DISSERTATION

INDEPENDENCE COMPLEXES OF FINITE GROUPS

Submitted by

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ABSTRACT

INDEPENDENCE COMPLEXES OF FINITE GROUPS

Understanding generating sets for finite groups has been explored previously via the generating graph of a group, where vertices are group elements and edges are given by pairs of group elements that generate the group. We generalize this idea by considering minimal generating sets (with respect to inclusion) for subgroups of finite groups. These form a simplicial complex, which we call the independence complex. The vertices of the independence complex are nonidentity group elements and the faces of size k correspond to minimal generating sets of size k. We give a complete characterization via constructive algorithms, together with enumeration results, for the independence complexes of cyclic groups whose order is a squarefree product of primes, finite abelian groups whose order is a product of powers of distinct primes, and the nonabelian class of semidirect products $C_{p_1 p_3 \cdots p_{2n-1}} \rtimes C_{p_2 p_4 \cdots p_{2n}}$ where $p_1, p_2, \ldots, p_{2n}$ are distinct primes with $p_{2i-1} > p_{2i}$ for all $1 \leq i \leq n$. In the latter case, we introduce a tool called a combinatorial diagram, which is a multipartite simplicial complex under certain numerical and minimal covering conditions. Combinatorial diagrams seem to be an interesting area of study on their own. We also include GAP and Polymake code which generates the facets of any (small enough) finite group, as well as visualize the independence complexes in small dimensions.

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DEDICATION

For Grandpa

Your continual encouragement

to educate myself to my heart's content,

complete my PhD, and enjoy life

lives inside me always.

TABLE OF CONTENTS

Chapter 1

Introduction

The focus of our study is to understand minimal generating sets for subgroups of finite groups via an associated simplicial complex. We will call this simplicial complex the independence complex of the group. Often certain generating sets are better suited than others to solve problems involving group computation. If one can understand all minimal generating sets of a finite group in a concise way, then choosing a generating set for a group suitable to the problem at hand becomes much simpler.

Group generation has been studied in various contexts. A classical example is the Cayley graph of a group G, whose vertices are group elements and, for a particular choice of generating set S of G and $g \in G$, whose edges are of the form $\{g, gs\}$ for all $s \in S$. Generated groups (groups together with a specified set of generators) are widely studied in, for instance, geometric group theory. Tao has studied the geometry of the Cayley graph of a generated group by assigning a metric to the group and investigating the asymptotics of the growth of the resulting metric balls [17].

Closely related to our work is the notion of the generating graph $\Gamma(G)$ of a group G, as defined in [11]. The generating graph of a finite group G has as its vertex set the elements of G, and has an edge between any two vertices which generate G (note that G must have a generating set of size two in order for the generating graph to be nonempty, so typically one restricts to 2-generated groups). The authors in [11] show that if H is a sufficiently large simple group with $\Gamma(G) \cong \Gamma(H)$ for some finite group G, then $G \cong H$. It is also shown in [11] that if H is a symmetric group with $\Gamma(G) \cong \Gamma(H)$ for some finite group G, then $G \cong H$. In some sense, in these cases the original group is determined by its generating graph, given the appropriate assumptions.

We generalize the notion of a generating graph by loosening the requirement that the whole group be generated, restricting our attention to minimal generating sets of subgroups, and considering generating sets of size larger than 2. The structure that arises is a simplicial complex, which we call the independence complex of the group (note this is different than the more standard independence complex of a graph). The overall theme of this paper is to describe the structure and combinatorics of the independence complexes of certain finite groups. This paper draws some inspiration from the authors' results in [11], in that it seeks to develop results of a similar flavor (though branching off into different questions) in the context of independence complexes.

In this paper, we define the **independence complex** of a finite group to be the set of all minimal generating sets for subgroups of G. We refer to the facets (maximal faces) of an independence complex as independent facets, and the faces as independent sets. In geometric combinatorics, quantities such as the number of faces of each dimension (called the f-numbers) of simplicial complexes having some prescribed structure are widely studied. These lead to statements about a generalized Euler Characteristic, which gives information about the topology of the complex. Often a goal is to classify the f-numbers and explore numerical phenomena such as unimodality (i.e. Does the sequence of f-numbers rise and then fall with no other increases?). In this paper, in addition to determining the structure of certain independence complexes, we will be interested in face enumeration.

The authors in [1] classify all finite groups G for which $m(G) = d(G)$, where $m(G)$ is the maximum size of a minimal generating set of G and $d(G)$ is the minimum size of a minimal generating set of G. In the setting of the independence complex, this implies all independent facets have the same dimension, and thus the complex is pure (its maximal faces all have the same dimension). This result is in the context of independent sets that generate the full group G. In our work, we will also allow for generation of proper subgroups of G.

See, for instance, [1], [11], and [12] for additional related results about generating graphs and minimal generating sets for the full group G.

We will be interested in studying combinatorial and topological properties of the independence complexes of certain finite groups and determining constructions for these complexes based on the original group structure. An ultimate, yet ambitious, goal is to develop results stating that the independence complex of a group (under the right conditions) gives information about the structure of the group from which it came, such as the previous results mentioned about generating graphs from [11]. Another ultimate yet ambitious goal is to understand the independence complexes of all finite groups.

This paper is structured as follows. In Section 2, we provide background on groups, simplicial complexes, and independence complexes. In Section 3, we give a complete characterization for the order independence complex (a structure associated to the independence complex) of cyclic groups whose orders are squarefree products of distinct primes. We then enumerate the number of faces of the corresponding independence complex. These groups already give rise to interesting structure and involved enumeration, even though they are among the easiest to study of the abelian groups. A similar structure to independence complexes was studied by the author in [20] in counting the number of irredundant generating sets for products of elementary abelian groups. Our results focus on a different but related structure, as we allow for generation of proper subgroups rather than requiring an independent set to generate the whole group.

We also give a complete characterization and facet enumeration of the order independence complexes of groups $C_{{p_1}^{k_1}} \times C_{{p_2}^{k_2}} \times \cdots \times C_{{p_n}^{k_n}}$ where the p_i are distinct primes in Section 3 (here, C_m represents the cyclic group of order m). We then enumerate the number of independent sets in this order independence complex. One of the highlights of our enumerative results is that the enumeration arises directly from the constructions for both of these classes of groups, with no correction for overcounting necessary. Section 3 constitutes our main results about these classes of cyclic and finite abelian groups.

Another prominent component of this thesis is computation and visualization, which we discuss in Section 4. The author has written code which uses GAP [6], Polymake [7], and SageMath [18] to compute and visualize the independence complexes of finite groups in small dimensions. This code is included in the Appendices. In Section 4.2 we include several examples and images which we computed with this code.

Our main theoretical results for a class of nonabelian groups are found in Section 5. Here we give a complete characterization of the independence complexes of groups of the form $\mathcal{G}_n = G_1 \times G_2 \times \cdots \times G_n$ where each G_i is a nonabelian finite group and $|G_1| = p_1p_2, |G_2| =$

$p_3p_4, \ldots, |G_n| = p_{2n-1}p_{2n}$ for distinct primes $p_1, p_2, \ldots, p_{2n}$ with $p_{2i-1} > p_{2i}$ for all $1 \leq i \leq n$. This class of nonabelian groups can be realized as a semidirect product of cyclic groups, namely $\mathcal{G}_n \cong C_{p_1p_3\cdots p_{2n-1}} \rtimes C_{p_2p_4\cdots p_{2n}}$. We introduce a new tool called a combinatorial diagram which we use as a main component of this characterization. We provide an algorithm to generate all combinatorial diagrams explicitly, which allows one to generate and enumerate all independence complexes of $\mathcal{G}_n$ for the desired value of n.

In Section 6, we apply the results from Section 5 and compute the full independence complex for $\mathcal{G}_3$. In Section 7, we give an example in a particular case of how one would enumerate the number of independent sets for the independence complex of $\mathcal{G}_3$. Section 8 states conjectures and future directions.

Chapter 2

Background

2.1 Group Theory

We begin with some standard definitions and results from group theory. Additional group theory background can be found in the standard literature, for instance [5].

2.1.1 Notation and terminology

Let G be a finite group. We write the subgroup generated by $g_1, g_2, \ldots, g_s \in G$ as $\langle\{g_1, \ldots, g_s\}\rangle$ or simply $\langle g_1, \ldots, g_s\rangle$. Let C_n denote the cyclic group of order n, and D_{2n} denote the dihedral group of order $2n$ (rotations and reflections of an n-gon). Recall that $C_{ab} \cong C_a \times C_b$ if and only if gcd$(a, b) = 1$. We use $A \subset B$ to denote a proper subset (i.e. $A \neq B$ but $A = \emptyset$ is possible) and $A \subseteq B$ to allow for subset equality; this distinction will be particularly important throughout the paper. We denote the set $\{1, 2, \ldots, n\}$ by $[n]$. For sets A, B we write $A \sqcup B$ to denote the disjoint union of A and B. We denote the identity of G by e_G, 1, or 1_G depending on the context.

2.1.2 Standard group theoretic results

Corollary 2.1.1 (Corollary to Lagrange's Theorem)**.** *Let G be a finite group with $x \in G$. Then $|x|$ divides $|G|$.*

Corollary 2.1.2. *For any $x \in G$, we have $|x| = |\langle x\rangle|$.*

Definition 2.1.3. *Let N be a subgroup of a group G. Let $gNg^{-1} = \{gng^{-1} : n \in N, g \in G\}$. We say N is **normal** in G, denoted $N \trianglelefteq G$, if for all $g \in G$ we have $gNg^{-1} = N$.*

Definition 2.1.4. *Let S be a subset of A. The **normalizer** of S in G is the set $N_G(S) = \{g \in G | gSg^{-1} = S\}$. Here, $gSg^{-1} = \{gsg^{-1} | s \in S\}$.*

A standard group theoretic argument shows that $N_G(S)$ is a subgroup of G.

2.1.3 Sylow Subgroups

Definition 2.1.5. *Let G be a finite group of order $p^\alpha m$ where p is prime and $p \nmid m$. A Sylow p-subgroup of G is any subgroup of G of order p^α.*

Theorem 2.1.6 (Sylow's Theorem)**.** *Let G be a finite group of order $p^\alpha m$ where p is prime and $p \nmid m$.*

1. *Sylow p-subgroups exist*
2. *Any two Sylow p-subgroups of G are conjugates of each other in G*
3. *G has n_p Sylow p-subgroups, where $n_p \equiv 1 (\text{mod}\, p)$. Additionally, if P is a Sylow p-subgroup, then n_p equals the index of $N_G(P)$ in G, so $n_p \mid m$.*

The following propositions are shown using standard group theoretic arguments (see [5]) and will be useful in our later work.

Proposition 2.1.7. *A Sylow p-subgroup P of G is normal in G if and only if $n_p = 1$ (i.e. if and only if P is the unique Sylow p-subgroup of G).*

Proposition 2.1.8. *Let $|G| = p_1 p_2$ for primes $p_1 > p_2$. The unique Sylow p_1-subgroup is normal in G; and if Q is any Sylow p_2-subgroup which is also normal in G, then G is cyclic.*

Thus if G is not cyclic, its unique Sylow p_1-subgroup is the only normal subgroup of G.

Definition 2.1.9. *A group G is **solvable** if there is a chain of subgroups $e_G = G_0, G_1, G_2, \ldots, G_m = G$ such that*

$$e_G = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_m = G$$

where for each $1 \leq i \leq m$, G_i/G_{i-1} is abelian.

2.1.4 Group Actions and Automorphisms

Let G, H be groups. A **homomorphism** from G to H is a structure preserving map, i.e. a map $\phi : G \to H$ where $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in G$. An **automorphism** of G is a bijective homomorphism $\phi : G \to G$.

Definition 2.1.10. *The **automorphism group*** $\mathrm{Aut}(G)$ *of a finite group* G *is the set of all automorphisms of* G*.*

The group operation in $\mathrm{Aut}(G)$ is composition of automorphisms. Denote by ϕ_{id} the identity element of $\mathrm{Aut}(G)$ satisfying $\phi_{\text{id}}(g) = g$ for all $g \in G$. One prominent property of automorphisms $\phi \in \mathrm{Aut}(G)$ is that they map generators to generators.

Definition 2.1.11. *Let* G *be a group and* S *be a set. We say* G ***acts*** *on* S *if there is a map* $\cdot : G \times S \to S$ *such that*

1. $e_G \cdot s = s$ *for all* $s \in S$
2. $(gh) \cdot s = g \cdot (h \cdot s)$ *for all* $g, h \in G$*,* $s \in S$

If a group G acts on a set S, we call $\{g \cdot s : g \in G\}$ the **orbit** of $s \in S$ under the group action. The group $\mathrm{Aut}(G)$ acts on G, since the evaluation map $\cdot_v : \mathrm{Aut}(G) \times G \to G$ defined by $\phi \cdot_v g = \phi(g)$ satisfies $\phi_{\text{id}} \cdot_v g = \phi_{\text{id}}(g) = g$ for all $g \in G$ and $(\phi_1\phi_2) \cdot_v g = (\phi_1\phi_2)(g) = \phi_1(\phi_2(g)) = \phi_1 \cdot_v (\phi_2(g))$ for all $g \in G$ and all $\phi_1, \phi_2 \in \mathrm{Aut}(G)$. We will be interested in the action of $\mathrm{Aut}(G)$ on sets of group elements; in particular, on the facets of the independence complex of G, which we will discuss in detail in a later section.

2.1.5 Bounding sizes of independent generating sets

In our goal to describe independence complexes, we highlight a result in a slightly different context aiming to bound the sizes of independent facets. In the case when G is a finite solvable group, the authors in [12] provide an upper bound on the largest size $m(G)$ of a minimal generating set for the whole group G. This result is as follows. Let $d_p(G)$ be the minimal number of generators of a Sylow p-subgroup of G, and let $\pi(G)$ be the number of distinct primes which divide $|G|$. If G is finite and nilpotent, then $m(G) = \sum_{p\in\pi(G)} d_p(G)$. Keith Dennis conjectured, in private communication with Lucchini, that $m(G)$ does not exceed this sum for any finite group. While this conjecture does not hold for S_n for certain values of n (see [12] for a description), the authors in this paper show that if G is a finite solvable group, then $m(G) \leq \sum_{p\in\pi(G)} d_p(G)$.

Note that this result pertains to generation of the full group G. In the context of our work, to determine the largest independent facet in the independence complex of G, we must also check $m(G')$ for subgroups G' of G.

2.1.6 Semidirect and Direct Products

A standard construction in group theory is the semidirect product, which gives a way to take two abstract groups and form a new group. Let N and K be groups and let $\phi : K \to \text{Aut}(N)$ be a homomorphism. Define a group by the set of elements $G := \{(n, k) : n \in N, k \in K\}$ and a multiplication by $(n_1, k_1)(n_2, k_2) = (n_1\phi_{k_1}(n_2), k_1k_2)$ where $\phi_k(n) = knk^{-1}$ for $k \in K$ and $n \in N$. In this case, N will be normal in G. This construction is called the **semidirect product**, which we notate by $G = N \rtimes K$.

Semidirect products can also be recognized by starting with two subgroups of a group G, one of which is normal in G, and combining them to make a semidirect product.

Proposition 2.1.12. *Let N, K be subgroups of a group G, such that*

1. $N \trianglelefteq G$
2. $N \cap K = 1_G$

Let $\phi : K \to$ Aut(N) be the homomorphism that maps $k \in K$ to the automorphism of N which conjugates $n \in N$ on the left by k (i.e. $\phi = \phi_k$). Then $NK \cong N \rtimes K$

We say for subgroups H, K of a group G that K is a **complement** for H in G if $G = HK$ and $H \cap K = 1_G$. From Proposition 2.1.12, given a group G and subgroups H and K, if K is a complement for H in G and $H \trianglelefteq G$ then G is the semidirect product $H \rtimes K$.

Lemma 2.1.13. *Let G_1, G_2 be groups. If $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$, then $N_1 \times N_2 \trianglelefteq G_1 \times G_2$.*

Proof: Let $(n_1, n_2) \in N_1 \times N_2$ and $(g_1, g_2) \in G_1 \times G_2$. Observe $(g_1, g_2)(n_1, n_2)(g_1{}^{-1}, g_2{}^{-1}) = (g_1n_1g_1{}^{-1}, g_2n_2g_2{}^{-1}) \in N_1 \times N_2$ since $N_1 \trianglelefteq G_1$ and $N_2 \trianglelefteq G_2$. Thus $N_1 \times N_2 \trianglelefteq G_1 \times G_2$. $\square$

A similar approach can be used to identifying direct products.

Proposition 2.1.14. *If a group G has normal subgroups K and H such that $G = KH$ and $K \cap H = \mathrm{id}_G$ then $G \cong K \times H$, the direct product of K and H.*

2.2 Simplicial Complexes

2.2.1 Simplicial complexes

Definition 2.2.1. *An abstract simplicial complex Δ on a finite set of vertices $V = V(\Delta)$ is a collection of subsets $F \subseteq V(\Delta)$, called faces of Δ, such that if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.*

For example, consider the simplicial complex in Figure 2.1 with vertex set $V = \{a, b, c, d, e, f, g\}$ and faces:

$$\{a,b,c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,d\}, \{b,e\}, \{c,f\}, \{d,e\}, \{d,f\}, \{e,f\},$$

$$\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \emptyset$$

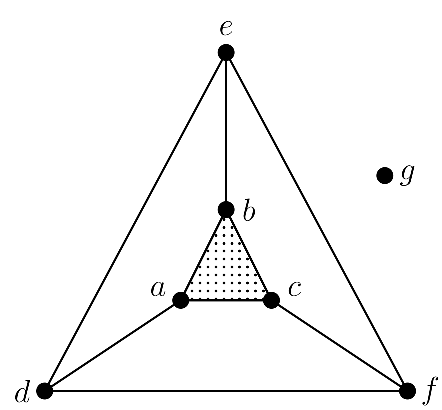


Figure 2.1: Non-pure simplicial complex

The **dimension** of a face F of a simplicial complex Δ is defined as $\dim F := |F| - 1$, where $|F|$ denotes the number of vertices in F. The **dimension of** Δ is defined to be $\dim \Delta := \max\{\dim F : F \in \Delta\}$. If $\dim F = k$ we say F is a k-face. For example, the triangular face $\{a, b, c\}$ in Figure 2.1 consisting of three distinct vertices has dimension two and is called a 2-face. We call 1-faces edges and 0-faces vertices. We say a face is a **maximal face**, or **facet**, of Δ if it is not properly contained in any other face of Δ. It is convenient to describe a simplicial complex only in terms of its facets, as proper subsets of facets are implied to be in the complex by definition. The complex in Figure 2.1 can thus be described by its shorter list of facets:

$$\{a, b, c\}, \{a, d\}, \{b, e\}, \{c, f\}, \{d, e\}, \{d, f\}, \{e, f\}, \{g\}$$

A simplicial complex Δ is **pure** if all of its facets have the same dimension. The simplicial complex in Figure 2.1 is not pure since it has edges and a 2-simplex which are facets. The boundary of a tetrahedron (see Figure 2.2) is pure, since all of its facets

$$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$$

are 2-simplices and thus have the same dimension.

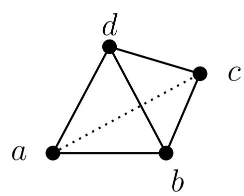


Figure 2.2: The boundary of a tetrahedron is a pure simplicial complex.

Definition 2.2.2. *A **subcomplex** B of a simplicial complex A is a simplicial complex with $B \subseteq A$. In other words, every face of B is also a face of A.*

Let Δ and Γ be simplicial complexes.

Definition 2.2.3. *The $(k-1)$**-skeleton** of a simplicial complex Δ is the subcomplex of Δ consisting of all faces of dimension $k-1$ or less (i.e. of size k or less).*

Definition 2.2.4. *A map $f : V(\Delta) \to V(\Gamma)$ is a **simplicial map** if for every simplex $\sigma = \{v_1, v_2, \ldots, v_m\} \in \Delta$, we have that $f(\sigma) = \{f(v_1), f(v_2), \ldots, f(v_m)\}$ is a simplex of Γ.*

Definition 2.2.5. *A simplicial map $f : V(\Delta) \to V(\Gamma)$ is a* ***simplicial isomorphism*** *if f is bijective and its inverse $f^{-1} : V(\Gamma) \to V(\Delta)$ is a simplicial map.*

The complexes Δ and Γ are **isomorphic** if there is a simplicial map $f : V(\Delta) \to V(\Gamma)$.

Definition 2.2.6. *A simplicial isomorphism $f : V(\Delta) \to V(\Delta)$ is called a* ***simplicial automorphism****.*

If the vertices of Δ are labeled, we denote by $l(v)$ the label of a vertex $v \in V(\Delta)$ and say Δ is **labeled**. We say a simplicial automorphism $f : V(\Delta) \to V(\Delta)$ is **label-preserving** if for every $v \in V(\Delta)$, $l(v) = l(f(v))$. We say two labeled simplicial complexes Δ, Γ are **label-equivalent** if Δ, Γ are isomorphic and there exists a label-preserving simplicial automorphism $f : V(\Delta) \to V(\Gamma)$. In this case we write $\Delta \sim_l \Gamma$.

2.2.2 Joins of simplicial complexes

New simplicial complexes can be constructed from existing ones by taking a union of all of the respective faces, as follows.

Definition 2.2.7. *Let K and L be simplicial complexes on vertex sets A,B, respectively. The* ***join*** *of K and L, denoted $K * L$, is the simplicial complex on vertex set $A \cup B$ consisting of all simplices of K and L together with the simplices $\{\sigma \cup \tau : \sigma \in K \text{ and } \tau \in L\}$.*

For example, consider the following join of two simplicial complexes:

Example 2.2.8. *Let $K = \{\{a, b\}, \{a\}, \{b\}, \emptyset\}$ and $L = \{\{c\}, \{d\}, \emptyset\}$ be simplicial complexes on vertex sets $A = \{a, b\}$ and $B = \{c, d\}$. Then the join of K and L is the 2-dimensional simplicial complex shown in Figure 2.3 and having the form*

$$K * L = \{\{a, b, c\}, \{a, b, d\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b\}, \{a\}, \{b\}, \{c\}, \{d\}, \emptyset\}$$

If K, L have dimensions m and n, respectively, then $K * L$ has dimension $m + n + 1$. This is because the largest simplex in K has size $m + 1$, the largest simplex in L has size $n + 1$, and the largest simplex in the join has size $m + n + 2$. Thus the join has dimension $m + n + 1$. For more information on joins, see [4].

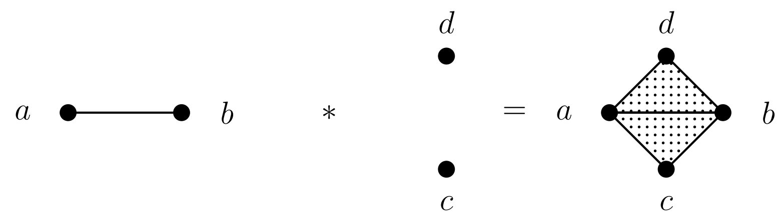


Figure 2.3

2.2.3 Face numbers and h-numbers

A natural and fundamental combinatorial invariant of a simplicial complex is known as its f**-vector**, which lists the number of faces of each dimension in the complex. The f-vector of a $(d-1)$-dimensional simplicial complex Δ is the integer vector

$$f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), f_1(\Delta), \ldots, f_{d-1}(\Delta))$$

where $f_i(\Delta)$ equals the number of faces of dimension i in Δ. We call the $f_i(\Delta)$ the **face numbers** or f-numbers of Δ. By definition, $f_{-1}(\Delta) = 1$ which corresponds to the empty face. Note that $f_0(\Delta)$ is the number of vertices in Δ, $f_1(\Delta)$ is the number of edges in Δ, and $f_2(\Delta)$ is the number of 2-simplices in Δ. The boundary of the tetrahedron in Figure 2.2 has f-vector $(1, 4, 6, 4)$ since it consists of four vertices, six edges, and four triangular faces. Likewise, the simplicial complex in Figure 2.1 has f-vector $(1, 7, 9, 1)$ since it consists of seven vertices, nine edges, and one triangular face.

A common technique in geometric combinatorics is to work with a certain linear transformation of the f-numbers that gives rise to a number of elegant combinatorial relationships (see [4] for additional details). This linear transformation gives rise to another family of combinatorial invariants of a simplicial complex, known as its h-numbers. The h-numbers of a $(d-1)$-dimensional simplicial complex are defined by the relation

$$h_j(\Delta) = \sum_{i=0}^{j} (-1)^{j-i} \binom{d-i}{j-i} f_{i-1}(\Delta), \tag{2.1}$$

where $0 \leq j \leq d$. The h-numbers are organized in a vector $h(\Delta) = (h_0(\Delta), h_1(\Delta), \ldots, h_d(\Delta))$, which is called the **$h$-vector** of Δ. It can be shown that

$$f_{i-1}(\Delta) = \sum_{j=0}^{i} \binom{d-j}{i-j} h_j(\Delta),$$

for $0 \leq i \leq d$. Thus the f-numbers can be determined by the h-numbers, so knowing the h-numbers of a simplicial complex is equivalent to knowing its face numbers. In practice, computing the h-numbers using Equation (2.1) directly can be tedious, so we present a shortcut introduced by Stanley [16] that allows for quick computation of h-numbers given f-numbers. We present Stanley's shortcut only for h-vectors of 2-dimensional simplicial complexes, but the method can easily be generalized for higher-dimensional simplicial complexes.

Given the f-vector $f(\Delta) = (1, f_0, f_1, f_2)$ of a simplicial complex Δ, we may compute its h-vector in the following way:

1. Write a diagonal of 1's on the left diagonal. On the right diagonal, starting directly after the first entry at the top of the first diagonal, write f_0, f_1, f_2.
2. Subtract each pair of entries on each line from right to left.
3. When all subtractions have been made, fill in one more 1 in the left-hand diagonal of 1's. This will become the first component of the h-vector. Subtract each of the remaining pairs on the fourth row from right to left to obtain the remaining components of the h-vector.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & f_0 & & \\ & 1 & & f_0 - 1 & & f_1 & \\ 1 & & f_0 - 2 & & f_1 - f_0 + 1 & & f_2 \\ \hline \end{array}$$

$$h(\Delta) = (1, \quad f_0 - 3, \quad f_1 - 2f_0 + 3, \quad f_2 - f_1 + f_0 - 1) \tag{2.2}$$

The h-vector is computed easily from the f-vectors by either implementing this process or directly using the form of the h-vector in Equation (2.2). For example, in Figure 2.2 the boundary of the tetrahedron has f-vector $(1, 4, 6, 4)$ so by Equation (2.2) its h-vector is $(1, 1, 1, 1)$. The components of an h-vector need not be nonnegative: the complex in Figure 2.1 has f-vector $(1, 7, 9, 1)$ and h-vector $(1, 4, -2, -2)$.

2.2.4 Matroids

Matroids aim to generalize the notion of linear independence in vector spaces and independence in graphs to other settings. For additional information about matroids beyond the examples provided here, see Oxley [14] and [19].

Definition 2.2.9. *A matroid M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E, called the ground set of M, and a collection $\mathcal{I}$ of subsets of E, called independent sets, satisfying the following three conditions:*

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there exists an element e of $I_2 - I_1$ such that $I_1 \cup \{e\} \in \mathcal{I}$.

Here $|I_k|$ denotes the cardinality of I_k and $I_2 - I_1$ denotes the complement of I_1 with respect to I_2.

Condition *(I2)* is often called the **hereditary property** and condition *(I3)* the **independence augmentation property** or **exchange property**. If M is a matroid $(E, \mathcal{I})$, we say that M is a matroid on E. Any subset of E that is not a member of $\mathcal{I}$ is said to be **dependent**. The terminology “independent set" for matroids has an unfortunate overlap with those which we define for independence complexes. Outside of our discussion of matroids, we return to our notion of an independent set as stated in Definition 2.4.1.

Condition *(I3)* is analogous to the linear algebraic property that every linearly independent collection of vectors can be extended to form a basis for a vector space. Motivated by this observation, a facet of a matroid is called a **basis**. The **rank** of a matroid is defined to be the cardinality of a basis in the matroid. Just as all bases for a vector space have the same dimension, all bases for a matroid also have the same dimension, so matroid complexes are pure.

A matroid can be thought of as a simplicial complex with additional structure imposed by condition *(I3)*. The next example illustrates this.

Example 2.2.10. *Consider the simplicial complex whose faces are the empty set, the vertices* $\{x_1\}$,$\{x_2\}$,$\{x_3\}$,$\{x_4\}$,$\{x_5\}$*, every possible edge connecting those vertices except the edge* $\{x_4, x_5\}$*, and the triangular 2-faces* $\{x_1, x_2, x_4\}$, $\{x_1, x_2, x_5\}$, $\{x_1, x_3, x_4\}$, $\{x_1, x_3, x_5\}$, $\{x_2, x_3, x_4\}$*, and* $\{x_2, x_3, x_5\}$*. This collection forms the boundary of a bipyramid (see Figure 2.4).*

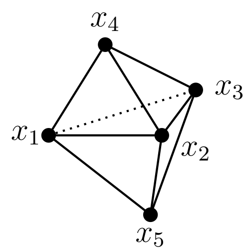


Figure 2.4: The boundary of a bipyramid

We can recognize this simplicial complex as a matroid as follows. Take the independent sets of $\mathcal{M}$ *to be all sets of vertices that form a face in the simplicial complex. Form a* 3×5 *matrix*

$$\begin{array}{c} \quad\;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \\ B = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

such that every face in the simplicial complex corresponds to a collection of linearly independent column vectors. Then the independent sets of $\mathcal{M}$ *arise from* B*. Observe that the set of column vectors* $\{4, 5\}$ *and any of its supersets is linearly dependent, corresponding to the missing edge* $\{4, 5\}$ *and all 2-faces that would contain that edge if they were present. Additionally, the column vectors whose labels are* $1, 2$ *and* 3 *are pairwise linearly independent but collectively dependent,* *corresponding to the edges* $\{1,2\},\{1,3\}$ *and* $\{2,3\}$ *present in the simplicial complex and the missing face* $\{1,2,3\}$.

In the case of elementary abelian groups, which can be viewed as vector spaces over finite fields, minimal generating sets correspond to linear algebraic bases for subspaces. This correspondence does not hold for other groups, showing that independence complexes have a more intricate structure that specializes to vector spaces in an easy case. One might ask whether independence complexes are matroids. However, not every independent set of an independence complex can be extended to form a larger independent set, so not every independence complex is a matroid. This observation together with several GAP [6] and SageMath [18] calculations indicates that independence complexes of finite groups have additional rich structure not found in matroids (for example, the complexes need not be pure, meaning that not all maximal faces have the same dimension). In this paper, our goal is to describe this rich structure in some restricted classes of groups.

