

Spectral characterizations of dumbbell graphs

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Abstract

A *dumbbell graph*, denoted by $D_{a,b,c}$, is a bicyclic graph consisting of two vertex-disjoint cycles C_a , C_b and a path P_{c+3} ($c \geq -1$) joining them having only its end-vertices in common with the two cycles. In this paper, we study the spectral characterization w.r.t. the adjacency spectrum of $D_{a,b,0}$ (without cycles C_4) with $\gcd(a, b) \geq 3$, and we complete the research started in [J.F. Wang *et al.*, A note on the spectral characterization of dumbbell graphs, *Linear Algebra Appl.* 431 (2009) 1707–1714]. In particular we show that $D_{a,b,0}$ with $3 \leq \gcd(a, b) < a$ or $\gcd(a, b) = a$ and $b \neq 3a$ is determined by the spectrum. For $b = 3a$, we determine the unique graph cospectral with $D_{a,3a,0}$. Furthermore we give the spectral characterization w.r.t. the signless Laplacian spectrum of all dumbbell graphs.

1 Introduction

Let $G = (V(G), E(G))$ be a graph with order $|V(G)| = n(G) = n$ and size $|E(G)| = m(G) = m$. Let $A(G)$ be the $(0,1)$ -adjacency matrix of G and $d_G(v) = d(v)$ the degree of the vertex v . The polynomial $\phi(G, \lambda) = \det(\lambda I - A(G))$ or simply $\phi(G)$, where I is the

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identity matrix, is defined as the *characteristic polynomial* of G , which can be written as $\phi(G) = \lambda^n + a_1(G)\lambda^{n-1} + a_2(G)\lambda^{n-2} + \dots + a_n(G)$. Since $A(G)$ is real and symmetric, its eigenvalues are all real numbers. Assume that $\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_n(G)$ are the *adjacency eigenvalues* of the graph G . The *adjacency spectrum* of G , denoted by $\text{Spec}(G)$, is the multiset of its adjacency eigenvalues.

Together with the adjacency spectrum, shortly denoted by *A-spectrum*, we will consider the *Q-spectrum*, defined similarly but with respect to the signless Laplacian matrix $Q(G) = A(G) + D(G)$, where $D(G)$ is the diagonal matrix of vertex degrees (of G). The same applies for eigenvalues, characteristic polynomial, and the corresponding notation differs by a prefix (*A-* or *Q-*, respectively). The characteristic polynomials of the matrices $A(G)$ and $Q(G)$ will be denoted by $\phi(G, \lambda)$ and $\varphi(G, \lambda)$, respectively; we will omit the variable if it is clear from the context. According to [3, 4, 5], all these approaches (with different matrices M) fit into the so called *M-theory* of graph spectra, and moreover there are some very helpful analogies between them.

In this paper, let M be the adjacency matrix A or the signless Laplacian matrix Q . Two graphs are said to be *M-cospectral* (or that they are *M-cospectral mates*) if they have equal M -spectrum, i.e. equal M -characteristic polynomial. A graph is said to be *determined by its M-spectrum*, or shortly *DMS*, if there is no other non-isomorphic graph with the same M -spectrum. Numerous examples of *M-cospectral* but non-isomorphic graphs, known as *M-PINGS*, are reported in the literature (see Chapter 6 in [2] for example). On the other hand, only a few graphs with very special structure have been proved to be determined by their M -spectra. For the background and some known results about this problem and related topics, we refer the readers to the excellent surveys [6, 7] and the references therein.

As usual, let C_n and P_n be, respectively, the *cycle*, and the *path* of order n . For two graphs G and H , $G \cup H$ denotes the disjoint union of G and H . Let $T_{a,b,c}$ denote the tree with exactly one vertex v having maximum degree 3 such that $T_{a,b,c} - v = P_a \cup P_b \cup P_c$. The *lollipop graph*, denoted by $L_{g,p}$ (note, in [10] $L_{g,p}$ is denoted by $H_{g+p,g}$), is obtained by appending a cycle C_g to a pendant vertex of a path P_{p+1} . The θ -*graph*, denoted by θ_{a_1,b_1,c_1} ($a_1 \leq b_1 \leq c_1$ and $(a_1, b_1) \neq (0, 0)$), is a graph consisting of two given vertices joined by three vertex disjoint paths whose orders are a_1 , b_1 and c_1 , respectively. The *dumbbell graph* $D_{a,b,c}$ consists of two vertex-disjoint cycles C_a, C_b and a path P_{c+3} ($c \geq -1$) joining them having only its end-vertices in common with the cycles (see Fig. 1). A graph G is said to be *almost regular* if $|d(v_i) - d(v_j)| \leq 1$ for any $v_i, v_j \in V(G)$. Clearly, there are two types of such graphs: one is the regular graph and the other one is called $(r, r+1)$ -*almost regular* graph, i.e., its vertex set can be partitioned into two subsets V_1 and V_2 such that $d(v_i) = r$ for $v_i \in V_1$ and $d(v_j) = r+1$ for $v_j \in V_2$. Note, there are exactly two kinds of $(2,3)$ -almost regular graphs such that $m = n+1$, and such graphs are the dumbbell graphs or the θ -graphs with eventually cycles as connected components.

In [10] and [1], the authors shown that all lollipop graphs are DAS. In [11] the authors shown that all θ -graphs with no unique cycle C_4 are DAS. In [12], we investigated the *A-spectral* characterization of dumbbell graphs without cycles C_4 and we left the special case $D_{a,b,0}$ with $\delta = \gcd(a, b) \geq 3$.

In this paper, we will show that $D_{a,b,0}$ with $\delta = \gcd(a,b) \geq 3$ is DAS if and only if $\delta \neq a$ or $\delta = a$ and $b \neq 3a$. For $b = 3a$ ($a \neq 4$) we determine the unique graph A -cospectral with $D_{a,3a,0}$, that is $\theta_{1,a-1,2a-1} \cup C_a$.

Furthermore we deduce from our main result the Q -spectral characterization of dumbbell graphs. In particular we prove that all dumbbell graphs $D_{a,b,c} \neq D_{a,3a,-1}$ are DQS, while $D_{a,3a,-1}$ is Q -cospectral just with $\theta_{0,a-1,2a-1} \cup C_a$.

The paper is organized as follows. In Section 2 we give a few basic results that will be used later. In Section 3 we restrict the structure of tentative A -cospectral mates with $D_{a,b,0}$. In Section 4 we give the A -spectral characterization of $D_{a,b,0}$ and we give the general result on the A -spectral characterization of $D_{a,b,c}$ without cycle C_4 as subgraph. Finally in Section 5, we give the Q -spectral characterization of $D_{a,b,c}$. Note that in order to keep the notation easier to read, we will omit the prefix A - in Sections 2, 3 and 4 since the latter sections are concerning just with the A -theory of graph spectra, while we again make use of the prefixes A - and Q - in Section 5.

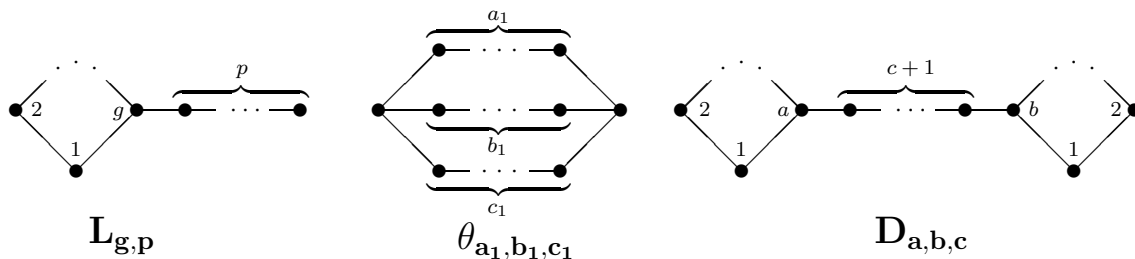


Fig. 1: The graphs $L_{g,p}$, θ_{a_1,b_1,c_1} and $D_{a,b,c}$.

