ embedded into $\mathrm{PG}(n, q^k)$ is called a *subgeometry*. Using cyclic projectivities one can prove that any $\mathrm{PG}(n, q^k)$ can be partitioned by subgeometries $\mathrm{PG}(n, q)$. For a detailed description of cyclic projectivities, subgeometries, and the proofs of the following three theorems we refer to [22], Section 4.

Theorem 4.8 ([22], Lemma 4.20). *Let $s(n, q, q^k)$ denote the number of subgeometries* $\mathrm{PG}(n, q)$ *in* $\mathrm{PG}(n, q^k)$. *Then*

$$s(n, q, q^k) = q^{\binom{n+1}{2}(k-1)} \prod_{i=2}^{n+1} \frac{q^{ki}-1}{q^i-1}.$$

Theorem 4.9 ([22], Theorem 4.29). $\mathrm{PG}(n, q^k)$ *can be partitioned into* $\theta(n, q, q^k) = \frac{(q^{k(n+1)}-1)(q-1)}{(q^k-1)(q^{n+1}-1)}$ *disjoint subgeometries* $\mathrm{PG}(n, q)$ *if and only if* $\gcd(k, n+1) = 1$.

Theorem 4.10 ([22], Theorem 4.35). *Suppose that* $\gcd(k, n + 1) = 1$. *Let* $p_0(n, q, q^k)$ *denote the number of projectivities which act cyclically on a* $\mathrm{PG}(n, q)$ *of* $\mathrm{PG}(n, q^k)$ *such that determine different partitions. Then*

$$p_0(n, q, q^k) = q^{k\binom{n+1}{2}} \frac{\prod\limits_{i=1}^{n}(q^{ki} - 1)}{n + 1}.$$

Any given subgeometry $\mathrm{PG}(n, q)$ *is contained in*

$$\rho_0(n, q) = q^{\binom{n+1}{2}} \frac{\prod\limits_{i=1}^{n}(q^{i} - 1)}{n + 1}$$

of these partitions.

We can consider the partitions of the point-set of $\mathrm{PG}(n, q^k)$ by subgeometries $\mathrm{PG}(n, q)$.

Each partition of $\mathrm{PG}(n, q^k)$ into subgeometries $\mathrm{PG}(n, q)$ defines a $\left(\frac{q(q^n-1)}{q-1} \right)$-factor of K_v, with $v = \frac{q^{k(n+1)}-1}{q^k-1}$. Each projectivity which acts cyclically on a $\mathrm{PG}(n, q)$ defines a $\left(\frac{q(q^n-1)}{q-1} \right)$-factorizations of the corresponding complete multigraph.

Theorem 4.11. *Let q be a prime power, $1 < k$ and n be positive integers for which* $\gcd(k, n+1) = 1$ *holds. Let* $\lambda = \frac{q^{\binom{n+1}{2}k}(q^k-1)(q^n-1)}{q^{k-1}(n+1)(q-1)} \prod\limits_{i=1}^{n-1} (q^{ki} - 1)$ *and* $v = \frac{q^{k(n+1)}-1}{q^k-1}$. *Then there exist a simple $\left(q \frac{q^n-1}{q-1} \right)$-factorization of λK_v induced by the set of those projectivities which act cyclically on a* $\mathrm{PG}(n, q)$ *of* $\mathrm{PG}(n, q^k)$ *such that they determine different partitions.*

Proof. It follows from Theorem 4.8 that the number S_e of subgeometries $\mathrm{PG}(n, q)$ through two points of $\mathrm{PG}(n, q^k)$ is

$$S_e = \frac{s(n, q, q^k) \times |\{\text{points in } \mathrm{PG}(n, q)\}| \times (|\{\text{points in } \mathrm{PG}(n, q)\}| - 1)}{|\{\text{points in } \mathrm{PG}(n, q^k)\}| \times (|\{\text{points in } \mathrm{PG}(n, q^k)\}| - 1)}$$
$$= \frac{q^{\binom{n+1}{2}(k-1)}(q^k - 1)}{q^{k-1}(q - 1)} \prod_{i=1}^{n-1} \frac{q^{ki} - 1}{q^i - 1}.$$

Each cyclic projectivity determines different partitions, hence it determines different factors. Thus $\lambda = S_e \times \rho_0(n, q)$. $\qquad\square$

We cannot decide the decomposability of the factorization construted in the previous theorem in general, but we can prove the existence of indecomposable factorizations in some cases. To do this we need the following result from number theory.