2.2.5 Multipartite simplicial complexes and clutters

Definition 2.2.11. *A simplicial complex* Δ *on vertex set* V *(the ground set) is* ***multipartite*** *(or* n***-partite****) if its vertices can be partitioned into* n *sets* $V_1 \sqcup V_2 \sqcup \cdots \sqcup V_n$ *such that the vertices contained in each facet of* Δ *are from distinct vertex sets.*

Understanding a simplicial complex's maximal faces (or facets) is equivalent to understanding the complex, as once one knows all maximal faces of a simplicial complex, all subsets of those facets are also faces by definition. As such, often we restrict our attention to only the maximal faces of a simplicial complex. In the literature, the resulting structure is called a clutter. See [2] for example, in which the author views clutters from a context particularly relevant to our work.

Definition 2.2.12. *A* ***clutter*** *is a pair* (V,E) *where* V *is a set of vertices (the ground set) and* E *is a collection of subsets* $I \subseteq V$ *called edges, such that for any* $I_1, I_2 \in E$*, we have* $I_1 \not\subseteq I_2$.

A clutter (V,E) can be interpreted as the set of maximal faces E of a simplicial complex on vertex set V. Clutters are also known as Sperner families or Sperner systems, and are a family of simple hypergraphs [13].

Definition 2.2.13. *A clutter is **k-uniform** if all edges consist of exactly k vertices.*

If a simplicial complex is pure and its facets have size k, its maximal faces form a k-uniform clutter. An n-partite clutter is a generalization of a bipartite graph (which is a 2-partite, 2-uniform clutter), and consists of the maximal faces of what we call a multipartite simplicial complex in this paper. One can also define n-partite n-uniform clutters, as in [2]. We will study a class of multipartite non-uniform clutters (i.e. the edges of the clutter have differing sizes, or equivalently, the simplicial complex is not pure). The language of multipartite simplicial complexes is more natural for our context, so we will maintain this description while noting that clutters are in themselves a topic of interest in the literature.

2.3 Combinatorics

2.3.1 Partitions

Definition 2.3.1. *Let S be a finite set. A **partition** of S is a sequence $S_1, S_2, \ldots, S_m$ of nonempty subsets of S, which we call **blocks**, such that $\cup_{i=1}^m S_i = S$ and $S_i \cap S_j = \emptyset$ for every $i \neq j$.*

Note that such a partition is ordered according to the choice of sequence. Throughout the paper we will signify when we require a specific ordering of blocks.

2.3.2 Stirling numbers

Let $\mathrm{St}(l, k)$ be the number of ways to partition l objects into k nonempty unordered subsets. These are the Stirling numbers of the second kind.

2.4 Independence Complexes

Independent sets and independence complexes

We now define independence complexes, which will be our main topic of study.

Definition 2.4.1. *Let G be a finite group. An **independent set** of G is a set S of group elements such that removing any element from S generates a smaller subgroup than that generated by S.*

In other words, S is independent if $S' \subset S$ implies $\langle S' \rangle \subset \langle S \rangle$. Thus an independent set is a minimal generating set for some subgroup of G. We will show the collection of all independent sets of a finite group forms an abstract simplicial complex whose vertices are the non-identity elements of the group. In this complex, a face of dimension d corresponds to an independent set of size $d + 1$. For instance, an independent set $\{g_1, g_2, g_3\}$ of G of size three corresponds to a 2-simplex (a filled in triangle). We call the resulting simplicial complex the **independence complex** of the group, and denote this by In(G).

Lemma 2.4.2. *The independence complex of a finite group is a finite simplicial complex.*

Proof: Let $F \in$ In(G), so that F is an independent set of G. Let $H \subset F$ and suppose to the contrary that H is not independent. Then there exists $h_1 \in H$ such that $\langle H \setminus h_1 \rangle = \langle H \rangle$. Then $\langle F \setminus h_1 \rangle = \langle F \rangle$, since we have the following containments:

1. $\langle F \setminus h_1 \rangle \subseteq \langle F \rangle$ as any word in the elements of F not including h_1 is still a word in F.

2. $\langle F \rangle \subseteq \langle F \setminus h_1 \rangle$ as if $x \in \langle F \rangle$, then x can be written as a word in F. If this word does not involve h_1 we are done. If this word does involve h_1, we may write h_1 as a word in elements of H not involving h_1, since $h_1 \in H \subseteq \langle H \rangle \subseteq \langle H \setminus h_1 \rangle$. Thus x can be written as a word in F not involving h_1, so $x \in \langle F \setminus h_1 \rangle$.

Thus F is not independent, a contradiction. For finiteness, note there are finitely many elements in G, and thus finitely many vertices of the simplicial complex. □

Note that the group identity e_G is not in any independent set, as it could always be removed. In particular, $\{e_G\}$ is not an independent set since $\langle \{e_G\} \rangle = \langle \emptyset \rangle = \{e_G\}$, as $\langle \emptyset \rangle$ is defined to be the smallest group containing the empty set, so must contain the group identity.

Proposition 2.4.3. Aut(G) *acts on the set of all facets of* In(G).

Proof: Let M be the set of all facets (maximal independent sets) of In(G). We claim that Aut(G) acts on M. Let $m = \{g_1, g_2, \ldots, g_k\}$ be a maximal independent set of In(G). Let

$\cdot_{v'} : \text{Aut}(G) \times M \to M$ be the evaluation map on maximal sets defined by $\phi \cdot_{v'} m = \phi(m) = \{\phi(g_1), \phi(g_2), \ldots, \phi(g_k)\}$. Since ϕ is an automorphism, it will map generators to generators, so the image of the map $\phi\cdot_{v'}$ is a maximal independent set, and thus in M. The remaining properties of group actions are inherited since $\text{Aut}(G)$ acts on G. Thus $\text{Aut}(G)$ acts on M. □

We will be interested in describing the structure of the independence complexes of some particular finite groups. We will focus our attention first on cyclic groups.

Example 2.4.4. *Consider the cyclic group $G = C_6 \cong C_2 \times C_3 = \langle(1,1)\rangle$ of order 6 under addition. Any nonidentity singleton element forms an independent set of size 1, so the independence complex has exactly five vertices. Any independent set of size 2 cannot contain the elements (1,1) or (1,2) as these generate the whole group. Independent sets of size two must have the form $\{(\star, 0), (0, \star)\}$ where $\star$ represents a nonzero element in its respective component. So there are $(p_1 - 1)(p_2 - 1) = 1 \cdot 2 = 2$ independent sets of size 2, where p_1 and p_2 are the prime group orders 2 and 3, respectively. Thus there are two edges in the independence complex. There are no independent sets of size three, as any two independent elements must form an independent set of the form $\{(\star, 0), (0, \star)\}$, which already generates G. The independence complex of C_6 is 1-dimensional and is shown in Figure 2.5.*

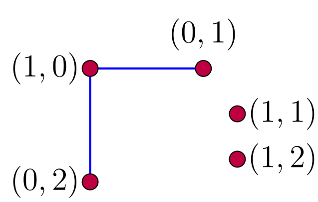


Figure 2.5: Independence complex of C_6

Theorem 2.4.5. *If G is a group with subgroup $H \leq G$, then In(H) $\subseteq$ In(G).*

Proof: Let H be a subgroup of G. We show that every independent set of H is also an independent set of G. Since independents sets of a group are faces of its independence complex, it will follow that every face of In(H) is also a face of In(G). Let I be an independent set of H. Then I is a set of elements of H (and thus of G) that generates a subgroup K of H (and thus a subgroup of G). Let g be any element of I. Then $I \setminus \{g\}$ is a proper subgroup of K, since I generates K minimally. So I is an independent set of G. Thus In(H) $\subseteq$ In(G). □

Figure 2.6 illustrates that subgroup inclusion for independence complexes is not facet-preserving. (In general inclusion doesn't preserve top-dimensional objects, so this is not surprising behavior.)

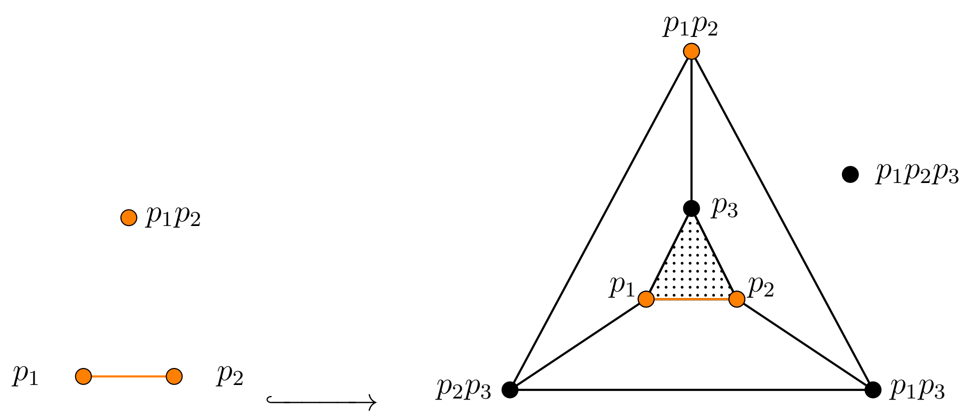


Figure 2.6: Inclusion is not facet-preserving. Here, elements of the same order are represented by single vertices; see later discussion of order independence complexes

Example 2.4.6. *Let $H = C_2 \times C_3$ and $G = C_2 \times C_3 \times C_5$. Then $H \leq G$ and In(H) $\subseteq$ In(G). However, the image of each facet of In(H) under the inclusion mapping In(H) $\hookrightarrow$ In(G) is no longer a facet. See Figure 2.6, which is built from Examples 2.4.13 and 2.4.15.*

2.4.1 Frattini subgroups

Example 2.4.7. *The independence complex for $C_3 \times C_4$ is shown in Figure 2.7. The elements $(0,1)$ and $(0,3)$, which have order 4, can be paired with the elements $(1,0)$ and $(2,0)$, which have order 3, or with the elements $(1,2)$ and $(2,2)$, which have order 6. These pairings generate $C_3 \times C_4$. The element $(0,2)$, which has order 2, can be paired with $(1,0)$ and $(2,0)$, each of which have order 3. However $(0,2)$ cannot be paired with either elements of order 3 or order 6, as the second component of those elements makes the second component of $(0,2)$ unnecessary. In Figure 2.7,* *group elements (represented by vertices) are arranged and labeled according to their order, and sets of edges are labeled with the subgroup generated by their boundary vertices.*

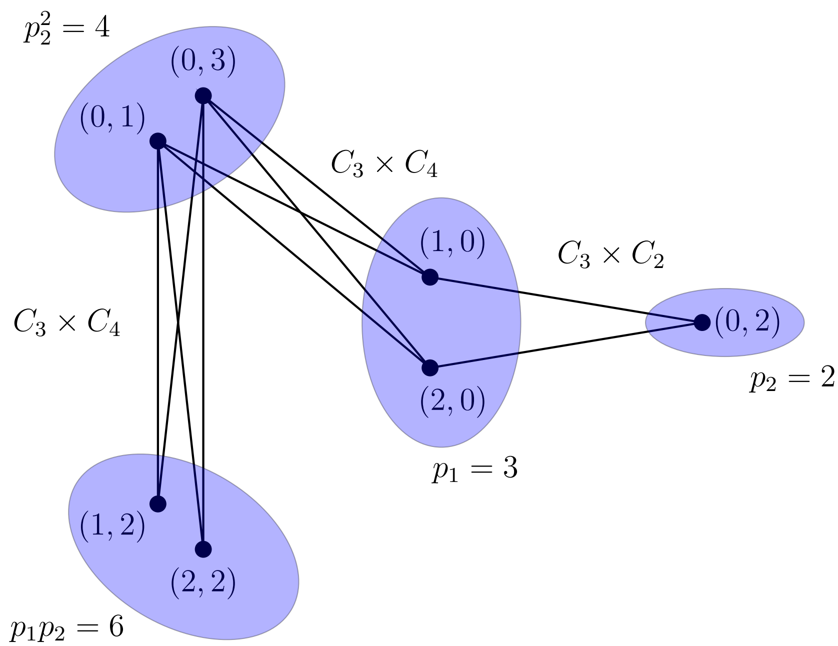


Figure 2.7: In($(C_3 \times C_4)$

This example exhibits the phenomenon that some elements cannot be paired independently with others to generate the entire group. Namely, the element $(0, 2)$ does not appear as a vertex of any edge that generates $C_3 \times C_4$. The element $(0, 2)$ is called a Frattini element in $C_3 \times C_4$, and showcases the behavior of such elements in independence complexes. Recall that a subgroup M of G is **maximal** if there does not exist a subgroup K of G with $M \subset K \subset G$.

Definition 2.4.8. *Let G be a finite group. The* ***Frattini subgroup*** *of G, denoted $\Phi(G)$, is the intersection of all maximal subgroups of G.*

A standard group theory argument shows that Frattini elements of a group G will always be redundant in any generating set for G.

Lemma 2.4.9. *Let G be a finite group. If an independent set I generates G, then I contains no Frattini element.*

Proof: Let $g \in \Phi(G)$ and $a_1, a_2, \ldots, a_m \in G$. Suppose $\langle \{g, a_1, a_2, \ldots, a_m\} \rangle = G$ with $I = \{g, a_1, a_2, \ldots, a_m\}$ an independent set. Since I is independent, $\langle \{a_1, a_2, \ldots, a_m\} \rangle$ is a proper subgroup of G, and thus lies in some maximal subgroup M of G. But g is in every maximal subgroup of G since it is a Frattini element, so in particular $g \in M$. Thus $\langle \{g, a_1, a_2, \ldots, a_m\} \rangle \subseteq M$. Hence I cannot generate G, so I must not have contained a Frattini element of G. $\square$

Example 2.4.10. *The maximal subgroups of the group $G = C_3 \times C_4$ are $H_1 \cong C_3 \times C_2$ and $H_2 \cong C_4$. The intersection $H_1 \cap H_2$ is the Frattini subgroup*

$$\begin{aligned}\Phi(G) &= \{(0,0),(0,2),(1,0),(1,2),(2,0),(2,2)\} \cap \{(0,0),(0,1),(0,2),(0,3)\} \\ &= \{(0,0),(0,2)\}\end{aligned}$$

Example 2.4.11. *The group $G = C_3 \times Q_8$ (where Q_8 is the quaternion group of order 8) has three maximal subgroups isomorphic to $C_3 \times C_4$, namely $\langle (1,1),(0,i) \rangle, \langle (1,1),(0,j) \rangle,$ and $\langle (1,1),(0,k) \rangle$ and one maximal subgroup isomorphic to Q_8, namely $\langle (0,i),(0,j) \rangle$. The intersection of all these subgroups is a subgroup isomorphic to C_2, namely $\Phi(G) = \{(0,1),(0,-1)\}$.*

The existence of Frattini elements in a group adds additional structure to its independence complex, as Frattini elements will only occur as vertices of facets that generate proper subgroups, and never the full group. Independent faces that contain a Frattini element and thus cannot be extended to an independent set that generates the whole group are one obstruction to the independence complex being a matroid. This is not the only obstruction, however: as shown later in Example 2.4.15, the independence complex of $C_{p_1} \times C_{p_2} \times C_{p_3}$ for distinct primes p_1, p_2, p_3 is not a matroid since the simplicial complex is not pure, and in this case the Frattini subgroup is trivial. Independence complexes of finite groups will not be pure except in a small number of cases, because minimal generating sets of groups (and their subgroups) are in general not all the same size.

2.4.2 Order independence complexes

The independence complex for $C_3 \times C_4$ from Figure 2.7 suggests a way to describe the complex more succinctly by grouping elements of the same order together. For instance, in this example the element $(0, 1)$, which has order 4, could be paired with either $(1, 0)$ or $(2, 0)$, both of which have order 3, to form an independent facet. For cyclic groups, recording only connections between sets of elements according to their order gives sufficient information about the independence complex, since elements of the same order behave in the same way in the independence complex. Motivated by this observation, we form a new simplicial complex that records only the orders of group elements, collapsing group elements of the same order into an equivalence class represented by a single vertex. We will call this new simplicial complex the **order independence complex** of G.

Definition 2.4.12. *The **order independence complex** of a group G, denoted $\mathrm{Oinc}(G)$, is the complex whose vertices are equivalence classes of elements of G according to element order and whose faces are collections of these vertices corresponding to the independent sets of the independence complex.*

Example 2.4.13. *Let $p_1 = 2$ and $p_2 = 3$. The order independence complex for $C_2 \times C_3$ is shown in Figure 2.8. This is also the order independence complex of $C_{p_1} \times C_{p_2}$ for any distinct primes p_1, p_2.*

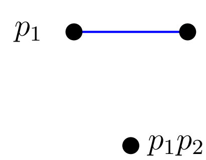


Figure 2.8: Order independence complex for $C_2 \times C_3$

Example 2.4.14. *The order independence complex of $C_3 \times C_4$ is shown in Figure 2.9. Each vertex label represents the order of the elements in its equivalence class. Each edge label represents the subgroup generated by that edge's boundary vertices.*

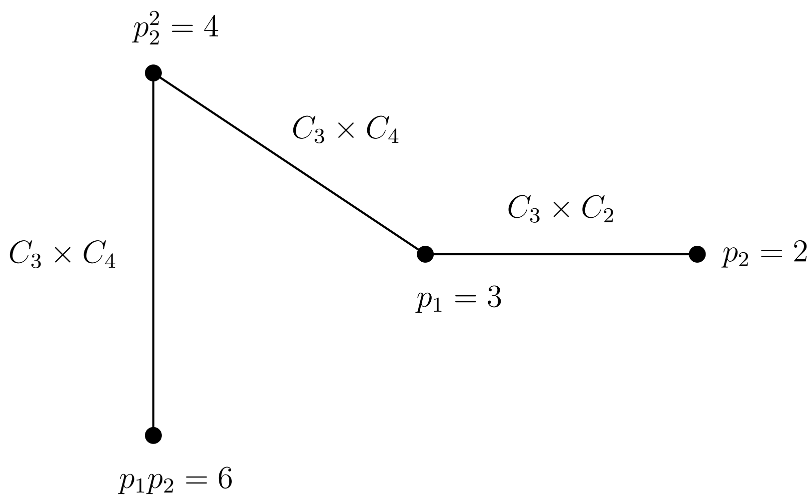


Figure 2.9: Oinc$(C_3 \times C_4)$

The benefit of studying the order independence complex is that it has significantly fewer faces than the independence complex, yet still encodes informative information about independent sets. The face numbers of an independence complex can get very large when working with groups of larger orders, so working with a smaller object with fewer faces is preferable. The order independence complex is a natural object to study when working with cyclic groups and groups of the form $C_{p_1}^{k_1} \times C_{p_2}^{k_2} \times \cdots \times C_{p_n}^{k_n}$ where the p_i are distinct primes. However, for more general finite abelian groups $C_{p_1}^{k_1} \times C_{p_2}^{k_2} \times \cdots \times C_{p_n}^{k_n}$ with nondistinct p_i for example, the order independence is not a natural construction. To see this, consider the independence complex for $C_2 \times C_4$, shown in Figures 2.10 and 2.11. Here the primes $p_1 = p_2 = 2$ are not distinct. Imposing the structure of the order independence complex would place the elements $(0, 1)$ and $(1, 2)$ in the same equivalence class, since both have order $p_1p_2 = p_2^2 = 4$. However these elements have very different behavior in the independence complex: $(1, 2)$ appears in an independent set with the Frattini element $(0, 2)$, but no independent set containing $(0, 1)$ has a Frattini element. Thus, for nondistinct primes p_i in $C_{p_1}^{k_1} \times C_{p_2}^{k_2} \times \cdots \times C_{p_n}^{k_n}$, the order independence complex removes too much information to usefully condense the independence complex.

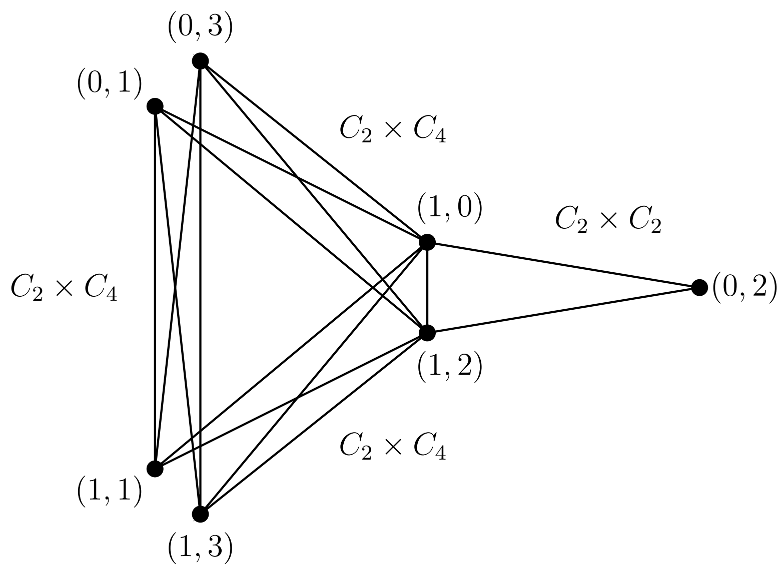


Figure 2.10: In($C_2 \times C_4$)

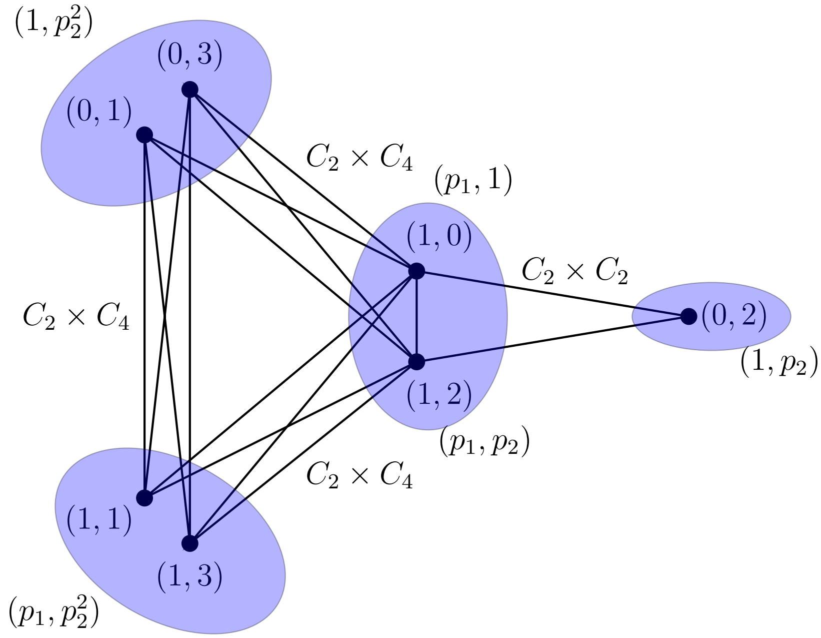


Figure 2.11: The order independence complex is not a natural construction for In($C_2 \times C_4$)

Example 2.4.15. *Let $G = C_{p_1} \times C_{p_2} \times C_{p_3}$ where p_1, p_2, p_3 are distinct primes. The order independence complex of G is shown in Figures 2.12 and 2.13 with two different labellings. In the first diagram, vertices are labeled according to element order. There is exactly one independent set of* *size 3, namely* $\{p_1, p_2, p_3\}$*. There are six maximal independent sets of size 2 (edges), namely*

$$\{p_1, p_2p_3\}, \{p_2, p_1p_3\}, \{p_3, p_1p_2\}$$

$$\{p_1p_2, p_1p_3\}, \{p_1p_2, p_2p_3\}, \{p_1p_3, p_2p_3\}$$

There is one isolated vertex $\{p_1p_2p_3\}$*. The second diagram uses star notation as in the earlier discussion in Example 2.4.4.*

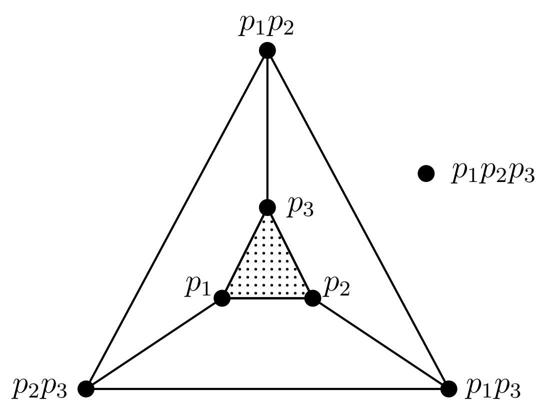


Figure 2.12: Order independence complex using order notation

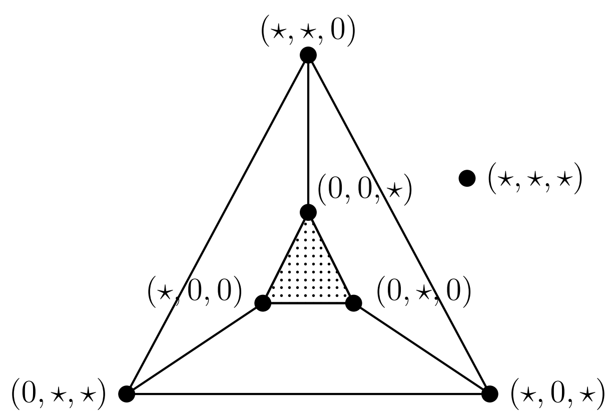


Figure 2.13: Order independence complex using star notation

Chapter 3

Cyclic and Finite Abelian Results (Coprime)

We will describe a constructive algorithm that lists all of the maximal independent sets in the order independence complex of the cyclic group $\mathcal{H} = C_{p_1 p_2 \cdots p_n} \cong C_{p_1} \times C_{p_2} \times \cdots \times C_{p_n}$ where the p_i are distinct primes. We then enumerate the number of faces of the corresponding independence complex. After this, we provide a more general constructive algorithm to compute all facets of the order independence complex of $C_{{p_1}^{k_1}} \times C_{{p_2}^{k_2}} \times \cdots \times C_{{p_n}^{k_n}}$ for distinct primes p_i. We then enumerate the number of independent sets in this order independence complex. We begin with a necessary and sufficient condition for a set of elements to be an independent set of $\mathcal{H}$ by introducing the notion of unique selling points.

3.1 Unique Selling Points

Let $(a_1, a_2, \ldots, a_n) \in \mathcal{H}$. If a_i is nonzero for some $1 \leq i \leq n$, then $\langle a_i \rangle = C_{p_i}$ since the order of a_i is p_i in C_{p_i}.

For example, if $G \cong C_2 \times C_3 \times C_5$, then the collection of tuples

$$\{(\star, 0, 0), (\star, 0, \star), (0, 0, \star)\} \tag{3.1}$$

(where the $\star$ in the i^{th} component of a tuple represents a nonzero element of the group C_{p_i}) generates the subgroup $C_2 \times C_5$ of G. However, the collection (3.1) is not an independent set, for a tuple (in fact, any tuple in this example) can be removed and the remaining tuples still generate the same subgroup. In contrast, the collection

$$\{(\star, \star, 0), (\star, 0, \star)\} \tag{3.2}$$

generates G and is an independent set, since $\langle(\star,\star,0)\rangle = C_2 \times C_3$ and $\langle(\star,0,\star)\rangle = C_2 \times C_5$, both of which are proper subgroups of G. The following observation leads us to a more general description of independence in sets of elements of $\mathcal{H}$. Let $t_1 = (\star,\star,0)$ and $t_2 = (\star,0,\star)$. There is a nonzero entry in the second component of t_1, and the second entry of t_2 is zero. Likewise, the third entry of t_2 is nonzero and the third entry of t_1 is zero. In essence, the set (3.2) is independent because each tuple has a component entry which makes that tuple necessary. This brings us to the following definition.

Definition 3.1.1. Let I be a set of k tuples $t_1, t_2, \ldots, t_k$ of $C_{p_1} \times C_{p_2} \times \cdots \times C_{p_n}$. Denote each of these tuples by $t_i = (a(i,1), a(i,2), \ldots, a(i,n))$ where $a(i,j) \in C_{p_j}$, $1 \leq i \leq k$, and $1 \leq j \leq n$. Suppose $a(i,j)$ is a nonzero entry of t_i and that all other tuples $t_m \in I$ with $m \neq i$ have $a(m,j) = 0$. Then we say t_i has a **unique selling point** $a(i,j)$, or that $a(i,j)$ is a **unique selling point in the set** $\{t_1, t_2, \ldots, t_k\}$.

Lemma 3.1.2. *Let I be a set of tuples of $\mathcal{H}$. Then I is an independent set if and only if every tuple of I has a unique selling point.*

Proof: Let I be a set of tuples $\{t_1, t_2, \ldots, t_k\}$ such that for each $1 \leq i \leq k$, t_i has a unique selling point $a(i,j)$ for some $1 \leq j \leq n$. Each unique selling point $a(i,j)$ generates a copy of C_{p_j}. Since $a(i,j)$ is a unique selling point, $a(m,j) = 0$ for all $m \neq i$, so none of the elements in the j^{th} projection of $I \setminus \{t_i\}$ generate C_{p_j}. Thus $\langle I \setminus \{t_i\}\rangle \subset \langle I\rangle$ for all $1 \leq i \leq k$, so I is independent.

Conversely, let $I = \{t_1, t_2, \ldots, t_k\}$ be an independent set. Suppose to the contrary that there exists some $1 \leq m \leq k$ for which $t_m \in I$ has no unique selling point. Then for each $1 \leq j \leq n$ for which the entry $a(m,j)$ of t_m is nonzero, there is a tuple $t_s = (a(s,1), a(s,2), \ldots, a(s,n))$ of I with $s \neq m$ for which $a(s,j)$ is nonzero and generates C_{p_j} (otherwise, removing t_m would decrease generation in the j^{th} component and $a(m,j)$ would be a unique selling point). Thus $\langle I\rangle = \langle I \setminus \{t_m\}\rangle$, so I is not independent, a contradiction. Therefore every tuple of I has a unique selling point. $\square$

For example, in (3.2) the unique selling point of t_1 is $a(1,2)$ and the unique selling point of t_2 is $a(2,3)$. Since each tuple has a unique selling point, this is an independent set. Observe that an element of an independent set may have more than one unique selling point. For example, in the independent set

$$\{(\star,0,0),(0,\star,\star)\}$$

where $t_1=(\star,0,0)$ and $t_2=(0,\star,\star)$, both $a(2,2)$ and $a(2,3)$ are unique selling points.

3.2 Oinc($C_{p_1} \times C_{p_2} \times \cdots C_{p_n}$) for distinct primes p_i

We now construct the order independence complex for cyclic groups whose order is a product of distinct primes and compute the f-vector of the associated independence complex. The reader has the luxury of skipping the construction leading up to and the proof of Theorem 3.2.1, as these results are constructed and proved more generally in Algorithm 3.3.4 and in the proof of Theorem 3.3.5, which gives a constructive algorithm for Oinc$(C_{{p_1}^{k_1}} \times C_{{p_2}^{k_2}} \times \cdots \times C_{{p_n}^{k_n}})$ where the p_i are distinct primes. We include the content surrounding Theorem 3.2.1 as it provides a different, but equivalent, combinatorial method of building independent sets.