Remark 1. Due to the symmetry, let $0 \leq a_1 \leq b_1 \leq c_1$ in the graph θ_{a_1,b_1,c_1} and $3 \leq a \leq b$ and $c \geq -1$ in the graph $D_{a,b,c}$.

2 Basic results

Some useful established results about the (A -)spectrum are presented in this section, which will play an important role throughout this paper. Recall that the prefix A - is omitted in this section.

Lemma 2.1 (Interlacing Theorem). *Let the eigenvalues of graphs G and $G - v$ be, respectively, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1}$, then $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n$.*

Lemma 2.2 (Schwenk's formulas). [2] *Let G be a (simple) graph. Denote by $\mathcal{C}(v)$ ($\mathcal{C}(e)$) the set of all cycles in G containing a vertex v (resp. an edge $e = uv$). Then we have:*

$$(i) \quad \phi(G, x) = x\phi(G - v, x) - \sum_{w \sim v} \phi(G - v - w, x) - 2 \sum_{C \in \mathcal{C}(v)} \phi(G - V(C), x)$$

$$(ii) \quad \phi(G, x) = \phi(G - e, x) - \phi(G - v - u, x) - 2 \sum_{C \in \mathcal{C}(e)} \phi(G - V(C), x).$$

We assume that $\phi(G, x) = 1$ if G is the empty graph (i.e. with no vertices).

Lemma 2.3. [2] *Let C_n and P_n be the cycle and the path on n vertices, respectively. Then*

- (i) $\phi(C_n) = \prod_{j=1}^n (\lambda - 2 \cos \frac{2\pi j}{n})$ and $\lambda_1(C_n) = 2$,
- (ii) $\phi(P_n) = \prod_{j=1}^n (\lambda - 2 \cos \frac{\pi j}{n+1})$ and $\lambda_1(P_n) < 2$.

Lemma 2.4. [6] *Let G and H be two graphs with the same spectrum w.r.t. A or Q . Then*

- (i) $n(G) = n(H)$;
- (ii) $m(G) = m(H)$.

Lemma 2.5. [8] $\phi(P_n, 2) = n + 1$ and $\phi(T_{a,b,c}, 2) = a + b + c + 2 - abc$.

From the above lemma, in [12] we got the following result.

Lemma 2.6. $2 \in \text{Spec}(D_{a,b,c})$ if and only if $c = 0$. Moreover, the multiplicity of 2's is one.

The following result describes the structure of tentative cospectral mates of almost regular graphs non containing cycles C_4 as subgraphs.

Theorem 2.7. [12] *Let two graphs H and G such that $\text{Spec}(H) = \text{Spec}(G)$, where G contains no the cycle C_4 as its subgraph. If G is a $(r, r + 1)$ -almost regular graph, then*

- (i) H contains no the cycle C_4 as its subgraph;
- (ii) H is a $(r, r + 1)$ -almost regular graph with the same degree sequence as G .

3 Preliminary results

In this section we will restrict the structure of H , the tentative (A -)cospectral mate of $D_{a,b,0}$. Recall that the prefix A - is omitted in this section. Note that from Theorem 2.7, H can be a dumbbell graph, a θ -graph, a disjoint union of a dumbbell graph and cycles, a disjoint union of a θ -graph and cycles. Since $D_{a,b,0}$ has 2 as an eigenvalue of multiplicity 1 (cf. Lemma 2.6) then H contains at most one cycle as connected component. Furthermore, the tentative connected cospectral mates are immediately discarded by the two following lemmas (see, for example, [12]).

Lemma 3.1. [11] *There is no θ -graph cospectral with a dumbbell graph.*

Lemma 3.2. [12] *No two non-isomorphic dumbbell graphs are cospectral.*

In [12] we considered the spectral characterization of dumbbell graphs. Our main result reads:

Theorem 3.3. *The graphs $D_{a,b,c}$, without cycles C_4 , with $c \neq 0$ or $c = 0$ and $\gcd(a, b) \leq 2$ are determined by their adjacency spectrum.*

Our aim in this paper is to study the spectral characterization of the remaining cases of Theorem 3.3, i.e. $D_{a,b,0}$ with $\gcd(a, b) \geq 3$. So in the rest of the paper we set $\delta = \gcd(a, b)$ and $\delta \geq 3$.

To prove the next lemmas we will rely on the Schwenk's formulas and the Interlacing Theorem. The main idea is the following: if a graph has some eigenvalues of multiplicity greater than 2, then these eigenvalues must appear at least once in all subgraphs obtained by deleting a vertex (from Interlacing Theorem). Hence, we can check the multiplicity of these eigenvalues of vertex deleted subgraphs by substituting them into the characteristic polynomial of the parent graph (by using the Schwenk's formulas). The following lemma characterizes the spectrum of $D_{a,b,0}$ (with $\delta \geq 3$).

Lemma 3.4. *The spectrum of $D_{a,b,0}$ with $\delta = \gcd(a, b) \geq 3$ consists of the eigenvalues of C_δ (except 2 and -2) with multiplicity 3, the eigenvalues of C_a and C_b not in C_δ with multiplicity 1 and all the other eigenvalues must strictly interlace the eigenvalues of $C_a \cup C_b$ and have multiplicity 1 as well.*

Proof. If we consider the Interlacing Theorem (Lemma 2.1) applied to the unique cut-vertex u of degree 2 in $D_{a,b,0}$ we get that if $\bar{\lambda}$ is of multiplicity ≥ 2 then $\bar{\lambda} \in \text{Spec}(C_a) \cup \text{Spec}(C_b)$. Consider now the Lemma 2.2(i) applied to u . We get:

$$\phi(D_{a,b,0}) = x\phi(C_a)\phi(C_b) - \phi(C_a)\phi(P_{b-1}) - \phi(P_{a-1})\phi(C_b). \quad (1)$$

Now take $\bar{\lambda} \in \text{Spec}(C_\delta)$ and $\bar{\lambda} \neq \pm 2$, it is easy to check that such a $\bar{\lambda}$ is 4 times solution of $\phi(C_a)\phi(C_b)$, 3 times solution of $\phi(C_a)\phi(P_{b-1})$ and 3 times solution of $\phi(P_{a-1})\phi(C_b)$. Consequently $\bar{\lambda} \in \text{Spec}(C_\delta)$ ($\lambda \neq \pm 2$) implies that $\bar{\lambda}$ is of multiplicity 3 for $D_{a,b,0}$. If $\bar{\lambda} = 2$, then 2 is a simple root of (1) (see also Lemma 2.6); note also that $\lambda_2(D_{a,b,0}) = 2$ (by Interlacing Theorem). If $\bar{\lambda} = -2 \in \text{Spec}(C_\delta)$, then -2 is a simple root of (1) as well.

