

## The Total Irregularity Strength of $m$ Copies of the Friendship Graph

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**Abstract:** This paper deals with the totally irregular total labeling of the disjoint union of friendship graphs. The results shows that the disjoint union of  $m$  copies of the friendship graph is a totally irregular total graph with the exact values of the total irregularity strength equals to its edge irregular total strength.

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**Key words:** disjoint union, friendship graph, total irregularity strength, totally irregular total labeling

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### 1. Introduction

In [1], Baca, *et al.* introduced a *vertex irregular total  $k$ -labeling* and an *edge irregular total  $k$ -labeling* of a graph as a modification of the well-known *irregular assignment* given by Chartrand, *et al.* [2].

A vertex irregular total  $k$ -labeling of a graph  $G$  is a function that map all the vertices and edges of  $G$  to  $k$  positive integer  $\lambda: V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$  such that all the vertex weights are distinct, where  $w(v) = \lambda(v) + \sum_{vx \in E(G)} \lambda(vx)$ . The minimum value  $k$  for which  $G$  has a vertex irregular total  $k$ -labeling is called the *total vertex irregularity strength* of  $G$ , denoted by  $tvs(G)$ . The boundaries of  $tvs(G)$  is given in [1] as follow.

*Theorem 1.* [1] Let  $G$  be a graph  $(p, q)$  with minimum degree  $\delta$  and maximum degree  $\Delta$ . Then

$$\left\lceil \frac{p+r}{r+1} \right\rceil \leq tvs(G) \leq p - r + 1 \quad (1)$$

An edge irregular total  $k$ -labeling of a graph  $G$  is a function that map all the vertices and edges of  $G$  to  $k$  positive integer  $\lambda: V(G) \cup E(G) \rightarrow \{1, 2, \dots, k\}$  such that all the edge weights are distinct, where  $w(vx) = \lambda(v) + \lambda(vx) + \lambda(x)$ . The minimum value  $k$  for which  $G$  has an edge irregular total  $k$ -labeling is called the *total edge irregularity strength* of  $G$ , denoted by  $tes(G)$ . The lower bound of  $tes(G)$  is given in [1] as follow.

*Theorem 2.* [1] Let  $G = (V, E)$  be any graph with maximum degree  $\Delta$ . Then

$$tes(G) \geq \max \left\{ \left\lceil \frac{|E(G)|+2}{3} \right\rceil, \left\lceil \frac{\Delta(G)+1}{2} \right\rceil \right\} \quad (2)$$

Later, many results given for improving the boundaries and the exact values of  $tvs(G)$  and  $tes(G)$  of certain types of graphs are determined instead of finding the exact values of any graph. In [1], Baca, *et al.* have determined the  $tvs(G)$  of complete graph, cycle graph, star graph, and prism graph. Nurdin, *et al* [3], Simanjuntak, *et al.* [4], and Susilawati, *et al.* [5], focused on the boundaries and exact values of the total vertex irregularity strength of trees. For the disjoint union of friendship graph, Ahmad, *et al.* [6] have showed that its total vertex irregularity strength equals to the lower bound given in equation (1) and provided corollary for  $m$  copies of friendship graph as follow.

*Corollary 3.* [6] Let  $F_n$  be a friendship graph with  $n$  triangles,  $n \geq 3$ , and let  $mF_n$  be the disjoint union of  $m$  copies of  $F_n$ ,  $m \geq 2$ . Then

$$tvs(mF_n) = \left\lceil \frac{2(mn+1)}{3} \right\rceil. \quad (3)$$

Many results of the total edge irregularity strength of some certain class of graphs also be provided by several authors, namely Baca, *et al.* for path graph, cycle graph, star graph, wheel graph, and friendship graph in [1], Ivanco and Jendrol [7] for trees, and many others. In [6], Ahmad, *et al*, also provided the total edge irregularity strength of the disjoint union of friendship graph, and the following corollary, which is equal to the lower bound in equation (2).