Fix n and let $\mathcal{H}=C_{p_1} \times C_{p_2} \times \cdots C_{p_n}$ for distinct primes p_i and $p_1<p_2<\cdots<p_n$. For fixed $1 \le k \le n$, we will list all facets of size k in the order independence complex of $\mathcal{H}$. Choose $k \le l \le n$ of the n primes, say $p_{i_1}, p_{i_2}, \ldots, p_{i_l}$. There are $\binom{n}{l}$ ways to do this. Partition these l primes into k sets. There are St(l,k) ways to do this. For each of these sets, form the product of its elements where the primes in each product are listed in increasing order. Call the set of these products A. Place these products into the entries of an ordered k-tuple $\vec{w}$ such that the smallest prime in each entry is larger than the smallest prime in each of the entries to its left. For a set A of nonempty products of squarefree primes (meaning that the product does not equal 1), we will denote the corresponding k-tuple ordered in this way by Ord(A). Once the l primes have been chosen, $n-l$ primes remain, say $R=\{p_{j_1}, p_{j_2}, \ldots, p_{j_{n-l}}\}$.

Choose a vector

$$\vec{c}=(c_1,c_2,\ldots,c_{n-l}) \in \{2,\ldots,k\}^{n-l}$$

Choose a vector $\vec{v}_{\vec{c}}$ of length k whose entries are squarefree (possibly empty, in which case fill the entry with the value 1) products of primes in R, such that prime p_{j_m} appears in exactly c_m of the products. For fixed $\vec{c}$, there are $\prod_{m=1}^{n-l}\binom{k}{c_m}$ choices for $\vec{v}_{\vec{c}}$ (since there are $\binom{k}{c_m}$ ways to put p_{j_m} into c_m of the k entries of $\vec{v}_{\vec{c}}$). Define an operator Unord which places the entries of a finite-length tuple into an (unordered) set. We will show that the maximal independent sets of size k are

$$\{\text{Unord}(\vec{w} \circ \vec{v}_{\vec{c}})\}_{\vec{c} \in \{2,\ldots,k\}^{n-l}}$$

where $\circ$ is the Hadamard (component-wise) product on vectors. Summing over all choices of $\vec{c}$, we then obtain a count for the number of maximal independent sets of size k, as stated in the following theorem.

Theorem 3.2.1. *Let $\mathcal{H} = C_{p_1} \times C_{p_2} \times \cdots C_{p_n}$ for distinct primes p_i with $p_1 < p_2 < \cdots < p_n$. Fix $1 \leq k \leq n$. Then the maximal independent sets of size k in the order independence complex of $\mathcal{H}$ are given by*

$$\{\text{Unord}(\vec{w} \circ \vec{v}_{\vec{c}})\}_{\vec{c} \in \{2,\ldots,k\}^{n-l}}$$

There are exactly

$$\sum_{l=k}^{n} \binom{n}{l} \text{St}(l,k) \sum_{\vec{c} \in \{2,\ldots,k\}^{n-l}} \prod_{m=1}^{n-l} \binom{k}{c_m}$$

such independent sets.

Proof: We will show that the sets

$$\{\text{Unord}(\vec{w} \circ \vec{v}_{\vec{c}})\}_{\vec{c} \in \{2,\ldots,k\}^{n-l}}$$

are in bijection with the collection of all maximal independent sets of size k of the order independence complex of $\mathcal{H}$.

($\Leftarrow$) We show first that every maximal independent set of size k of the order independence complex of G can be realized as the set $\text{Unord}(\vec{w} \circ \vec{v}_{\vec{c}})$ for some choice of $\vec{w}$, $\vec{c}$, and $\vec{v}_{\vec{c}}$. Let I be a maximal independent set of size k of the order independence complex of $\mathcal{H}$, and write

$\text{Ord}(I) = (f_1(p_1, \ldots, p_n), \ldots, f_k(p_1, \ldots, p_n))$ where for each i, $f_i = {p_1}^{a_1}{p_2}^{a_2} \cdots {p_n}^{a_n}$ for some $a_1, \ldots, a_n \in \{0, 1\}$. Let $\mathcal{U} \subseteq \{p_1, \ldots, p_n\}$ be the set of all unique selling points of I. Define a map

$$\begin{aligned} \mathcal{L} : &\{\text{ordered tuples coming from indep sets of size } k \text{ of Oinc}(G)\} \\ \to &\{\text{ordered tuples of nonempty squarefree products of primes } p_i \in \mathcal{U}\} \end{aligned}$$

$$\begin{aligned} \mathcal{L}(\text{Ord}(I)) &= \mathcal{L}((f_1(p_1, \ldots, p_n), \ldots, f_k(p_1, \ldots, p_n))) \\ &= (f_1(S(p_1), \ldots, S(p_n)), \ldots, f_k(S(p_1), \ldots, S(p_n))) \end{aligned}$$

where

$$\begin{aligned} S(p_i) &= p_i \text{ if } p_i \in \mathcal{U} \\ &= 1 \text{ if } p_i \notin \mathcal{U} \end{aligned}$$

Let $\vec{w} = \mathcal{L}(\text{Ord}(I))$. We now describe $\vec{v}_{\vec{c}}$. Define a map

$$\begin{aligned} \mathcal{L}_2 : &\{ \text{ ordered tuples coming from independent sets of size } k \text{ of Oinc}(G)\} \to \\ &\{ \text{ ordered tuples of (possibly empty) squarefree products of primes } p_i \notin \mathcal{U}\} \end{aligned}$$

$$\begin{aligned} \mathcal{L}_2(\text{Ord}(I)) &= \mathcal{L}_2((f_1(p_1, \ldots, p_n), \ldots, f_k(p_1, \ldots, p_n))) \\ &= (f_1(S_2(p_1), \ldots, S_2(p_n)), \ldots, f_k(S_2(p_1), \ldots, S_2(p_n))) \end{aligned}$$

where $S_2 : \{p_1, \dots, p_n\} \to \{p_1, \dots, p_n, 1\}$ and

$$\begin{aligned} S_2(p_i) &= p_i \text{ if } p_i \notin \mathcal{U} \\ &= 1 \text{ if } p_i \in \mathcal{U} \end{aligned}$$

Let $\vec{v}_{\vec{c}} = \mathcal{L}_2(\mathrm{Ord}(I))$. Relabel the primes not in $\mathcal{U}$ in increasing order with labels $p_{j_1} < \cdots < p_{j_{n-l}}$. Let $\vec{c} = (c_1, \dots, c_{n-l})$ be the vector of length $n - \#\mathcal{U} = n - l$ such that c_m is the number of products in which prime p_{j_m} appears in $\mathcal{L}_2(\mathrm{Ord}(I))$. Then I can be recognized as $\mathrm{Unord}(\mathcal{L}(\mathrm{Ord}(I)) \circ \mathcal{L}_2(\mathrm{Ord}(I)))$.

($\Rightarrow$) We now show that every set of the form $\mathrm{Unord}(\vec{w} \circ \vec{v}_{\vec{c}})$ for some $\vec{w}$, some $\vec{c}$, and some $\vec{v}_{\vec{c}}$ is a maximal independent set of size k of the order independence complex of $\mathcal{H}$. To do this, we must show three things:

(a) The entries of the set

$$\mathrm{Unord}(\vec{w} \circ \vec{v}_{\vec{c}})$$

are nonempty (i.e. not equal to 1) squarefree products of primes among $p_1, \dots, p_n$

(b) For all choices of $\vec{w}$, $\vec{c}$, and $\vec{v}_{\vec{c}}$, every element of $\mathrm{Unord}(\vec{w} \circ \vec{v}_{\vec{c}})$ has a unique selling point (so that by Lemma 3.1.2, removing an element results in generation of a smaller subgroup).

(c) $\mathrm{Unord}(\vec{w} \circ \vec{v}_{\vec{c}})$ is a facet of size k

We prove these claims as follows.

(a) Consider, for some choice of $\vec{w}, \vec{c}$, and $\vec{v}_{\vec{c}}$, the set $\mathrm{Unord}(\vec{w} \circ \vec{v}_{\vec{c}})$. Since the vector $\vec{w}$ was formed by partitioning $l \geq k$ primes into k nonempty sets, the entries of $\vec{w}$ are nonempty squarefree products of primes in $\mathcal{U}$ (in particular, no entry of $\vec{w}$ is 1). Thus the Hadamard product $\vec{w} \circ \vec{v}_{\vec{c}}$ does not have any entries equal to 1 (even though $\vec{v}_{\vec{c}}$ could have entries equal to 1 (for example, take $n = 5, k = 3, \mathrm{Ord}(I) = (p_1p_2p_5, p_3p_5, p_4), \vec{c} = (2), \vec{v}_{\vec{c}} = (p_5, p_5, 1))$.

Since the set of primes from $\mathcal{U}$ occurring in $\vec{w}$ is disjoint from the set of primes occuring in $\vec{v}_{\vec{c}}$, and the entries of $\vec{w}$ and $\vec{v}_{\vec{c}}$ are each squarefree, their Hadamard product $\vec{w} \circ \vec{v}_{\vec{c}}$ consists of squarefree products of primes.

(b) We now show that every element of Unord$(\vec{w} \circ \vec{v}_{\vec{c}})$ has a unique selling point. Since every entry of $\vec{w}$ is a product of unique selling points, and the set of primes in $\vec{v}_{\vec{c}}$ are disjoint from the set of primes in $\vec{w}$, we know that every prime that was a unique selling point in Unord$(\vec{w})$ stays a unique selling point in Unord$(\vec{w} \circ \vec{v}_{\vec{c}})$.

(c) An independent set in the order independence complex of G will be maximal if every prime among $p_1, \ldots, p_n$ appears at least once (otherwise the independent set could be extended by an element involving one of the missing primes). All l primes in $\mathcal{U}$ occur in $\vec{w}$, and each remaining prime occurs at least twice in $\vec{v}_{\vec{c}}$ since $\vec{c} = (c_1, \ldots, c_m) \in \{2, \ldots, k\}^{n-l}$. Thus every prime among $p_1, \ldots, p_n$ appears at least once in $\vec{w} \circ \vec{v}_{\vec{c}}$, so Unord$(\vec{w} \circ \vec{v}_{\vec{c}})$ is a facet of size k.

□

Example 3.2.2. *Let $n = 6$, $k = 3$ and $l = 4$. One choice for $\vec{w}$ is (p_1p_2, p_3, p_5). We have $R = \{p_4, p_6\}$ and so $j_1 = 4$ and $j_2 = 6$. We have $n - l = 2$ so $\vec{c} = (c_1, c_2)$ is of length 2, and the possible choices for $\vec{c}$ are $(2, 2)$, $(2, 3)$, and $(3, 3)$. The resulting vectors $\vec{v}_{\vec{c}}$ are given by:*

If $\vec{c} = (2, 3)$, then $\vec{v}_{\vec{c}}$ is one of the following: (p_4p_6, p_4p_6, p_6),(p_4p_6, p_6, p_4p_6),(p_6, p_4p_6, p_4p_6)

If $\vec{c} = (3, 3)$, then $\vec{v}_{\vec{c}}$ is (p_4p_6, p_4p_6, p_4p_6)

If $\vec{c} = (2, 2)$, then $\vec{v}_{\vec{c}}$ is one of the following:

$(p_4p_6, p_4p_6, 1)$,(p_4p_6, p_6, p_4),(p_6, p_4p_6, p_4)

(p_4p_6, p_4, p_6),$(p_4p_6, 1, p_4p_6)$,(p_6, p_4, p_4p_6)

(p_4, p_4p_6, p_6),(p_4, p_6, p_4p_6),$(1, p_4p_6, p_4p_6)$

Note that for $\vec{c} = (2,2)$ *there are* $\binom{3}{2}\binom{3}{2} = 9$ *vectors* $\vec{v}_{\vec{c}}$

For $\vec{c} = (2,3)$ *there are* $\binom{3}{2}\binom{3}{3} = 3$ *vectors* $\vec{v}_{\vec{c}}$

For $\vec{c} = (3,3)$ *there is* $\binom{3}{3}\binom{3}{3} = 1$ *vector* $\vec{v}_{\vec{c}}$

The corresponding independent sets are

For $\vec{c} = (2,3)$*:* $\{p_1p_2p_4p_6, p_3p_4p_6, p_5p_6\}, \{p_1p_2p_4p_6, p_3p_6, p_4p_5p_6\}, \{p_1p_2p_6, p_3p_4p_6, p_5p_4p_6\}$

For $\vec{c} = (3,3)$*:* $\{p_1p_2p_4p_6, p_3p_4p_6, p_4p_5p_6\}$

For $\vec{c} = (2,2)$*:*

$\{p_1p_2p_4p_6, p_3p_4p_6, p_5\}, \{p_1p_2p_4p_6, p_3p_6, p_4p_5\}, \{p_1p_2p_6, p_3p_4p_6, p_4p_5\}$

$\{p_1p_2p_4p_6, p_3p_4, p_5p_6\}, \{p_1p_2p_4p_6, p_3, p_4p_5p_6\}, \{p_1p_2p_6, p_3p_4, p_4p_5p_6\}$

$\{p_1p_2p_4, p_3p_4p_6, p_5p_6\}, \{p_1p_2p_4, p_3p_6, p_4p_5p_6\}, \{p_1p_2, p_3p_4p_6, p_4p_5p_6\}$

Figure 3.1 shows the independence complex for $C_2 \times C_3 \times C_4$ generated using GAP and Polymake. We discuss this code in detail in Section 4. Figure 3.2 shows a paper model of the order independence complex of $C_{p_1} \times C_{p_2} \times C_{p_3} \times C_{p_4}$ where p_1, p_2, p_3, p_4 are distinct primes. Some maximal 2-faces and edges have been omitted to better show the internal structure of the complex. A 3D-printed version of this complex is shown in Figure 3.3 (note that an isolated vertex is omitted). Many thanks to Graham Harper for generously designing and printing this model.

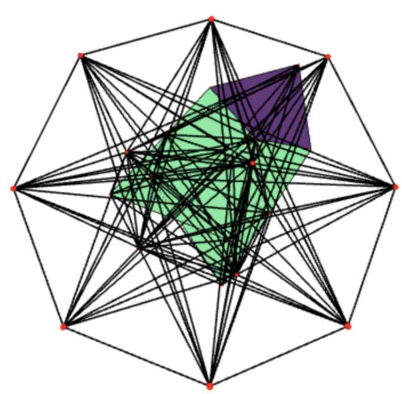


Figure 3.1: Independence Complex for $C_2 \times C_3 \times C_4$

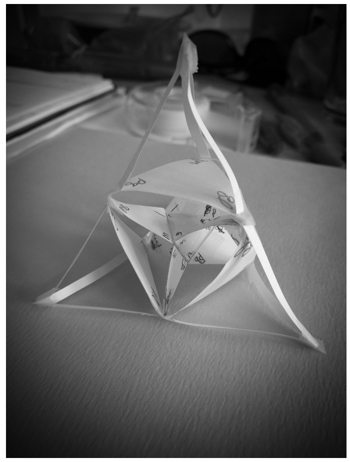


Figure 3.2: Paper model of Oinc($C_{p_1} \times C_{p_2} \times C_{p_3} \times C_{p_4}$)

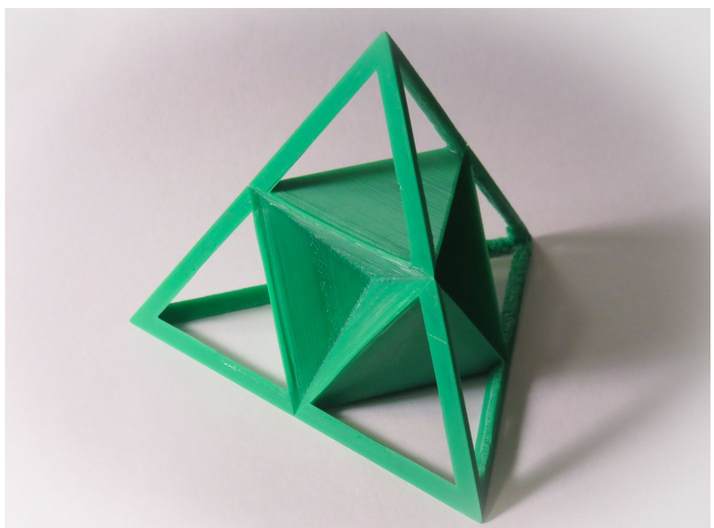


Figure 3.3: 3D-printed Oinc($C_{p_1} \times C_{p_2} \times C_{p_3} \times C_{p_4}$) model, designed and printed by Graham Harper

We now enumerate the faces in each dimension of the independence complex of $\mathcal{H}$, using the construction of the order independence complex.

Theorem 3.2.3 (Cyclic Group Enumeration)**.** *For fixed n, the number of faces of size k of the independence complex of $\mathcal{H} = C_{p_1} \times C_{p_2} \times \cdots \times C_{p_n}$ where the p_i are distinct primes is given by*

$$\sum_{m=k}^{n} \sum_{\substack{S\subseteq[n] \\ |S|=m}} St(m,k) \prod_{i\in S} A_i \prod_{j\notin S} \left(1+\binom{k}{2}A_j^2+\cdots+\binom{k}{k}A_j^k\right)$$

where $A_i = p_i - 1$ and $St(m,k)$ is the number of ways to partition an m-element set into k parts.

The proof of Theorem 3.2.3 follows the construction of the order independence complex given in the text surrounding Theorem 3.2.1, but takes into account the values of the p_i and counts faces rather than only facets. In proving this counting result, we walk through the aforementioned construction and indicate what changes in this setting.

Proof: Let I_k be an independent set of size k of G. Define $A_i := p_i - 1$, the number of choices for nonzero elements of C_{p_i}. We describe all the possible forms of I_k, thus giving a count for the number of sets I_k for a fixed value of k. Since I_k is independent, every element of I_k has at least one unique selling point. Let S be the set of all unique selling points. There are at minimum k unique selling points (as each element of I_k has a unique selling point) and at maximum n unique selling points (as there are only n primes). Partition the m unique selling points into k blocks. There are $\mathrm{St}(m,k)$ ways to do this. Each choice of S contributes $\prod A_i$ where $i \in S$ to the count. The tuple entries that are outside of S, i.e. those that appear in at least two blocks (and at most k blocks), must be accounted for. There are $\binom{k}{l}$ ways that a non-unique variable $A_j := p_j - 1$ could appear in l blocks (where $0 \leq l \leq k$), and when this occurs that variable contributes ${A_j}^l$ to the overall count. Summing over all choices of S of m unique selling points where $k \leq m \leq n$, we obtain the desired enumeration. Since we allow for A_j to appear in 0 blocks, we obtain a count of faces, rather than facets, of the independence complex. □

The counting techniques used in this proof generalize combinatorial techniques of Hearne and Wagner [8] and Clarke [3].

Note that according to the formula in Theorem 3.2.3, the independence complex of $C_{210} \cong C_2 \times C_3 \times C_5 \times C_7$ has f-vector $(1, 209, 6232, 4988, 48)$. We can determine the f-vector of the order independence complex for C_{210} by studying the order independence complex for $\mathrm{Oinc}(C_{p_1} \times$

$C_{p_2} \times C_{p_3} \times C_{p_4}$); a paper model is shown in Figure 3.2. The order independence complex has as its facets one empty set, 15 vertices, 22 edges, 22 filled-in triangles, and one 3-simplex. Note that the model in Figure 3.2 omits some two-dimensional faces so one can view its internal face structure. Adding to these counts all faces that are not facets, we obtain the f-vector $(1, 15, 52, 26, 1)$. This reduction in the number of faces motivates studying the order independence complex, as, having fewer faces, it is easier to work with.

If the cyclic group order is ${p_1}^{k_1}{p_2}^{k_2}\cdots{p_n}^{k_n}$ for distinct primes p_i and values of k_i other than 1, enumeration becomes significantly more difficult. We provide a complete characterization for this case in the next section. In the more general case of finite abelian groups, which have the form $C_{{p_1}^{k_1}} \times C_{{p_2}^{k_2}} \times \cdots \times C_{{p_n}^{k_n}}$ where the primes p_i are not necessarily distinct, enumeration becomes even more complicated. Enumerating the faces of independence complexes for all finite abelian groups is outside of the scope of this paper.

3.3 Oinc($C_{{p_1}^{k_1}} \times C_{{p_2}^{k_2}} \times \cdots \times C_{{p_n}^{k_n}}$) for distinct primes p_i

The following is a constructive algorithm to compute the facets of size k of the order independence complex of $\mathcal{J} = C_{{p_1}^{k_1}} \times C_{{p_2}^{k_2}} \times \cdots \times C_{{p_n}^{k_n}}$ where the p_i are distinct primes with $p_1 < p_2 < \cdots < p_n$.

Definition 3.3.1. *In the group $\mathcal{J}$, we say a group element a of order p^r* ***dominates*** *an element b of order p^t if $r > t$.*

In the context of order independence complexes, such an element order p^r dominates the element order p^t. In the group $C_{{p_1}^{k_1}} \times C_{{p_2}^{k_2}} \times \cdots \times C_{{p_n}^{k_n}}$, if a dominates b, then the subgroup $\langle a \rangle \cong C_{p^r}$ contains as a subgroup an isomorphic copy of $\langle b \rangle \cong C_{p^t}$. For example, in $C_3 \times C_4$, the element a of order 2 is dominated by any element of order 4, and the latter generates a subgroup isomorphic to C_4, which contains the subgroup $\langle a \rangle \cong C_2$. We say an independent set B is an extension of an independent set A, or that A can be extended to B, if $A \subset B$.

We define a specific ordering on the sequence $S_1, S_2, \ldots, S_m$ of partition blocks in Definition 2.3.1 which will come into play in Algorithm 3.3.4.

Definition 3.3.2. *Let S be any finite set* $\{{p_i}^{a_i} : p_i \text{ is prime}, a_i \leq k_i \text{ is a non-negative integer}\}$ *of (not necessarily distinct) powers of primes p_i, and let $S_1, S_2, \ldots, S_m$ be a partition of S. Define an ordering $\prec_1$ on the blocks $S_1, S_2, \ldots, S_m$ and an ordering $\prec_2$ on the sets in each block as follows.*

1. *Within each block S_r, order the prime powers so that*

   (a) *If $p_i < p_j$, then ${p_i}^{a_i} \prec_2 {p_j}^{a_j}$.*

   (b) *If $p_i = p_j$ and $a_i < a_j$, then ${p_i}^{a_i} \prec_2 {p_j}^{a_j}$.*

2. *For each $1 \leq r \leq m$, define* $\operatorname{minp}(S_r)$ *to be the smallest prime base p_i occurring in the subset S_r. Then*

   (a) *If* $\operatorname{minp}(S_r) < \operatorname{minp}(S_t)$ *then $S_r \prec_1 S_t$.*

   (b) *If $p_i = \operatorname{minp}(S_r) = \operatorname{minp}(S_t) = p_j$ and $a_i < a_j$ where a_i, a_j are the exponents on p_i, p_j respectively, then $S_r \prec_1 S_t$.*

Recall that $\mathcal{J} = C_{{p_1}^{k_1}} \times C_{{p_2}^{k_2}} \times \cdots \times C_{{p_n}^{k_n}}$ where the p_i are distinct primes with $p_1 < p_2 < \cdots < p_n$.

Lemma 3.3.3. *Let T be the tuple whose entries are the elements of an independent facet of size k of the order independence complex of $\mathcal{J}$. Order the entries in this tuple according to the orders $\prec_1$ and $\prec_2$ in Definition 3.3.2. Then for each of the k slots of T, there is at most one i for which that slot contains ${p_i}^{l_i}$ when $l_i < k_i$. Furthermore, any slot containing ${p_i}^{l_i}$ with $l_i < k_i$ does not also contain any ${p_j}^{l_j}$ with $l_j = k_j$ where $i \neq j$.*

Proof: If some slot of I contains both ${p_i}^{l_i}$ and ${p_j}^{l_j}$ with $i \neq j$, $l_i < k_i$, and $l_j < k_j$, then no matter what other elements that slot contains, I could be extended by a $(k+1)^{\text{st}}$ slot $\{p_i^{t_i}\}$ where $l_i < t_i < k_i$, implying that I is not a facet. Likewise, if any slot contains both ${p_i}^{l_i}$ with $l_i < k_i$ and ${p_j}^{l_j}$ with $l_j = k_j$ for $i \neq j$, then I can also be extended by $\{p_i^{t_i}\}$ where $l_i < t_i < k_i$, implying that I is not a facet. $\square$

Algorithm 3.3.4. *We compute the facets of size k for $1 \leq k \leq n$ of the order independence complex of $\mathcal{J} = C_{{p_1}^{k_1}} \times C_{{p_2}^{k_2}} \times \cdots \times C_{{p_n}^{k_n}}$, for p_i distinct primes with $p_1 < p_2 < \cdots < p_n$.*

1. *Choose an ordered n-tuple $L = (l_1, l_2, \ldots, l_n) \in [k_1] \times [k_2] \times \cdots \times [k_n]$ such that*

   (a) *$l_i < k_i$ for all i only if $k = n$*

   (b) *If $l_j = k_j$ for at least one j, then $0 \leq m \leq k$ where m is the number of entries l_i with $l_i < k_i$.*

   The tuple L will be the exponent vector encoding the highest prime powers that appear in an independent set. Let $\mathcal{N} = \{{p_i}^{l_i} : l_i < k_i\}$. Let $\mathcal{D} = \{{p_i}^{d_i} : d_i < l_i\}$.

2. *Choose a subset $\mathcal{U} \subseteq \{p_i^{l_i} : l_i = k_i\}$ such that*

   (a) *If $m = k$ then $\mathcal{U} = \emptyset$*

   (b) *If $0 \leq m < k$ then $k \leq |\mathcal{U}| + m \leq n$*

3. *Partition the elements of $\mathcal{N} \sqcup \mathcal{U}$ into k blocks as follows. Place each element of $\mathcal{N}$ into a block of its own. Partition the entries of $\mathcal{U}$ into the remaining $k - m$ blocks. If necessary, reorder the k blocks according to the partition ordering described in Definition 3.3.2. Label these ordered blocks by $1, 2, \ldots, k$ in increasing order from left to right. Let λ be the chosen partition $\lambda_1, \lambda_2, \ldots, \lambda_{k-m}$ of the elements of $\mathcal{U}$ after reordering.*

4. *Let $\mathcal{U}^c = \{{p_j}^{l_j} : l_j = k_j \text{ and } {p_j}^{l_j} \notin \mathcal{U}\}$. For each j with ${p_j}^{l_j} \in \mathcal{U}^c$, choose a subset $A_j \subseteq [k]$ with $|A_j| \geq 2$. Insert ${p_j}^{l_j}$ into the blocks whose labels are the elements of A_j. The products of the entries in each of the resulting k blocks yield the orders of elements of an independent set.*

5. *For each positive integer $d_i < l_i$ and for each $1 \leq i \leq n$, choose a nonempty subset $B_{d_i} \subseteq [k]$ corresponding to any subset of blocks from Step 4 that do not contain a power of p_i. Insert ${p_i}^{d_i}$ into the blocks whose labels are the elements of B_{d_i}. The products of the entries of each of the resulting k blocks yield the orders of elements of an independent set.*

Theorem 3.3.5. *The sets arising from Algorithm 3.3.4 in Steps 4 and 5 are in bijection with the facets of size* $1 \leq k \leq n$ *of the order independence complex of* $\mathcal{J}$.

Proof: We first show that every set A arising from Algorithm 3.3.4 is an independent facet of $\text{oinc}(\mathcal{J})$.

Claim: The set A is independent.

To show that A is independent, we show that removing an element a from A results in the generation of a smaller subgroup. Every element of A has, in particular, either an element of $\mathcal{N}$ or an element of $\mathcal{U}$ (and does not contain elements from both). If a has an element of $\mathcal{N}$, then removing a removes some power ${p_i}^{l_i}$ with $l_i < k_i$, resulting in the generation of a smaller subgroup since $A \setminus \{a\}$ contains no element of order ${p_i}^{l_i}$ or any of its higher powers. If a has instead an element of $\mathcal{U}$, then removing a removes some power ${p_i}^{k_i}$, resulting in the generation of a smaller subgroup since ${p_i}^{k_i}$ does not appear in any other element of A. Other prime powers occurring in a are either elements of $\mathcal{U}^c$ (and thus appear in another slot of A) or elements ${p_i}^{d_i}$ of $\mathcal{D}$ (each of which is dominated by ${p_i}^{l_i}$, which appears in another slot of A). In either case, these elements do not contribute to any new generation. Thus $\langle A \setminus \{a\} \rangle$ is a proper subgroup of A, so A is independent.

Claim: The set A is a facet.

Let A be a set arising from Algorithm 3.3.4. We show that A cannot be extended by any element of G and remain independent. Any slot that extends A contains some combination of elements of $\mathcal{U}, \mathcal{N}, \mathcal{U}^c, \mathcal{D}$, or strictly higher powers than the elements of N. If the new slot contains an element of $\mathcal{U}$, $\mathcal{N}$, or $\mathcal{U}^c$, this element does not generate anything new since this element already occurs in another slot. If the new slot contains an element ${p_i}^{d_i}$ of $\mathcal{D}$, this element does not contribute anything new since it is dominated by the higher power ${p_i}^{l_i}$ in another slot.

Suppose the new slot contains ${p_i}^{h_i}$, a strictly higher power than ${p_i}^{l_i} \in \mathcal{N}$. Then ${p_i}^{h_i}$ dominates ${p_i}^{l_i}$ which appears in another slot $\mathcal{B}$. We show that the remaining elements of $\mathcal{B}$ also become redundant in the extension of A. The only other elements $\mathcal{B}$ could contain are elements of $\mathcal{U}^c$ or $\mathcal{D}$ ($\mathcal{B}$ contains no elements of $\mathcal{U}$ or other elements of $\mathcal{N}$ by construction). Any element of $\mathcal{U}^c$ in $\mathcal{B}$ already appears in another slot of A than $\mathcal{B}$. Any ${p_i}^{d_i} \in \mathcal{D}$ in $\mathcal{B}$ is dominated by the higher power

${p_i}^{l_i}$ which occurs, by construction, in a slot of A other than $\mathcal{B}$. Thus the addition of the new slot containing ${p_i}^{h_i}$ makes $\mathcal{B}$ redundant, so this extension is not independent. Thus any extension of A results in a set that is not independent. So A is a facet.

We now show that every independent facet I has the form of a set arising from Algorithm 3.3.4. Let I be an independent facet of size k. Order the slots in the tuple corresponding to I and its individual slots according to the orderings in Definition 3.3.2. Label the ordered slots by $1, 2, \ldots, k$ in increasing order from left to right.

- **Identify L and show that (a) if $m = n$ then $k = n$, and (b) if $m < n$ then $0 \leq m \leq k$.**

  For each $1 \leq i \leq n$, identify the highest exponent l_i such that ${p_i}^{l_i}$ occurs in some slot of I. Let $L = (l_1, l_2, \ldots, l_n)$. Let $\mathcal{N} = \{{p_i}^{l_i} : l_i < k_i\}$.

  To show (a), suppose $l_i < k_i$ for all $1 \leq i \leq n$. By Lemma 3.3.3, each slot contains at most one element of $\mathcal{N}$. Since there are n such prime powers, this implies $k = n$.

  To show (b), suppose $l_j = k_j$ for at least one j. If $m > k$ then some slot contains more than one element of $\mathcal{N}$, contradicting Lemma 3.3.3. Thus $0 \leq m \leq k$.