Take now $\bar{\lambda} \in \text{Spec}(C_a) \cup \text{Spec}(C_b) \setminus \text{Spec}(C_\delta)$, note that $\text{Spec}(C_a) \cap \text{Spec}(C_b) = \text{Spec}(C_\delta)$ (see Lemma 2.3(i)). Similarly to above we can say that such a $\bar{\lambda}$ is an eigenvalue of multiplicity 1 for $D_{a,b,0}$.

Since all multiple eigenvalues of $D_{a,b,0}$ must come from $\text{Spec}(C_a) \cup \text{Spec}(C_b)$, then, by Interlacing Theorem, all remaining eigenvalues must interlace the eigenvalues of $C_a \cup C_b$ and be of multiplicity 1.

This ends the proof. □

From the above lemma we know to some extent the spectrum of $D_{a,b,0}$. If H is a tentative cospectral mate of $D_{a,b,0}$, then H cannot be connected (by Theorem 2.7 and Lemmas 3.1 and 3.2). Furthermore by Lemma 2.6 (cf. also Lemma 3.4), we know that 2 is simple and the second largest eigenvalue of $D_{a,b,0}$. The latter implies that H can be of two kinds: a θ -graph with a cycle or a dumbbell graph with a cycle. The eigenvalues of multiplicity 3 of $D_{a,b,0}$ (recall that they belong to C_δ , by Lemma 3.4) will force the latter mentioned cycles to be C_δ . This fact will be proved in the following lemmas.

Lemma 3.5. *If $H = D_{a',b',c'} \cup C_{p'}$ is cospectral with $D_{a,b,0}$, then $c' = -1$ and $p' = \delta$.*

Proof. Recall that, by Lemma 2.6, 2 is a simple eigenvalue of $D_{a,b,0}$. Assume that $\phi(D_{a,b,0}) = \phi(H)$. Since H contains a cycle then 2 appears already as an eigenvalue and, consequently, $D_{a',b',c'}$ cannot have 2 as its eigenvalue. By Lemma 2.6 we get $c' \neq 0$.

Assume that $c' > 0$. Considering Lemma 2.1 applied to the cut-vertex of degree 2 of $D_{a,b,0}$, we get that $\lambda_1(D_{a,b,0}) > \lambda_1(C_b) \geq \lambda_2(D_{a,b,0}) \geq \lambda_1(C_a)$. Consider now H , it is easy to see, by using the above argument, that its second largest eigenvalue is (strictly) greater than 2 whenever $c' > 0$, which is a contradiction.

Take now $c' = -1$. So $H = D_{a',b',-1} \cup C_{p'}$. Recall that by Lemma 3.4, we know that the spectrum of $D_{a,b,0}$ contains the eigenvalues of C_δ (except ± 2) with multiplicity 3 and the remaining eigenvalues are simple. It is easy to see that p' divides δ , otherwise H has some eigenvalues of multiplicity at least 2 not appearing in $D_{a,b,0}$. Assume, for a contradiction, that $p' < \delta$. If so, $H \setminus C_{p'} = D_{a',b',-1}$ has at least an eigenvalue $\bar{\lambda}$ of multiplicity 3. By Lemma 2.2(ii) applied at the (unique) bridge of $D_{a',b',-1}$, we have

$$\phi(D_{a',b',-1}) = \phi(C_{a'})\phi(C_{b'}) - \phi(P_{a'-1})\phi(P_{b'-1}). \quad (2)$$

By Lemma 2.1 applied at the vertex of degree 3 in $C_{b'}$, we have that $\bar{\lambda} \in \text{Spec}(C_{a'})$, and by the same lemma applied at the other vertex of degree 3 we have that $\bar{\lambda} \in \text{Spec}(C_{b'})$. Hence from (2), $\bar{\lambda}$ is exactly of multiplicity 2 in $D_{a',b',-1}$, that is a contradiction. So the eigenvalues of $H \setminus C_{p'} = D_{a',b',-1}$ are simple and, consequently, it must be $\delta = p'$. \square

Lemma 3.6. *Let $L_{g,p}$ be a lollipop. If $\bar{\lambda} \in \text{Spec}(L_{g,p})$ is of multiplicity greater than 1, then its multiplicity is exactly 2 and $\bar{\lambda} \in \text{Spec}(C_g) \cap \text{Spec}(P_{p-1})$.*

Proof. Recall that from Lemma 2.3 we have the following facts: if $\lambda \in \text{Spec}(C_n)$ then $\lambda \in \text{Spec}(P_{n-1})$; if $\lambda \in \text{Spec}(P_n)$ then $\lambda \notin \text{Spec}(P_{n-1})$; if $\lambda \in \text{Spec}(P_n)$ then λ is of multiplicity 1.

Assume that $\bar{\lambda}$ is of multiplicity at least 2 for $L_{g,p}$. By the Interlacing Theorem applied at the vertex of degree 2 in the path adjacent to the vertex of degree 3, $\bar{\lambda}$ must be an eigenvalue of C_g or of P_{p-1} . Consider now the Schwenk formula for edges (Lemma 2.2(ii)) at the bridge between the path and the cycle in $L_{g,p}$. We have

$$\phi(L_{g,p}) = \phi(C_g)\phi(P_p) - \phi(P_{g-1})\phi(P_{p-1}). \quad (3)$$

It is easy to see that if $\bar{\lambda}$ is an eigenvalue of C_g in (3) then such an eigenvalue is an eigenvalue of P_{p-1} (recall that $\bar{\lambda}$ is of multiplicity at least 2) as well, while if $\bar{\lambda}$ is an eigenvalue of P_{p-1} then (3) holds if and only if such an eigenvalue belongs to $\text{Spec}(C_g)$ as well. So we can conclude that $\bar{\lambda} \in \text{Spec}(C_g) \cap \text{Spec}(P_{p-1})$. Finally, it is easy to observe that such a $\bar{\lambda} \in \text{Spec}(C_g) \cap \text{Spec}(P_{p-1})$ is a solution of (3) exactly twice.

This ends the proof. \square

Lemma 3.7. *Let $\bar{\lambda}$ be an eigenvalue of multiplicity at least 3 for θ_{a_1,b_1,c_1} . Then the multiplicity of $\bar{\lambda}$ is exactly 3, a_1 , b_1 and c_1 are odd integers and $\bar{\lambda} = 0$.*

Proof. Let $\bar{\lambda}$ be an eigenvalue of multiplicity at least 3 for θ_{a_1, b_1, c_1} , then, by the Interlacing theorem (Lemma 2.1), $\bar{\lambda}$ is an eigenvalue of multiplicity (at least) 2 in all vertex deleted subgraphs of θ_{a_1, b_1, c_1} . Assume that the multiplicity of $\bar{\lambda}$ is strictly greater than 3, then $\bar{\lambda}$ is of multiplicity at least 3 in all vertex deleted subgraphs, including the lollipop graphs and cycles, but by Lemmas 3.6 and 2.3 we have that this is impossible. So in the rest we assume that $\bar{\lambda}$ is of multiplicity exactly 3.

Assume first that $a_1 > 2$ and consider the three lollipops coming from θ_{a_1, b_1, c_1} by deleting a vertex. It is easy to see that these three lollipops are indeed $L_{a_1+b_1+2, c_1-1}$, $L_{a_1+c_1+2, b_1-1}$ and $L_{b_1+c_1+2, a_1-1}$.