*Corollary 4.* [6] Let  $F_n$  be a friendship graph with  $n$  triangles,  $n \geq 3$ , and let  $mF_n$  be the disjoint union of  $m$  copies of  $F_n$ ,  $m \geq 2$ . Then

$$tes(mF_n) = mn + 1 \quad (4)$$

Motivated by both labeling, Marzuki, *et al.* [8] introduced a *totally irregular total  $k$ -labeling* of a graph  $G$ , as a total  $k$ -labeling such that for every two distinct vertices  $v$  and  $x$ , their weights  $w(v)$  and  $w(x)$  are distinct, and for every two distinct edges  $v_1v_2$  and  $x_1x_2$ , their weights  $w(v_1v_2)$  and  $w(x_1x_2)$  are distinct. The minimum  $k$  for which  $G$  has a totally irregular total  $k$ -labeling is called the total irregularity strength of  $G$ , denoted  $ts(G)$ .

They [8] provided the lower bound of  $ts(G)$  as follow.

*Observation 5.* [8] For every graph  $G$ ,

$$ts(G) \geq \max\{tes(G), tvs(G)\} \quad (5)$$

They [8] proved that the lower bound is sharp for path graph of order  $n \neq 2, 5$  and cycle graphs. Ramdani and Salman [9], Ramdani, *et al* [10], Tilukay, *et al.* [11-13], also confirmed the sharpness of the lower bound of the total irregularity of several types of graphs, including friendship graph  $F_n$ . More results can be seen in a survey provided by Galian in [14].

In this paper, we investigate the total irregularity strength of the disjoint union of  $m$  copies of friendship graph.

## 2. The Total Irregularity Strength of $m$ Copies of the Friendship Graph

The friendship graph  $F_n$  is a set of  $n$  copies of a triangle having a common vertex as a center and the other mutually disjoint vertices.

*Theorem 6.* Let  $mF_n$  be the disjoint union of  $m$  copies of a friendship graph  $F_n$ , where  $m \geq 2$  and  $n \geq 2$ . Then

$$ts(mF_n) = mn + 1.$$

*Proof.* Since the friendship graph  $F_n$  has  $2n + 1$  vertices and  $3n$  edges, the disjoint union of  $m$  copies of the friendship graph  $F_n$  is a graph of order  $2mn + m$ , size  $3mn$ , and maximum degree  $2n$ . Follow from equation (3-5), we obtain  $ts(mF_n) \geq mn + 1$ , for  $m, n \geq 2$ . Next, to conclude that it is the exact value of  $ts(mF_n)$ , we need to prove that there is a totally irregular total  $(mn + 1)$ -labeling of  $mF_n$ . Let  $V(mF_n) = \{v_i | 1 \leq i \leq m\} \cup \{x_{i_1}^j, y_{i_1}^j | 1 \leq i \leq m, 1 \leq j \leq r\} \cup \{x_i^{r+1}, y_i^{r+1} | 1 \leq i \leq m\} \cup \{x_{i_2}^j, y_{i_2}^j | 1 \leq i \leq m, 1 \leq j \leq n - r - 1\}$  and  $E(mF_n) = \{v_i x_{i_1}^j, v_i y_{i_1}^j, x_{i_1}^j y_{i_1}^j | 1 \leq i \leq m, 1 \leq j \leq r\} \cup \{v_i x_i^{r+1}, v_i y_i^{r+1}, x_i^{r+1} y_i^{r+1} | 1 \leq i \leq m\} \cup \{v_i x_{i_2}^j, v_i y_{i_2}^j, x_{i_2}^j y_{i_2}^j | 1 \leq i \leq m, 1 \leq j \leq n - r - 1\}$ . Let  $t_i = ni + 1$  and  $r = \lfloor \frac{n-1}{2} \rfloor$ . Consider that  $mF_n$  consists of  $m$  component of form  $F_n$ . We partition  $n$  triangles of each component of form  $F_n$  into 3 parts, as follow:

- i.  $r$  first triangles  $v_i x_{i_1}^j y_{i_1}^j v_i$ , where  $1 \leq i \leq m$  and  $1 \leq j \leq r$ ;
- ii. Triangle  $v_i x_i^{r+1} y_i^{r+1} v_i$ , where  $1 \leq i \leq m$ ;
- iii.  $n - r - 1$  triangles  $v_i x_{i_2}^j y_{i_2}^j v_i$ , where  $1 \leq i \leq m$  and  $1 \leq j \leq n - r - 1$ .