- **Identify $\mathcal{U}$ and show that (a) If $m = k$ then $\mathcal{U} = \emptyset$, and (b) if $0 \leq m < k$ then $k \leq |\mathcal{U}| + m \leq n$.**

  Let $\mathcal{U}$ be the set of all prime powers ${p_i}^{l_i}$ for which $l_i = k_i$, each of which occur in exactly one slot of I.

  If $m = k$, then by Lemma 3.3.3 each element of $\mathcal{N}$ appears in one of the k slots, leaving no slots to contain elements of $\mathcal{U}$, so $\mathcal{U} = \emptyset$.

  Now suppose $0 \leq m < k$. Certainly $|\mathcal{U}| + m \leq n$ as there are only n primes p_i. Suppose to the contrary that $|\mathcal{U}| + m \leq k - 1$. By Lemma 3.3.3, each element of N is in a different slot. We claim the remaining $k - m$ slots must be filled with the entries of $\mathcal{U}$ with no slots left empty. If there is some slot B that does not contain an element of $\mathcal{N} \sqcup \mathcal{U}$, then since k is fixed, B must contain at least one of ${p_i}^{l_i}$ where $l_i = k_i$ and ${p_i}^{l_i} \notin \mathcal{U}$, or ${p_j}^{a_j}$ where $a_j < l_j$.

If B contains ${p_i}^{l_i}$, then ${p_i}^{l_i}$ appears in some slot other than B since ${p_i}^{l_i} \notin \mathcal{U}$. If B contains ${p_j}^{a_j}$, then ${p_j}^{l_j}$, a higher power of p_j, appears in some slot other than B. Thus B is redundant, so I is not independent, so all $k - m$ slots are filled with elements of $\mathcal{U}$. Thus $|\mathcal{U}| \geq k - m$, or $|\mathcal{U}| + m \geq k$.

Let $U = \{u_1, u_2, \ldots, u_{k-m}\} \subseteq [k]$ be the set of labels of slots that contain elements of $\mathcal{U}$. Let λ be the partition $\lambda_{u_1}, \lambda_{u_2}, \ldots, \lambda_{u_{k-m}}$ of $\mathcal{U}$ where λ_{u_i} is the set of all elements of $\mathcal{U}$ that occur in slot u_i of I.

- **For each j with $l_j = k_j$ and ${p_j}^{l_j} \notin \mathcal{U}$, identify A_j.**

  For each j for which $l_j = k_j$ and ${p_j}^{l_j}$ appears in more than one slot of I, let $A_j \subseteq [k]$ be the set of labels of all slots containing ${p_j}^{l_j}$. Then $|A_j| \geq 2$.

- **For each $1 \leq i \leq n$ and each $a_i < l_i$, identify B_{a_i}.**

  For each $1 \leq i \leq n$ and each integer $1 \leq a_i < l_i$ such that ${p_i}^{a_i}$ occurs in at least one slot of I, let $B_{a_i} \subseteq [k]$ be the set of labels of all slots that contain ${p_i}^{a_i}$. Since only one power of a given prime p_i can appear in each slot, no other powers of p_i appear in B_{a_i}.

$\square$

We now count the number of independent facets I in $\operatorname{oinc}(C_{{p_1}^{k_1}} \times C_{{p_2}^{k_2}} \times \cdots \times C_{{p_n}^{k_n}})$. First, we count the number of ways to choose L with $|\mathcal{N}| = m$. There are $k_{i_j} - 1$ ways to choose the ${i_j}^{th}$ entry of L since this entry can be one of $1, 2, 3, \ldots, k_{i_j} - 1$. So there are $\prod_{j=1}^{m}(k_{i_j} - 1)$ ways to choose L with $|\mathcal{N}| = m$. The set $\mathcal{N}$ is determined by the choice of L, and there are $\binom{k}{m}$ ways of placing the m elements of $\mathcal{N}$ into m distinct slots of the k slots of I. There are $n - m$ primes ${p_j}^{l_j}$ with $l_j = k_j$, and $\binom{n-m}{s}$ ways to choose s of these to comprise $\mathcal{U}$. Since $k \leq s + m \leq n$ by Step 2 of Algorithm 3.3.4, we have $k - m \leq s \leq n - m$. Thus there are $\sum_{s=k-m}^{n-m} \binom{n-m}{s}$ ways to choose $\mathcal{U}$ where $|\mathcal{U}| = s$. There are $St(s, k - m)$ ways to partition the elements of $\mathcal{U}$ into the remaining $k - m$ slots. Each of the $n - m - s$ elements of $\mathcal{U}^c$ appears in at least two slots of the k slots of I, so there are $\sum_{t=2}^{k} \binom{k}{t}$ ways to place each of these elements. For each $1 \leq i \leq n$ and each integer $0 \leq d_j < l_i$, there are three cases to consider:

1. If ${p_i}^{l_i} \in \mathcal{N}$ then ${p_i}^{l_i}$ appears in exactly one slot of I, so there are $\sum_{r=0}^{k-1} \binom{k-1}{r} = 2^{k-1}$ ways to insert ${p_i}^{d_i}$ into a slot that contains no power of p_i already.

2. If ${p_i}^{l_i} \in \mathcal{U}$ then as in Step 1 ${p_i}^{l_i}$ appears in exactly one slot of I, so there are $\sum_{r=0}^{k-1} \binom{k-1}{r} = 2^{k-1}$ ways to insert ${p_i}^{d_i}$.

3. If ${p_i}^{l_i} \in \mathcal{U}^c$ and appears in $2 \leq t \leq k$ slots, then there are $\sum_{r=0}^{k-t} \binom{k-t}{t} = 2^{k-t}$

So there are $2 \cdot 2^{k-1} + 2^{k-t} = 2^k + 2^{k-t}$ ways to insert ${p_i}^{d_i}$ into a slot that doesn't already contain another power of p_i. Summing over all $0 \leq m \leq k$ and over every choice of m elements to belong to $\mathcal{N}$, we obtain the number of independent facets of size k:

$$\sum_{\substack{0 \leq m \leq k \\ i_1, \ldots, i_m \in [n] \\ i_1 < \cdots < i_m}} \Big(\prod_{j=1}^{m} (k_{i_j} - 1) \binom{k}{m} \sum_{s=k-m}^{n-m} \binom{n-m}{s} St(s, k-m)(n-m-s) \sum_{t=2}^{k} \binom{k}{t} (2^k + 2^{k-t}) \Big)$$

Chapter 4

Computation

4.1 GAP, Sage, and Polymake Algorithms

The author has written GAP [6] and SageMath [18] code which takes as input a finite group and computes the facets of the independence complex, organizing the output according to orbits under the automorphism group. In practice, the most efficient ways to store and compute with a group in GAP are by defining the group using a polycyclic representation (Pc Group) if the group is solvable, or by a permutation representation (Permutation Group) if the group is not solvable. If examples are relatively small, one can directly represent the group in GAP as a finitely presented group (FpGroup) or by using GAP's Small Groups library. After computing the facets of the independence complex in GAP, we use this information to compute properties of the simplicial complex using GAP's simpcomp package REF and SageMath's functionality for simplicial complexes. We refer to the independence complex resulting from this GAP code as the Automorphism Independence Complex. Namely, we define an **automorphism independence complex** as an independence complex arising from a finite group G together with an identification of its facet orbits under the action of Aut(G).

The author also has written GAP code to format the automorphism independence complex in a way that is readable by Poymake, thus allowing for visualization of the resulting simplicial complexes in low-dimensional cases (or restricting one's attention to low-dimensional facets). This reformatted version is stored in the record from the file mainGAP.txt in Appendix A. The author includes code which can be used in Polymake's topaz application in Appendix B. This code plots a visual representation of the independence complex where the orbits are distinguished by different colors and which one can manipulate by magnifying and rotating to view the complex from different angles. One file is produced for each orbit. For technical reasons (see comments in the appendices), one additional orbit is generated in all the images, which is artificial and should be ignored. This additional orbit only appears in the transition from GAP to Polymake, and not in the output of mainGAP.txt. Appendix B shows some examples of how to relabel the vertices in Polymake with group element names, according to the chosen group representation in GAP, so that the independence complex as it relates to the group can be more readily identified.

At a high level, the GAP algorithm, code for which can be found in Appendix A, proceeds as follows:

1. Compute set orbits of the sets of group elements of desired facet size k, under the action of Aut(G).

2. For each orbit, choose a representative (without loss of generality, the first set in the orbit) and check whether that set is independent by removing one element at a time and checking whether the subgroup generated by the remaining elements reduces in size or stays the same.

3. Return a record which contains a list of all independent facets

The algorithm checks for independent sets of size $2 \leq k \leq 5$ by brute force, but working up to group automorphism instead of element-by-element. The code can easily be modified to include facets of size 1 (isolated vertices). The code can also be modified to compute orbits up to conjugacy instead of the action of Aut(G). To make computation more efficient when enumerating tuples up to automorphism, one can also expand on the GAP code in the appendices and move to the semidirect product of G with Aut(G) to perform enumeration calculations. In this setting, testing for equivalence under automorphism becomes testing for conjugacy equivalence. For more information on this algorithm, see [9].

4.2 Non-abelian Independence Complexes

All tables in this section were translated by hand into recognizable group elements from the output from GAP code.

4.2.1 Independence Complex of $S_3 \times C_5$

Table 4.1 shows a list of representatives of independent facets of In($S_3 \times C_5$) together with their corresponding orbit sizes. Here, $(1,2,3)$ represents any element from the equivalence class $\{(1,2,3),(1,3,2)\}$; the element $(1,2)$ (and likewise the element $(1,3)$) represent the equivalence class $\{(1,2),(1,3),(2,3)\}$ and a and b are any distinct elements of order 5 in C_5. The calculations in Table 4.1 match GAP calculations for the sizes of orbits.

Table 4.1: Facet Representatives for In($S_3 \times C_5$)

Facet Size	Orbit Size	Facet Representatives
2	$24 = 2 \cdot 3 \cdot 4$	$\{((1,2,3),a),((1,2),1)\}$
	$24 = 2 \cdot 3 \cdot 4$	$\{((1,2,3),1),((1,2),a)\}$
	$24 = 2 \cdot 3 \cdot 4$	$\{((1,2,3),a),((1,2),a)\}$
	$72 = 2 \cdot 3 \cdot 4 \cdot 3$	$\{((1,2,3),a),((1,2),b)\}$
	$24 = \binom{3}{2} \cdot 4 \cdot 2$	$\{((1,2),a),((1,3),1)\}$
	$12 = \binom{3}{2} \cdot 4$	$\{((1,2),a),((1,3),a)\}$
	$36 = \binom{3}{2} \cdot 4 \cdot 3$	$\{((1,2),a),((1,3),b)\}$
3	$24 = 2 \cdot 3 \cdot 4$	$\{((1,2,3),1),((1,2),1),((),a)\}$
	$12 = \binom{3}{2} \cdot 4$	$\{((1,2),1),((1,3),1),((),a)\}$

The facets of In($S_3 \times C_5$) each generate the whole group (the facets of S_3 all generate S_3, and the cyclic component does not contribute a significant change). For $S_4 \times C_5$ this is not the case. For example, consider the set $I = \{((1,2)(3,4),id),((1,3)(2,4),id),((),a)\}$ where a is a cyclic generator of C_5. Then I is an independent facet in $S_4 \times C_5$; here $\{(1,2)(3,4),(1,3)(2,4)\}$ is a facet of S_4 that generates $C_2 \times C_2$, so I generates $C_2 \times C_2 \times C_5$. Many Polymake-generated images become difficult to parse: see Figure 4.1, which depicts all 1-dimensional facets of In($S_3 \times C_5$).

4.2.2 Independence Complex of $S_3 \times C_3$

The facets of an independence complex have different and seemingly more involved behavior when the component group orders are not coprime (as they were in In($S_3 \times C_5$)). Table 4.2 lists the

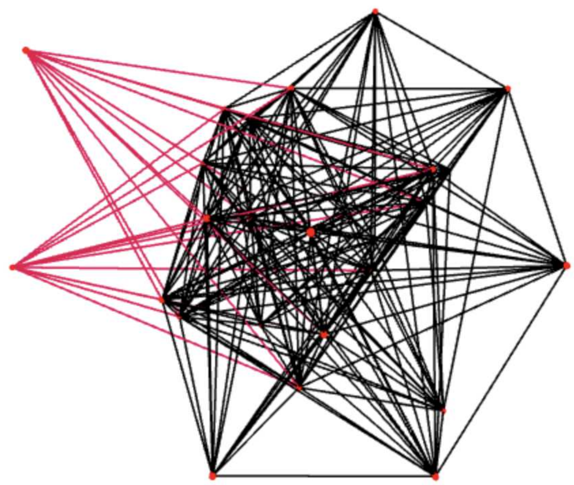


Figure 4.1: 1-dimensional facets of In($S_3 \times C_5$)

orbit representatives from GAP of the action of the automorphism group of $S_3 \times C_3$ on the facets of In($S_3 \times C_3$). In the table, a and b are distinct nonidentity elements of C_3.

Table 4.2: Facet Representatives for In($S_3 \times C_3$)

Facet Size	Orbit Size	Facet Representatives
2	4	$\{((1,2,3),a),((),a)\}$
	4	$\{((1,2,3),a),((),b)\}$
	12	$\{((1,2),a),((2,3),\text{id})\}$
	12	$\{((1,2),\text{id}),((1,2,3),a)\}$
	6	$\{((1,2),a),((2,3),a)\}$
	6	$\{((1,2),a),((2,3),b)\}$
	12	$\{((1,2),a),((1,2,3),\text{id})\}$
	12	$\{((1,2),a),((1,2,3),a)\}$
	12	$\{((1,2),a),((1,2,3),b)\}$
	4	$\{((1,2,3),\text{id}),((1,2,3),a)\}$
	4	$\{((1,2,3),\text{id}),((1,3,2),a)\}$
	2	$\{((1,2,3),a),((1,2,3),b)\}$
	2	$\{((1,2,3),a),((1,3,2),a)\}$
3	6	$\{((),a),((2,3),()),((1,2),())\}$
	12	$\{((),a),((2,3),()),((1,2,3),())\}$

4.2.3 Automorphic Images of In($C_3 \times Q_8$)

Figure 4.2 illustrates the automorphic images of In($C_3 \times Q_8$), computed in GAP and displayed in Polymake (see Appendix). In the two top left images, the orbits are highlighted in blue and dark green, respectively. In the other images, each orbit is highlighted in red. Figure 4.3 shows the element labels.

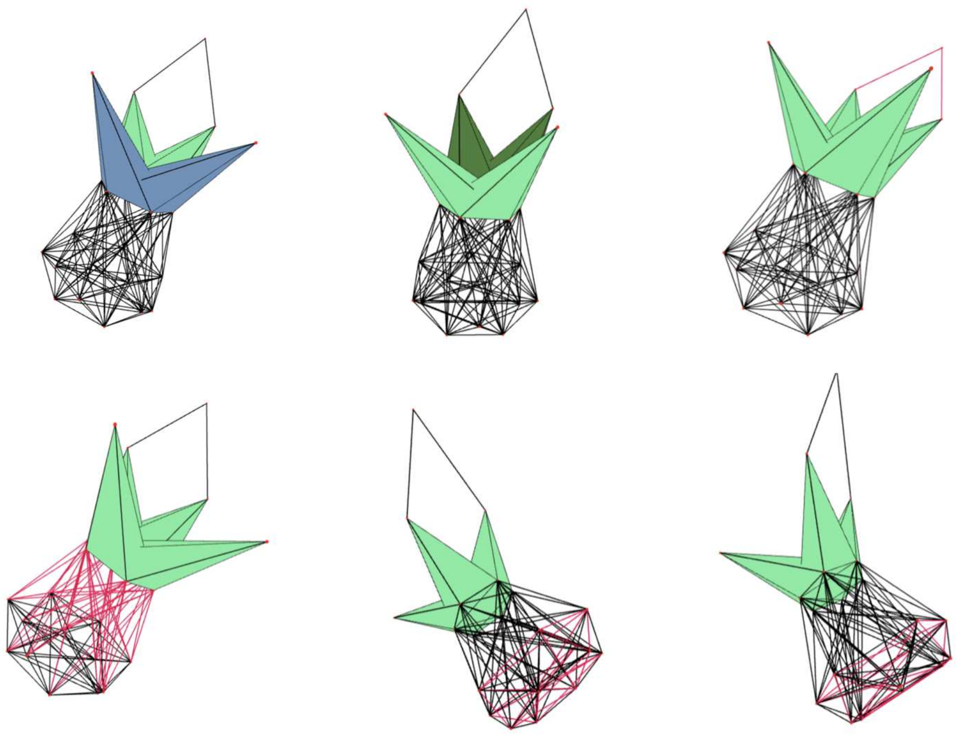


Figure 4.2: Automorphic Images of In($C_3 \times Q_8$)

4.2.4 Larger examples

Example 4.2.1. *The independence complex of* $S_3 \times (C_{11} \rtimes C_5)$ *has 23760 facets of size 2, 198000 facets of size 3, and 11880 facets of size 4. This example is computable in a reasonable amount of time in GAP.*

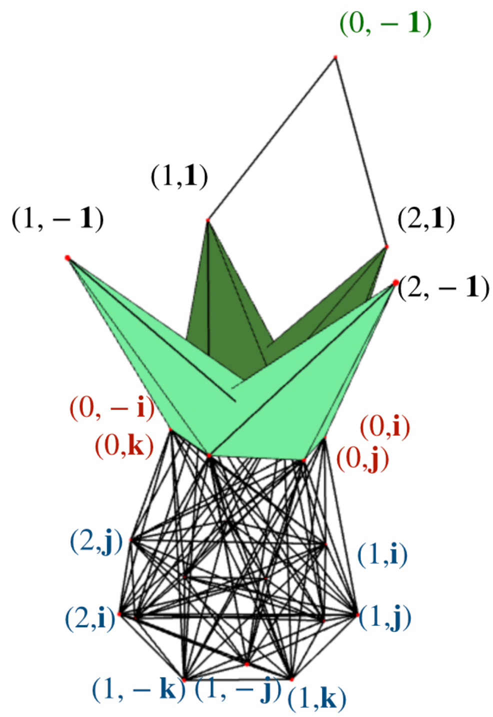


Figure 4.3: Automorphic Images of In($C_3 \times Q_8$)

4.2.5 Representations of Independence Complex of $C_4 \times Q_8$

The 1-dimensional facets of In($C_4 \times Q_8$) are shown in Figure 4.4. The 2-dimensional facets of In($C_4 \times Q_8$) are listed in Tables 4.3 and 4.4.

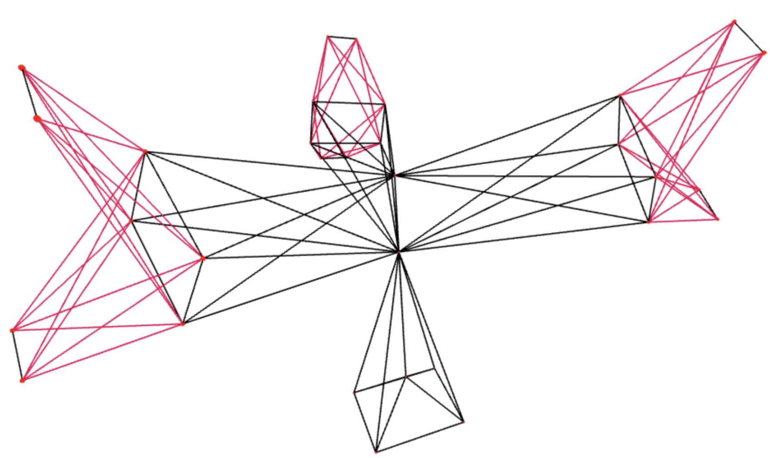


Figure 4.4: 1-dimensional facets of In($C_4 \times Q_8$)

Table 4.3: 2-dimensional Facet Representatives for In($C_4 \times Q_8$) (Part I)

Orbit	Size	Facet Representatives	
12	192	$(3, \mathbb{1}), (0, i), (0, j)$	$(1, \mathbb{1}), (0, i), (0, j)$
		$(3, \mathbb{1}), (0, i), (2, j)$	$(1, \mathbb{1}), (0, i), (2, j)$
		$(3, \mathbb{1}), (2, i), (2, j)$	$(1, \mathbb{1}), (2, i), (2, j)$
		$(3, -\mathbb{1}), (0, i), (0, j)$	$(1, -\mathbb{1}), (0, i), (0, j)$
		$(3, -\mathbb{1}), (0, i), (2, j)$	$(1, -\mathbb{1}), (0, i), (2, j)$
		$(3, -\mathbb{1}), (2, i), (2, j)$	$(1, -\mathbb{1}), (2, i), (2, j)$
13	384	$(3, \mathbb{1}), (0, i), (1, j)$	$(1, \mathbb{1}), (0, i), (1, j)$
		$(3, \mathbb{1}), (0, i), (3, j)$	$(1, \mathbb{1}), (0, i), (3, j)$
		$(3, \mathbb{1}), (2, i), (1, j)$	$(1, \mathbb{1}), (2, i), (1, j)$
		$(3, \mathbb{1}), (2, i), (3, j)$	$(1, \mathbb{1}), (2, i), (3, j)$
		$(3, -\mathbb{1}), (0, i), (1, j)$	$(1, -\mathbb{1}), (0, i), (1, j)$
		$(3, -\mathbb{1}), (0, i), (3, j)$	$(1, -\mathbb{1}), (0, i), (3, j)$
		$(3, -\mathbb{1}), (2, i), (1, j)$	$(1, -\mathbb{1}), (2, i), (1, j)$
		$(3, -\mathbb{1}), (2, i), (3, j)$	$(1, -\mathbb{1}), (2, i), (3, j)$
14	192	$(3, \mathbb{1}), (1, i), (1, j)$	$(1, \mathbb{1}), (1, i), (1, j)$
		$(3, \mathbb{1}), (1, i), (3, j)$	$(1, \mathbb{1}), (1, i), (3, j)$
		$(3, \mathbb{1}), (3, i), (3, j)$	$(1, \mathbb{1}), (3, i), (3, j)$
		$(3, -\mathbb{1}), (1, i), (1, j)$	$(1, -\mathbb{1}), (1, i), (1, j)$
		$(3, -\mathbb{1}), (1, i), (3, j)$	$(1, -\mathbb{1}), (1, i), (3, j)$
		$(3, -\mathbb{1}), (3, i), (3, j)$	$(1, -\mathbb{1}), (3, i), (3, j)$
15	384	$(3, i), (0, \pm i), (0, j)$	$(1, i), (0, \pm i), (0, j)$
		$(3, i), (2, \pm i), (0, j)$	$(1, i), (2, \pm i), (0, j)$
		$(3, i), (0, \pm i), (2, j)$	$(1, i), (0, \pm i), (2, j)$
		$(3, i), (2, \pm i), (2, j)$	$(1, i), (2, \pm i), (2, j)$
16	192	$(3, i), (0, j), (0, k)$	$(1, i), (0, j), (0, k)$
		$(3, i), (0, j), (2, k)$	$(1, i), (0, j), (2, k)$
		$(3, i), (2, j), (2, k)$	$(1, i), (2, j), (2, k)$
17	48	$(2, \mathbb{1}), (0, i), (0, j)$	
		$(2, \mathbb{1}), (0, i), (2, j)$	
		$(2, \mathbb{1}), (2, i), (2, j)$	
18	48	$(2, i), (0, i), (0, j)$	
		$(2, i), (0, i), (2, j)$	
19	48	$(2, i), (0, -i), (0, j)$	
		$(2, i), (0, -i), (2, j)$	

Table 4.4: 2-dimensional Facet Representatives for In($C_4 \times Q_8$) (Part II). Orbit 22 is a very large orbit, and it is tedious to discern by hand whether a completely accurate representative was obtained without accidentally missing information. Hence we mark one orbit by ? to note this uncertainty.

Orbit	Size	Facet Representatives	
20	32	$(2,i),(0,j),(0,k)$	
		$(2,i),(2,j),(2,k)$	
21	48	$(2,-\mathbb{1}),(0,i),(0,j)$	
		$(2,-\mathbb{1}),(0,i),(2,j)$	
		$(2,-\mathbb{1}),(2,i),(2,j)$	
22	384	$(0,\pm i),(1,i),(1,j)$	$(2,i),(3,i),(1,j)$
		$(0,\pm i),(3,i),(1,j)$	$(2,i),(3,i),(3,j)$
		$(0,\pm i),(3,i),(3,j)$	$(2,i),(1,j),(1,j)$?
23	64	$(1,i),(1,j),(1,k)$	
		$(1,i),(1,j),(3,k)$	
		$(1,i),(3,j),(3,k)$	
		$(3,i),(3,j),(3,k)$	

Chapter 5

Main Results (Semidirect product)

Let $\mathcal{G}_n = G_1 \times G_2 \times \cdots \times G_n$ where each G_i is a nonabelian finite group and $|G_1| = p_1p_2, |G_2| = p_3p_4, \ldots, |G_n| = p_{2n-1}p_{2n}$ for distinct primes $p_1, p_2, \ldots, p_{2n}$ with $p_{2i-1} > p_{2i}$ for all $1 \leq i \leq n$. Our main contribution is to describe the independence complex of $\mathcal{G}_n$. To do so, we introduce a simplicial complex structure, called a combinatorial diagram, which translates a difficult enumeration problem into one which is made tractable using the language of simplicial complexes. We first specialize to the cases when $n = 1$ and $n = 2$ and describe the structure of the groups $\mathcal{G}_1$ and $\mathcal{G}_2$. We describe all facets in the independence complexes of $\mathcal{G}_1$ and $\mathcal{G}_2$, using a group theoretic argument in the former and a combinatorial argument in the latter. We next describe the structure of the groups $\mathcal{G}_n$ for arbitrary n and their independence complexes, and provide an algorithm using combinatorial diagrams with which one can generate the facets of the independence complex of $\mathcal{G}_n$. Additionally, we implement this algorithm in GAP.

5.1 Structure of In$(\mathcal{G}_1)$

Recall $\mathcal{G}_1$ is a nonabelian finite group of order p_1p_2 for distinct primes $p_1 > p_2$. We describe the group structure of $\mathcal{G}_1$ and give a description and enumerate the facets of its independence complex.

5.1.1 Structure of $\mathcal{G}_1$

Proposition 5.1.1. *Any independent facet of $\mathcal{G}_1$ has either the form $I = \{x, y\}$, where $|x| = p_1$ and $|y| = p_2$ or the form $\{y, z\}$ where $|z| = p_2$ and y, z are elements from two different Sylow p_2-subgroups.*

Proof: Since $\mathcal{G}_1$ is nonabelian, it is not cyclic. So $\mathcal{G}_1$ has no elements of order p_1p_2, for such an element would generate $\mathcal{G}_1$ by Corollary 2.1.2 and $\mathcal{G}_1$ is not cyclic. By Corollary 2.1.1, possible element orders of $\mathcal{G}_1$ are p_1, p_2, or 1. Let I be an independent facet of $\mathcal{G}_1$. Since I does not contain the identity element, elements of I have possible orders p_1 or p_2. Note that since $n_{p_1} \equiv 1 \mod p_1$, $n_{p_1}|p_2$, and $p_1 > p_2$, we have $n_{p_1} = 1$, so C_{p_1} is the unique Sylow p_1-subgroup.

Suppose first that I has an element x of order p_1. Then x generates the unique Sylow p_1-subgroup P of $\mathcal{G}_1$. No other element of P can be added without redundancy. A second element, y, of order p_2 from some Sylow p_2-subgroup of $\mathcal{G}_1$ can be added to $\{x\}$ without redundancy. Since y lies in a different subgroup than P, the addition of y generates a strictly larger subgroup than $\langle x \rangle = P$, so $\langle x, y \rangle = \mathcal{G}_1$. Since $\langle x \rangle \cong C_{p_1}$ and $\langle y \rangle \cong C_{p_2}$, $I = \{x, y\}$ is independent. No additional elements can be added without redundancy, so I is a facet.

Now suppose I has no elements of P. Then I consists of elements of order p_2. If $I = \{y\}$ for some y in some Sylow p_2-subgroup Q of $\mathcal{G}_1$, then $I \cup \{x\}$ is an independent set for any nontrivial $x \in P$, so I is not a facet. Thus I contains another element z of order p_2. If $z \in Q$, then z is redundant since $\langle y \rangle \cong Q$. Thus z must be in a different Sylow p_2-subgroup of $\mathcal{G}_1$ than y. Since $\mathcal{G}_1$ is not cyclic, none of the Sylow p_2-subgroups are normal in $\mathcal{G}_1$, so $n_{p_2} \neq 1$. Since $n_{p_2}|p_1$, we have $n_{p_2} = p_1$. Thus at least two Sylow p_2-subgroups exist. Since $\langle y \rangle = Q$ and $z \notin Q$, we have $\langle y, z \rangle \cong \mathcal{G}_1$. Since $\langle y \rangle \cong \langle z \rangle \cong C_{p_2}$ and no additional elements can be added to $I = \{y, z\}$ without redundancy, I is an independent facet.

Observe that if $|x| = p_1$ and $|y| = p_2$ then $\{x\}$ and $\{y\}$ are not facets, as each is contained in a larger independent set. Thus every independent facet of In($\mathcal{G}_1$) has size 2, so In($\mathcal{G}_1$) is pure of dimension 1.

□

Note that edges of the independence complex of $\mathcal{G}_1$ are given by the collection of all sets of two elements each coming from distinct subgroups. Also observe that $\mathcal{G}_1$ has a unique Sylow p_1-subgroup and p_1 Sylow p_2-subgroups.

We will see in the following text that Lucchini's bound (see subsection 2.1.5) states that the maximal size of independent sets of $\mathcal{G}_1$ is two. Thus Lucchini's bound is met here. The following, due to Burnside, is a classical result in group theory.

Theorem 5.1.2 (Burnside, see [5]). *Let G be a finite group of order $p^\alpha q^\beta$ where p, q are prime and $\alpha, \beta \in \mathbb{Z}_{\geq 0}$. Then G is solvable.*

The group $\mathcal{G}_1$ in Proposition 5.1.1 is solvable by Burnside's Theorem. Thus by Lucchini's bound, the largest size of an independent set that generates $\mathcal{G}_1$ is given by summing over the sizes of the minimal generating sets for the Sylow p-subgroups of $\mathcal{G}_1$. Observe that $\mathcal{G}_1$ has Sylow p_1-subgroup C_{p_1} and Sylow p_2-subgroup C_{p_2}. Since $m(C_{p_1}) = m(C_{p_2}) = 1$, we have

$$m(\mathcal{G}_1) \leq \sum_{p \in \pi(\mathcal{G}_1)} d_p(\mathcal{G}_1) = d_{p_1}(\mathcal{G}_1) + d_{p_2}(\mathcal{G}_1) = 2$$

As shown in Proposition 5.1.1, this bound is tight. Note that this provides a class of examples which are not necessarily nilpotent (take for example S_3 or $C_{11} \rtimes C_5$) and achieve the bound (see for comparison the comments in subsection 2.1.5).