From Lemma 3.6, if $L_{g, m-1}$ has an eigenvalue of multiplicity 2 then such an eigenvalue belongs to $\text{Spec}(C_g) \cap \text{Spec}(P_{m-2})$, and in particular $\bar{\lambda} \in \text{Spec}(P_{m-2})$. If we look to $\bar{\lambda}$ as an eigenvalue of multiplicity 3 in θ_{a_1, b_1, c_1} we get the following condition:

$$\bar{\lambda} \in \text{Spec}(P_{a_1-2}) \cap \text{Spec}(P_{b_1-2}) \cap \text{Spec}(P_{c_1-2}) \quad (4)$$

Consider now the vertex deleted subgraph of θ_{a_1, b_1, c_1} , i.e. T_{a_1, b_1, c_1} . By reasoning in a similar way as above we get that $\bar{\lambda}$ is an eigenvalue of multiplicity 2 of T_{a_1, b_1, c_1} and, consequently, $\bar{\lambda}$ is an eigenvalue of any vertex deleted subgraph of T_{a_1, b_1, c_1} , including $P_{a_1} \cup P_{b_1} \cup P_{c_1}$. So we get:

$$\bar{\lambda} \in \text{Spec}(P_{a_1}) \cup \text{Spec}(P_{b_1}) \cup \text{Spec}(P_{c_1}). \quad (5)$$

By combining (4) and (5), we get that the only possibility is that a_1, b_1 and c_1 are odd integers and $\bar{\lambda} = 0$. In fact, if $\bar{\lambda} \in \text{Spec}(P_{a_1})$ (if $\bar{\lambda} \in \text{Spec}(P_{b_1})$ or $\bar{\lambda} \in \text{Spec}(P_{c_1})$ the proof is analogous) then $\bar{\lambda} \in \text{Spec}(P_{a_1-2})$ if and only if a_1 is odd and $\bar{\lambda} = 0$, but this implies that $0 \in \text{Spec}(P_{b_1-2}) \cap \text{Spec}(P_{c_1-2})$ which means that b_1 and c_1 are odd numbers as well.

Assume now that $a_1 = 0$, then $\bar{\lambda}$ cannot be an eigenvalue of multiplicity 2 for $T_{a_1, b_1, c_1} = P_{b_1+c_1+1}$, so we must consider only the cases $a_1 = 1$ and $a_1 = 2$. Suppose first that $a_1 = 1$. If $\bar{\lambda} \in \text{Spec}(P_{a_1})$ then $\bar{\lambda} = 0$ and b_1, c_1 are odd integers. Otherwise, if $\bar{\lambda} \in \text{Spec}(P_{b_1}) \cup \text{Spec}(P_{c_1})$, then we can proceed as above. Finally, let us consider the case $a_1 = 2$. By applying (3) at this situation we obtain that $L_{b_1+c_1+2, 1}$ cannot have any eigenvalue $\bar{\lambda}$ of multiplicity 2. \square

Lemma 3.8. *If $H = \theta_{a_1, b_1, c_1} \cup C_{d_1}$ is cospectral with $D_{a, b, 0}$, then $d_1 = \delta$.*

Proof. Since C_{d_1} contributes to $\text{Spec}(H)$ with eigenvalues of multiplicity 2, we have that d_1 divides δ . Note that if $\delta \leq 5$ then $d_1 = \delta$ (otherwise $d_1 = 1$ or $d_1 = 2$, impossible). If $\delta = 6$ and $d_1 = 3$, then $H = \theta_{a_1, b_1, c_1} \cup C_3$ has 1 as an eigenvalue of multiplicity 3, with $1 \in \text{Spec}(\theta_{a_1, b_1, c_1})$, impossible by Lemma 3.7. So let $\delta \geq 7$, if so any $\lambda \in \text{Spec}(H) \cap \text{Spec}(C_\delta) \setminus \{\pm 2\}$ is of multiplicity three and all other eigenvalues of H are simple. If $d_1 < \delta$, then θ_{a_1, b_1, c_1} must have at least two eigenvalues of multiplicity 3. The latter fact is a contradiction, since from Lemma 3.7 we have that at most one eigenvalue (i.e. 0) can be of multiplicity 3 in θ_{a_1, b_1, c_1} . This means that all eigenvalues of multiplicity 3 in H must be eigenvalues of C_{d_1} , which implies $d_1 = \delta$. \square

4 A -spectral characterization of dumbbell graphs

Recall that the prefix A - is omitted in this section. By Lemmas 3.5 and 3.8 and we have that a tentative (A -)cospectral mate with $D_{a,b,0}$ reduces to $H = D_{a',b',-1} \cup C_\delta$ or $H = \theta_{a_1,b_1,c_1} \cup C_\delta$. Furthermore $\phi(C_\delta)$ divides both $\phi(D_{a,b,0})$ and $\phi(H)$, then we can just compare $\phi(D_{a,b,0})/\phi(C_\delta)$ with $\phi(D_{a',b',-1})$ and $\phi(\theta_{a_1,b_1,c_1})$. To make such comparisons, we will follow the idea of Ramezani et al. (see [11]), that is to express the latter mentioned polynomials through the characteristic polynomials of paths. Let us pose $a = \delta\bar{a}$ and $b = \delta\bar{b}$.

By Lemma 2.2, we obtain

$$\phi(C_a) = \phi(P_a) - \phi(P_{a-2}) - 2;$$

$$\frac{\phi(D_{\delta\bar{a},\delta\bar{b},0})}{\phi(C_\delta)} = \lambda \frac{\phi(C_{\delta\bar{a}})}{\phi(C_\delta)} \phi(C_{\delta\bar{b}}) - \frac{\phi(C_{\delta\bar{a}})}{\phi(C_\delta)} \phi(P_{\delta\bar{b}-1}) - \frac{\phi(C_{\delta\bar{b}})}{\phi(C_\delta)} \phi(P_{\delta\bar{a}-1});$$

$$\phi(D_{a',b',-1}) = \phi(C_{a'})\phi(C_{b'}) - \phi(P_{a'-1})\phi(P_{b'-1});$$

$$\begin{aligned} \phi(\theta_{a_1,b_1,c_1}) &= \lambda^2 \phi(P_{a_1})\phi(P_{b_1})\phi(P_{c_1}) - 2\lambda(\phi(P_{a_1-1})\phi(P_{b_1})\phi(P_{c_1}) + \phi(P_{a_1})\phi(P_{b_1-1})\phi(P_{c_1}) \\ &\quad + \phi(P_{a_1})\phi(P_{b_1})\phi(P_{c_1-1})) + 2(\phi(P_{a_1-1})\phi(P_{b_1-1})\phi(P_{c_1}) + \phi(P_{a_1-1})\phi(P_{b_1})\phi(P_{c_1-1}) \\ &\quad + \phi(P_{a_1})\phi(P_{b_1-1})\phi(P_{c_1-1})) + \phi(P_{a_1-2})\phi(P_{b_1})\phi(P_{c_1}) + \phi(P_{a_1})\phi(P_{b_1-2})\phi(P_{c_1}) \\ &\quad + \phi(P_{a_1})\phi(P_{b_1})\phi(P_{c_1-2}) - 2(\phi(P_{a_1}) + \phi(P_{b_1}) + \phi(P_{c_1})). \end{aligned}$$