Next, we construct a total labeling  $\lambda: V \cup E \rightarrow \{1, 2, 3, \dots, mn + 1\}$  defined by

$$\begin{aligned} \lambda(v_i) &= n(i - 1) + r + 1, & 1 \leq i \leq m; \\ \lambda(x_{i_1}^j) &= n(i - 1) + 1, & 1 \leq i \leq m, 1 \leq j \leq r; \\ \lambda(y_{i_1}^j) &= n(i - 1) + 1, & 1 \leq i \leq m, 1 \leq j \leq r; \\ \lambda(x_i^{r+1}) &= n(i - 1) + r + 1, & 1 \leq i \leq m; \\ \lambda(y_i^{r+1}) &= n(i - 1) + r + 1, & 1 \leq i \leq m; \\ \lambda(x_{i_2}^j) &= ni + 1, & 1 \leq i \leq m, 1 \leq j \leq n - r - 1; \\ \lambda(y_{i_2}^j) &= ni + 1, & 1 \leq i \leq m, 1 \leq j \leq n - r - 1; \\ \lambda(x_{i_1}^j y_{i_1}^j) &= n(i - 1) + j & 1 \leq i \leq m, 1 \leq j \leq r; \\ \lambda(v_i x_{i_1}^j) &= n(i - 1) + 2j - 1, & 1 \leq i \leq m, 1 \leq j \leq r; \\ \lambda(v_i y_{i_1}^j) &= n(i - 1) + 2j, & 1 \leq i \leq m, 1 \leq j \leq r; \\ \lambda(x_i^{r+1} y_i^{r+1}) &= n(i - 1) + r + 1, & 1 \leq i \leq m; \\ \lambda(v_i x_i^{r+1}) &= n(i - 1) + r + 2, & 1 \leq i \leq m; \\ \lambda(v_i y_i^{r+1}) &= n(i - 1) + r + 3, & 1 \leq i \leq m; \\ \lambda(x_{i_2}^j y_{i_2}^j) &= n(i - 1) + r + j + 1, & 1 \leq i \leq m, 1 \leq j \leq n - r - 1. \end{aligned}$$

$$\lambda(v_i x_{i_2}^j) = \begin{cases} n(i-1) + 2j, & \text{even } n, 1 \leq i \leq m, 1 \leq j \leq n-r-1; \\ n(i-1) + 2j + 1, & \text{odd } n, 1 \leq i \leq m, 1 \leq j \leq n-r-1; \end{cases}$$

$$\lambda(v_i y_{i_2}^j) = \begin{cases} n(i-1) + 2j + 1, & \text{even } n, 1 \leq i \leq m, 1 \leq j \leq n-r-1; \\ n(i-1) + 2j + 2, & \text{odd } n, 1 \leq i \leq m, 1 \leq j \leq n-r-1; \end{cases}$$

From the construction above, one can check that the maximum label is  $mn + 1$ , as on vertex  $x_{i_2}^n$  for example. Next, we evaluate all the vertex-weights and all the edge-weights as follow.