Proposition 5.1.3. *The number of independent facets of $\mathcal{G}_1$ is $\binom{p_1}{2}(p_2^2 - 1)$.*

Proof: There are $(p_1 - 1)p_1(p_2 - 1)$ facets of the form $\{x, y\}$ where $|x| = p_1$ and $|y| = p_2$, since there are $p_1 - 1$ ways to choose a nonidentity element from the unique Sylow p_1-subgroup, and $p_2 - 1$ ways to choose a nonidentity element from any of the p_1 Sylow p_2-subgroups. There are $\binom{p_1}{2}(p_2 - 1)^2$ independent facets of the form $\{y, z\}$ where y and z have order p_2 and are in two different Sylow p_2-subgroups; one first chooses two of the p_1 Sylow p_2-subgroups, and then selects from among the $p_2 - 1$ nonidentity elements of each chosen subgroup. Observe

$$\begin{aligned}
(p_1 - 1)p_1(p_2 - 1) + \binom{p_1}{2}(p_2 - 1)^2 &= 2\binom{p_1}{2}(p_2 - 1) + \binom{p_1}{2}(p_2 - 1)^2 \\
&= \binom{p_1}{2}(p_2 - 1)(p_2 + 1) \\
&= \binom{p_1}{2}(p_2^2 - 1)
\end{aligned}$$

□

Example 5.1.4. *In S_3 (here $p_1 = 3$ and $p_2 = 2$), one can have independent facets of the form $\{(1,2,3),(1,2)\}$ where the elements have orders p_1 and p_2, as well as independent facets of the form $\{(1,2),(1,3)\}$ where both elements have order p_2 and lie in different Sylow p_2-subgroups. In S_3, there are two elements of order three and there are three Sylow 2-subgroups. There are $2 * 3 = 6$ independent facets of the form $\{p_1, p_2\}$ (here, we list only the orders of elements) and $\binom{3}{2} = 3$ independent sets of the form $\{p_2, p_2\}$ (representing the orders of two elements from different Sylow p_2-subgroups). Thus S_3 has a total of 9 (in this case, we can count this as $\binom{5}{2} - 1$) independent facets of size 2, and is pure-dimensional.*

Example 5.1.5. *Let $\mathcal{G}_1 = C_{11} \rtimes C_5$, $p_1 = 11$ and $p_2 = 5$. Then $\mathrm{In}(\mathcal{G}_1)$ has $\binom{11}{2} \cdot 24 = 1320$ facets.*

5.2 Structure of In$(\mathcal{G}_2)$

Our next step is to describe and enumerate the facets of the independence complex $\mathcal{G}_2 := \mathcal{G} \times \mathcal{H}$ where $\mathcal{G}, \mathcal{H}$ are nonabelian finite groups of orders p_1p_2, p_3p_4, respectively, for distinct primes p_1, p_2, p_3, p_4 with $p_1 > p_2$ and $p_3 > p_4$.

5.2.1 Structure of $\mathcal{G}_2$ and facet bound

We begin by describing all subgroups of $\mathcal{G}_2$ and obtaining an upper bound on the size of its independent facets by showing that $\mathcal{G}_2$ is solvable and using Lucchini's upper bound (see subsection 2.1.5 and [12]). A standard group theory result tells us:

Proposition 5.2.1. *[[10], Exercise 5C.4] If all Sylow subgroups of a group G are cyclic, then for every divisor m of $|G|$, there is a subgroup of order m. Furthermore, any two subgroups of G having order m are conjugate.*

Since $\mathcal{G}_2$ has order $p_1p_2p_3p_4$, its Sylow subgroups have orders p_1, p_2, p_3 and p_4. Such subgroups exist by Theorem 2.1.6 (Sylow's Theorem), and are cyclic. Thus by Proposition 5.2.1, $\mathcal{G}_2$ has subgroups of orders given by all proper divisors of $p_1p_2p_3p_4$, and any two subgroups of the same order are conjugate in $\mathcal{G}_2$. Since the order of a subgroup divides the order of the group, these account for all possible subgroups of $\mathcal{G}_2$.

It is a standard group theoretic result (see [15], Corollary 5.18) that the direct product of two solvable groups is solvable. Since $\mathcal{G}$ and $\mathcal{H}$ are both solvable by Theorem 5.1.2 (Burnside's Theorem), the direct product $\mathcal{G}_2 = \mathcal{G} \times \mathcal{H}$ is also solvable. Thus we can apply Lucchini's upper bound. Since the Sylow p_i-subgroups for all $1 \leq i \leq 4$ of $\mathcal{G}_2$ are all cyclic, and $m(C_{p_i}) = 1$ for all $1 \leq i \leq 4$, by Lucchini's result we have $m(\mathcal{G}_2) \leq \sum_{p \in \pi(\mathcal{G}_2)} d_p(\mathcal{G}_2) = 4$. Thus any minimal generating set for $\mathcal{G}_2$ will have size no larger than 4 (note that this statement only applies directly to independent sets which generate the full group $\mathcal{G}_2$). Indeed, this bound is met; an independent set of size 4 that generates $\mathcal{G}_2$ by Corollary 2.1.1 is

$$\{(g_1, 1_{\mathcal{H}}), (g_2, 1_{\mathcal{H}}), (1_{\mathcal{G}}, h_1), (1_{\mathcal{G}}, h_2)\} \tag{5.1}$$

where g_1, g_2 have orders p_1, p_2 and h_1, h_2 have orders p_3, p_4 respectively.

Note that if $\mathcal{G}_2$ is a product of *cyclic* groups, say $\mathcal{G}_2 = C_{p_1p_2} \times C_{p_3p_4}$ for distinct primes p_1, p_2, p_3, p_4, then in any facet of $\mathcal{G}_2$, elements of all orders p_1, p_2, p_3, p_4 must appear in their appropriate component. If some p_i is missing, the independent set can be extended by the tuple with an element of order p_i in the appropriate component and the identity in the other component. Hence the only independent facets of $\mathcal{G}_2$ having size 4 are of the form given in Expression (5.1) when the direct product components are cyclic. We get other independent sets, however, when the component groups of $\mathcal{G}_2$ are nonabelian. For example, if $\mathcal{G}_2 = G \times H$ where G, H are nonabelian groups with $|G| = p_1p_2$, $|H| = p_3p_4$ for distinct primes p_i with $p_1 > p_2, p_3 > p_4$, the set in Expression (5.1) is also independent when g_1, g_2 both have order p_2 and h_1, h_2 both have order p_4 (each element taken from a distinct Sylow subgroup). We describe the independent facets of groups with two nonabelian components in Section 5.2.3.

5.2.2 Unique selling points

Lemma 5.2.2. *Let $\mathcal{G}_n = G_1 \times G_2 \times \cdots \times G_n$ where the G_i are finite groups with relatively prime orders. Then in any independent set I, for every tuple $t \in I$ there is some $1 \leq i \leq n$ such that $\langle \pi_i(I \setminus t) \rangle < \langle \pi_i(I) \rangle$ in G_i. We call the element $\pi_i(t)$ a* ***unique selling point*** *of t.*

Proof: Let I be an independent set of $G_1 \times G_2 \times \cdots \times G_n$ where the G_i are finite groups whose orders are relatively prime. Suppose to the contrary that for some tuple $t \in I$, we have $\langle \pi_i(I \setminus t) \rangle = \langle \pi_i(I) \rangle$ for every $1 \leq i \leq n$. We show that $\langle I \setminus \{t\} \rangle = \langle I \rangle$. First, we determine $|\langle I \setminus \{t\} \rangle|$. For all $1 \leq i \leq n$, we have

$$\pi_i(\langle I \setminus \{t\} \rangle) = \langle \pi_i(I \setminus \{t\}) \rangle = \langle \pi_i(I) \rangle = \pi_i(\langle I \rangle).$$

For all $1 \leq i \leq n$, $|\pi_i(\langle I \setminus \{t\} \rangle)|$ divides $|\langle I \setminus \{t\} \rangle|$ since π_i is a homomorphism. Thus

$$\begin{aligned} |\langle I \setminus \{t\} \rangle| &\geq \text{lcm}(|\pi_1(\langle I \setminus \{t\} \rangle)|, \ldots, |\pi_n(\langle I \setminus \{t\} \rangle)|) \\ &= \text{lcm}(|\langle \pi_1(I) \rangle|, \ldots, |\langle \pi_n(I) \rangle|) \\ &= |\langle \pi_1(I) \rangle| \cdot |\langle \pi_2(I) \rangle| \cdot \cdots \cdot |\langle \pi_n(I) \rangle| \end{aligned} \tag{5.2}$$

where the last equality holds because the factors G_i have relatively prime orders. We now show that $\langle I \rangle$ is a direct product of the same order as the product in line (5.2).

We claim $\langle I \rangle = \langle \pi_1(I) \rangle \times \cdots \times \langle \pi_n(I) \rangle$. By Proposition 2.1.14, we must show that:

1. $\langle \pi_i(I) \rangle$ is normal in $\langle I \rangle$ for all $1 \leq i \leq n$
2. $\langle I \rangle = \langle \pi_1(I) \rangle \langle \pi_2(I) \rangle \cdots \langle \pi_n(I) \rangle$
3. $\langle \pi_i(I) \rangle \cap \langle \pi_j(I) \rangle = \text{id}_{\mathcal{G}_n}$ for all $1 \leq i, j \leq n$ with $i \neq j$

We do so as follows.

1. Let $g = (g_1, g_2, \ldots, g_n) \in \langle I \rangle$ and let $n = (1, \ldots, 1, n_i, 1, \ldots, 1) \in \langle \pi_i(I) \rangle$ where n_i is in the i^{th} component of n. Then

$$\begin{aligned} gng^{-1} &= (g_1, g_2, \ldots, g_n)(1, \ldots, 1, n_i, 1, \ldots, 1)(g_1, g_2, \ldots, g_n)^{-1} \\ &= (g_1 g_1^{-1}, g_2 g_2^{-1}, \ldots, g_i n_i g_i^{-1}, \ldots, g_n g_n^{-1}) \\ &= (1, \ldots, 1, g_i n_i g_i^{-1}, 1, \ldots, 1) \in \langle \pi_i(I) \rangle \end{aligned}$$

where inclusion in the last line holds since $g_i, g_i^{-1} \in \pi_i(\langle I \rangle) = \langle \pi_i(I) \rangle$ and $n_i \in \langle \pi_i(I) \rangle$. So $\langle \pi_i(I) \rangle$ is normal in $\langle I \rangle$ for each $1 \leq i \leq n$.

2. Let $g = (g_1, \ldots, g_n) \in \langle I \rangle$. Then g can be represented as an element in the product in the righthand side of (2). If $g \in \langle \pi_1(I) \rangle \langle \pi_2(I) \rangle \cdots \langle \pi_n(I) \rangle$, then g can be written as $g = (g_1, 1, \ldots, 1)(1, g_2, 1, \ldots, 1) \cdots (1, \ldots, 1, g_n) = (g_1, \ldots, g_n) \in \langle I \rangle$.

3. $\langle \pi_i(I) \rangle \cap \langle \pi_j(I) \rangle = \mathrm{id}_{\mathcal{G}_n}$ for all $1 \leq i, j \leq n$ with $i \neq j$ since the $|G_i|$ (and the orders of subgroups of G_i) are relatively prime, so there is no interaction between different component projections.

Hence $\langle I \rangle$ is a direct product, namely $\langle I \rangle = \langle \pi_1(I) \rangle \times \cdots \times \langle \pi_n(I) \rangle$. Thus

$$|\langle I \rangle| = |\langle \pi_1(I) \rangle| \cdot |\langle \pi_2(I) \rangle| \cdot \cdots \cdot |\langle \pi_n(I) \rangle|$$

By line (5.2) and the fact that $\langle I \setminus \{t\} \rangle \subseteq \langle I \rangle$, we have that $\langle I \setminus \{t\} \rangle = \langle I \rangle$. Hence I is not independent, a contradiction. $\square$

For example, let $G = C_2 \times C_3 \cong C_6$. Then $I = \{(1, 0), (0, 1)\}$ is an independent set, with unique selling points given by the first tuple's first component and the second tuple's second component. If the component group orders are not relatively prime, the independent sets need not have the same notion of unique selling points. For instance, let $G = C_3 \times C_3$ and $I = \{(1, 1), (1, 2)\}$, which is an independent set generating G. Removing either tuple does not decrease generation in the individual group components, yet both tuples are required to generate G (for if one is removed, the result generates only a cyclic subgroup). Thus the notion of unique selling points does not generalize directly to direct products whose component groups have orders with common factors.

Lemma 5.2.3. *Let $G = G_1 \times G_2 \times \cdots \times G_n$ where the G_i are finite groups. Let I be a set of group elements of G such that every tuple $t \in I$ has a unique selling point. Then I is an independent set.*

Proof: Let I be a set of tuples of G, each of which has a unique selling point. Let $t \in I$. Then for some $1 \leq i \leq n$, $\langle \pi_i(I \setminus t) \rangle < \langle \pi_i(I) \rangle$. Since subgroup generation is reduced in the i^{th} component group, we have that $\langle I \setminus t \rangle < \langle I \rangle$. $\square$

5.2.3 Structure description of In$(\mathcal{G}_2)$

Proposition 5.2.4. *Let A, B be distinct nontrivial subgroups of $\mathcal{G}$ and C, D distinct nontrivial subgroups of $\mathcal{H}$. The independent facets of* In$(\mathcal{G}_2)$ *have size $k = 2, 3$ and 4 and have the form:*

1. $(k = 2)$ $\{(a, c), (b, d)\}$
2. $(k = 3)$
   - *(i)* $\{(a, c), (1_{\mathcal{G}}, d), (b, 1_{\mathcal{G}})\}$
   - *(ii)* $\{(a, c), (a', d), (b, 1_{\mathcal{H}})\}$
   - *(iii)* $\{(a, c), (b, c'), (1_{\mathcal{G}}, d)\}$
3. $(k = 4)$ $\{(a, 1_{\mathcal{H}}), (b, 1_{\mathcal{H}}), (1_{\mathcal{G}}, c), (1_{\mathcal{G}}, d)\}$

where $a, a' \in A$, $b \in B$, $c, c' \in C$, $d \in D$.

Proof: We first show that a facet cannot have size one, and we then describe all independent facets of $\mathcal{G}_2$ of sizes $k = 2, 3$, and 4.

$(k \neq 1)$ Observe there are no independent facets of $\mathcal{G}_2$ of size one, as such a independent facet $I = \{(g_1, h_1)\}$ could only support one element $g_1 \in A$ of and one element $h_1 \in C$ where A, C are subgroups of $\mathcal{G}, \mathcal{H}$, respectively. Since $\mathcal{G}_2$ is not cyclic, $\{(g_1, h_1)\}$ generates some proper subgroup of $\mathcal{G}_2$, and can be extended (for instance) to an independent set $\{(g_1, h_1), (g_2, h_2)\}$ where $g_2 \notin A$ and $h_2 \notin C$. In each component projection, the full component group is generated, so $\langle (g_1, h_1), (g_2, h_2) \rangle = \mathcal{G}_2$. Since $\mathcal{G}, \mathcal{H}$ are both noncyclic, $\mathcal{G}_2$ is noncyclic, so neither tuple can be removed from $I \cup \{(g_2, h_2)\}$ and have the remaining element generate $\mathcal{G}_2$, so $I \cup \{(g_2, h_2)\}$ is an independent facet containing I. Thus there no independent facets of size one.

$(k = 2)$ Any independent facet of $\mathcal{G}_2$ of size 2 will contain one element of the form (g_1, h_1) where g_1 is in some subgroup of $\mathcal{G}$ and h_1 in some subgroup of $\mathcal{H}$. If in the second element (g_2, h_2), we have that g_2, g_1 are in the same subgroup and h_2, h_1 are in the same subgroup, then the second element can be removed as it is redundant. If g_2, g_1 are in the same subgroup and h_2, h_1 are in different subgroups, then $\{(g_1, h_1), (g_2, h_2), (g_3, 1_{\mathcal{H}})\}$, of size 3, is an independent facet where g_3 is any element from a different subgroup than g_1 and g_2. So g_1, g_2 must be in different subgroups of $\mathcal{G}$ and h_1, h_2 in different subgroups of $\mathcal{H}$. Indeed, removing either tuple from $I = \{(g_1, h_1), (g_2, h_2)\}$ results in generation of a smaller group, so I is an independent facet.

$(k = 3)$ Let I be an independent facet of $\mathcal{G}_2$ of size 3. First observe that each component projection $\pi_i(I)$ for $i \in \{1, 2\}$ must contain elements from (at least) two distinct subgroups. If, without loss of generality, $\pi_1(I)$ contains elements from only one nontrivial subgroup A of $\mathcal{G}$, then $I \cup \{(b, 1_{\mathcal{H}})\}$, where $b \notin A$, is independent, a contradiction. So $\pi_i(I)$ contains elements from (at least) two distinct subgroups for $i \in \{1, 2\}$. Every tuple of I has (at least one) unique selling point by Lemma 5.2.2, say without loss of generality a, b in $\pi_1(I)$ and d in $\pi_2(I)$. Without loss of generality, I has the form $I = \{(a, \_), (b, c), (\_, d)\}$ where $a \in A, b \in B, c \in C, d \in D$, A, B are distinct subgroups of $\mathcal{G}$, and C, D are distinct subgroups of $\mathcal{H}$ since (at least) two subgroups must be represented in each component. Here $\{a, b\}$ is a facet of $\mathcal{G}$ and $\{c, d\}$ is a facet of $\mathcal{H}$.

We now describe the options for the empty slots of I.

1. Both empty slots could be the identity elements. Note that $\{(a, 1_{\mathcal{H}}), (b, c), (1_{\mathcal{G}}, d)\}$ is independent and generates $\mathcal{G}_2$.

2. One empty slot could be the identity element, and the other nonidentity. Without loss of generality, suppose I has the form $\{(a, c'), (b, c), (1_{\mathcal{G}}, d)\}$ for some $c' \in \mathcal{H}$. Then $c' \in C$, for if c' was in a different subgroup than C, the facet $\{c, c'\}$ would generate $\mathcal{H}$, making the tuple $(1_{\mathcal{G}}, d)$ redundant. A similar argument shows that I could be of the form $(a, 1_{\mathcal{H}}), (b, c), (b', d)$ where $b' \in B$.

At least one of the empty slots must be the identity. If not, then $I = \{(a, c'), (b, c), (b', d)\}$ where $c' \in \mathcal{H}$ and $b' \in \mathcal{G}$ are nonidentity elements. There are four cases:

1. $c' \in C$ and $b' \in B$
2. $c' \notin C$ and $b' \notin B$
3. $c' \in C$ and $b' \notin B$
4. $c' \notin C$ and $b' \in B$

We show a contradiction is reached in the first case; we leave the other three similar arguments to the reader. If $c' \in C$ and $b' \in B$, then $\langle a, b' \rangle = \mathcal{G}$ and $\langle c', d \rangle = \mathcal{H}$. Thus the tuple (b, c) is redundant, and hence I is not independent, a contradiction. Contradictions are reached in the other three cases as well. Thus at least one of the empty slots must be the identity.

$(k = 4)$ By Lemma 5.2.2, every tuple has (at least one) unique selling point, and each projection has (at least) two distinct nontrivial subgroups represented. So I has the form

$$\{(a, \_), (b, \_), (\_, c), (\_, d)\}$$

where $a \in A, b \in B, c \in C, d \in D$ are unique selling points for distinct nontrivial subgroups A, B of $\mathcal{G}$ and C, D of $\mathcal{H}$. We claim the remaining entries are identity elements. Assume to the contrary that there is a nonidentity element in one of the remaining slots; without loss of generality, a nonidentity element $x \in \mathcal{H}$ in the last tuple. If $x \in A$ then $\langle a, b, x \rangle = \langle b, x \rangle$ so a is not a unique selling point of $(a, \_)$, a contradiction. Thus $x \notin A$, and similarly $x \notin B$, so x is in some other subgroup of $\mathcal{G}$ distinct from A and B. Thus, since any two nonidentity elements from distinct subgroups generate $\mathcal{G}$, $\langle a, b, x \rangle = \langle b, x \rangle$. Hence a is not a unique selling point of $(a, \_)$, a contradiction. Therefore $x = 1_{\mathcal{G}}$. Thus I has the form

$$\{(a, 1_{\mathcal{H}}), (b, 1_{\mathcal{H}}), (1_{\mathcal{G}}, c), (1_{\mathcal{G}}, d)\}$$

Later we present a more efficient way of describing the independent facets of $\mathcal{G}_n$ which does not rely on case analysis and brute force. $\square$

Table 5.1: Independent facets of $\mathcal{G}_2$. For $k=3$, facets (a)–(f) are Type (ii); facets (g)–(l) are Type (iii); and facets (m)–(r) are Type (i), as in Proposition 5.2.4. For facets of size $k=2$ and $k=4$, the values in the fourth column are equal to the sum of the values in the corresponding rows in the third column. In the superscripts, i,j,k,m are all distinct so that, for instance, $C_{p_2}^{(i)}$ and $C_{p_2}^{(j)}$ represent different Sylow p_2-subgroups. We omit superscripts where there is no ambiguity as to which subgroup is meant.

k	Facet	Number of Facets by Type
2	(a) $\{(C_{p_1},C_{p_3}),(C_{p_2},C_{p_4})\}$	$(p_1-1)p_1(p_2-1)(p_3-1)p_3(p_4-1)$
	(b) $\{(C_{p_1},C_{p_4}),(C_{p_2},C_{p_3})\}$	$(p_1-1)p_1(p_2-1)(p_3-1)p_3(p_4-1)$
	(c) $\{(C_{p_1},C_{p_4}^{(i)}),(C_{p_2},C_{p_4}^{(j)})\}$	$2(p_1-1)p_1(p_2-1)\binom{p_3}{2}(p_4-1)^2$
	(d) $\{(C_{p_2}^{(i)},C_{p_3}),(C_{p_2}^{(j)},C_{p_4})\}$	$2\binom{p_1}{2}(p_2-1)^2(p_3-1)p_3(p_4-1)$
	(e) $\{(C_{p_2}^{(i)},C_{p_4}^{(k)}),(C_{p_2}^{(j)},C_{p_4}^{(m)})\}$	$2\binom{p_1}{2}(p_2-1)^2\binom{p_3}{2}(p_4-1)^2$
3	(a) $\{(C_{p_1},\langle 1_{\mathcal{H}}\rangle),(C_{p_2}^{(i)},C_{p_3}),(C_{p_2}^{(i)},C_{p_4})\}$	See Equation 5.4 for total count
	(b) $\{(C_{p_1},\langle 1_{\mathcal{H}}\rangle),(C_{p_2}^{(i)},C_{p_4}^{(j)}),(C_{p_2}^{(i)},C_{p_4}^{(k)})\}$	
	(c) $\{(C_{p_2},\langle 1_{\mathcal{H}}\rangle),(C_{p_1}^{(i)},C_{p_3}),(C_{p_1}^{(i)},C_{p_4})\}$	
	(d) $\{(C_{p_2},\langle 1_{\mathcal{H}}\rangle),(C_{p_1}^{(i)},C_{p_4}^{(j)}),(C_{p_1}^{(i)},C_{p_4}^{(k)})\}$	
	(e) $\{(C_{p_2}^{(j)},\langle 1_{\mathcal{H}}\rangle),(C_{p_2}^{(i)},C_{p_3}),(C_{p_2}^{(i)},C_{p_4})\}$	
	(f) $\{(C_{p_2}^{(j)},\langle 1_{\mathcal{H}}\rangle),(C_{p_2}^{(i)},C_{p_4}^{(k)}),(C_{p_2}^{(i)},C_{p_4}^{(m)})\}$	
	(g) $\{(\langle 1_{\mathcal{G}}\rangle,C_{p_3}),(C_{p_1},C_{p_4}^{(i)}),(C_{p_2},C_{p_4}^{(i)})\}$	
	(h) $\{(\langle 1_{\mathcal{G}}\rangle,C_{p_3}),(C_{p_2}^{(j)},C_{p_4}^{(i)}),(C_{p_2}^{(k)},C_{p_4}^{(i)})\}$	
	(i) $\{(\langle 1_{\mathcal{G}}\rangle,C_{p_4}),(C_{p_1},C_{p_3}^{(i)}),(C_{p_2},C_{p_3}^{(i)})\}$	
	(j) $\{(\langle 1_{\mathcal{G}}\rangle,C_{p_4}),(C_{p_2}^{(j)},C_{p_3}^{(i)}),(C_{p_2}^{(k)},C_{p_3}^{(i)})\}$	
	(k) $\{(\langle 1_{\mathcal{G}}\rangle,C_{p_4}^{(j)}),(C_{p_1},C_{p_4}^{(i)}),(C_{p_2},C_{p_4}^{(i)})\}$	
	(l) $\{(\langle 1_{\mathcal{G}}\rangle,C_{p_4}^{(j)}),(C_{p_2}^{(k)},C_{p_4}^{(i)}),(C_{p_2}^{(m)},C_{p_4}^{(i)})\}$	
	(m) $\{(C_{p_1},C_{p_3}),(\langle 1_{\mathcal{G}}\rangle,C_{p_4}),(C_{p_2},\langle 1_{\mathcal{H}}\rangle)\}$	
	(n) $\{(C_{p_1},C_{p_4}),(\langle 1_{\mathcal{G}}\rangle,C_{p_3}),(C_{p_2},\langle 1_{\mathcal{H}}\rangle)\}$	
	(o) $\{(C_{p_1},C_{p_4}^{(i)}),(\langle 1_{\mathcal{G}}\rangle,C_{p_4}^{(j)}),(C_{p_2},\langle 1_{\mathcal{H}}\rangle)\}$	
	(p) $\{(C_{p_2}^{(i)},C_{p_3}),(\langle 1_{\mathcal{G}}\rangle,C_{p_4}),(C_{p_2}^{(j)},\langle 1_{\mathcal{H}}\rangle)\}$	
	(q) $\{(C_{p_2}^{(i)},C_{p_4}),(\langle 1_{\mathcal{G}}\rangle,C_{p_3}),(C_{p_2}^{(j)},\langle 1_{\mathcal{H}}\rangle)\}$	
	(r) $\{(C_{p_2}^{(i)},C_{p_4}^{(k)}),(\langle 1_{\mathcal{G}}\rangle,C_{p_4}^{(m)}),(C_{p_2}^{(j)},\langle 1_{\mathcal{H}}\rangle)\}$	
4	$\{(\langle 1_{\mathcal{G}}\rangle,C_{p_3}),(\langle 1_{\mathcal{G}}\rangle,C_{p_4}),(C_{p_1},\langle 1_{\mathcal{H}}\rangle),(C_{p_2},\langle 1_{\mathcal{H}}\rangle)\}$	$(p_1-1)p_1(p_2-1)(p_3-1)p_3(p_4-1)$
	$\{(\langle 1_{\mathcal{G}}\rangle,C_{p_4}^{(i)}),(\langle 1_{\mathcal{G}}\rangle,C_{p_4}^{(j)}),(C_{p_1},\langle 1_{\mathcal{H}}\rangle),(C_{p_2},\langle 1_{\mathcal{H}}\rangle)\}$	$(p_1-1)p_1(p_2-1)\binom{p_3}{2}(p_4-1)^2$
	$\{(\langle 1_{\mathcal{G}}\rangle,C_{p_3}),(\langle 1_{\mathcal{G}}\rangle,C_{p_4}),(C_{p_2}^{(i)},\langle 1_{\mathcal{H}}\rangle),(C_{p_2}^{(j)},\langle 1_{\mathcal{H}}\rangle)\}$	$\binom{p_1}{2}(p_2-1)^2(p_3-1)p_3(p_4-1)$
	$\{(\langle 1_{\mathcal{G}}\rangle,C_{p_4}^{(k)}),(\langle 1_{\mathcal{G}}\rangle,C_{p_4}^{(m)}),(C_{p_2}^{(i)},\langle 1_{\mathcal{H}}\rangle),(C_{p_2}^{(j)},\langle 1_{\mathcal{H}}\rangle)\}$	$\binom{p_1}{2}(p_2-1)^2\binom{p_3}{2}(p_4-1)^2$

5.2.4 Counting facets of In($\mathcal{G}_2$)

We now enumerate the independent facets of $\mathcal{G}_2$ of sizes $k=2,3$, and 4. A summary of these counts are included in Table 5.1. For subgroups $G_i \leq \mathcal{G}, H_i \leq \mathcal{H}$ for $1 \leq i \leq m$, define the following:

$$(G_1,H_1),(G_2,H_2),\ldots,(G_m,H_m) :=$$
$$\{(g_1,h_1),(g_2,h_2),\ldots,(g_m,h_m) : g_i \in G_i, h_i \in H_i \text{ for all } 1 \le i \le m\} \tag{5.3}$$

In Table 5.1, we use this notation to list representatives of the independent facets of $\mathcal{G}_2$.

Denote the p_1+1 nontrivial proper subgroups of $\mathcal{G}$ by $C_{p_1}, C_{p_2}^{(1)}, C_{p_2}^{(2)}, \ldots, C_{p_2}^{(p_1)}$, and similarly $C_{p_3}, C_{p_4}^{(1)}, C_{p_4}^{(2)}, \ldots, C_{p_4}^{(p_3)}$ for $\mathcal{H}$. Here $C_{p_r}^{(i)}$ and $C_{p_r}^{(j)}$ represent different Sylow p_r-subgroups if $i \neq j$ and the same subgroup if $i=j$. If the context is clear and there is only one subgroup to reference, we sometimes omit the superscript (i) on $C_{p_r}^{(i)}$.

(k=2) The counts for the independent facets of size 2 are listed in Table 5.1. Here we highlight the counts for (d) and (e). The rest are similar.

Facets (d) have the form $\{(C_{p_2}^{(i)}, C_{p_3}), (C_{p_2}^{(j)}, C_{p_4})\}$, or namely, all sets of the form $\{(a,c),(b,d)\}$ where a,b are from two different Sylow p_2-subgroups, c is from the unique Sylow p_3-subgroup, and d is from some Sylow p_4-subgroup (here, i and j are distinct). There are $(p_3-1)p_3(p_4-1)$ ways to choose the element c from the Sylow p_3-subgroup and the element d from the p_3 Sylow p_4-subgroups. Once these are chosen, there are $\binom{p_1}{2}(p_2-1)^2$ ways to choose elements a and b from two of the p_1 Sylow p_2-subgroups, and then 2 ways to order the elements a and b. Thus there are a total of $2\binom{p_1}{2}(p_2-1)^2(p_3-1)p_3(p_4-1)$ independent facets of size 2 having form (d).

Facets (e) have the form $\{(C_{p_2}^{(i)}, C_{p_4}^{(k)}), (C_{p_2}^{(j)}, C_{p_4}^{(m)})\}$, namely all sets of the form $\{(a,c),(b,d)\}$ where a,b are from two different Sylow p_2-subgroups (there are $\binom{p_1}{2}(p_2-1)^2$ ways to choose these) and c,d are from two different Sylow p_4-subgroups ($\binom{p_3}{2}(p_4-1)^2$ ways). Finally, there are 2 ways to order a and b. Thus there are a total of $2\binom{p_1}{2}(p_2-1)^2\binom{p_3}{2}(p_4-1)^2$ independent facets of size 3 having form (e).