Lemma 4.12 ([22], Lemma 4.24). *If r, s and x are positive integers with $x > 1$, then $\frac{(x^{rs}-1)(x-1)}{(x^r-1)(x^s-1)}$ is an integer if and only if $\gcd(r, s) = 1$.*

We apply it in a particular case.

Proposition 4.13. *Let q be a prime power, $1 < k$ and n be positive integers for which $\gcd(k, n + 1) = 1$ and $\gcd(k, n) \neq 1$ hold. Let $d = \gcd\left(\frac{q^{kn}-1}{q^k-1}, \frac{q^n-1}{q-1}\right)$, $v = \frac{q^{k(n+1)}-1}{q^k-1}$ and $m = q\frac{q^n-1}{q-1}$. Suppose that $\mathcal{F}$ is an m-factorization of λK_v for some λ such that each factor is the disjoint union of $\theta(n, q, q^k)$ complete graphs on $(q^{n+1} - 1)/(q - 1)$ vertices. If f denotes the number of m-factors in $\mathcal{F}$ then $\frac{q^n-1}{d(q-1)}$ divides λ and $q^{k-1}\frac{q^{kn}-1}{d(q^k-1)}$ divides f.*

Proof. The standard double counting gives

$$\lambda \times \binom{v}{2} = \binom{m+1}{2} \times \theta(n, q, q^k) \times f,$$

thus $\lambda \times q^{k-1}\frac{q^{kn}-1}{d(q^k-1)} = f \times \frac{q^n-1}{d(q-1)}$. Because of Lemma 4.12, $\frac{q^n-1}{d(q-1)}$ divides λ, hence $q^{k-1}\frac{q^{kn}-1}{d(q^k-1)}$ divides f. $\qquad\square$

As a direct corollary of the previous proposition we get the following result about the indecomposibility of the factorizations constructed in Theorem 4.11 .

Theorem 4.14. *Let q be a prime power, $1 < k$ and n be positive integers for which $\gcd(k, n+1) = 1$ and $\gcd(k, n) \neq 1$ hold. Let $d = \gcd\left(\frac{q^{kn}-1}{q^k-1}, \frac{q^n-1}{q-1}\right)$, $v = \frac{q^{k(n+1)}-1}{q^k-1}$ and $m = q\frac{q^n-1}{q-1}$. Then there exist a simple and indecomposable m-factorization of λK_v, where $\lambda = t\frac{q^n-1}{d(q-1)}$ for some t in $\{1, \ldots, d\frac{q^{\binom{n+1}{2}k}(q^k-1)}{q^{k-1}(n+1)} \prod_{i=1}^{n-1}(q^{ki} - 1)\}$.*

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References

[1] P. Adams, D. Bryant and B. Maenhaut, Cube factorizations of complete graphs, *J. Combin. Des.* **12** (2004), 381–388.

[2] B. Alspach, The Oberwolfach problem, in: C. J. Colbourn and J. H. Dinitz (eds), *CRC Handbook of Combinatorial Designs,* CRC Press, Boca Raton, FL, 1996, 394–395.

[3] D. Archdeacon and J. H. Dinitz, Constructing indecomposable 1-factorizations of the complete multigraph, *Discrete Math.* **92** (1991), 9–19.

[4] A. H. Baartmans and W. D. Wallis, Indecomposable factorizations of multigraphs, *Discrete Math.* **78** (1989), 37–43.

[5] R. D. Baker, Partitioning the planes of $AG_{2m}(2)$ into 2-designs, *Discrete Math.* **15** (1976), 205–211.