From $\phi(P_m) = \lambda\phi(P_{m-1}) - \phi(P_{m-2})$, we get, by solving the latter recurrence equation (see [11]), that for $m \geq -2$,

$$\phi(P_m) = \frac{x^{2m+2} - 1}{x^{m+2} - x^m},$$

where x satisfies $x^2 - \lambda x + 1 = 0$. So we can express the above characteristic polynomials in terms of x . Note also that $n(\theta_{a_1,b_1,c_1}) = n(D_{a,b,0}) - n(C_\delta) = n(D_{a',b',-1}) = a + b + 1 - \delta$. After some computations, we have (we used *Derive* to make such computations):

$$\phi(C_a) = x^a + x^{-a} - 2$$

$$D_1(a, b, 0; x) = (x^2 - 1)^3 x^{m+2} \frac{\phi(D_{a,b,0})}{\phi(C_\delta)}, \quad (6)$$

where $m = a + b - 1 - \delta$ and

$$\begin{aligned} D_1(a, b, 0; x) &= (x^2 - 1)^2 (x^{\delta\bar{a}} - 1)(x^{\delta\bar{b}} - 1) [(x^{\delta\bar{a}}(x^{\delta\bar{b}}(x^4 - 2x^2 - 1) - x^4 + 1) + x^{\delta\bar{b}}(1 - x^4) \\ &\quad + x^4 + 2x^2 - 1)](x^\delta - 1)^{-2}. \end{aligned}$$

Note that,

$$\frac{x^{\delta t} - 1}{x^{\delta} - 1} = \sum_{i=0}^{t-1} x^{i\delta}$$

Then, if $\bar{a} = 1$ (so $\delta = \delta\bar{a} = a$ and $b = \delta\bar{b} = ka$, for some integer k), $D_1(a, ka, 0; x)$ becomes

$$(x^2 - 1)^2 \left[\sum_{i=0}^{k-1} x^{ia} [x^a (x^{ka} (x^4 - 2x^2 - 1) - x^4 + 1) + x^{ka} (1 - x^4) + x^4 + 2x^2 - 1] \right], \quad (7)$$

specially if $k = 1$ (so $b = a$) (7) reduces to

$$x^{2(a+4)} - 4x^{2(a+3)} + 4x^{2(a+2)} - x^{2a} - 2x^{a+8} + 4x^{a+6} - 4x^{a+2} + 2x^a + x^8 - 4x^4 + 4x^2 - 1; \quad (8)$$

otherwise if $\bar{a} > 1$ (so $\delta < a$) we have that $D_1(\delta\bar{a}, \delta\bar{b}, 0; x)$ becomes

$$(x^2 - 1)^2 \left[\sum_{i=0}^{\bar{a}-1} x^{i\delta} \right] \left[\sum_{i=0}^{\bar{b}-1} x^{i\delta} [x^{\delta\bar{a}} (x^{\delta\bar{b}} (x^4 - 2x^2 - 1) - x^4 + 1) + x^{\delta\bar{b}} (1 - x^4) + x^4 + 2x^2 - 1] \right] \quad (9)$$

$$D_2(a', b', -1; x) = (x^2 - 1)^3 x^{m+2} \phi(D_{a', b', -1}), \quad (10)$$

where $m = a' + b' - 2 = a + b - 1 - \delta$ and

$$\begin{aligned} D_2(a', b', -1; x) = & x^{2(a'+b')+6} (x^2 - 2)^2 - x^{2a'+2b'} - 2x^{2a'+b'+6} + 6x^{2a'+b'+4} - 6x^{2a'+b'+2} \\ & + 2x^{2a'+b'} + x^{2(a'+3)} - 2x^{2(a'+2)} + 2x^{2(a'+1)} - x^{2a'} - 2x^{a'+2b'+6} \\ & + 6x^{a'+2b'+4} - 6x^{a'+2b'+2} + 2x^{a'+2b'} + 4x^{a'+b'+6} - 12x^{a'+b'+4} + 12x^{a'+b'+2} \\ & - 4x^{a'+b'} - 2x^{a'+6} + 6x^{a'+4} - 6x^{a'+2} + x^{2(b'+3)} - 2x^{2(b'+2)} + 2x^{2(b'+1)} \\ & - x^{2b'} - 2x^{b'+6} + 6x^{b'+4} - 6x^{b'+2} + 2x^{a'} + 2x^{b'} + x^6 - 4x^4 + 4x^2 - 1. \end{aligned}$$

$$T(a_1, b_1, c_1; x) = (x^2 - 1)^3 x^{m+2} \phi(\theta_{a_1, b_1, c_1}), \quad (11)$$

where $m = a_1 + b_1 + c_1 = a + b - 1 - \delta$ and

$$\begin{aligned} T(a_1, b_1, c_1; x) = & x^{2(a_1+b_1+c_1)+6} (x^2 - 2)^2 - 4x^{a_1+b_1+4} - 4x^{a_1+c_1+4} - 4x^{b_1+c_1+4} + 2x^{a_1+b_1+6} \\ & + 2x^{a_1+c_1+6} + 2x^{b_1+c_1+6} - x^{2a_1+2b_1+4} - x^{2a_1+2c_1+4} - x^{2b_1+2c_1+4} \\ & + 4x^{2a_1+b_1+c_1+6} + 4x^{a_1+2b_1+c_1+6} + 4x^{a_1+b_1+2c_1+6} - 2x^{2a_1+b_1+c_1+4} \\ & - 2x^{a_1+2b_1+c_1+4} - 2x^{a_1+b_1+2c_1+4} - 2x^{2a_1+b_1+c_1+8} - 2x^{a_1+2b_1+c_1+8} \\ & - 2x^{a_1+b_1+2c_1+8} + x^{2a_1+6} + x^{2b_1+6} + x^{2c_1+6} + 2x^{a_1+b_1+2} + 2x^{a_1+c_1+2} \\ & + 2x^{b_1+c_1+2} - 4x^4 + 4x^2 - 1. \end{aligned}$$

If $D_{a,b,0}$ is cospectral with $D_{a',b',-1} \cup C_\delta$ or $\theta_{a_1,b_1,c_1} \cup C_\delta$, then the polynomials (6) and (10) or (6) and (11) must be the same, respectively. Next, we compare the monomials with lowest exponent of the above polynomials. Unfortunately in some particular cases, from the lowest exponent monomial we cannot distinguish whether the graphs are cospectral or not, so we will compare the rest of the polynomial. Note that $-(1 - 4x^2 + 4x^4)$ is common to all of them, so we will not consider the latter polynomial during the comparisons.

If we look to the lowest exponent monomial (other than $-4x^4 + 4x^2 - 1$) of the above polynomials, we get for $D_1(a, b, 0; x)$:

- ($\delta < a$) the monomial with minimum exponent is either $-2x^\delta$ if $3 \leq \delta < 8$, or $-x^8$ if $\delta = 8$, or x^8 if $\delta > 8$;
- ($\delta = a$ and $k = 1$) the monomial with minimum exponent is either $2x^a$ if $3 \leq a < 8$, or $3x^8$ if $a = 8$, or x^8 if $a > 8$;
- ($\delta = a$ and $k \geq 2$) the monomial with minimum exponent is either $2x^{a+2}$ if $3 \leq a < 6$, or $3x^8$ if $a = 6$, or x^8 if $a > 6$.

For $D_2(a', b', -1; x)$, the monomial with minimum exponent can be deduced from $x^6 + 2x^{a'} + 2x^{b'}$. Then we have that it is either x^6 if $a' > 6$, or $3x^6$ if $a' = 6 < b'$, or $5x^6$ if $a' = b' = 6$, or $2x^{a'}$ if $b' \neq a' \leq 5$, or $4x^{a'}$ if $a' = b' \leq 5$.

