Linear Separation of Connected Dominating Sets in Graphs

(Extended Abstract)

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Abstract

A *connected dominating set* in a graph is a dominating set of vertices that induces a connected subgraph. We introduce and study the class of *connected-domishold graphs*, which are graphs that admit non-negative real weights associated to their vertices such that a set of vertices is a connected dominating set if and only if the sum of the corresponding weights exceeds a certain threshold. We show that these graphs form a non-hereditary class of graphs properly containing two well known classes of chordal graphs: block graphs and trivially perfect graphs. We characterize, in several ways, the graphs every induced subgraph of which is connected-domishold: in terms of forbidden induced subgraphs, in terms of 2-asummability of certain derived Boolean functions, and in terms of the dually Sperner property of certain derived hypergraphs.

Introduction

A possible approach for dealing with the intractability of a given decision or optimization problem is to identify restrictions on input instances under which the problem can still be solved efficiently. One generic framework for describing a kind of such restrictions in case of graph problems is the following: Given a graph G, does G admit non-negative integer weights on its vertices (or edges, depending on the problem) and a set T of integers such that a subset X of its vertices (or edges) has property P if and only if the sum of the weights of elements of X belongs to T? Property P can denote any of the desired substructures we are looking for, such as matchings, cliques, stable sets, dominating sets, etc.

The above framework provides a unified way of describing characteristic properties of several graph classes studied in the literature, such as threshold graphs (Chvátal and Hammer 1977), domishold graphs (Benzaken and Hammer 1978), total domishold graphs (Chiarelli and Milanič 2013a; 2013b) and equistable graphs (Payan 1980; Mahadev, Peled, and Sun 1994). If weights as above exist and are given with the graph, and the set T is given by a membership oracle, then a dynamic programming algorithm can be employed to find a subset with property P of either maximum or minimum cost (according to a given cost function on the vertices/edges) in $O(nM)$ time and with M calls of the membership oracle, where n is the number of vertices (or edges) of G and M is a given upper bound for T (Milanič, Orlin, and Rudolf 2011). The framework can also be applied more generally, in the context on Boolean optimization (Milanič, Orlin, and Rudolf 2011).

In general, the advantages of the above framework depend both on the choice of property P and on the constraints (if any) imposed on the structure of the set T. For example, if P denotes the property of being a stable (independent) set, and set T is restricted to be an interval unbounded from below, we obtain the class of *threshold graphs* (Chvátal and Hammer 1977), which is very well understood and admits several characterizations and linear time algorithms for recognition and for several optimization problems (see, e.g., (Mahadev and Peled 1995)). If P denotes the property of being a dominating set and T is an interval unbounded from above, we obtain the class of *domishold graphs* (Benzaken and Hammer 1978), which enjoy similar properties as the threshold graphs. On the other hand, if P is the property of being a *maximal* stable set and T is restricted to consist of a single number, we obtain the class of *equistable graphs* (Payan 1980), for which the recognition complexity is open (see, e.g., (Levit, Milanič, and Tankus 2012)), no structural characterization is known, and several NP-hard optimization problems remain intractable on this class (Milanič, Orlin, and Rudolf 2011).

Notions and results from the theory of Boolean functions (Crama and Hammer 2011) and hypergraph theory (Berge 1989) can be useful for the study of graph classes defined within the above framework. For instance, the characterization of hereditary total domishold graphs in terms of forbidden induced subgraphs from (Chiarelli and Milanič 2013b) is based on the facts that every threshold Boolean function is 2-asummable (Chow 1961) and that every dually Sperner hypergraph is threshold (Chiarelli and Milanič 2013a). Moreover, the fact that threshold Boolean functions are closed under dualization and can be recognized in polynomial time (Peled and Simeone 1985) leads to efficient algorithms for recognizing total domishold graphs, and for finding a minimum total dominating set in a given total domishold graph (Chiarelli and Milanič 2013a).