For the edge-weights we have:

$$w(x_{i_1}^j y_{i_1}^j) = 3n(i-1) + j + 2, \quad 1 \leq i \leq m, 1 \leq j \leq r;$$

$$w(v_i x_{i_1}^j) = 3n(i-1) + r + 2j + 1, \quad 1 \leq i \leq m, 1 \leq j \leq r;$$

$$w(v_i y_{i_1}^j) = 3n(i-1) + r + 2j + 2, \quad 1 \leq i \leq m, 1 \leq j \leq r;$$

$$w(x_i^{r+1} y_i^{r+1}) = 3n(i-1) + 3r + 3, \quad 1 \leq i \leq m;$$

$$w(v_i x_i^{r+1}) = 3n(i-1) + 3r + 4, \quad 1 \leq i \leq m;$$

$$w(v_i y_i^{r+1}) = 3n(i-1) + 3r + 5, \quad 1 \leq i \leq m;$$

$$w(x_{i_2}^j y_{i_2}^j) = 3n(i-1) + 2n + r + j + 3, \quad 1 \leq i \leq m, 1 \leq j \leq n-r-1.$$

$$w(v_i x_{i_2}^j) = \begin{cases} 3n(i-1) + n + r + 2j + 2, & \text{even } n, 1 \leq i \leq m, 1 \leq j \leq n-r-1; \\ 3n(i-1) + n + r + 2j + 3, & \text{odd } n, 1 \leq i \leq m, 1 \leq j \leq n-r-1; \end{cases}$$

$$w(v_i y_{i_2}^j) = \begin{cases} 3n(i-1) + n + r + 2j + 3, & \text{even } n, 1 \leq i \leq m, 1 \leq j \leq n-r-1; \\ 3n(i-1) + n + r + 2j + 4, & \text{odd } n, 1 \leq i \leq m, 1 \leq j \leq n-r-1. \end{cases}$$

Investigating all these edge-weights, we obtain that there is no two vertices of the same weight. Specifically, this total labeling is very optimal such that all the edge-weights for a consecutive sequence of difference 1 from 3 to  $|E(mF_n)| + 2$ .

For the vertex-weights we have:

$$w(v_i) = \begin{cases} (2n^2 + n)(i-1) + 2n^2 + 4r^2 - 4nr - n + 5r + 5, & \text{even } n, 1 \leq i \leq m; \\ (2n^2 + n)(i-1) + 2n^2 + 4r^2 - 4nr + n + 3r + 3, & \text{odd } n, 1 \leq i \leq m; \end{cases}$$

$$w(x_{i_1}^j) = 3n(i-1) + 3j, \quad 1 \leq i \leq m, 1 \leq j \leq r;$$

$$w(y_{i_1}^j) = 3n(i-1) + 3j + 1, \quad 1 \leq i \leq m, 1 \leq j \leq r;$$

$$w(x_i^{r+1}) = 3n(i-1) + 3r + 4, \quad 1 \leq i \leq m;$$

$$w(y_i^{r+1}) = 3n(i-1) + 3r + 5, \quad 1 \leq i \leq m;$$

$$w(x_{i_2}^j) = \begin{cases} 3ni + r + 3j + 2, & \text{even } n, 1 \leq i \leq m, 1 \leq j \leq n-r-1; \\ 3ni + r + 3j + 3, & \text{odd } n, 1 \leq i \leq m, 1 \leq j \leq n-r-1; \end{cases}$$

$$w(y_{i_2}^j) = \begin{cases} 3ni + r + 3j + 3, & \text{even } n, 1 \leq i \leq m, 1 \leq j \leq n-r-1; \\ 3ni + r + 3j + 4, & \text{odd } n, 1 \leq i \leq m, 1 \leq j \leq n-r-1. \end{cases}$$

Investigating all these vertex-weights, we obtain the following conditions.

1.  $w(x_{i_1}^j) < w(y_{i_1}^j) < w(x_i^{r+1}) < w(y_i^{r+1}) < w(x_{i_2}^j) < w(y_{i_2}^j)$ ;
2.  $w(v_i) = w(x_{k_1}^j)$  or  $w(v_i) = w(y_{k_1}^j)$  or  $\dots$  or  $w(v_i) = w(y_{k_2}^j)$ , for some  $i < k \leq m$ .