For $T(a_1, b_1, c_1; x)$, we can deduce, similarly to above, the monomial with minimum exponent from $x^{2a_1+6} + x^{2b_1+6} + x^{2c_1+6} + 2x^{a_1+b_1+2} + 2x^{a_1+c_1+2} + 2x^{b_1+c_1+2}$.

Lemma 4.1. $D_{a,b,0}$ is not cospectral with $H = D_{a',b',-1} \cup C_\delta$.

Proof. We will consider three cases depending on δ and k . Recall that $a' \leq b'$ and $a' + b' + \delta = a + b + 1$.

Case 1: $\delta < a$

It is easy to see that if $3 \leq \delta \leq 8$, then the lowest exponent monomial for $D_1(a, b, 0; x)$ has a negative coefficient, while the lowest exponent monomial for $D_2(a', b', -1; x)$ (which comes from $x^6 + 2x^{a'} + 2x^{b'}$) has a positive coefficient. If $\delta > 8$, then x^8 is the lowest exponent monomial for $D_1(a, b, 0; x)$, while for $D_2(a', b', -1; x)$ it is rx^t with $t \leq 6$.

Case 2: $\delta = a$ and $k = 1$

It is easy to observe that for $a \geq 7$ the two polynomials are different, indeed in $D_1(a, a, 0)$ we have that the minimum exponent is greater than or equal to 7, while in $D_2(a', b', -1)$ the minimum exponent is less than or equal to 6. If $a = 6$, then the coefficient related to x^6 in $D_1(a, a, 0)$ is 2, while in $D_2(a', b', -1)$ the coefficient related to x^6 is either 1, or 3 or 5. If $3 \leq a \leq 5$, then $a' = a < b'$. Since $a' + b' = a + 1$, we obtain that $b' = 1$, impossible.

Case 3: $\delta = a$ and $k \geq 2$

The lowest exponent monomial for $D_1(a, ka, 0; x)$ is either x^8 (when $a > 6$) or $3x^8$ (when $a = 6$) or $2x^{a+2}$ (when $a < 6$), while for $D_2(a', b', -1; x)$ the lowest exponent monomial is either x^6 (for $a' > 6$) or $rx^{a'}$ (for $a' \leq 6$), with $r = 2, 3, 4, 5$.

If $a \geq 5$, clearly the two polynomials are different and this implies that $D_{a,ka,0}$ can not be cospectral with $D_{a',b',-1} \cup C_a$.

Assume $a = 4$, then the lowest exponent monomial for $D_1(4, 4k, 0; x)$ is $2x^6$, but for $D_2(a', b', -1; x)$ we have either x^6 (when $a' > 6$) or $3x^6$ (when $a' = 6 < b'$) or $5x^6$ (when $a' = b' = 6$), a contradiction.

Finally assume $a = 3$, then the lowest exponent monomial for $D_1(3, 3k, 0; x)$ is $2x^5$. Then it must be $a' = 5$ and $a' < b'$. Consequently from $a' + b' = 5 + b' = 3k + 1$ we have $b' = 3k - 4$ (so $k \geq 4$, otherwise $b' \leq a'$). If so, by comparing $D_1(3, 3k, 0)$ and $D_2(5, 3k - 4, -1; x)$, we have that the two polynomials are different for any $k \geq 4$.

This completes the proof. □

Now we will compare the lower exponent monomials of $D_1(a, b, 0; x)$ and $T(a_1, b_1, c_1; x)$. In Lemma 4.2 we will consider that $\delta < a$, while in Lemma 4.3 we will consider that $\delta = a$ and, consequently, $b = ka$ for some k .

Lemma 4.2. *For $\delta < a$, $D_{a,b,0}$ is not cospectral with $H = \theta_{a_1,b_1,c_1} \cup C_\delta$.*

Proof. If $3 \leq \delta \leq 8$, then the lowest exponent monomial in $D_1(a, b, 0; x)$ has a negative coefficient, while in $T(a_1, b_1, c_1; x)$ it has a positive coefficient.

If $\delta > 8$ then the lowest exponent monomial in $D_1(a, b, 0; x)$ is x^δ , then in $T(a_1, b_1, c_1; x)$ it must be x^{2a_1+6} with $a_1 = 1 < b_1$. If we look to the second lowest exponent monomial in $D_1(a, b, 0; x)$, i.e. $-2x^\delta$, we have that it has a negative coefficient while in $T(a_1, b_1, c_1; x)$ it is positive. □

Lemma 4.3. *Let $k \geq 1$ be an integer. Then*

- i) for $k = 1$ or $k = 2$, $D_{a,ka,0}$ is determined by the spectrum;*
- ii) for $k \geq 3$, if $D_{a,ka,0}$ is cospectral with $H = \theta_{a_1,b_1,c_1} \cup C_a$ then $a_1 = 1$, $b_1 = a - 1$ and $c_1 = (k - 1)a - 1$.*

Proof. The proof is based on two cases according to k . Recall that $a_1 \leq b_1 \leq c_1$ and $a_1 + b_1 + c_1 = ka - 1$.

Case 1: $k = 1$

If $a > 8$, then x^8 is the lowest exponent monomial in $D_1(a, a, 0)$. The lowest exponent monomial of $T(a_1, b_1, c_1; x)$ must be x^{2a_1+6} . So we have $a_1 = 1$ and $b_1 > a_1$. If we look to the second lowest exponent monomial we have for $D_1(a, a, 0; x)$ that it is $2x^a$. If we assume that $2b_1 + 6 = 2c_1 + 6 = a$ we get that $a_1 + b_1 + c_1 = a - 5 \neq a - 1$, a contradiction. So it must be $a_1 + b_1 + 2 = a$, and $b_1 = a - 3$. The latter is a contradiction since $c_1 = 1 < b_1$.

If $a = 8$, then $3x^8$ is the lowest exponent monomial in $D_1(a, a, 0)$. It is easy to check that $2a_1 + 6 = 2b_1 + 6 = 2c_1 + 6 = 8$ leads to a contradiction. So it must be $2a_1 + 6 = a_1 + b_1 + 2 = 8$, which implies $a_1 = 1$ and $b_1 = 5$, then $c_1 = 1$, a contradiction.

Considering (8), if $a = 7$, from $a_1 + b_1 + 2 = 7$, $2a_1 + 6 = 8$, $a_1 + b_1 + c_1 = 6$, we obtain $a_1 = 1$, $b_1 = 4$ and $c_1 = 1$, that is a contradiction.

Consider $a = 6$, then the lowest exponent monomial in $D_1(a, a, 0)$ is $2x^6$. So we have that either $2a_1 + 6 = 2b_1 + 6 = 6$ or $a_1 + b_1 + 2 = 6$ with $c_1 > b_1$. From the former we get $a_1 = b_1 = 0$, impossible (it is a multigraph), from the latter, by considering that $a_1 + b_1 + c_1 = 5$, we get a contradiction. The cases $a = 3, 4, 5$ can be done similarly.

Case 2: $k \geq 2$

Take $a > 6$. Then the lowest exponent monomial for $D_1(a, ka, 0; x)$ is x^8 . The lowest exponent monomial of $T(a_1, b_1, c_1; x)$ must be x^{2a_1+6} . So we have $a_1 = 1$ and $b_1 > a_1$. If we look to the second lowest exponent monomial we have for $D_1(a, ka, 0; x)$ that it is $2x^{a+2}$. Suppose first that $2b_1 + 6 = 2c_1 + 6 = a + 2$, since $a_1 + b_1 + c_1 = ka - 1$, we get $(k - 1)a = -2$, impossible. So the unique possibility is that $a + 2 = a_1 + b_1 + 2$ and it must be $b_1 = a - 1$ and $c_1 > b_1$. Since $a_1 + b_1 + c_1 = ka - 1$, $a_1 = 1$ and $b_1 = a - 1$, we get $c_1 = (k - 1)a - 1$. Clearly we have that $k \neq 2$ (otherwise $c_1 = b_1$ and the second lowest exponent monomial is different from $2x^{a+2}$).