It is the aim of this note to present another application of the notions of threshold Boolean functions/hypergraphs to the above graph theoretic framework. More specifically, we introduce and study the case when P is the property of being a connected dominating set and T is an interval unbounded from above. Given a graph $G = (V, E)$, a *connected dominating set* (*c-dominating set* for short) is a subset S of the vertices of G that is *dominating*, that is, every vertex of G is either in S or has a neighbor in S, and *connected*, that is, the subgraph of G induced by S, henceforth denoted by $G[S]$, is connected. The notion of connected dominating sets in graphs is one of the many variants of domination. It finds applications in modeling wireless network connectivity, and has been extensively studied in the literature, see, e.g., the books (Du and Wan 2013; Haynes, Hedetniemi, and Slater 1998b; 1998a), and recent papers (Ananchuen, Ananchuen, and Plummer 2012; Butenko, Kahruman-Anderoglu, and Ursulenko 2011; Chandran et al. 2012; Duckworth and Mans 2009; Fomin, Grandoni, and Kratsch 2008; Karami et al. 2012; Schaudt 2012a; 2012b; Schaudt and Schrader 2012).

Definition 1. *A graph $G = (V, E)$ is said to be* connected-domishold *(*c-domishold *for short) if there exists a pair (w, t) where $w : V \to \mathbb{R}_+$ is a weight function and $t \in \mathbb{R}_+$ is a threshold such that for every subset $S \subseteq V$, $w(S) := \sum_{x\in S} w(x) \geq t$ if and only if S is a connected dominating set in G. A pair (w, t) as above will be referred to as a* connected-domishold (c-domishold) structure *of G.*

Observe that if G is disconnected, then G does not have any c-dominating sets and is thus trivially c-domishold (just set $w(x) = 1$ for all $x \in V(G)$ and $t = |V(G)| + 1$).

Example 1. *The complete graph of order n is c-domishold. Indeed, any nonempty subset $S \subseteq V(K_n)$ is a connected dominating set of K_n, and the pair $(w, 1)$ where $w(x) = 1$ for all $x \in V(K_n)$ is a c-domishold structure of K_n.*

Example 2. *The 4-cycle C_4 is not c-domishold: Denoting its vertices by v_1, v_2, v_3, v_4 in the cyclic order, we see that a subset $S \subseteq V(C_4)$ is c-domishold if and only if it contains an edge. Therefore, if (w, t) is a c-domishold structure of C_4, then $w(v_i) + w(v_{i+1}) \geq t$ for all $i \in \{1, 2, 3, 4\}$ (indices modulo 4), which implies $w(V(C_4)) \geq 2t$. On the other hand, $w(v_1) + w(v_3) < t$ and $w(v_2) + w(v_4) < t$, implying $w(V(C_4)) < 2t$.*

Example 3. *The graph G obtained from C_4 by adding to it a new vertex, say v_5, and making it adjacent exactly to one vertex of the C_4, say to v_4, is c-domishold: the (inclusion-wise) minimal c-dominating sets of G are $\{v_1, v_4\}$ and $\{v_3, v_4\}$, hence a c-domishold structure of G is given by $w(v_2) = w(v_5) = 0$, $w(v_1) = w(v_3) = 1$, $w(v_4) = 2$, and $t = 3$.*

A graph class is said to be *hereditary* if it is closed under vertex deletion. The above examples show that, contrary to the classes of threshold and domishold graphs, the class of connected-domishold graphs is not hereditary. This motivates the following definition:

Definition 2. *A graph G is said to be* hereditary connected-domishold *(*hereditary c-domishold *for* *short) if every induced subgraph of it is connected-domishold.*

In general, for a graph property Π, we will say that a graph is *hereditary* Π if every induced subgraph of it satisfies Π.

Our main result (Theorem 2) is a characterization of hereditary connected-domishold graphs in terms of forbidden induced subgraphs, in terms of 2-asummability of certain derived Boolean functions, and in terms of the dually Sperner property of certain derived hypergraphs. Before stating the result, we give in the next section some preliminary definitions and results.

Preliminary results

Boolean functions. Let n be positive integer. Given two vectors $x, y \in \{0,1\}^n$, we write $x \leq y$ if $x_i \leq y_i$ for all $i \in [n] = \{1, \ldots, n\}$. A Boolean function $f : \{0,1\}^n \to \{0,1\}$ is *positive* (or: *monotone*) if $f(x) \leq f(y)$ holds for every two vectors $x, y \in \{0,1\}^n$ such that $x \leq y$. A positive Boolean function f is said to be *threshold* if there exist non-negative real weights $w = (w_1, \ldots, w_n)$ and a non-negative real number t such that for every $x \in \{0,1\}^n$, $f(x) = 0$ if and only if $\sum_{i=1}^{n} w_i x_i \leq t$. Such a pair (w, t) is called a *separating structure* of f. Every threshold positive Boolean function admits an integral separating structure.