The total number of facets of Type (ii) of all types (a)–(f) is:

$$2\binom{p_1}{2}(p_2^2-1)\binom{p_3}{2}(p_4^2-1)$$

This value can be obtained, for instance, by summing the quantities in the third row of Table 5.1 for $k = 2$.

(k=3) Now we enumerate the independent facets of $\mathcal{G}_2$ of size 3. We begin by counting the number of independent facets of Type (i) in Proposition 5.2.4. By Proposition 5.1.3, the number of independent facets of $\mathcal{G}$ is $G = \binom{p_1}{2}(p_2^2 - 1)$, and the number of independent facets of $\mathcal{H}$ is $H = \binom{p_3}{2}(p_4^2 - 1)$. Since A, B, C, D are distinct subgroups, choices of a, b, c, d are in bijection with the edge facets of $\mathcal{G}$ and of $\mathcal{H}$. There are G facets to choose for $\{a, b\}$ and there are H facets to choose for $\{c, d\}$. Once these are chosen, there are two ways to order each, for a total of $4GH$ facets of Type (i).

In Type (ii), we know a and a' are from the same subgroup from Proposition 5.2.4, so $\{a, a'\}$ can be from either a Sylow p_1-subgroup or a Sylow p_2-subgroup. In the former case, first choose a Sylow p_1-subgroup (1 way), then choose an ordered pair of non-identity elements a, a' from that subgroup ($(p_1 - 1)^2$ ways). Next, choose a Sylow p_2-subgroup (p_1 ways) and choose b from that subgroup ($p_2 - 1$ ways). Finally, choose any facet $\{c, d\}$ of In$(\mathcal{H})$ (H ways). In the latter case, first choose a Sylow p_2-subgroup A (p_1 ways), then an ordered pair $\{a, a'\}$ of non-identity elements from that subgroup ($(p_2 - 1)^2$ ways). Next, choose a non-identity element b outside of A; since you can choose from any of the $p_1 - 1$ remaining p_2-subgroups or from the unique p_1-subgroup, there are $(p_1 - 1)(p_2 - 1) + (1)(p_1 - 1) = (p_1 - 1)(p_2 - 1 + 1) = (p_1 - 1)p_2$ ways to choose b. Finally, choose a facet $\{c, d\}$ of In$(\mathcal{H})$ (H ways). Thus there are a total of

$$(1)(p_1 - 1)^2 p_1(p_2 - 1)H + p_1(p_2 - 1)^2(p_1 - 1)p_2 H$$

facets of Type (ii).

The count for facets of Type (iii) is analogous. Here the ordered pair $\{c, c'\}$ can be from the unique Sylow p_3-subgroup or some Sylow p_4-subgroup. In the former case, there are $(1)(p_3 - 1)^2$ ways to choose the ordered pair $\{c, c'\}$. There are $p_3(p_4 - 1)$ ways to choose d, and G choices for $\{a, b\}$. In the latter case, first choose a Sylow p_4-subgroup C (p_3 ways), then an ordered pair $\{c, c'\}$ from C ($(p_4 - 1)^2$ ways). Next, choose a non-identity element $d \notin C$, for which there are

$(p_3-1)(p_4-1)+(1)(p_3-1)=(p_3-1)(p_4-1+1)=(p_3-1)p_4$ ways. Finally, there are G choices for $\{a,b\}$. Thus there are a total of

$$(1)(p_3-1)^2p_3(p_4-1)G+p_3(p_4-1)^2(p_3-1)p_4G$$

facets of Type (iii).

Taking the three types together, the total number of facets of size 3 is given by the following formula:

$$\begin{aligned}
&4GH+(1)(p_1-1)^2p_1(p_2-1)H+p_1(p_2-1)^2(p_1-1)p_2H+\\
&(1)(p_3-1)^2p_3(p_4-1)G+p_3(p_4-1)^2(p_3-1)p_4G\\
&=4GH+H(p_1-1)p_1(p_2-1)(p_1-1+(p_2-1)p_2)+\\
&G(p_3-1)p_3(p_4-1)(p_3-1+(p_4-1)p_4)\\
&=4GH+2H\binom{p_1}{2}(p_2-1)(p_1-1+2\binom{p_2}{2})+2G\binom{p_3}{2}(p_4-1)(p_3-1+2\binom{p_4}{2})
\end{aligned} \tag{5.4}$$

(k=4) Let $G=\binom{p_1}{2}(p_2^2-1)$ be the number of independent facets of $\mathcal{G}$, and let $H=\binom{p_3}{2}(p_4^2-1)$ be the number of independent facets of $\mathcal{H}$ (see Proposition 5.1.3). Observing the structure of all possible independent facets, as listed in Table 5.1, we see that the total number of independent facets of size 4 is:

$$GH=\binom{p_1}{2}(p_2^2-1)\binom{p_3}{2}(p_4^2-1)$$

The numbers of independent facets of each individual type are listed in the third column of Table 5.1; these values sum to GH.

5.3 Group structure of $\mathcal{G}_n$

Proposition 5.3.1. *Let $G = G_1 \times G_2 \times \cdots \times G_n$ where each G_i is nonabelian and $|G_1| = p_1q_1, |G_2| = p_2q_2, \ldots, |G_n| = p_nq_n$ for distinct primes $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n$ with $p_i > q_i$ for all $1 \leq i \leq n$. Then $G \cong C_{p_1p_2\cdots p_n} \rtimes C_{q_1q_2\cdots q_n}$*

Proof: We show that

1. $\mathcal{G} = C_{p_1p_2\cdots p_n} C_{q_1q_2\cdots q_n}$

2. $C_{p_1p_2\cdots p_n} \cap C_{q_1q_2\cdots q_n} = 1_{\mathcal{G}}$

3. $C_{p_1p_2\cdots p_n} \trianglelefteq \mathcal{G}$

It will follow that $\mathcal{G} = C_{p_1p_2\cdots p_n} \rtimes C_{q_1q_2\cdots q_n}$.

1. ($\Rightarrow$) Let $g = (g_1, g_2, \ldots, g_n) \in \mathcal{G}$. The elements of G_i have order p_i, q_i, or 1 (G_i has no elements of order p_iq_i, since by Corollary 2.1.1, if a was an element of order p_iq_i in G_i then $|\langle a\rangle| = |a| = p_iq_i$ so $\langle a\rangle \cong G_i$, but G_i is not cyclic). Thus for each $1 \leq i \leq n$, g_i has order p_1, q_1 or 1. Write g as a product of tuples t_1t_2 such that

   - if the i^{th} component of g has order p_i, g_i is placed into the i^{th} component of t_1
   - if the i^{th} component of g has order q_i, g_i is placed into the i^{th} component of t_2
   - if the i^{th} component of g is trivial, 1_{G_i} is placed into the i^{th} components of both t_1 and t_2

   Then $t_1 \in C_{p_1} \times C_{p_2} \times \cdots \times C_{p_n}$ since its i^{th} entries contain only elements of order p_i and the identity, and $t_2 \in C_{q_1} \times C_{q_2} \times \cdots \times C_{q_n}$ since its i^{th} entries contain only elements of order q_i and the identity. Thus $g \in C_{p_1p_2\cdots p_n} C_{q_1q_2\cdots q_n}$.

   ($\Leftarrow$) Let $g = (a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) \in C_{p_1p_2\cdots p_n} C_{q_1q_2\cdots q_n}$. Then for each $1 \leq i \leq n$, a_i has order p_i or 1 and b_i has order q_i or 1. Then $g = (a_1b_1, a_2b_2, \ldots, a_nb_n)$. Now for each $1 \leq i \leq n$, a_ib_i is a product of an element of order p_i (or 1) and an element of order q_i (or 1), and thus $a_ib_i \in G_i$. Thus $g \in \mathcal{G}$.

2. Let $h = (h_1, h_2, \ldots, h_n) \in C_{p_1} \times C_{p_2} \times \cdots \times C_{p_n} \cap C_{q_1} \times C_{q_2} \times \cdots \times C_{q_n}$. Then for each $1 \leq i \leq n$, h_i must be trivial, for if not then h_i has order p_i and q_i for distinct primes p_i, q_i, an impossibility. Thus $h = 1_{\mathcal{G}}$.

3. Let $(g_1, g_2, \ldots, g_n) \in \mathcal{G}$. We show that $(g_1, g_2, \ldots, g_n)C_{p_1p_2\cdots p_n}(g_1, g_2, \ldots, g_n)^{-1} \subseteq C_{p_1p_2\cdots p_n}$. Let $h = (h_1, h_2, \ldots, h_n) \in C_{p_1} \times C_{p_2} \times \cdots \times C_{p_n}$. Then

$$\begin{aligned}(g_1, g_2, \ldots, g_n)(h_1, h_2, \ldots, h_n)(g_1, g_2, \ldots, g_n)^{-1} &= (g_1h_1g_1^{-1}, g_2h_2g_2^{-1}, \ldots, g_nh_ng_n^{-1}) \\ &= (g_1g_1^{-1}h_1, g_2g_2^{-1}h_2, \ldots, g_ng_n^{-1}h_n) \\ &= (h_1, h_2, \ldots, h_n) \in C_{p_1p_2\cdots p_n}\end{aligned}$$

where the second-to-last step holds because the C_{p_i} are abelian for all $1 \leq i \leq n$. Thus $C_{p_1p_2\cdots p_n} \trianglelefteq \mathcal{G}$. Therefore, $\mathcal{G} = C_{p_1p_2\cdots p_n} \rtimes C_{q_1q_2\cdots q_n}$.

□

Let $\mathcal{G}_n = G_1 \times G_2 \times \cdots \times G_n$ where each G_i is nonabelian and $|G_1| = p_1p_2, |G_2| = p_3p_4, \ldots, |G_n| = p_{2n-1}p_{2n}$ for distinct primes $p_1, p_2, \ldots, p_{2n}$ with $p_{2i-1} > p_{2i}$ for all $1 \leq i \leq n$.

Lemma 5.3.2. *For any independent set I of $\mathcal{G}_n$, the projection $\pi_j(I)$ onto the j^{th} component of I contains at most two unique selling points for all $1 \leq j \leq n$.*

Proof: Suppose I is an independent set and that for some $1 \leq j \leq n$, $\pi_j(I)$ has three unique selling points a, b, c (in other words, three tuples of I have unique selling points in their j^{th} component). By definition, $\langle \pi_j(I) \setminus g \rangle < \langle \pi_j(I) \rangle$ for all $g \in \{a, b, c\}$ since a, b, c are unique selling points and the direct product component orders of $\mathcal{G}_n$ are relatively prime. We claim $\{a, b, c\}$ is independent in G_j. Without loss of generality, assume to the contrary that $\langle a, b, c \rangle = \langle a, b \rangle$. Then $c \in \{a, b\}$, so $\langle \pi_j(I) \setminus c \rangle = \langle \pi_j(I) \rangle$ since $a, b \in \pi_j(I)$. This is a contradiction, since $\langle \pi_j(I) \setminus c \rangle < \langle \pi_j(I) \rangle$. Thus $\{a, b, c\}$ is an independent set in G_j. But any independent set of In(G_j) has size at most two by Proposition 5.1.1, so $\{a, b, c\}$ cannot be independent in In(G_j). □

In a projection $\pi_i(I)$ of $\mathcal{G}_n$, we say a subgroup H_i of G_i is **represented in** $\pi_i(I)$ if one or more elements of $\pi_i(I)$ generate H_i. The following lemma provides some intuition for the structure of independent facets of $\mathcal{G}_n$. Broadly speaking, the lemma states that if a projection contains at least one unique selling point, then exactly two distinct subgroups are represented in that projection. We will later see that components which contain no unique selling points can contain elements from a much larger variety of subgroups (see Remark 5.4.3). This will be an important observation when we introduce combinatorial diagrams in Section 5.4 (see Theorem 5.4.7).

Lemma 5.3.3. *Let I be an independent facet of $\mathcal{G}_n$. If $\pi_i(I)$ contains exactly one unique selling point, then $k \geq 3$. If $\pi_i(I)$ contains exactly two unique selling points, then $k \geq 2$. In both cases, exactly two distinct subgroups are represented in $\pi_i(I)$.*

Proof: Let I be an independent facet of $\mathcal{G}_n$. Throughout this proof, each occurrence of the elements u, v, w are assumed to be nonidentity.

Case 1: Suppose $\pi_i(I)$ has exactly one unique selling point, u. If $\pi_i(I)$ has only one subgroup, $\langle u \rangle$, represented, then $I \cup \{(1, \ldots, 1, g, 1, \ldots, 1)\}$ where g is an element of a subgroup of G_i distinct from $\langle u \rangle$, is an independent facet, a contradiction. Therefore $\pi_i(I)$ has at least two subgroups represented. If $\pi_i(I)$ has exactly two distinct subgroups represented, one by u and one by an element v generating a subgroup H which is distinct from $\langle u \rangle$, then $k \geq 3$ since $\pi_i(I)$ requires a second element from H, or else v is a unique selling point. If $\pi_i(I)$ has strictly more than two distinct subgroups represented, say by u and by elements v, w, all three generating distinct subgroups, then $\langle v, w \rangle = G_i$, so u is not a unique selling point. This is a contradiction. Thus $\pi_i(I)$ has exactly two subgroups represented.

Case 2: Suppose $\pi_i(I)$ has exactly two unique selling points, u, v. Then u, v represent two distinct subgroups H, K of G_i, we require $k \geq 2$, and $\langle u, v \rangle = G_i$. If a third subgroup is represented in $\pi_i(I)$, say by a third element w from a subgroup distinct from H and K, then $\langle v, w \rangle = G_i$ so u is not a unique selling point, a contradiction. Thus only two distinct subgroups are represented in $\pi_i(I)$. □

Lemma 5.3.4. *Let I be an independent facet of $\mathcal{G}_n$. Then I contains at most $2n$ tuples.*

Proof: Let I be an independent facet of $\mathcal{G}_n$. By Lemma 5.3.2, each of the n component projections of I contains at most two unique selling points, for a maximum of $2n$ possible unique selling points in I. Since I is independent, every tuple $t \in I$ contains at least one unique selling point by Lemma 5.2.2. Thus the maximum number of tuples in I is $2n$. □

The following corollary shows that the maximum independent facet size in Lemma 5.3.4 is achieved.

Corollary 5.3.5. *Every independent facet I of maximum size $k = 2n$ of $\mathcal{G}_n$ has the form:*

$$\{(g_1, 1, \ldots, 1), (h_1, 1, \ldots, 1), (1, g_2, 1, \ldots, 1), (1, h_2, 1, \ldots, 1), \ldots, (1, \ldots, 1, g_n), (1, \ldots, 1, h_n)\}$$

where $\{g_i, h_i\}$ is an independent facet of G_i for all $1 \leq i \leq n$ and the 1's represent identity elements from the appropriate component groups.

Proof: Let I be an independent facet of $\mathcal{G}_n$ of size $k = 2n$. Since every tuple of I has a unique selling point by Lemma 5.2.2, and there are at most $2n$ unique selling points (see the proof of Lemma 5.3.4), each tuple contains exactly one unique selling point (and there are exactly $2n$ unique selling points). Each of the n component projections has no more than two unique selling points by Lemma 5.3.2, so every component projection has exactly two unique selling points, and I has the form:

$$\{(g_1, \_, \ldots, \_), (h_1, \_, \ldots, \_), (\_, g_2, \_, \ldots, \_), (\_, h_2, \_, \ldots, \_), \ldots, (\_, \ldots, \_, g_n), (\_, \ldots, \_, h_n)\}$$

where g_i and h_i are unique selling points. For each $1 \leq i \leq n$, g_i and h_i are nontrivial elements from distinct subgroups of G_i since both are unique selling points (if g_i and h_i were from the same subgroup and g_i is the i^{th} component entry of tuple $t \in I$, then $\langle \pi_i(I) \setminus t \rangle = \langle \pi_i(I) \rangle$ in G_i, contradicting the assumption that g_i is a unique selling point. Trivial elements cannot be unique selling points, as they do not contribute to subgroup generation). So by Proposition 5.1.1, $\{g_i, h_i\}$ is an independent facet of G_i for all $1 \leq i \leq n$.

The remaining entries are not unique selling points, since there are at most $2n$ unique selling points. We claim all of the remaining entries must be identity elements. Suppose some remaining entry $x \in \pi_i(I) \setminus \{g_i, h_i\}$ in some tuple $t \in I$ is a nonidentity element of G_i. If x is in the same subgroup as a unique selling point $u \in \{g_i, h_i\}$, where u is the i^{th} component of some other tuple t^*, then t^*, whose only unique selling point is u, is redundant in I, so $\langle \pi_i(I) \setminus t^* \rangle = \langle \pi_i(I) \rangle$. This is a contradiction since u is a unique selling point. If x is in a different subgroup than either of the unique selling points g_i and h_i (which are each in different subgroups), then we reach a contradiction: Lemma 5.3.3 states that any component projection containing a unique selling point has at most two subgroups represented.

Thus x must be the identity element, so all of the independent sets have the form

$$(g_1, 1, \dots, 1), (h_1, 1, \dots, 1), (1, g_2, 1, \dots, 1), (1, h_2, 1, \dots, 1), \dots, (1, \dots, 1, g_n), (1, \dots, 1, h_n)$$

□

5.4 Main Theorem: Structure of In$(\mathcal{G}_n)$

Definition 5.4.1. *A multipartite simplicial complex $\mathcal{D}$ is a* ***minimal cover*** *of its vertices if*

1. *$\mathcal{D}$ contains at least two vertices in each component, and*
2. *Removing a facet of $\mathcal{D}$ and all of its subsets not shared by another facet results in some component having fewer than two vertices.*

In the following figures, we group vertices of multipartite simplicial complex visually according to how their vertices are partitioned. For instance, in Figures 5.1 and 5.2, there are three vertex components, each consisting of two vertices: (1) The top two vertices, (2) The leftmost two vertices, and (3) the rightmost two vertices. We use the same illustration scheme throughout similar examples in this paper. For example, Figure 5.1 shows a multipartite simplicial complex $\mathcal{D}$ whose facets consist of a single vertex, an edge, and a 2-simplex and which is a minimal cover. Figure 5.2 shows this same complex with an extra edge between the bottom two vertex components. This edge could be removed and still two vertices are covered in every component, so the complex in Figure 5.2 does not minimally cover its vertices. In Section 7, we will see examples of combinatorial diagrams which have more than two vertices in each component.

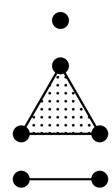


Figure 5.1: A complex which is a minimal cover of its vertices

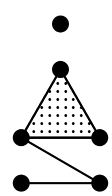


Figure 5.2: A complex which is not a minimal cover of its vertices

Lemma 5.4.2 (Numerical Condition)**.** *Let I be an independent facet of size k of In($\mathcal{G}_n$). Then the maximum number of component projections of I which contain no unique selling points is $n - \frac{k}{2}$ if k is even, and $n - \frac{k+1}{2}$ if k is odd.*

Proof: Let I be an independent facet of size k of In($\mathcal{G}_n$). By Lemma 5.2.2, since the component groups of $\mathcal{G}_n$ have relatively prime orders, each of the k tuples has a unique selling point. Thus the minimum number of unique selling points required among the tuples of I is k. By Lemma 5.3.2, no component projection has more than two unique selling points. We will show that if I has exactly k unique selling points (one in each tuple), then the statement of the lemma holds. If I has strictly more than k unique selling points, then fewer component projections than the maximum stated by the lemma will contain no unique selling points.

If k is even and each of the k tuples has exactly one unique selling point, we claim that exactly $\frac{k}{2}$ component groups are fully generated. First we show that each component projection of I contains either exactly two unique selling points which come from distinct subgroups, or no unique selling points. Suppose for some $1 \leq j \leq n$ that $\pi_j(I)$ contains only one unique selling point u, so that G_j is not fully generated. Then $I \cup \{(\ldots, v, \ldots)\}$, where v is an element of a subgroup of G_j distinct from the subgroup containing u, is an independent set containing I. Hence I would not be a facet, a contradiction. Thus each component projection of I contains exactly two or exactly zero unique selling points. Thus, among these k tuples which contain exactly one unique selling point each, these unique selling points must come in pairs, so exactly $\frac{k}{2}$ component groups are fully generated. So the maximum number of components which could contain no unique selling points is $n - \frac{k}{2}$ (this maximum occurs in the case where I has exactly k unique selling points).

If k is odd and each of the k tuples has exactly one unique selling point, then there is some $1 \leq j \leq n$ for which $\pi_j(I)$ has only one unique selling point u (otherwise, if every component projection had either 0 or 2 unique selling points, then k would be even), so the component G_j is not fully generated. We claim I must have a $(k+1)^{st}$ unique selling point, v, in $\pi_j(I)$, from a subgroup of G_j distinct from the subgroup containing u. If not, then $I \cup \{(\ldots, v, \ldots)\}$, where v is in a subgroup of G_j which is distinct from u's subgroup, is an independent set containing I, a contradiction since I is a facet. Thus I contains exactly $k+1$ unique selling points in this case. Using the same argument as in the case when k is even, the component projections which contain unique selling points must have exactly two unique selling points each. Thus $\frac{k+1}{2}$ component groups are generated, so there are $n - \frac{k+1}{2}$ remaining components with no unique selling points. Thus the maximum number of tuple components which do not contain any unique selling points is $n - \frac{k+1}{2}$. □

For example, let $n = 3$, $k = 4$ and

$$I = (\underline{a_1}, 1, c_1), (\underline{a_2}, 1, c_2), (1, \underline{b_1}, c_3), (1, \underline{b_2}, c_4)\} \tag{5.5}$$

where a_1, a_2 are elements of distinct subgroups of G_1 and b_1, b_2 are elements of distinct subgroups of G_2. Here the unique selling points a_1, a_2, b_1, b_2 are underlined. Then $n - \frac{k}{2} = 3 - \frac{4}{2} = 1$, and we see that $\pi_3(I)$ is the only component which contains no unique selling points. The elements c_1, c_2, c_3, c_4 must be chosen from at least two distinct subgroups (or else, using an argument analogous to those in the proof of Lemma 5.4.2, I is not a maximal face). These elements can come from as many as k or $p_5 + 1$ distinct subgroups, whichever is smaller (recall that $|G_3| = p_5 p_6$ with $p_5 > p_6$, and G_3 has exactly 1 subgroup of order p_5 and p_5 subgroups of order p_6, for a total of $p_5 + 1$ subgroups).

Remark 5.4.3. *In general, if I is an independent facet of size k of In($\mathcal{G}_n$) and $\pi_i(I)$ contains no unique selling points, the maximum number of distinct subgroups represented in $\pi_i(I)$ is $\min\{k, p_{2i-1} + 1\}$ (since $|G_i| = p_{2i-1}p_{2i}$ where $p_{2i-1} > p_{2i}$, and G_i has exactly 1 subgroup of order p_{2i-1} and p_{2i-1} subgroups of order p_{2i}, for a total of $p_{2i-1} + 1$ distinct subgroups). The minimum number of subgroups represented in $\pi_i(I)$ is two; if fewer subgroups were represented, then another tuple could be added with an additional distinct subgroup represented in its i^{th} component and the other components identity elements while retaining independence, so that I would not be a facet.*

Definition 5.4.4. *Let I be an independent facet of In($\mathcal{G}_n$). The tuple $(T_1, T_2, \ldots, T_n) \in \mathbb{Z}_{\geq 0}$ where T_i is the number of distinct subgroups represented among the entries of $\pi_i(I)$ is called the **type** of I.*

For example, since the entries c_1, c_2, c_3, c_4 can represent a minimum of 2 and a maximum of $\min\{k, p_5 + 1\}$ distinct subgroups, the facet in Equation (5.5) has possible types

$$(2, 2, 2), (2, 2, 3), \ldots, (2, 2, \min\{k, p_5 + 1\})$$

Definition 5.4.5. *Let S be a set of group elements (tuples) of $\mathcal{G}_n$. The **group superstructure** of S is the set of tuples whose entries consist of the containing subgroups of the corresponding group elements in tuples of S. We write 1 to represent the trivial group of G_i for any $1 \leq i \leq n$.*

The notion of independence for a facet and for its corresponding group superstructure remains the same. An independent group superstructure is simply a collection of a certain family of independent sets.

Example 5.4.6. *For example, if*

$$I = \{(a_1, 1, c_1), (a_2, 1, c_1'), (1, b_1, c_2), (1, b_2, c_3)\}$$

where a_1, a_2 are from distinct subgroups A_1, A_2 of G_1; b_1, b_2 are from distinct subgroups B_1, B_2 of G_2; c_2, c_3 are from distinct subgroups C_2, C_3 of G_3; and c_1, c_1' are elements from a subgroup C_1 which is distinct from C_2 and C_3 (here, all subgroups are nontrivial), then the group superstructure of I is:

$$\{(A_1, 1, C_1), (A_2, 1, C_1), (1, B_1, C_2), (1, B_2, C_3)\}$$

Similarly, one can start with the group superstructure and work backwards to form a corresponding set of group elements (there are many such choices), keeping in mind that the nonzero elements chosen from the repeated occurrences of C_1 can be distinct or equal. When the context is clear, we refer to independent sets and independent group superstructures simply as **independent**. Since there is an easy translation in both directions between a group superstructure and a choice of independent set, we sometimes refer to these synonymously unless one format in particular is needed.

We will refer to components of a multipartite simplicial complex $\mathcal{D}$ as **vertex components**, and components of a tuple which is an element of $\mathcal{G}_n$ as **tuple components**, or when the context is clear, simply as **components**. If we have distinguished the vertex components of $\mathcal{D}$ by labelling them in some order, we say $\mathcal{D}$ has **vertex type** $(T_1, T_2, \ldots, T_n)$ if the i^{th} vertex component of $\mathcal{D}$ contains exactly T_i vertices.

The following theorem describes a bijection between multipartite simplicial complexes on the appropriate numbers of vertices which are minimal covers, and independent facets arising from the group $\mathcal{G}_n$ for some n.

Theorem 5.4.7 (Main Theorem)**.** *Let $\mathcal{G}_n = G_1 \times G_2 \times \cdots \times G_n$ where each G_i is nonabelian and $|G_1| = p_1p_2, |G_2| = p_3p_4, \ldots, |G_n| = p_{2n-1}p_{2n}$ for distinct primes $p_1, p_2, \ldots, p_{2n}$ with $p_{2i-1} > p_{2i}$ for all $1 \leq i \leq n$. Let $\mathcal{S}$ be the set of all multipartite simplicial complexes of vertex type $(T_1, T_2, \ldots, T_n)$ with k facets where r of the T_i satisfy $2 \leq T_i \leq \min\{k, p_{2i-1}+1\}$ where $0 \leq r \leq m$ with*

$$m = \begin{cases} n - \frac{k}{2} & \text{if } k \text{ is even} \\ n - \frac{k+1}{2} & \text{if } k \text{ is odd} \end{cases}$$

and all other T_i's equal 2, listed up to simplicial isomorphism. Then the members of $\mathcal{S}$ which are minimal covers of their vertex sets are in bijection with the independent facets of In($\mathcal{G}_n$).

Proof:

($\Rightarrow$) Let D be a member of $\mathcal{S}$ which is a minimal cover of its vertices. We show D gives rise to an independent facet $I(D)$ of In($\mathcal{G}_n$). First, we construct this set $I(D)$. If D has n vertex components and its i^{th} vertex component has m vertices, label the vertices in component i by $v(i,1), v(i,2), \ldots, v(i,m)$ for all $1 \leq i \leq n$. For each facet Δ of D, form an n-tuple t whose i^{th} component is defined to be:

- $v(i,j)$ if Δ contains vertex $v(i,j)$ for some j
- 1_{G_i} if Δ has no vertices in the i^{th} vertex component

for all $1 \leq i \leq n$. Note that since D is multipartite, Δ contains at most one vertex from each vertex component, so the i^{th} component of t is uniquely determined. Let $S = S(D)$ be the set of the resulting n-tuples (so that each tuple of S corresponds to a facet Δ of D). We can realize S as a group superstructure by considering distinct vertices in the i^{th} vertex component of D as representing distinct, nontrivial subgroups of G_i, so that distinct labels in the i^{th} tuple components of the tuples of S correspond to distinct, nontrivial subgroups of G_i. Note that S is uniquely determined, up to the labelling chosen for the vertex components (the labels chosen for individual vertices within each vertex component is irrelevant for our purposes, since these only record whether the corresponding subgroups are distinct or equal). Let I represent any set of group elements corresponding to S (see Example 5.4.6 and the surrounding comments). The following does not depend on the choice of I.

We show I is independent. Let $t \in I$. We show that $\langle I \setminus t\rangle < \langle I\rangle$. Recall that removing the tuple t from I corresponds to removing a facet $\Delta(t)$ from D. Since D is a minimal cover, for some $1 \leq i \leq n$ the i^{th} vertex component of $D \setminus \Delta(t)$ has fewer than two vertices, by definition. Thus, $\pi_i(I)$ has fewer than two distinct subgroups represented, so $\langle \pi_i(I \setminus t)\rangle < \langle \pi_i(I)\rangle$. Thus every tuple of I has a unique selling point by definition, so by Lemma 5.2.3, I is independent.

Next, we show I is a facet. Since D is a minimal cover of its vertices, each vertex component of D contains at least two vertices by definition. Thus at least two distinct, nontrivial subgroups are covered in $\pi_i(I)$ for every $1 \leq i \leq n$. Thus $\langle \pi_i(I)\rangle = G_i$ for all $1 \leq i \leq n$. Thus $\langle I\rangle = G$ so any tuple added to I must be redundant. Hence I is a facet.

($\Leftarrow$) Let I be an independent facet of In($\mathcal{G}_n$) of size k. We show I gives rise to a unique member of $\mathcal{S}$ which is a minimal cover of its vertex set. We build a multipartite simplicial complex $D(I)$ whose i^{th} vertex component consists of vertices which correspond to the distinct subgroups represented among the nonidentity elements in $\pi_i(I)$ for all $1 \leq i \leq n$ (here, if two elements come from the same subgroup, they are represented by a single vertex in $D(I)$). For each tuple of I, $D(I)$ has a facet consisting of the vertices which correspond to the containing subgroups of the nonidentity elements in that tuple. (Note that information about existence of identity elements in the components of a tuple is retained by the corresponding facet not being full-dimensional.) Let $(T_1, T_2, \ldots, T_n)$ be the vertex type of $D(I)$. We show $D(I)$ (a) is in $\mathcal{S}$ and (b) minimally covers its vertices.

(a) Since the vertices of $D(I)$ come from distinct component projections, $D(I)$ is multipartite with its vertices partitioned into blocks corresponding to each component projection. By Lemma 5.4.2, the maximum number of component projections of I which contain no unique selling points is $n - \frac{k}{2}$ if k is even, and $n - \frac{k+1}{2}$ if k is odd. If $\pi_i(I)$ has no unique selling points for some $1 \leq i \leq n$, there is a minimum of two and a maximum of $\min\{\text{number of distinct subgroups of } G_i, k\} = \min\{p_{2i-1} + 1, k\}$ distinct subgroups represented in $\pi_i(I)$. To see this, recall that the number of distinct subgroups of G_i, where $|G_i| = p_{2i-1}p_{2i}$ with $p_{2i-1} > p_{2i}$, is $p_{2i-1} + 1$, since G_i has p_{2i-1} subgroups of order p_{2i} and one subgroup of order p_{2i-1} (see also Remark 5.4.3 for the minimum bound of two). Certainly, a set of k tuples cannot contain more than k distinct entries in its i^{th} component projection.