Take $a = 6$. Then the lowest exponent monomial for $D_1(6, 6k, 0; x)$ is $3x^8$. Similarly to above we get that $a_1 = 1$, $b_1 = 5 = a - 1$ and $c_1 > b_1$. So again it is $c_1 = (k - 1)a - 1$, with $k \neq 2$.

Take $a = 5$. Then the lowest exponent monomial for $D_1(5, 5k, 0; x)$ is $2x^7$. Similarly to above we get that it must be $a_1 + b_1 + 2 = 7$, $c_1 > b_1$. The second lowest exponent monomial for $D_1(5, 5k, 0; x)$ is x^8 so it must be $a_1 = 1$ and $b_1 > a_1$. So we again get $b_1 = 4 = a - 1$ and $c_1 = (k - 1)5 - 1 = (k - 1)a - 1$, with $k \neq 2$.

Take $a = 4$. Then the lowest exponent monomial for $D_1(4, 4k, 0; x)$ is $2x^6$. Suppose that $2a_1 + 6 = 2b_1 + 6 = 6$, then we get $a_1 = b_1 = 0$, impossible (it is a multigraph). So it must be $a_1 + b_1 + 2 = 6$ and $b_1 < c_1$. If $k = 2$, then $c_1 = 3 > b_1$ and the unique possibility is that $H = \theta_{2,2,3} \cup C_4$, but this graph is not cospectral with $D_{4,8,0}$. So $k \neq 2$. If $k \geq 3$, then the second lowest exponent monomial is $-3x^8$. From the latter we have that the unique possibility (note the coefficient is odd) is that $2a_1 + 6 = a_1 + b_1 + 4 = 8$, and consequently $a_1 = 1$, $b_1 = 3 = a - 1$ and $c_1 = (k - 1)4 - 1 = (k - 1)a - 1$.

Finally take $a = 3$. Then the lowest exponent monomial for $D_1(3, 3k, 0; x)$ is $2x^5$. The unique possibility is that $a_1 + b_1 + 2 = 5$, so $a_1 + b_1 = 3$. If $k = 2$, then $c_1 = 2 > b_1$, impossible. So $k \neq 2$. If $k \geq 3$, the second lowest exponent monomial is $-4x^7$ which comes from $-4x^{a_1+b_1+4}$ (cf. Formula (11)). The third lowest exponent monomial is $3x^8$. Similarly to above we get that $2a_1 + 6 = 8$ (in order to get an odd coefficient) and the unique possibility is $a_1 = 1$, $b_1 = 2$ and $c_1 = 3(k - 1) - 1$. \square

The following lemma establishes which are the unique graphs for $a \neq 4$ cospectral with $D_{a,ka,0}$ (note, the result holds also for $a = 4$). See also Fig 2.

Lemma 4.4. *The graph $D_{a,ka,0}$ ($k \geq 3$) is cospectral with $H = \theta_{1,a-1,(k-1)a} \cup C_a$ if and only if $k = 3$.*

Proof. We will directly compare their characteristic polynomials through the Schwenk's formulas.

$$\begin{aligned} \phi(D_{a,ka,0}) &= \phi(C_a)\phi(C_{ka}) - \phi(C_a)\phi(P_{ka-1}) - \phi(C_{ka})\phi(P_{a-1}); \\ \phi(C_a) \cdot \phi(\theta_{1,a-1,(k-1)a-1}) &= \phi(C_a)[\lambda\phi(C_{ka}) - 2\phi(P_{ka-1}) - 2\phi(P_{(k-1)a-1}) - 2\phi(P_{a-1})] \end{aligned}$$

If we substitute $\lambda = \frac{x^2 + 1}{x}$, $\phi(P_m, \lambda) = \frac{x^{2m+2} - 1}{x^{m+2} - x^m}$ and $\phi(C_m) = x^m + x^{-m} - 2$, and by equating the above polynomials we get that:

$$x^a(x^{2k+1} - x^{2k} - x^{k+4} + x^{k+3} - x^{k+1} + x^k + x^4 - x^3) = 0 \quad (12)$$

If $D_{a,ka,0}$ and $H = \theta_{1,a-1,(k-1)a} \cup C_a$ are cospectral then the above polynomial must reduce to the zero polynomial. It is easy to check that (12) is zero for any x if and only if $k = 3$. This completes the proof. \square

By collecting the above results we finally get our main result:

Theorem 4.5. *The dumbbell graph $D_{a,b,0}$ (without cycle C_4) with $\delta = \gcd(a, b) \geq 3$ and $\delta \neq a$ is DAS. For $\delta = a$ (then $b = ka$), the graph $D_{a,ka,0}$ ($a \neq 4$) is DAS if and only if $k \neq 3$. For $k = 3$, the graph $D_{a,3a,0}$ ($a \neq 4$) is A-cospectral only with $\theta_{1,a-1,2a-1} \cup C_a$.*

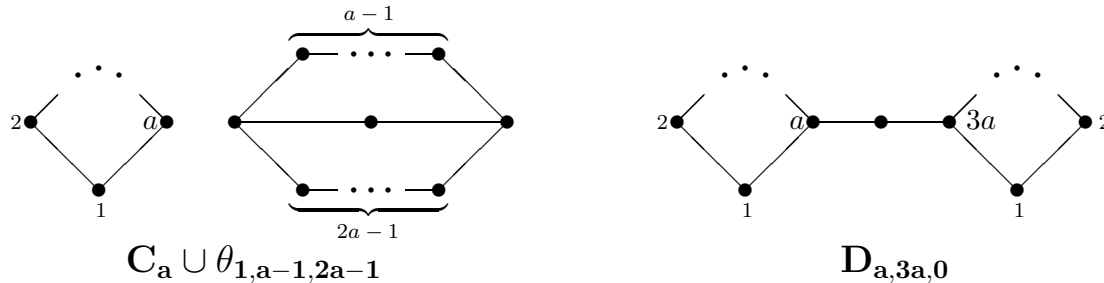


Fig. 2: The A-cospectral graphs described in Lemma 4.4.

By combining the above result and the main result from [12], we obtain:

Theorem 4.6. *All dumbbell graphs $D_{a,b,c}$, without cycle C_4 , different from $D_{a,3a,0}$ are determined by the spectrum of the adjacency matrix.*

Remark 2. *In [12] we proved that most dumbbell graphs are DAS, but it was left to consider dumbbell graphs $D_{a,b,0}$ with $3 \leq \gcd(a, b) \leq a$ and $D_{a,b,c}$ with a or b equal to 4. There is a question: are all dumbbell graphs DAS? Now we have from Theorem 4.5 that the answer is negative. However it remains to consider the case when $D_{a,b,c}$ contains cycle C_4 , i.e. a or b are equal to 4. Note that $D_{4,12,0}$ is A-cospectral with $\theta_{1,3,7} \cup C_4$ but the latter graph could possibly be not the unique A-cospectral graph with $D_{4,12,0}$.*

Remark 3. *In [9] the authors proved that all dumbbell graphs not containing cycle C_4 are DAS, clearly their result is not correct since in this paper we detected an exception for $D_{a,3a,0}$. Furthermore the authors of [9] proved that all ∞ -graphs (denoted in their paper by $b(r, s)$) not containing C_4 are DAS, but we got an exception for $b(2r, 2r + 2)$ (cf. Proof of Lemma 6.12 in [13]).*

5 Q -spectral characterization of dumbbell graphs

In [13] we showed that from the A -spectral characterization of a graph, we can deduce its Q -spectral characterization. Since in our papers we got that, for $a \geq 6$ all dumbbell graphs $D_{a,b,c}$ with $c \neq 0$ or $c = 0$ and $b \neq 3a$, are DAS, then we are able to easily extend such results to the Q -theory of graph spectra.