Threshold Boolean functions have been characterized in (Chow 1961) and (Elgot 1960), as follows. For $k \geq 2$, a positive Boolean function $f : \{0,1\}^n \to \{0,1\}$ is said to be k*-summable* if, for some $r \in \{2, \ldots, k\}$, there exist r (not necessarily distinct) false points of f, say, $x^1, x^2, \ldots, x^r$, and r (not necessarily distinct) true points of f, say $y^1, y^2, \ldots, y^r$, such that $\sum_{i=1}^{r} x^i = \sum_{i=1}^{r} y^i$. (A *false point* of f is an input vector $x \in \{0,1\}^n$ such that $f(x) = 0$; a *true point* is defined analogously.) Function f is said to be k*-asummable* if it is not k-summable, and it is *asummable* if it is k-asummable for all $k \geq 2$.

Theorem 1 ((Chow 1961), (Elgot 1960), see also Theorem 9.14 in (Crama and Hammer 2011))**.** *A positive Boolean function f is threshold if and only if it is asummable.*

Hypergraphs. A *hypergraph* is a pair $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a finite set of *vertices* and $\mathcal{E}$ is a set of subsets of $\mathcal{V}$, called *(hyper)edges* (Berge 1989). A *transversal* of $\mathcal{H}$ is a set $S \subseteq \mathcal{V}$ such that $S \cap e \neq \emptyset$ for all $e \in \mathcal{E}$. A hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is *threshold* if there exist a weight function $w : \mathcal{V} \to \mathbb{R}_+$ and a threshold $t \in \mathbb{R}_+$ such that for all subsets $X \subseteq \mathcal{V}$, it holds that $w(X) \leq t$ if and only if X contains no edge of $\mathcal{H}$ (Golumbic 2004). A hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is said to be *Sperner* (or: a *clutter*) if no edge of $\mathcal{H}$ contains another edge, or, equivalently, if for every two distinct edges e and f of $\mathcal{H}$, it holds that $\min\{|e \setminus f|, |f \setminus e|\} \geq 1$. In (Chiarelli and Milanič 2013a), the following class of threshold hypergraphs was introduced.

Definition 3. *(Chiarelli and Milanič 2013a) A hypergraph $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ is said to be* dually Sperner *if for every two distinct edges e and f of $\mathcal{H}$, it holds that* $\min\{|e \setminus f|, |f \setminus e|\} \leq 1$.

Lemma 1. *(Chiarelli and Milanič 2013a) Every dually Sperner hypergraph is threshold.*

The main notion that will provide the link between threshold Boolean functions and hypergraphs is that of separators in graphs. A *separator* in a graph $G = (V, E)$ is a subset $S \subseteq V(G)$ such that $G - S$ is not connected. A separator is *minimal* if it does not contain any other separator. The following characterization of c-dominating sets was proved in (Kanté et al. 2012).

Proposition 1. *(Kanté et al. 2012) In every connected graph $G = (V, E)$, a subset $D \subseteq V$ is a c-dominating set if and only if $D \cap S \neq \emptyset$ for every minimal separator S in G.*

In other words, c-dominating sets are exactly the transversals of the minimal separators. Based on this fact and the fact that threshold Boolean functions are closed under dualization (Crama and Hammer 2011), a similar technique as that used to prove Proposition 4.1 in (Chiarelli and Milanič 2013b) can be used to characterize c-domishold graphs in terms of the thresholdness properties of a derived Boolean function and of a derived hypergraph. Given a graph $G = (V, E)$, its *minimal separator function* is the positive Boolean function $ms_G : \{0,1\}^V \to \{0,1\}$ that takes value 1 precisely on vectors $x \in \{0,1\}^V$ whose *support set* $S(x) := \{v \in V : x_v = 1\}$ contains some minimal separator of G. The *hypergraph of minimal separators* of G is the hypergraph $HmS(G) = (V(G), \mathcal{S}(G))$, where $\mathcal{S}(G) = \{S : S \subseteq V(G)$ and S is a minimal separator in $G\}$.