Since the distinct subgroups represented in $\pi_i(I)$ correspond to distinct vertices in the i^{th} vertex component of $D(I)$ for each $1 \leq i \leq n$, we have that r of the T_i satisfy $2 \leq T_i \leq \min\{p_{2i-1}+1, k\}$ where $0 \leq r \leq m$ with

$$m = \begin{cases} n - \frac{k}{2} & \text{if } k \text{ is even} \\ n - \frac{k+1}{2} & \text{if } k \text{ is odd} \end{cases}$$

for each $1 \leq i \leq n$.

In the component projections $\pi_i(I)$ which contain either one or two unique selling points, exactly two distinct subgroups are represented, by Lemma 5.3.3. By Lemma 5.3.2, no other cases are possible. Thus the remaining T_i's are equal to 2. Thus $\mathcal{D}(I)$ is in $\mathcal{S}$.

(b) Now we show that $D(I)$ is minimally covering. First, note that $D(I)$ contains at least two vertices in each component, as in part (a) we saw that the T_i satisfy $T_i \geq 2$ for all $1 \leq i \leq n$. Now we show that removing a facet of $D(I)$ results in some component having fewer than two vertices. Let F be a facet of $D(I)$ and consider the multipartite simplicial complex $D(I) \setminus F$ (where the removal of F is accomplished by removing the facet F and all of its subsets not shared by another facet of $D(I)$). The facet F corresponds to a tuple $t \in I$. By Lemma 5.2.2, t has a unique selling point $\pi_j(t)$ for some $1 \leq j \leq n$. By Lemmas 5.3.3 and 5.3.2, since $\pi_j(I)$ contains a unique selling point, exactly two distinct nontrivial subgroups are represented in $\pi_j(I)$. Thus the

j^{th} vertex component of $D(I)$ has exactly two distinct vertices. One of these vertices, v, is the vertex in $D(I)$ which corresponds to the unique selling point $\pi_j(t)$, and is a vertex of F. Thus removing F from $D(I)$ removes v from the j^{th} vertex component of $D(I)$, which contained only two vertices before removing F. Thus the j^{th} vertex component of $D(I) \setminus F$ contains fewer than two vertices. Therefore, $D(I)$ is a minimal cover of its vertices. Hence I gives rise to a multipartite simplicial complex which satisfies the numerical conditions in $\mathcal{S}$ and is a minimal cover. □

We call a multipartite simplicial complex $\mathcal{D}(F)$ arising from an independent facet F of In(G) as in Theorem 5.4.7 a **combinatorial diagram** (i.e. a combinatorial diagram is a multipartite simplicial complex that is a member of $\mathcal{S}$ and is a minimal cover of its vertex set in the sense of Definition 5.4.1). Note that the first direction of the proof of Theorem 5.4.7 does not rely on D having membership in $\mathcal{S}$; indeed the more general statement in one direction is true that any multipartite simplicial complex which is a minimal cover of its vertices gives rise to an independent facet.

5.5 Main Algorithm: Computing the combinatorial diagrams for $\mathcal{G}_n$

We have devised an algorithm, namely Algorithm 5.5.4, to make a list of all combinatorial diagrams for $\mathcal{G}_n$. This is the main topic of the current section. Algorithm 5.5.4 will generate redundant combinatorial diagrams, so to correct for this we will consider this list up to simplicial isomorphism. Though one can attempt to write down all combinatorial diagrams naïvely, the process quickly becomes tedious and it is easy to miss cases. We will later illustrate Theorem 5.4.7 and Algorithm 5.5.4 in Section 6 by computing all combinatorial diagrams for $\mathcal{G}_3$ up to simplicial isomorphism. This will be followed by an example in Section 7 of how to attach a count to a combinatorial diagram to obtain the total number of independent sets (at the level of elements) which arise from that diagram.

One of our first steps in Algorithm 5.5.4 will be to determine all possible ways that unique selling points can be distributed among the components of the tuples of I. To aid with this, we introduce dot diagrams.

Definition 5.5.1. *Let I be an independent set of $\mathcal{G}_n = G_1 \times G_2 \times \cdots \times G_n$ where each G_i is nonabelian and $|G_1| = p_1p_2, |G_2| = p_3p_4, \ldots, |G_n| = p_{2n-1}p_{2n}$ for distinct primes $p_1, p_2, \ldots, p_{2n}$ with $p_{2i-1} > p_{2i}$ for all $1 \leq i \leq n$. The* ***dot diagram*** *$\mathcal{U}(I)$ of I is a 1-dimensional multipartite simplicial complex with n vertex components, where the vertices in the j^{th} vertex component correspond to the unique selling points in the projection $\pi_j(I)$, for each $1 \leq j \leq n$.*

Example 5.5.2. *Let $I = \{(\underline{a_1}, b_1', c_1'), (\underline{a_2}, b_1'', c_1''), (1, \underline{b_2}, c_1'''), (1, b_1''', \underline{c_2})\}$, with $a_1 \in A_1$, $a_2 \in A_2$, $b_1', b_1'', b_1''' \in B_1$, $b_2 \in B_2$, $c_1', c_1'', c_1''' \in C_1$, and $c_2 \in C_2$ where $A_1, A_2 \leq G_1, B_1, B_2 \leq G_2$, and $C_1, C_2 \leq G_3$ are distinct Sylow subgroups of their respective groups G_i and the elements listed in each subgroup are nontrivial but are not necessarily distinct. Note that I has group superstructure $\{(\underline{A_1}, B_1, C_1), (\underline{A_2}, B_1, C_1), (1, \underline{B_2}, C_1), (1, B_1, \underline{C_2})\}$. The unique selling points of each tuple in I and the corresponding subgroups in its group superstructure are underlined. The dot diagram $\mathcal{U}(I)$ is shown in Figure 5.3 (left), with vertices labeled by the containing subgroups of each unique selling point (i.e. the underlined subgroups in the group superstructure of I).*

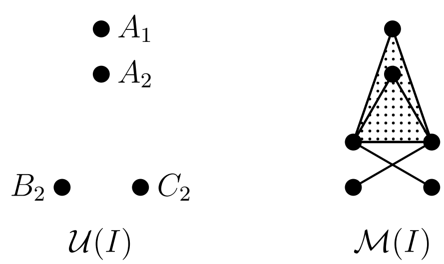


Figure 5.3: Dot diagram $\mathcal{U}(I)$ and multipartite simplicial complex $\mathcal{M}(I)$ for I

When the context is clear and we are working only with group superstructures and not independent sets directly, we refer to the subgroups in group superstructures which correspond to unique selling points of I also as unique selling points. We will often focus our attention only on the combinatorial distribution of unique selling points among vertex components and omit labels. Dot diagrams allow us to visualize unique selling points as distinguished vertices in a combinatorial diagram. We will later organize the list of combinatorial diagrams for $\mathcal{G}_3$ in terms of their numbers of facets and their underlying dot diagrams.

The following remark relates will help us generate all combinatorial diagrams for $\mathcal{G}_n$ in Algorithm 5.5.4.

Remark 5.5.3. *Let I be an independent facet of $\mathcal{G}_n$ of size k. By Lemma 5.2.2, each tuple of $t \in I$ has at least one unique selling point. By Lemma 5.3.2, for each $1 \leq j \leq n$, $\pi_j(I)$ contains at most two unique selling points. Let u be the number of unique selling points in I. Since each of the k tuples of I has at least one unique selling point and there are no more than two unique selling points in each of the n components, we know $k \leq u \leq 2n$.*

We say a partition $\mathcal{P}$ of the labeled vertices of a dot diagram is **valid** if either it corresponds directly to an independent facet, or if there is some way to insert other labels into the blocks of $\mathcal{P}$ (later, we call these partition fillings and describe these fillings in Steps 6 and 7 of Algorithm 5.5.4) in such a way that the resulting partition corresponds to an independent facet.

Algorithm 5.5.4. *Let $\mathcal{G}_n = G_1 \times G_2 \times \cdots \times G_n$ where each G_i is nonabelian and $|G_1| = p_1p_2, |G_2| = p_3p_4, \ldots, |G_n| = p_{2n-1}p_{2n}$ for distinct primes $p_1, p_2, \ldots, p_{2n}$ with $p_{2i-1} > p_{2i}$ for all $1 \leq i \leq n$.*

1. *For all $1 \leq k \leq 2n$ **(Independent facet size)**, do the following:*
2. *For all $k \leq u \leq 2n$ **(Total number of unique selling points)**, do the following:*
3. *Draw all possible 1-dimensional multipartite simplicial complexes with n vertex components and u vertices, where each vertex component has 0, 1, or 2 vertices and the ordering of the vertex components does not matter. **(Dot diagrams)***
4. *Label and then partition the vertices from the previous step into k nonempty blocks, such that each block contains at most one vertex from the same vertex component. **(Partition of unique selling points)***

5. *If every vertex component of the dot diagram in Step 3 has two vertices, then the partition from the previous step is valid, and directly yields a combinatorial diagram.*

6. *Suppose $k \geq 3$. If some vertex component of the dot diagram in Step 3 has exactly one vertex, then insert into the partition blocks at least two occurrences of the new vertex (adhering to the rule that each partition block contains at most one subgroup from the same group component).*

7. *If some vertex component of the complex in Step 3 contains no vertices, then choose one of the following:*

   - *If $k \geq 4$, we can insert into the blocks of the partition two distinct nontrivial subgroups, where each subgroup is represented at least twice.*
   - *If $k \geq 3$, we can insert into the blocks of the partition three or more (and no more than $\min\{k, p_{2i-1} + 1\}$) nontrivial distinct subgroups, repeated in any fashion as long as all three occur at least once.*

We call the partitions formed at the end of Steps 6 and 7 ***partition fillings****. We also refer to these as* ***filled partitions*** *(in contrast with the partitions of the vertices of dot diagrams prior to filling). Now run over all ways to form dot diagrams in Step 3, all partitions of their vertices in Step 4, and all ways to fill the partitions in each of the cases described (Steps 5–7). Let $\mathcal{L}$ be the list of resulting partitions. For each $\mathcal{P}_f \in \mathcal{L}$, form the multipartite simplicial complex whose vertices correspond to the labels among the blocks of $\mathcal{P}_f$, organized into vertex components according to the dot diagram from Step 3 and associated partition fillings, and whose facets correspond to the blocks of $\mathcal{P}_f$. Let $\mathcal{C}$ be the list of all of the resulting multipartite simplicial complexes, up to simplicial isomorphism.*

When computing explicit examples, particular shortcuts can be taken; for instance, when forming $\mathcal{L}$, some partitions will have duplicate behavior so only one need to be written down. We later focus on the particular example $\mathcal{G}_3$; for this example, the resulting partitions and their corresponding combinatorial diagrams, both listed up to simplicial isomorphism of the combinatorial diagrams, are listed in the rightmost two columns of the tables in Chapter 6. In the following theorem, we exhibit a surjective map from the set of multipartite simplicial complexes arising from Algorithm 5.5.4 to the set of combinatorial diagrams. This will show that all combinatorial diagrams can be generated from Algorithm 5.5.4.

Theorem 5.5.5. *Let $\mathcal{G}_n = G_1 \times G_2 \times \cdots \times G_n$ where each G_i is nonabelian and $|G_1| = p_1 p_2, |G_2| = p_3 p_4, \ldots, |G_n| = p_{2n-1}p_{2n}$ for distinct primes $p_1, p_2, \ldots, p_{2n}$ with $p_{2i-1} > p_{2i}$ for all $1 \leq i \leq n$. There is a surjective map from the multipartite simplicial complexes arising from Algorithm 5.5.4 in $\mathcal{C}$ to the set of all combinatorial diagrams listed up to simplicial isomorphism.*

Proof: ($\Rightarrow$) **Claim:** Let $\mathcal{D}$ be a multipartite simplicial complex arising from Algorithm 5.5.4. We show that $\mathcal{D}$ is a combinatorial diagram (namely, we take the identity map ι from the set of multipartite simplicial complexes in $\mathcal{C}$ from Algorithm 5.5.4 to the set of combinatorial diagrams, and first show the image of this map is indeed a combinatorial diagram). To do so, we first show that $\mathcal{D}$ is a minimal cover of its vertices.

1. First, we show that $\mathcal{D}$ contains at least two vertices in each component. In Step 3 of Algorithm 5.5.4 (the dot diagram), every component contains exactly 0, 1, or 2 vertices.

    - If every component contains 2 vertices, we are done.
    - If some vertex component contains exactly 1 vertex in Step 3, then $k \geq 3$ and we fill the remaining partition blocks with at least 2 occurrences of a distinct subgroup corresponding to a second vertex. All such vertex components contain two vertices by the end of Step 6.
    - If some vertex component contains no vertices in Step 3, then by the end of Step 7 the partition blocks are filled with either exactly two or at least three distinct subgroups. These correspond to distinct vertices.

2. Second, we show that removing a facet from $\mathcal{D}$ results in some vertex component having fewer than two vertices. Every facet F of $\mathcal{D}$ corresponds to a block of a partition, and consists

of at least one element u^* from Step 4 (a unique selling point) since the blocks in the partition from Step 4 are nonempty. Removing F from $\mathcal{D}$ removes the vertex corresponding to the unique occurrence of u^* in the filled partition which corresponds to $\mathcal{D}$. In Algorithm 5.5.4, the only case where a vertex component can have strictly more than two distinct vertices is in the case where that vertex component contained no vertices in Step 3, i.e. no unique selling points (in the other cases, each vertex component has exactly two vertices after any partition fillings in Steps 5–7). So in the vertex component containing the vertex associated to u^*, there are a total of two distinct vertices in $\mathcal{D}$. Thus removing the facet containing u^* removes the vertex corresponding to u^*, leaving that vertex component with only one vertex.

Thus $\mathcal{D}$ is a minimal cover of its vertices.

We next show that $\mathcal{D} \in \mathcal{S}$ as in Theorem 5.4.7. Namely, we show that $\mathcal{D}$ with k facets has vertex type $(T_1, T_2, \ldots, T_n)$ where r of the T_i satisfy $2 \leq T_i \leq \min\{k, p_{2i-1}+1\}$ where $0 \leq r \leq m$ with

$$m = \begin{cases} n - \frac{k}{2} & \text{if } k \text{ is even} \\ n - \frac{k+1}{2} & \text{if } k \text{ is odd} \end{cases}$$

and all other T_i's equal 2.

1. If every vertex component in Step 3 has two vertices, then no additional vertices are introduced (as there is no additional filling in the later steps). So $T_i = 2$ for all $1 \leq i \leq n$. In other words, $\mathcal{D}$ has type $(2, 2, \ldots, 2)$.

2. If some vertex component in Step 3 has exactly one vertex, then exactly one more vertex is introduced in that component during the filling process. So $T_i = 2$ for all $1 \leq i \leq n$. Thus $\mathcal{D}$ has type $(2, 2, \ldots, 2)$.

3. If, for some $1 \leq i \leq n$, the i^{th} vertex component contains no vertices in Step 3, then in Step 7 we will fill the i^{th} vertex component with either vertices representing exactly two distinct subgroups (in which case $T_i = 2$), or with vertices representing at minimum three and maximum $\min\{k, p_{2i-1}+1\}$ distinct subgroups (in which case $2 < T_i \leq \min\{k, p_{2i-1}+1\}$).

We show that the maximum number of vertex components which contain no vertices in Step 3 is $n - \frac{k}{2}$ (if k is even) or $n - \frac{k+1}{2}$ (if k is odd). Let u be the number of vertices in the dot diagram, where $k \leq u \leq 2n$. If $u = k$, this corresponds to the case with the maximum number of empty vertex components in the dot diagram (for if $u > k$, fewer vertex components are empty). Assume $u = k$.

To maximize the number of empty vertex components in the dot diagram in Step 3, we fill as few as possible of the n vertex components with the $u = k$ vertices. To do so, if k (and thus u) is even we can fill a minimum of $\frac{k}{2}$ vertex components with two vertices each; in this case $n - \frac{k}{2}$ vertex components remain empty. If k (and thus u) is odd, we can fill a minimum of $\frac{k+1}{2}$ vertex components so that one of those components has exactly one vertex and the rest have two vertices. In this case, $n - \frac{k+1}{2}$ vertex components remain empty. Thus the maximum possible number of empty vertex components in the dot diagram in Step 3 is $n - \frac{k}{2}$ (if k is even) or $n - \frac{k+1}{2}$ (if k is odd).

Thus $\mathcal{D} \in \mathcal{S}$. It follows that $\mathcal{D}$ is a combinatorial diagram.

($\Leftarrow$) **Claim:** We now show the map ι is surjective. Let $\mathcal{D}$ be a combinatorial diagram with k facets. Then $\mathcal{D}$ corresponds to an independent facet $I = I(\mathcal{D})$ by Theorem 5.4.7. We show that $\mathcal{D}$ arises from Algorithm 5.5.4. In other words, we identify k, u, an associated dot diagram $\mathcal{U}(I)$, a partition $\mathcal{P}$ of the vertices in $\mathcal{U}(I)$, and a specific filling $\mathcal{P}_f$ of $\mathcal{P}$, as follows.

The number of facets k in $\mathcal{D}$ equals the number of tuples in I. Naïvely, $k \geq 1$. By Lemma 5.3.4, $k \leq 2n$ since I is an independent facet. This value of k corresponds to the value of k in Algorithm 5.5.4. Let $\mathcal{U}(I)$ be the dot diagram of I, as defined in Definition 5.5.1 (in practice, $\mathcal{U}(I)$ can be identified directly from $\mathcal{D}$ by identifying all vertices which are contained in only one facet of $\mathcal{D}$). Let u be the number of vertices in $\mathcal{U}(I)$, namely the total number of unique selling points of I. Then $u \geq k$ since I has k tuples and each tuple has at least one unique selling point by Lemma 5.2.2. By Lemma 5.3.2, $u \leq 2n$ since each of the n tuple components contains at most two unique selling points. Let $\mathcal{P}$ be the partition of the vertices in $\mathcal{U}(I)$ whose blocks are given by the set of unique selling points in each tuple of I. Let $\mathcal{P}_f$ be the partition of the vertices of $\mathcal{D}$ whose blocks are given by the set of all nonidentity elements in each tuple of I.

We show that $\mathcal{P}_f$ can be recognized as being obtained from $\mathcal{P}$ by one of the fillings described in Steps 5, 6, and 7 of Algorithm 5.5.4.

- **(Step 5)** If every vertex component of $\mathcal{U}(I)$ has two vertices, we are done.

- **(Step 6)** If for some $1 \leq i \leq n$, the i^{th} vertex component of $\mathcal{U}(I)$ has exactly one vertex, then since $\mathcal{D}$ corresponds to I, by Lemma 5.3.3 exactly one additional subgroup is represented in $\pi_i(\mathcal{U}(I))$. This additional subgroup is not a unique selling point, so occurs at least twice. Thus $k \geq 3$, so there is room to fill $\mathcal{P}$ as described in Step 6 of Algorithm 5.5.4.

- **(Step 7)** If for some $1 \leq i \leq n$, the projection $\pi_i(\mathcal{U}(I))$ has no vertices, then Remark 5.4.3 gives the number of possible subgroups represented in $\pi_i(\mathcal{U}(I))$, as follows:

    1. If $k \geq 4$, it is possible that $\pi_i(\mathcal{U}(I))$ contains exactly two distinct nontrivial subgroups. Here, we require $k \geq 4$ since each of these two subgroups must occur at least twice, since they are not unique selling points.

    2. If $k \geq 3$, $\pi_i(\mathcal{U}(I))$ can contain m distinct nontrivial subgroups, where $3 \leq m \leq \min\{k, p_{2i-1}+1\}$ (there is no requirement that any of these subgroups occur more than once, though they can; we only require that at least three distinct nontrivial subgroups are represented; hence $k \geq 3$).

Thus $\mathcal{P}_f$ arises from $\mathcal{P}$ in Algorithm 5.5.4. Forming the corresponding multipartite simplicial complex as described in the last paragraph of Algorithm 5.5.4, we obtain $\mathcal{D}$. Thus $\mathcal{D}$ arises from Algorithm 5.5.4, so ι is surjective. □

Chapter 6

Example: Independent facets for $G_1 \times G_2 \times G_3$

To illustrate Theorem 5.4.7 and Algorithm 5.5.4, we describe how to generate the independent facets of $G_1 \times G_2 \times G_3$ where the G_i are nonabelian with orders $|G_1| = p_1p_2$, $|G_2| = p_3p_4$, $|G_3| = p_5p_6$. Here, the p_i are distinct primes with $p_1 > p_2$, $p_3 > p_4$, $p_5 > p_6$, and $n = 3$ and $2 \leq k \leq 6$.

6.1 Applying Algorithm 5.5.4

Tables 6.1 and 6.2, appearing at the end of this chapter, record the steps to compute all combinatorial diagrams for $\mathcal{G}_3$. We describe the general process to obtain the entries in these tables for $\mathcal{G}_n$. Fix k, the number of tuples in an independent facet I. For each $k \leq u \leq 2n$ (see Remark 5.5.3), form a dot diagram with u vertices, arranged among n vertex components where each component has 0, 1, or 2 vertices. Label these vertices according to their vertex components. These u vertices will be the unique selling points in I. Then list all possible partitions of these u unique selling points into k blocks, where each block contains no more than one vertex label from each component. We list whether the resulting partitions are valid (correspond to some combinatorial diagram after filling) or invalid. In the penultimate column of the tables, we list all possible partition fillings for each partition. The last column refers to the corresponding combinatorial diagrams. These combinatorial diagrams are illustrated in Figures 6.5 and 6.6, which appear at the end of this chapter.

Algorithm 5.5.4 outlines numerical restrictions to ensure there is ample room for elements which are not unique selling points. For instance, if $k = 3$ and we have the dot diagram in Figure 6.1 with partition $A_1|A_2|B_1$, then although there is room to place a copy of B_2 into each of the first two blocks, the two distinct nontrivial subgroups C_1 and C_2 cannot be inserted each with multiplicity two. However, three distinct nontrivial subgroups C_1, C_2, and C_3 can be inserted, one in each block. This is a particular instance of Step 7 in Algorithm 5.5.4. The resulting partition filling and combinatorial diagram are listed in Table 6.2, corresponding to the combinatorial diagram (i).

Figure 6.1: Dot diagram with partition $A_1|A_2|B_1$

Some partitions do not give rise to valid partitions. For instance, suppose that $k = 2$, $u = 4$, and we have the dot diagram shown in Figure 6.2 with partition $A_1|A_2B_1C_1$ where two vertex components have only one vertex each (see Step 6 in Algorithm 5.5.4). In this case, there are not enough blocks in the partition to insert multiple occurrences of distinct nontrivial subgroups B_2 and C_2. The partition $A_1|A_2B_1C_1$ is invalid, as it will not give rise to an independent facet. We indicate in Tables 6.1 and 6.2 which partitions are valid and which are invalid, omitting dot diagrams for those which are invalid.

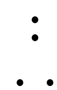


Figure 6.2: Dot diagram with partition $A_1|A_2B_1C_1$

Once a partition of unique selling points in the fourth column of Tables 6.1 and 6.2 is deemed valid, we insert any remaining variables from $\{A_1, A_2, B_1, B_2, C_1, C_2\}$ (etcetera for other values of n) which do not already occur into the partition according to Algorithm 5.5.4.

For example, consider the partition $A_1|B_1|B_2C_2|C_1$ (corresponding to the first entry $k = 4$ and $u = 5$ in Table 6.1). The only variable which does not occur is A_2, which can be inserted into the last three blocks in $\binom{3}{2} + \binom{3}{3} = 4$ different ways:

$$A_1|A_2B_1|A_2B_2C_2|C_1 \tag{6.1}$$

$$A_1|A_2B_1|B_2C_2|A_2C_1 \tag{6.2}$$

$$A_1|B_1|A_2B_2C_2|A_2C_1 \tag{6.3}$$

$$A_1|A_2B_1|A_2B_2C_2|A_2C_1 \tag{6.4}$$

This gives rise to the independent facets which have group superstructures

$$\{(A_1, 1, 1), (A_2, B_1, 1), (A_2, B_2, C_2), (1, 1, C_1)\}$$

$$\{(A_1, 1, 1), (A_2, B_1, 1), (1, B_2, C_2), (A_2, 1, C_1)\}$$

$$\{(A_1, 1, 1), (1, B_1, 1), (A_2, B_2, C_2), (A_2, 1, C_1)\}$$

$$\{(A_1, 1, 1), (A_2, B_1, 1), (A_2, B_2, C_2), (A_2, 1, C_1)\}$$

respectively. Partitions (6.1) and (6.3) (and their associated group superstructures) correspond to the combinatorial diagram (h), partition (6.2) to (j) and partition (6.4) to (p) in Tables 6.1 and 6.2 and Figures 6.5 and 6.6.

6.2 Simplicial Perspectives

We can view each of these combinatorial diagrams as being constructed directly as a simplicial complex from a simpler simplicial complex. For example, the complex (h) can be constructed by starting with the simplicial complex corresponding to the partition $A_1|B_1|B_2C_2|C_1$ of its unique selling points (Figure 6.3, left) and forming the convex hull of A_2 with the other vertices in each block of the partition where A_2 occurs. Figure 6.3 (right) shows one such resulting complex, associated with the partition $A_1|A_2B_1|A_2B_2C_2|C_1$.

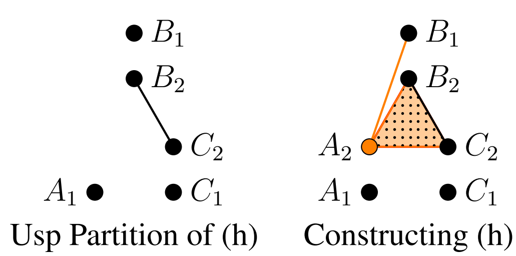


Figure 6.3: Construction of (h) from its Usp Partition

Figure 6.4 depicts all four of these deconstructions, using the same coloring scheme. Since only one vertex, A_2, is not a unique selling point, this construction can be interpreted as taking a join (see Definition 2.2.7) of the vertex A_2 with each allowable subset (i.e. respecting that filled partition blocks should not contain vertices from the same vertex component) of the vertices in the original partition.

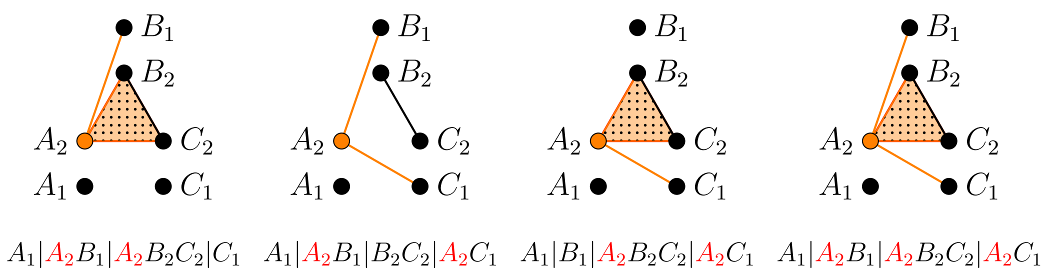
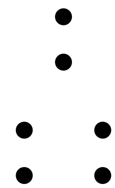


Figure 6.4: Simplicial complex join deconstructions

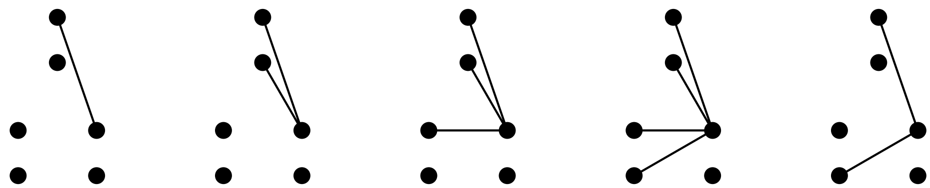
This gives an idea for translation of the partition descriptions of combinatorial diagrams (after the unique selling points are partitioned) to a constructive description given by joins of simplicial complexes. Listing all possible combinatorial diagrams becomes easier when viewed as simplicial complexes since they are much simpler to list up to automorphism than their partition counterparts; and the partition description allows us to check that we have generated all such complexes. Expanding on this translation seems promising based on examples computed, and would likely be a fruitful direction to explore in the future.