The following results can be found in [13] (cf. also [3, 4]). However in order to make this paper self-contained we report them here. Recall, we will say that two graphs G and H are A -cospectral if and only if their A -spectra are the same. Similarly, G and H are Q -cospectral if and only if their Q -spectra are the same. A graph G that is determined by the adjacency (signless Laplacian) spectrum will said to be a DAS (resp. DQS) graph. $\varphi(G)$ denotes the Q -characteristic polynomial of G , while $\phi(G)$ denotes the A -characteristic polynomial of $A(G)$. Finally if G is a graph, then $S(G)$ denotes the *subdivision graph* of G , obtained from G by inserting a vertex of degree 2 in each edge of G .

The following lemma can be found in many references, see [3, 14] for example.

Lemma 5.1. *Let G be a graph of order n and size m , and $S(G)$ be the subdivision graph of G . Then*

$$\phi(S(G), \lambda) = \lambda^{m-n} \varphi(G, \lambda^2).$$

Theorem 5.2. *Let G be a graph of order n and size m , and $S(G)$ be the subdivision graph of G .*

- (i) *Graphs G and H are Q -cospectral if and only if $S(G)$ and $S(H)$ are A -cospectral;*
- (ii) *Let G be a graph and $S(G)$ a DAS-graph. Then G is a DQS-graph;*
- (iii) *Let G be a DQS-graph. If any graph A -cospectral with $S(G)$ is a subdivision of some graph, then $S(G)$ is a DAS-graph.*

Proof. (i) Since G and H are Q -cospectral, then $\varphi(G, \lambda) = \varphi(H, \lambda)$. By Lemma 2.4, G and H have the same order and size, which implies that $m(G) - n(G) = m(H) - n(H)$. Thus,

$$\lambda^{m(G)-n(G)} \varphi(G, \lambda^2) = \lambda^{m(H)-n(H)} \varphi(H, \lambda^2),$$

which shows by Lemma 5.1 that $\phi(S(G), \lambda) = \phi(S(H), \lambda)$. This ends the necessity.

Conversely, assume that $S(G)$ and $S(H)$ are A -cospectral, then

$$\phi(S(G), \lambda) = \phi(S(H), \lambda), \quad n(S(G)) = n(S(H)), \quad m(S(G)) = m(S(H)).$$

Note that for any graph G we have

$$m(S(G)) = 2m(G), \quad n(S(G)) = m(G) + n(G).$$

Hence, from $n(S(G)) = n(S(H))$ and $m(S(G)) = m(S(H))$, we obtain that $m(G) = m(H)$ and $n(G) = n(H)$. So we get that

$$(\sqrt{\lambda})^{n(G)-m(G)} \phi(S(G), \sqrt{\lambda}) = (\sqrt{\lambda})^{n(H)-m(H)} \phi(S(H), \sqrt{\lambda}),$$

which shows from Lemma 5.1 that $\varphi(G, \lambda) = \varphi(H, \lambda)$.

(ii) Set $\varphi(H, \lambda) = \varphi(G, \lambda)$. Then by (i) we get $\phi(S(H), \lambda) = \phi(S(G), \lambda)$. Since $S(G)$ is a DAS-graph, then $S(H) \cong S(G)$ which shows that $H \cong G$.

(iii) Without loss of generality, let H and H' be two graphs such that $H = S(H')$ and $\phi(H, \lambda) = \phi(S(H'), \lambda) = \phi(S(G), \lambda)$, which implies from (i) that $\varphi(H', \lambda) = \varphi(G, \lambda)$. Since G is a DQS-graph, then $H' \cong G$, and so $H = S(H') \cong S(G)$ which shows that $S(G)$ is a DAS-graph. \square

Since the subdivision of a dumbbell graph $D_{a,b,c}$ is the dumbbell graph $D_{2a,2b,2(c+1)}$, by combining Theorems 4.6 and 5.2, we are able to state the following theorem:

Theorem 5.3. *All dumbbell graphs $D_{a,b,c}$ different from $D_{a,3a,-1}$, are determined by the spectrum of the signless Laplacian matrix. The graph $D_{a,3a,-1}$ is Q -cospectral only with $\theta_{0,a-1,2a-1} \cup C_a$.*

Proof. Any dumbbell graph $D_{a,b,c}$ with $c \geq 0$ is surely DQS. Indeed $S(D_{a,b,c}) = D_{2a,2b,2(c+1)}$ and from $a \geq 3$ and $c \geq 0$ we have that, by Theorem 4.6, $D_{2a,2b,2(c+1)}$ is DAS. Hence by Theorem 5.2, $D_{a,b,c}$ ($c \neq -1$) is determined by the signless Laplacian spectrum.

Take $c = -1$, then $S(D_{a,b,-1}) = D_{2a,2b,0}$. By Theorem 4.6 we have that if $2b \neq 3(2a)$, hence $b \neq 3a$, then, similarly to above, we get that also $D_{a,b,-1}$ is DQS. Finally, if $c = -1$ and $b = 3a$, then $\phi(S(D_{a,3a,-1})) = \phi(D_{2a,6a,0}) = \phi(\theta_{1,2a-1,4a-1})\phi(C_{2a}) = \phi(S(\theta_{0,a-1,2a-1} \cup C_a))$. The latter implies $\varphi(D_{a,3a,-1}) = \varphi(\theta_{0,a-1,2a-1} \cup C_a)$ (cf. Fig. 3).

This completes the proof. \square

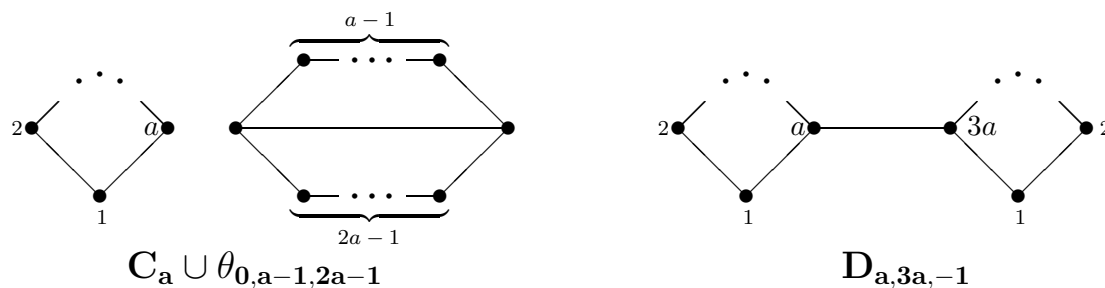


Fig. 3: A pair of non-isomorphic Q -cospectral graphs.

Remark 4. *Similarly to above, since the subdivision of a θ -graph is still a θ -graph but without cycle C_4 , we have from the main result of [11] that all θ -graphs are DQS-graphs.*

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