Proposition 2. *For a connected graph $G = (V, E)$, the following are equivalent:*

1. *G is c-domishold.*
2. *Its minimal separator function ms_G of G is threshold.*
3. *Its hypergraph of minimal separators $HmS(G)$ is threshold.*

Moreover, if $(w_1, \ldots, w_n, t)$ is an integral separating structure of ms_G or of $HmS(G)$, then $(w; \sum_{i=1}^{n} w_i - t)$ with $w(v_i) = w_i$ for all $i \in [n]$ is a c-domishold structure of G.

Definition 4. *A graph G is* ms-dually-Sperner *if its hypergraph of minimal separators $HmS(G)$ is dually Sperner, and* ms-2-asummable *if its minimal separator function ms_G is 2-asummable.*

Proposition 2, Theorem 1 and Lemma 1 imply the following result.

Proposition 3. *Every ms-dually-Sperner graph is c-domishold. Every c-domishold graph is ms-2-asummable.*

Neither of the two statements in Proposition 3 can be reversed. The reader can easily verify that the graph obtained from the complete graph K_4 by gluing a triangle on every edge is c-domishold graph but not ms-dually-Sperner. Moreover, there exists an ms-2-asummable graph G that is not c-domishold. This can be derived using the fact that not every 2-asummable positive Boolean function is threshold (Theorem 9.15 in (Crama and Hammer 2011)), results from (Chiarelli and Milanič 2013b) establishing the connections between threshold Boolean functions and total domishold split graphs, and the observation that a split graph without universal vertices is c-domishold if and only if it is total domishold. (A graph is *split* if its vertex set can be partitioned into a clique and an independent set, where a clique is a set of pairwise adjacent vertices, and an independent set is a set of pairwise non-adjacent vertices.)

Our main result (Theorem 2 below) implies a partial converse of Proposition 3: both statements can be reversed if we require the properties to hold in the stronger, hereditary sense. Our proof of Theorem 2 will rely on the following property of chordal graphs. Recall that a graph G is *chordal* if it does not contain any induced cycle of order at least 4.

Lemma 2. *(Kumar and Madhavan 1998) If S is a minimal separator of a chordal graph G, then each connected component of $G \setminus S$ has a vertex that is adjacent to all the vertices of S.*

Characterizations of hereditary c-domishold graphs

Now we state our main result. Due to space constraints, we only give a part of the proof here.

Theorem 2. *For every graph G, the following are equivalent:*

1. *G is hereditary c-domishold.*
2. *G is hereditary ms-2-asummable.*
3. *G is hereditary ms-dually-Sperner.*
4. *G is a $\{F_1, F_2, H_1, H_2, \ldots\}$-free chordal graph, where the graphs F_1, F_2, and a general member of the family $\{H_i\}$ are depicted in Fig. 1.*

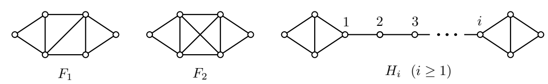

Figure 1: Graphs F_1, F_2, and H_i.

We now examine some of the consequences of the forbidden induced subgraph characterization of hereditary c-domishold graphs given by Theorem 2. The *diamond* and the *kite* (also known as the *co-fork* or the *co-chair*) are the graphs depicted in Fig. 2.

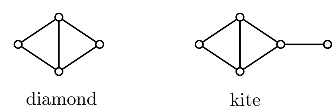

Figure 2: The diamond and the kite.

The equivalence between items (1) and (4) in Theorem 2 implies that the class of hereditary c-domishold graphs is a proper generalization of the class of kite-free chordal graphs.

Corollary 1. *Every kite-free chordal graph is hereditary c-domishold.*

Furthermore, Corollary 1 implies that the class of hereditary c-domishold graphs generalizes two well known classes of chordal graphs, the block graphs and the trivially perfect graphs. A graph is said to be a *block graph* if every block of it is complete. The block graphs are well known to coincide with the diamond-free chordal graphs. A graph G is said to be *trivially perfect* (Golumbic 1978) if for every induced subgraph H of G, it holds $\alpha(H) = |\mathcal{C}(H)|$, where $\alpha(H)$ denotes the *independence number* of H (that is, the maximum size of an independent set in H), and $\mathcal{C}(H)$ denotes the set of all maximal cliques of H. Trivially perfect graphs coincide with the so-called *quasi-threshold graphs* (Yan, Chen, and Chang 1996), and also with the $\{P_4, C_4\}$-free graphs (Golumbic 1978).