$k = 6$:



(a)

$k = 5$:



$k = 4$ (continued in Figure 6.6):

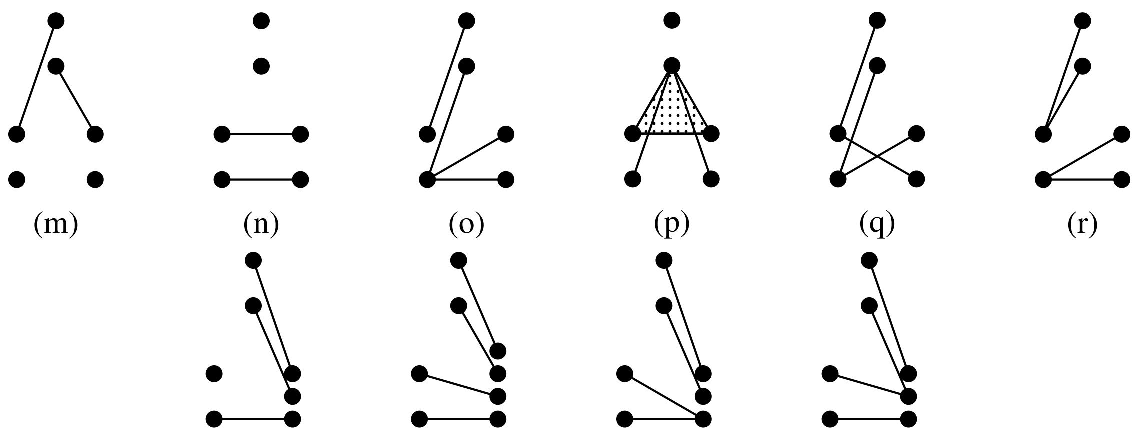
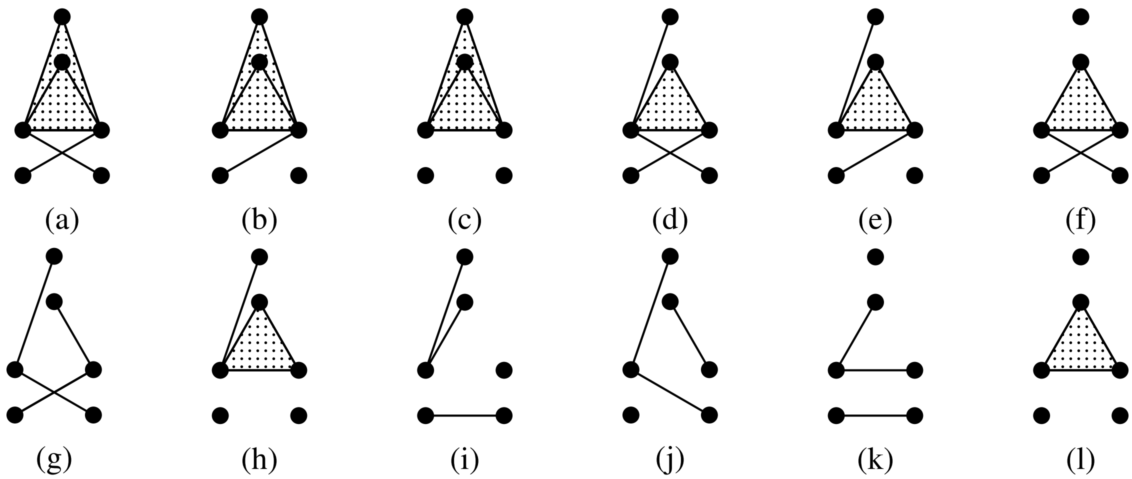
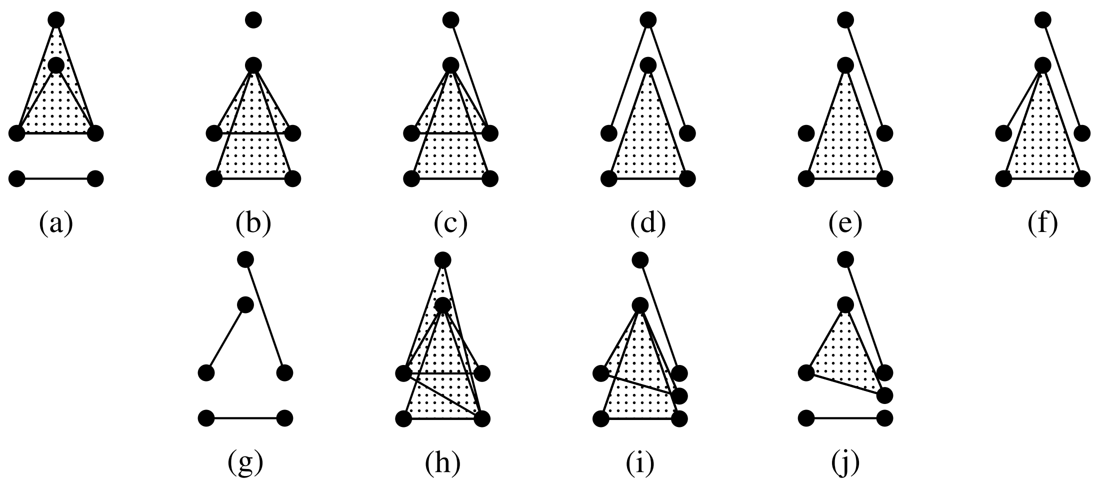


Figure 6.5: Combinatorial Diagrams for $\mathcal{G}_3$ (Part I)

$k = 4$ (continued from Figure 6.5):



$k = 3$:



$k = 2$:

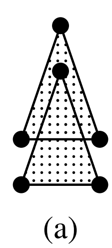


Figure 6.6: Combinatorial Diagrams for $\mathcal{G}_3$ (Part II)

Table 6.1: Partitions of unique selling points (Part I)

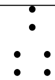
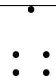
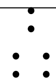
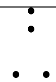
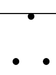
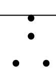
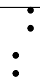
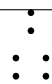
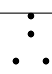
k	u	Dot Diagram	Partition of usps	Valid?	Partition Fillings	Diagram
6	6		$A_1\vert A_2\vert B_1\vert B_2\vert C_1\vert C_2$	yes	$A_1\vert A_2\vert B_1\vert B_2\vert C_1\vert C_2$	(a)
5	5		$A_1\vert B_1\vert B_2\vert C_1\vert C_2$	yes	$A_1\vert A_2B_1\vert A_2B_2\vert C_1\vert C_2$ $A_1\vert A_2B_1\vert B_2\vert A_2C_1\vert C_2$ $A_1\vert A_2B_1\vert A_2B_2\vert A_2C_1\vert C_2$ $A_1\vert A_2B_1\vert A_2B_2\vert A_2C_1\vert A_2C_2$	(b) (e) (c) (d)
	6		$A_1\vert A_2\vert B_1\vert B_2C_2\vert C_1$	yes	$A_1\vert A_2\vert B_1\vert B_2C_2\vert C_1$	(a)
4	4		$A_1\vert A_2\vert B_1\vert C_1$	yes	$A_1B_2C_2\vert A_2B_2C_2\vert B_1\vert C_1$ $A_1B_2C_2\vert A_2B_2C_2\vert B_1C_2\vert C_1B_2$ $A_1B_2C_2\vert A_2B_2C_2\vert B_1C_2\vert C_1$ $A_1B_2\vert A_2B_2C_2\vert B_1C_2\vert C_1B_2$ $A_1B_2\vert A_2B_2C_2\vert B_1C_2\vert C_1$ $A_1\vert A_2B_2C_2\vert B_1C_2\vert C_1B_2$ $A_1B_2\vert A_2C_2\vert B_1C_2\vert C_1B_2$	(c) (a) (b) (d) (e) (f) (g)
			$A_1\vert A_2\vert B_1\vert B_2$	yes	$A_1C_1\vert A_2C_1\vert B_1C_2\vert B_2C_2$ $A_1C_1\vert A_2C_2\vert B_1C_1\vert B_2C_2$ $A_1C_1\vert A_2C_2\vert B_1C_3\vert B_2$ $A_1C_1\vert A_2C_2\vert B_1C_3\vert B_2C_3$ $A_1C_1\vert A_2C_3\vert B_1C_2\vert B_2C_3$ $A_1C_1\vert A_2C_2\vert B_1C_3\vert B_2C_4$	(r) (q) (s) (u) (v) (t)
	5		$A_1\vert B_1\vert B_2C_2\vert C_1$	yes	$A_1\vert A_2B_1\vert A_2B_2C_2\vert C_1$ $A_1\vert A_2B_1\vert B_2C_2\vert A_2C_1$ $A_1\vert A_2B_1\vert A_2B_2C_2\vert A_2C_1$	(h) (j) (p)
			$A_1C_2\vert B_1\vert B_2\vert C_1$	yes	$A_1C_2\vert A_2B_1\vert A_2B_2\vert C_1$ $A_1C_2\vert A_2B_1\vert B_2\vert A_2C_1$ $A_1C_2\vert A_2B_1\vert A_2B_2\vert A_2C_1$	(i) (k) (o)
	6		$A_1\vert A_2\vert B_1C_1\vert B_2C_2$ $A_1C_1\vert A_2\vert B_1\vert B_2C_2$ $A_1\vert A_2B_2C_2\vert B_1\vert C_1$	yes yes yes	$A_1\vert A_2\vert B_1C_1\vert B_2C_2$ $A_1C_1\vert A_2\vert B_1\vert B_2C_2$ $A_1\vert A_2B_2C_2\vert B_1\vert C_1$	(n) (m) (l)

Table 6.2: Partitions of unique selling points (Part II)

k	u	Dot Diagram	Partition of usps	Valid?	Partition Fillings	Diagram
3	3		$A_1\|B_1\|C_1$	yes	$A_1B_2C_2\|A_2B_1C_2\|A_2B_2C_1$	(h)
			$A_1\|A_2\|B_1$	yes	$A_1B_2C_1\|A_2B_2C_2\|B_1C_3$	(i)
	4		$A_1\|A_2C_2\|B_1$	yes	$A_1B_2C_1\|A_2B_2C_2\|B_1C_1$	(c)
			$A_1\|A_2\|B_1C_2$	yes	$A_1B_2C_1\|A_2B_2C_1\|B_1C_2$	(a)
			$A_1\|A_2B_1\|B_2$	yes	$A_1C_1\|A_2B_1C_2\|B_2C_3$	(j)
	5		$A_1\|B_1C_1\|B_2C_2$	yes	$A_1\|A_2B_1C_1\|A_2B_2C_2$	(b)
			$A_1C_1\|B_1\|B_2C_2$	yes	$A_1C_1\|A_2B_1\|A_2B_2C_2$	(f)
			$A_1B_2C_2\|B_1\|C_1$	yes	$A_1B_2C_2\|A_2B_1\|A_2C_1$	(d)
	6		$A_1B_2C_1\|A_2C_2\|B_1$	yes	$A_1B_2C_1\|A_2C_2\|B_1$	(e)
			$A_1B_2\|A_2C_2\|B_1C_1$	yes	$A_1B_2\|A_2C_2\|B_1C_1$	(g)
2	2		$A_1\|B_1$	no		
			$A_1\|A_2$	no		
	3		$A_1B_2\|B_1$	no		
			$A_1\|B_1C_1$	no		
	4		$A_1B_1\|A_2C_1$	no		
			$A_1\|A_2B_1C_1$	no		
			$A_1B_1\|A_2B_2$	no		
	5		$A_1B_1C_1\|A_2B_2$	no		
	6		$A_1B_1C_1\|A_2B_2C_2$	yes	$A_1B_1C_1\|A_2B_2C_2$	(a)
1	1		A_1	no		
	2		A_1B_1	no		
	3		$A_1B_1C_1$	no		
	4		none	no		
	5		none	no		
	6		none	no		

Chapter 7

Example: Counting the number of independent facets for $\mathcal{G}_3$

Once a list of combinatorial diagrams is obtained for $\mathcal{G}_n$ for a particular choice of n, one is faced with the task of counting how many independent facets arise from each combinatorial diagram on the level of elements. We provide an example of how one might perform this count. The process feels very similar to graph (vertex) coloring or simplicial complex (vertex) coloring problems, yet with slightly different criteria.

The vertices in each vertex component of the combinatorial diagrams shown in Figures 6.6 and 6.5 can be labeled as follows. If there are exactly two vertices in a vertex component, the vertices can be labelled by p_i, p_{i+1} or p_{i+1}, p'_{i+1} where $i \in \{1, 3, 5\}$ according to the appropriate vertex component. Here, the vertex labeling p_{i+1}, p'_{i+1} indicates that each vertex represents an element from a distinct subgroup of order p_{i+1} (so that p_{i+1}, p'_{i+1} represent two distinct subgroups of order p_{i+1}). Depending on the symmetry of the combinatorial diagram, in some cases we will want to consider ordering to matter in these labelings. We provide an explicit example which illustrates this point and is indicative of the process involved to compute counts for generic combinatorial diagrams.

Consider the combinatorial diagram (b) for $k = 3$ where the vertices correspond to distinct subgroups of order p_2, p_4, and p_6 (in the appropriate components). We count the number of independent facets arising from (b). These facets have the form

$$\{(p_2, 1, 1), (p_2{}', p_4, p_6), (p_2{}', p_4{}', p_6{}')\}$$

where the p_i, $p_i{}'$, and 1 represent orders of the containing subgroups; p_i and $p_i{}'$ represent the orders of two distinct subgroups of order p_i, and the double occurrence of $p_2{}'$ indicates that two nonidentity elements (possibly distinct but not necessarily) are taken from a subgroup of order p_2 (this subgroup is represented by ${p_2}'$). Figure 7.1 shows the corresponding combinatorial diagram with labelled vertices.

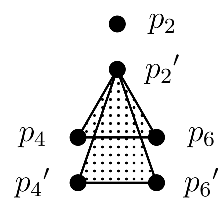


Figure 7.1: Combinatorial diagram with labelled vertices

We count the number of independents sets to which this diagram gives rise. First, choose a p_2-Sylow subgroup (there are p_1 subgroups of order p_2), and a non-identity element from that subgroup ($p_2 - 1$ choices). This subgroup corresponds to the vertex labelled p_2 in the figure. Next, choose a different p_2-Sylow subgroup (there are $p_1 - 1$ remaining subgroups). This subgroup corresponds to the vertex labelled ${p_2}'$ in the figure.

We consider two cases: (1) The two elements chosen from the subgroup represented by ${p_2}'$ are distinct and (2) these elements are identical.

(1) Choose two distinct nonidentity elements from the chosen subgroup represented by ${p_2}'$ (there are $\binom{p_2-1}{2}$ ways to do so).

(2) Choose one nonidentity element from the chosen subgroup represented by ${p_2}'$ (there are $p_2 - 1$) ways to do so.

In Case (1), fix an order for the two distinct elements from the subgroup ${p_2}'$ in which they appear in the independent set tuples. Choose a subgroup of order p_4 (there are p_3 such subgroups), and a non-identity element from that subgroup ($p_4 - 1$ ways). Then choose a different subgroup of order p_4 (there are $p_3 - 1$ remaining choices) and a non-identity element from that subgroup ($p_4 - 1$ ways). Then do the same for the two distinct subgroups of order p_6.

In Case (1), no double counting occurs since the entries corresponding to ${p_2}'$ are distinct.

In Case (2), we have only one element from ${p_2}'$. Proceed to choose elements of orders p_4 and p_6 as in Case (1). Now, since the tuple entries corresponding to ${p_2}'$ are identical, we obtain a double count when choosing elements of order p_4 and p_6, so must divide the result by two.

The total number of independent facets arising from the diagram is:

$$
\begin{aligned}
&p_1(p_2-1)(p_1-1)(\binom{p_2-1}{2})p_3(p_4-1)(p_3-1)(p_4-1)p_5(p_6-1)(p_5-1)(p_6-1)\\
&+\frac{1}{2}p_1(p_2-1)(p_1-1)(p_2-1)p_3(p_4-1)(p_3-1)(p_4-1)p_5(p_6-1)(p_5-1)(p_6-1)
\end{aligned} \tag{7.1}
$$

where the first term corresponds to Case (1) and the second term to Case (2). Expression (7.1) simplifies to

$$
\begin{aligned}
&p_1(p_1-1)(p_2-1)p_3(p_3-1)(p_4-1)^2p_5(p_5-1)(p_6-1)^2\Big(\binom{p_2-1}{2}+\frac{1}{2}(p_2-1)\Big)\\
&=p_1(p_1-1)(p_2-1)p_3(p_3-1)(p_4-1)^2p_5(p_5-1)(p_6-1)^2\Big(\frac{(p_2-1)(p_2-2)}{2}+\frac{1}{2}(p_2-1)\Big)\\
&=p_1(p_1-1)(p_2-1)p_3(p_3-1)(p_4-1)^2p_5(p_5-1)(p_6-1)^2\frac{1}{2}(p_2-1)^2\\
&=\frac{1}{2}p_1(p_1-1)(p_2-1)^3p_3(p_3-1)(p_4-1)^2p_5(p_5-1)(p_6-1)^2\\
&=4\binom{p_1}{2}(p_2-1)^3\binom{p_3}{2}(p_4-1)^2\binom{p_5}{2}(p_6-1)^2
\end{aligned}
$$

Counts for the remaining diagrams can be obtained using similar combinatorial arguments.

Chapter 8

Conjectures and Future Directions

Independence complexes of dihedral groups exhibit nice structural properties in small examples. We have preliminary results (not included in this paper) showing how the independence complex of the dihedral group D_{2pq} of order $2pq$ can be constructed from the cyclic group C_{pq} of order pq, where p and q are distinct odd primes. More generally, one could explore whether independence complexes for certain nonabelian groups can be built from the independence complexes of abelian groups. Classifying the independence complexes of dihedral groups of restricted order would be another interesting class of nonabelian examples to explore. From a group-theoretic viewpoint, dihedral groups naturally generalize to metacyclic and metabelian groups.

Dihedral groups have a cyclic normal subgroup N consisting of all rotations, and the factor group D_{2n}/N is cyclic. A natural question to ask is whether behavior and properties of the independence complexes arising from dihedral groups generalize to the class of finite groups G that have a cyclic normal subgroup N with G/N cyclic (such groups are called metacyclic), possibly with restricted order conditions. Metabelian groups (those having an abelian normal subgroup such that the quotient is abelian) are also a natural class of groups to consider. Recall the generating graph defined in [11]. In this context, it is known that if there is at most one isolated vertex (i.e. an element g such that when g is combined with any other element in the group, the pair still generates only a proper subgroup of G) in the generating graph, then for every non-trivial normal subgroup N of G, the factor group G/N is cyclic. It is possible that a related result could exist in the context of independence complexes.

Combinatorial diagrams are interesting objects in themselves. An interesting problem would be to count the number of combinatorial diagrams on certain vertex types; though this additionally has applications in group theory, it is an interesting combinatorial problem in itself.

As mentioned in Section 7, counting the number of independent facets which arise from a combinatorial diagram share flavors with graph (vertex) coloring and simplicial complex (vertex) coloring. It would be interesting to investigate some of these similarities, and to see whether one can draw parallels between these areas.

Other objects that would be interesting to investigate are the f-vectors and h-vectors of independence complexes. In general, it is common to look for unimodality of h-vectors. Unimodality of f-vectors, however, is relatively rare, and thus results about unimodal f-vectors are particularly interesting. We have computed some data with GAP which supports the following conjecture.

Conjecture 8.0.1. *The f-vector of the independence complex of $C_{p_1} \times C_{p_2} \times \cdots \times C_{p_n}$ where the primes p_i are distinct is unimodal.*

Expanding on the translation between generating partitions in Algorithm 5.5.4 and direct simplicial, combinatorial constructions is a fruitful area to explore. See Section 6.2 for more details.

Developing more code which severely restricts the group type and computes combinatorial diagrams or independence complexes utilizing these restrictions would be useful in exploring the independence complexes of other classes of finite groups.

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Appendix A

GAP Code

A.1 GAP: Ainc.txt

```
## Ainc.txt

## Contents: makeAinc and supporting functions makeAutoOrbitsGroup,

##   isIndependentSet

## Final 5/31/2021 version for Dissertation "Independence

## Complexes of Finite Groups"



## Generates orbits under automorphism group action of all sets

## of size 2...size of group elements of G

makeAutoOrbitsGroup:=function(G,size)

   local L, aut, ElmtsNZ, k, groupOrb;

   L:=[];


   aut:=AutomorphismGroup(G);

   ElmtsNZ:=Difference(Elements(G),[Identity(G)]);


   for k in [2..size] do

      groupOrb:=Orbits(aut,Combinations(ElmtsNZ,k),OnSets);

      Append(L,groupOrb);

   od;

   return L; #returns group orbits of all sets of group elements

end;
```

```
## Input: Finite group G, independent set S
## Output: List of all automorphic images of S under every automorphism
##    of G
## i.e. calculates the orbit of a set S under action of all elements of
##    Aut(G)

generateAutomorphicImages:=function(G,S)
   local aut, orb;
   aut:=AutomorphismGroup(G);
   orb:=Orbits(aut,[S],OnSets);
   return orb[1]; #removes outer brackets
end;



## A set I of group elements is independent if for all elements g in I,
## removing g results in a strictly smaller subgroup generated.
## isIndependentSet runs through elements g in I; if for some g in I
## we have <I>=<I/g>, then computation stops as I is not independent.
## If for all g in I, <I/g> < <I>, then I is independent and function
## returns true.
## Input: Finite group G, subset S of nonzero elements of G
## Output: true if S is an independent set of G, false otherwise

isIndependentSet:=function(G,S)
   local indep, g;
   indep:=true;
```

```
   for g in S do
      if Subgroup(G,S)=Subgroup(G,Difference(S,[g])) then
         indep:=false;
         break; #at the first redundant element, break
      fi;
   od;
   return indep;
end;



## Generates Ainc(G), the Automorphism Independence Complex of a finite
##   group G
## Generates all orbits of G under action of automorphism group
## Picks a single representative from each orbit (the first)
## Check if representatives are independent, keep the ones that are
## Input: Finite group G, desired size of independent sets size, bool
## generateFullIndepCplx (generates list of full orbits if true), bool
## showNonIndep (if true, displays non-independent sets. If false, does
##   not)
## Output: record with info about orbit representatives for Ainc(G):
##     r.allFacets: List of all facets of length size, organized as a
##         list of orbit lists
##     r.orbSizes: List of lengths of each orbit
##     r.orbitsPoly: Full list of facets, separated by orbit
##     r.polyFacets: Full list of facets, no orbit separation,
##         formatted for polymake input
## Prints (additionally, and optionally)
##     Total number of orbits
```

```
##     r.orbSizes: Size of each orbit
##     r.allFacets: All facets of independence complex, according to
##        orbit


## Example call for main function:
## makeAinc(G,3,true,false);

makeAinc:=function(G,size,generateFullIndepCplx,showNonIndep,r)
   local L, S, l, j, o, f, e, a, b, A, orbs, orbReps, keep,
   indepNonFacets, orbitOfS, polymakeL;
   r.allFacets:=[];
   r.orbSizes:=[];
   r.polyFacets:=[];
   r.orbitsPoly:=[];

   L:=[]; #list of independent facet orbit representatives
   orbReps:=[];
   #allFacets:=[];
   indepNonFacets:=[];
   polymakeL:=[];
   #orbSizes:=[];
   keep:=false;
   ##SetReducedMultiplication(G); #If G is defined via SmallGroup,
   #    then comment out line
   orbs:=makeAutoOrbitsGroup(G,size);

   for o in orbs do
      Append(orbReps,[o[1]]);
```

```
   #Append(indicesList,[Position(orbs,o)]);
od;



for S in orbReps do
   if isIndependentSet(G,S)=true then
      Append(L,[S]);
   fi;
od;

for S in L do #make sure you're only keeping facets
  for A in L do
   if (A in L and Length(A)<Length(S) and IsSubset(S,A)=true) then
      Remove(L,Position(L,A));
   elif (S in L and Length(S)<Length(A) and IsSubset(A,S)=true) then
      Remove(L,Position(L,S));
   fi;
  od;
od;




if showNonIndep=true then
   Print("Non-independent orbit reps and indep set reps that are not
      facets:");
   Print(Difference(orbReps,L));
   Print("\n");
fi;
if generateFullIndepCplx=true then
```

```
   for S in L do
      orbitOfS:=generateAutomorphicImages(G,S);
      Append(r.allFacets,[orbitOfS]);
   od;
   Print("All facets of independence complex, according to orbit:");
   Print(r.allFacets);
   Print("\n");
fi;
Print("There are ");
Print(Length(r.allFacets));
Print(" orbits.");
Print("\n");
Print("Sizes of orbits are: ");
for l in r.allFacets do
   Append(r.orbSizes,[Length(l)]);
   #Print(Length(l));
   #Print(",");
od;
Print(r.orbSizes);
Print("\n");
Print("List of representatives of each orbit (facet reps in Ainc):");
Print(L);
Print("\n");
Print("Record is returned with allFacets, orbSizes:");
#return L; #orbits
#return allFacets;

#makes list of facets to feed into polymake (good)
for o in r.allFacets do
```

```
      for f in o do
         for e in f do
            a:=List(f,e->Position(Elements(G),e));
         od;
         Append(r.polyFacets,[a]);
      od;
   od;



   #makes list of facets to feed into polymake, with labels 0,1,2,3,...

   #Makes list of orbits with polymake labels 9/5/19
   #For copy and paste into polymake
   for o in r.allFacets do
      for f in o do
         for e in f do
            r.orbitsPoly:=List(r.allFacets,o->List(o,f->List(f,e->
               Position(Elements(G),e))));
         od;
      od;
   od;
   return r; #returns record
end;
```

A.2 GAP: mainGAP.txt

```
## mainGAP.txt
## Final 5/31/2021 version for Dissertation "Independence
## Complexes of Finite Groups"
```

```
## To run in GAP in Terminal, first call Ainc.txt by typing:
Read(".../Ainc.txt");
## where ... represents the filepath


r:=rec();

## Sample input for GAP:
## Form a finitely presented group
## (a) by specifying element orders

#C2xC4
G:=AbelianGroup(IsFpGroup,[2,4]);

## or (b) by specifying generators and relations

## for C3xQ8
f:=FreeGroup("a","i","j","k");
rels:=ParseRelators(f,"a3,i4,j4,k4,i2=j2,i2=k2,
j2=k2,(ij)4,(ik)4,(jk)4,[i,a],[j,a],[k,a],ijk=i2");


## for C4xQ8
f:=FreeGroup("a","i","j","k");
rels:=ParseRelators(f,"a4,i4,j4,k4,i2=j2,i2=k2,
j2=k2,(ij)4,(ik)4,(jk)4,[i,a],[j,a],[k,a],ijk=i2");


## for C3xQ8xQ8
```

```
f:=FreeGroup("a","i","j","k","x","y","z");
rels:=ParseRelators(f,"a3,i4,j4,k4,x4,y4,z4,i2=j2,
i2=k2,j2=k2,x2=y2,x2=z2,y2=z2,(ij)4,(ik)4,(jk)4,
(xy)4,(xz)4,(yz)4,[i,a],[j,a],[k,a],[x,a],[y,a],[z,a],
ijk=i2,xyz=x2,[x,i],[x,j],[x,k],[y,i],[y,j],[y,k],[z,i],[z,j],[z,k]");


## for C3xC7xQ8xQ8
f:=FreeGroup("a","b","i","j","k","x","y","z");
rels:=ParseRelators(f,"a3,b7,i4,j4,k4,x4,y4,z4,i2=j2,i2=k2,
j2=k2,x2=y2,x2=z2,y2=z2,(ij)4,(ik)4,(jk)4,(xy)4,(xz)4,(yz)4,
[i,a],[j,a],[k,a],[x,a],[y,a],[z,a],[i,b],[j,b],[k,b],[x,b],[y,b],[z,b],
[a,b], ijk=i2,xyz=x2,[x,i],[x,j],[x,k],[y,i],[y,j],[y,k],[z,i],[z,j],
[z,k]");


G:=f/rels;
StructureDescription(G);
SetReducedMultiplication(G);
myrec:=makeAinc(G,2,true,false,r);
```

Appendix B

Polymake Code

B.1 Polymake: mainPolymake.txt

```
## mainPolymake.txt


## Input: Record output r.polyFacets from mainGAP.txt and Ainc.txt,
##     copy and pasted and padded with any unused vertices


## In Polymake:
application 'topaz';


## GAP outputs from mainGAP.txt and Ainc.txt a list of facets
## r.polyFacets, e.g. [[0,1,2],[2,3,4],[3,4,5],[5,6,7],[7,8,9]]
## and a list of orbits
## [[[0,1,2],[2,3,4]],[[3,4,5],[5,6,7],[7,8,9]]]
## Note that GAP outputs facets of one size at a time; to visualize the
## full complex in Polymake, generate facets of each size separately
## in GAP.


## In Polymake, copy and paste orbit list (r.orbitsPoly) and facet list
##   (r.polyFacets) into the following commands from GAP's record
##   output, padding entries according to the following note.
## NOTE: Polymake requires, for technical reasons, that all vertex
##   labels appear in the list of facets and orbits. To fix this, one
##   can manually add in an extra facet, and an orbit containing that,
##   or those, facets, with any unused vertices. For instance, if
##   G=C2xC4 then r.polyFacets is the list
```

```
##   [[2,4],[2,9],[2,5],[3,4],[2,6],.....] of facets, none of which
##   contains vertices 0 or 1, and one should add the facet [0,1]. The
##   orbit [[0,1]] should also be added to r.orbitsPoly; i.e. resulting
##   in [[[2,4],[2,9],[2,5],[3,4],[2,6],.....],[[0,1]]]. This new facet
##   and new orbit are artificial and can be disregarded when
##   viewing the final complex.

## Copy padded r.orbitsPoly here
$orbitList= new Array<Array<Array<Int>>>
([[[0,1,2],[2,3,4]],[[3,4,5],[5,6,7],[7,8,9]]]);

## Copy padded r.polyFacets here
$facetList= new Array<Array<Int>>
([[0,1,2],[2,3,4],[3,4,5],[5,6,7],[7,8,9]]);

$s=new SimplicialComplex(INPUT_FACES=>$facetList);

## View with no group element labels, just vertex labels 1,2,3,...
$coloredFacets = new Array<Array<Int>>;
foreach(@$orbitList){$coloredFacets=$_, $s->VISUAL->
   FACES($coloredFacets, FacetColor=> new RGB(rand,rand,rand));};


## One can also label the vertices with specified group element names.
## We include a few examples generated by hand, but this process
## could be automated.


## For C3xQ8, using SmallGroup(24,11)
```

```
foreach(@$orbitList){$coloredFacets=$_, $s->VISUAL(VertexLabels=>
["","","f1", "f2", "f3", "f4", "f1*f2", "f1*f3", "f1*f4", "f2*f3",
"f2*f4", "f3^2", "f3*f4", "f1*f2*f3", "f1*f2*f4", "f1*f3^2",
"f1*f3*f4", "f2*f3^2", "f2*f3*f4", "f3^2*f4", "f1*f2*f3^2",
"f1*f2*f3*f4", "f1*f3^2*f4", "f2*f3^2*f4", "f1*f2*f3^2*f4" ])->
FACES($coloredFacets,FacetColor=> new RGB(rand,rand,rand));};

## For C3xQ8, using SmallGroup(24,11), with cleaner labels
foreach(@$orbitList){$coloredFacets=$_, $s->VISUAL(VertexLabels=>
["","","f1", "f2", "f3", "f4", "f1f2", "f1f3", "f1f4", "f2f3",
"f2f4", "f3^2", "f3f4", "f1f2f3", "f1f2f4", "f1f3^2",
"f1f3f4", "f2f3^2", "f2f3f4", "f3^2f4", "f1f2f3^2", "f1f2f3f4",
"f1f3^2f4", "f2f3^2f4", "f1f2f3^2f4" ])->
FACES($coloredFacets,FacetColor=> new RGB(rand,rand,rand));};

## For C4xQ8, using SmallGroup(32,26)
foreach(@$orbitList){$coloredFacets=$_, $s->VISUAL(VertexLabels=>
["","","f1", "f2", "f3", "f4","f5","f1f2", "f1f3", "f1f4", "f1f5",
 "f2f3", "f2f4", "f2f5", "f3f4", "f3f5", "f4f5", "f1f2f3", "f1f2f4",
 "f1f2f5", "f1f3f4", "f1f3f5", "f1f4f5", "f2f3f4", "f2f3f5",
 "f2f4f5", "f3f4f5", "f1f2f3f4", "f1f2f3f5", "f1f2f4f5", "f1f3f4f5",
 "f2f3f4f5", "f1f2f3f4f5"])->FACES($coloredFacets,FacetColor=>
 new RGB(rand,rand,rand));};

## For C4xQ8, using the group element labels given by representation:
## f:=FreeGroup("a","i","j","k");
```

```
## rels:=ParseRelators(f,"a4,i4,j4,k4,i2=j2,i2=k2,j2=k2,
(ij)4,(ik)4,(jk)4,[i,a],[j,a],[k,a],ijk=i2");
## G:=f/rels;
foreach(@$orbitList){$coloredFacets=$_, $s->VISUAL(VertexLabels=>
["","", "a^-1", "a", "i^-1", "i", "j^-1", "j", "k^-1", "k", "a^-1*i",
"a^-1*j", "a^-1*k", "a^2", "i^-1*a^-1", "i^-1*a", "i*a", "i^2",
"j^-1*a^-1", "j^-1*a", "j*a", "k^-1*a^-1", "k^-1*a", "k*a", "a^-1*i^2",
"i^-1*a^2", "i*a^2", "i^2*a", "j^-1*a^2", "j*a^2","k^-1*a^2", "k*a^2",
"i^2*a^2"])->
   FACES($coloredFacets,FacetColor=> new RGB(rand,rand,rand));};


## Collections of facets in the final complex plotted in Polymake are
## colored according to orbit (there is one output file for each orbit)
```