[6] A. Beutelspacher, On parallelisms in finite projective spaces, *Geom. Dedicata* **3** (1974), 35–40.

[7] N. L. Biggs, T. P. Kirkman, mathematician, *Bull. London Math. Soc.* **13** (1981), 97–120.

[8] S. Bonvicini, G. Mazzuoccolo and G. Rinaldi, On 2-factorizations of the complete graph: from the k-pyramidal to the universal property, *J. Combin. Des.* **17** (2009), 211–228.

[9] R. C. Bose, A note on the resolvability of balanced incomplete block designs, *Sankhya* **6** (1942), 105–110.

[10] R. H. Bruck, Construction problems in finite projective spaces, in: A. Barlotti (ed.), *Finite geometric structures and their applications*, Edizioni Cremonese, Rome, 1973, 105–188.

[11] A. A. Bruen and J. W. P. Hirschfeld, Applications of line geometry over finite fields, I: The twisted cubic, *Geom. Dedicata* **6** (1977), 495–509.

[12] D. Bryant, D. Horsley, B. Maenhaut and B. R. Smith, Cycle decompositions of complete multigraphs, *J. Combin. Des.* **19** (2011), 42–69.

[13] M. Buratti and G. Rinaldi, 1-rotational k-factorizations of the complete graph and new solutions to the Oberwolfach problem, *J. Combin. Des.* **16** (2008), 87–100.

[14] P. Cameron and J. H. van Lint, *Designs, Graphs, Codes and their Links*, London Mathematical Society Student Texts 22, Cambridge University Press, Cambridge, 1991.

[15] W. Chu, New constructions of simple and indecomposable 1-factorizations of complete multigraphs, *J. Statist. Plann. Inference* **94** (2001), 181–196.

[16] C. J. Colbourn, M. J. Colbourn and A. Rosa, Indecomposable 1-factorizations of the complete multigraph, *J. Austral. Math. Soc. Ser. A* **39** (1985), 334–343.

[17] G. L. Ebert, Partitioning projective geometries into caps, *Canad. J. Math.* **37** (1985), 1163–1175.

[18] M. N. Ferencak and A. J. W. Hilton, Regular multigraphs and their semiregular factorizations, *Congr. Numer.* **209** (2011), 149–159.

[19] A. Hartman and A. Rosa, Cyclic one-factorization of the complete graph, *European. J. Combin.* **6** (1985), 45–48.

[20] J. W. P. Hirschfeld, *Finite projective spaces of three dimensions*, Clarendon Press, Oxford, 1985.

[21] J. W. P. Hirschfeld J. A. Thas, *General Galois geometries*, Clarendon Press, Oxford, 1991.

[22] J. W. P. Hirschfeld, *Projective geometries over finite fields*, second ed., Clarendon Press, Oxford, 1998.

[23] B. C. Kestenband, Projective geometries that are disjoint unions of caps, *Canad. J. Math.* **32** (1980), 1299–1305.

[24] G. B. Khosrovshahi, Ch. Maysoori and B. Tayfeh-Rezaie, A note on 3-factorizations of K_{10}, *J. Combin. Des.* **9** (2001), 379–383.

[25] Gy. Kiss, One-factorizations of complete multigraphs and quadrics in $\mathrm{PG}(n, q)$, *J. Combin. Des.* **10** (2002), 139–143.

[26] G. Korchmáros, A. Siciliano and A. Sonnino, 1-factorizations of complete multigraphs arising from finite geometry, *J. Combin. Theory Ser. A* **93** (2001), 385–390.

[27] B. R. Smith, Cycle decompositions of complete multigraphs, *J. Combin. Des.* **18** (2010), 85–93.

[28] A. Sonnino, One-factorizations of complete multigraphs arising from maximal $(k; n)$-arcs in $\mathrm{PG}(2, 2^h)$, *Discrete Math.* **231** (2001), 447–451.

[29] W. D. Wallis, *One-factorizations*, Mathematics and its Applications, 390, Kluwer Academic Publishers Group, Dordrecht, 1997.