Corollary 2. *Every block graph is hereditary c-domishold. Every trivially perfect graph is hereditary c-domishold.*

Sketch of proof of Theorem 2. The implications $(3) \Rightarrow (1) \Rightarrow (2)$ follow from Proposition 3.

For the implication $(2) \Rightarrow (4)$, we only need to verify that none of the graphs in the set $\mathcal{F} := \{C_k : k \geq 4\} \cup \{F_1, F_2\} \cup \{H_i : i \geq 1\}$ is ms-2-asummable. Let $F \in \mathcal{F}$. Suppose first that F is a cycle C_k for some $k \geq 4$, let u_1, u_2, u_3, u_4 be four consecutive vertices on the cycle. For a set $S \subseteq V(F)$, let $x^S \in \{0,1\}^{V(F)}$ denote the characteristic vector of S, defined by $x^S_i = 1$ if and only if $i \in S$. Let $A = \{u_1, u_3\}$, $B = \{u_2, u_4\}$, $C = \{u_1, u_2\}$ and $D = \{u_3, u_4\}$. Then, A and B are minimal separators of F, while C and D do not contain any minimal separator of F. Therefore, x^A and x^B are true points of the minimal separator function ms_F, while x^C and x^D are false points of ms_F. Since $x^A + x^B = x^C + x^D$, the minimal separator function ms_F is 2-summable. If $F \in \{F_1, F_2\} \cup \{H_i : i \geq 1\}$, then let a and b be the two vertices of degree 2 in F, let $N(a) = \{a_1, a_2\}$, $N(b) = \{b_1, b_2\}$, let $A = N(a)$, $B = N(b)$, $C = \{a_1, b_1\}$ and $D = \{a_2, b_2\}$. The rest of the proof is the same as above.

It remains to show the implication $(4) \Rightarrow (3)$. Since the class of $\{F_1, F_2, H_1, H_2, \ldots\}$-free chordal graphs is hereditary, it is enough to show that every $\{F_1, F_2, H_1, H_2, \ldots\}$-free chordal graph is ms-dually-Sperner. Suppose for a contradiction that there exists a $\{F_1, F_2, H_1, H_2, \ldots\}$-free chordal graph $G = (V, E)$ that is not ms-dually-Sperner. If G is disconnected, then $HmS(G) = (V(G), \{\emptyset\})$ and G is ms-dually-Sperner. Thus, G is connected. Since G is not ms-dually-Sperner, there exist two minimal separators in G, say S and S', such that $\min\{|S|, |S'|\} \geq 2$. Let $C = \{a, b\}$ for some $a, b \in S \setminus S'$ with $a \neq b$ and let $C' = \{a', b'\}$ for some $a', b' \in S' \setminus S$ with $a' \neq b'$. Further, let X, Y be two components of $G - S$ and X', Y' two components of $G - S'$. By Lemma 2, there exist vertices $x \in X$ and $y \in Y$ such that each of x and y dominates S and $x' \in X'$ and $y' \in Y'$ such that each of x' and y' dominates S'. Define $Z = \{x, y\}$ and $Z' = \{x', y'\}$.

Claim 1. *Either $N(x) \cap \{a', b'\} = \emptyset$ or $N(y) \cap \{a', b'\} = \emptyset$. Similarly, either $N(x') \cap \{a, b\} = \emptyset$ or $N(y') \cap \{a, b\} = \emptyset$.*

Proof. If $N(x) \cap \{a', b'\} \neq \emptyset$ and $N(y) \cap \{a', b'\} \neq \emptyset$, then there exists an (x, y)-path in $G - S$, contrary to the fact that S is an (x, y)-separator. The other statement follows similarly. □

Notice that $C \cap C' = C \cap Z = C' \cap Z' = \emptyset$ and so $|C \cup C'| = |C \cup Z| = |C' \cup Z'| = 4$. Further, since every minimal separator in a chordal graph is a clique (Dirac 1961), C and C' are cliques. On the other hand, Z and Z' are independent sets, therefore $|C \cap Z'| \leq 1$ and $|C' \cap Z| \leq 1$.

Claim 2. $|N(C') \cap Z| \leq 1$ *and* $|N(C) \cap Z'| \leq 1$.

Proof. If $|N(C') \cap Z| > 1$ then $Z \subseteq N(C')$. Since $C' \cap S = \emptyset$, this implies that x and y are in the same connected component of $G - S$, a contradiction. The other statement follows by symmetry. □

Claim 2 implies that $Z \neq Z'$. Up to symmetry, it remains to analyze five cases, depending whether the sets C, C', Z, Z' have vertices in common (where possible) or not. In what follows we use the notation $u \sim v$ (resp. $u \nsim v$) to denote the fact that two vertices u and v are adjacent (resp. non-adjacent).

Case 1. $|C \cap Z'| = |Z \cap Z'| = 1$.

Without loss of generality, we may assume that $a = x'$. Since $C \cap Z = \emptyset$ and $a = x'$ it follows that $x' \notin Z$, implying $Z \cap Z' = \{y'\}$. Without loss of generality, we may assume that $y' = y$. But the fact that $y \sim a$ implies $y' \sim x'$, leading to a contradiction.

The case $|C' \cap Z| = |Z \cap Z'| = 1$ is symmetric to Case 1.

Case 2. $|Z \cap Z'| = 1$ and $C \cap Z' = C' \cap Z = \emptyset$.

Without loss of generality, we may assume that $x = x'$. Since $a, b \notin S'$ and S' separates x' and y', we conclude that $N(y') \cap \{a, b\} = \emptyset$. By symmetry, $N(y) \cap \{a', b'\} = \emptyset$, and consequently, $y' \notin S$ and $y \notin S'$. Since S separates x and y and $\{a', b', y'\} \cap S = \emptyset$, we have $N(y) \cap \{a', b', y'\} = \emptyset$, and, similarly, $N(y') \cap \{a, b, y\} = \emptyset$.

We must have $y \nsim y'$ since otherwise G contains either an induced C_4 on the vertex set $\{y, a, a', y'\}$ (if $a \sim a'$) or an induced C_5 on the vertex set $\{y, a, x = x', a', y'\}$ (otherwise).

To avoid an induced copy of H_1 on the vertex set $\{y, a, b, x = x', a', b', y'\}$, we may assume, without loss of generality, that $a \sim a'$.

Suppose first that $b \sim b'$. Then also $a \sim b'$ or $a' \sim b$, since otherwise $\{a, a', b', b\}$ would induce a copy of C_4. But now (depending if we have one edge or both) the vertex set $\{y, a, b, a', b', y'\}$ induces a copy of either F_1 or of F_2. Therefore, $b \nsim b'$.

Suppose that $a' \sim b$. But now, either the vertex set $\{y, a, b, x = x', a', b'\}$ induces a copy of F_2 (if $a \nsim b'$), or the vertex set $\{y, a, b, a', b', y'\}$ induces a copy of F_1 (if $a \sim b'$). Therefore, $a' \nsim b$, and by symmetry, $a \nsim b'$. But now, the vertex set $\{y, a, b, x = x', a', b'\}$ induces a copy of F_1, a contradiction.

Case 3. $|C \cap Z'| = |C' \cap Z| = 1$ and $Z \cap Z' = \emptyset$.

Without loss of generality, we may assume that $a = x'$ and $a' = x$. By Claim 1 it follows that $b \nsim y'$. The fact that $y \nsim x = a'$ implies $y \notin S'$ (since S' is a clique) and consequently also $y \nsim y'$ (otherwise $x' = a$ and y' would be in the same component of $G - S'$). To avoid an (x, y)-path in $G - S$, we conclude that $y \nsim b'$. Now, the vertices $\{a = x', a' = x, b, b', y, y'\}$ induce either a copy of F_1 (if $b \nsim b'$) or of F_2 (otherwise). In either case, we reach a contradiction.

Case 4. $|C \cap Z'| = 1$ and $C' \cap Z = Z \cap Z' = \emptyset$.

Without loss of generality, we may assume that $a = x'$. By Claim 2, we have $|N(C') \cap Z| \leq 1$. Thus, we may assume that $x \notin N(C')$. Consequently, $N(x) \cap \{a', b'\} = \emptyset$ and therefore also $x \nsim y'$, for otherwise we would have a C_4 induced by $\{a, a', y', x\}$. To avoid an (x', y')-path in $G - S'$, we conclude that $b \nsim y'$. Moreover, we also have $N(b) \cap \{a', b'\} = \emptyset$, since otherwise the vertex set $\{x, a = x', a', b, b', y'\}$ induces either a copy of F_1 (if $|N(b) \cap \{a', b'\}| = 1$) or of F_2 (otherwise). If $y \sim y'$, then, to avoid an induced C_4 on $\{y, a = x', a', y'\}$, we conclude that $y \sim a'$. But now we have a copy of F_1 induced by $\{x, b, a = x', a', y, y'\}$, a contradiction. Thus, $y \nsim y'$, implying also $N(y) \cap \{a', b'\} = \emptyset$, since otherwise the vertex set $\{b, a = x', y, a', b', y'\}$ induces either a copy of F_1 (if $|N(y) \cap \{a', b'\}| = 1$) or of F_2 (otherwise).

Since neither of the vertices a', b' and y' is adjacent to b and S is a clique containing b, we conclude that $\{a', b', y'\} \cap S = \emptyset$. In particular, if K denotes the component of $G - S$ containing a', this implies $b', y' \in V(K)$. By Lemma 2, there exists a vertex $w \in V(K)$ that dominates S. Since S separates x from y, we have $\{x, y\} \nsubseteq K$; without loss of generality, we may assume that $y \notin V(K)$. Let $P = (w_1, w_2, \ldots, w_k)$ be a shortest $\{a', b', y'\}$–w path in K where $w_1 \in \{a', b', y'\}$ and $w_k = w$. Since w_1 is not adjacent to b but w_k is, we have $k > 1$.

Suppose that $k = 2$. If $w \nsim y'$, then the vertex set $\{b, a = x', w, a', b', y'\}$ induces either a copy of F_1 (if $|N(w) \cap \{a', b'\}| = 1$) or of F_2 (otherwise), a contradiction. Hence $w \sim y'$. To avoid an induced C_4 on the vertices $\{w, a = x', a', y'\}$, we conclude that $w \sim a'$. But now, the vertex set $\{y, a = x', b, w, a', y'\}$ induces a copy of F_1, a contradiction. Therefore, $k \geq 3$.

To avoid an induced copy of a cycle of order at least 4, we conclude that vertex $a = x'$ dominates P. If $y' \sim w_2$ then also $a' \sim w_2$ and $b' \sim w_2$ (or otherwise we would have an induced C_4 on the vertex set $\{a = x', w_2, b', y'\}$ or $\{a = x', w_2, a', y'\}$) but that gives us an induced F_1 on the vertex set $\{y', a', a = x', w_2, w_3, w_4\}$ (where $w_4 = b$ if $k = 3$). Therefore, $y' \nsim w_2$. Without loss of generality, we may assume that $w_1 = a'$. But now, the vertex set $\{y', a' = w_1, b', a = x', w_2, w_3\}$ induces a copy of either F_1 (if $b' \nsim w_2$) or of F_2 (otherwise), a contradiction.

The case $|C' \cap Z| = 1$ and $C \cap Z' = Z \cap Z' = \emptyset$ is symmetric to Case 4.

It remains to consider the case when $C' \cap Z = C \cap Z' = Z \cap Z' = \emptyset$. This case is analyzed depending on the number of edges between $\{x, y\}$ and $\{x', y'\}$, and on the number of edges between $\{a, b\}$ and $\{a', b'\}$. Due to space constraints, we omit here the analysis of this case. $\square$

Discussion

We conclude with a brief discussion on algorithmic aspects of (hereditary) c-domishold graphs. The characterization of hereditary c-domishold graphs in terms of forbidden induced subgraphs given by Theorem 2 can be used to develop a polynomial time recognition algorithm for this class. Moreover, since the set of all minimal separators of a given graph can be computed in output-polynomial time (Shen and Liang 1997), c-domishold graphs can be recognized in polynomial time in any class of graphs with only polynomially many minimal separators, by applying Proposition 2, and verifying whether the minimal separator function ms_G is threshold (which can be done in polynomial time using the algorithm from (Peled and Simeone 1985)). Since hereditary c-domishold graphs are chordal, and chordal graphs have a linear number of minimal separators (Chandran and Grandoni 2006), the above approach can also be used to compute a c-domishold structure of a given hereditary c-domishold graph.

We leave open the recognition problem for the general case.

Problem 1. *Determine the computational complexity of recognizing c-domishold graphs.*

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