For instance, Condition 2 above is occurred on graph  $12F_6$ . Under the total labeling  $\lambda$  on graph  $12F_6$ , we found that  $w(v_1) = w(x_{3_2}^2) = 49$ ,  $w(v_2) = w(x_{3_2}^3)$ , and  $w(v_3) = w(y_{12_1}^2) = 205$ . It means that we need to eliminate Condition 2 to have the appropriate total labeling  $\lambda$ .

Since the  $|V(mF_n)| < |E(mF_n)|$  while  $\lambda$  is very optimal for labeling all the edges, there are gaps in the sequence of vertex-weights  $w(x_{i_1}^j) < w(y_{i_1}^j) < w(x_i^{r+1}) < w(y_{i_1}^{r+1}) < w(x_{i_2}^j) < w(y_{i_2}^j)$ . It means that the weight of  $v_i$  can be changed to eliminate Condition 2. Consider the  $i$ -th friendship graph  $F_n$  of Condition 2 (where its center-weight  $w(v_i)$  equals to one of  $w(x_{i_1}^j), w(y_{i_1}^j), w(x_i^{r+1}), w(y_{i_1}^{r+1}), w(x_{i_2}^j), w(y_{i_2}^j)$ ). Since  $\lambda(x_{i_1}^j) = \lambda(y_{i_1}^j), \lambda(x_i^{r+1}) = \lambda(y_{i_1}^{r+1})$ , and  $\lambda(x_{i_2}^j) = \lambda(y_{i_2}^j)$ , then  $w(v_i)$  can be modified to have a distinct weight that fill the gap without changing the edge-weight sequence as follow.

Let  $a \neq 0$  be the minimum integer for which  $w(v_i) + a$  can fill the gap.

1. For even  $a > 0$ , choose some triangles  $v_i x_i^j y_i^j$ , for some  $j$  and define  $\lambda^*(x_i^j) = \lambda^*(y_i^j) = \lambda(x_i^j) - 1$ ,  $\lambda^*(v_i x_i^j) = \lambda(v_i x_i^j) + 1$ ,  $\lambda^*(v_i y_i^j) = \lambda(v_i y_i^j) + 1$ , and  $\lambda^*(x_i^j y_i^j) = \lambda(x_i^j y_i^j) + 2$ .
2. For even  $a < 0$ , choose some triangles  $v_i x_i^j y_i^j$ , for some  $j$  and define  $\lambda^*(x_i^j) = \lambda^*(y_i^j) = \lambda(x_i^j) + 1$ ,  $\lambda^*(v_i x_i^j) = \lambda(v_i x_i^j) - 1$ ,  $\lambda^*(v_i y_i^j) = \lambda(v_i y_i^j) + 1$ , and  $\lambda^*(x_i^j y_i^j) = \lambda(x_i^j y_i^j) - 2$ .
3. For odd  $a > 0$ , choose triangles  $v_i x_i^j y_i^j$  triangles, for some  $j$ , of the last  $n - r - 1$  triangles, and define  $\lambda^*(v_i x_i^j) = \lambda(x_i^j y_i^j) + \lambda(y_i^j) - \lambda(v_i)$  and  $\lambda^*(x_i^j y_i^j) = \lambda(v_i x_i^j) + \lambda(v_i) - \lambda(y_i^j)$ .
4. For odd  $a < 0$ , choose triangles  $v_i x_i^j y_i^j$  triangles, for some  $j$ , of the first  $r + 1$  triangles, and define  $\lambda^*(v_i x_i^j) = \lambda(x_i^j y_i^j) + \lambda(y_i^j) - \lambda(v_i)$  and  $\lambda^*(x_i^j y_i^j) = \lambda(v_i x_i^j) + \lambda(v_i) - \lambda(y_i^j)$ .
5. Set  $\lambda^*(v) = \lambda(v)$  and  $\lambda^*(e) = \lambda(e)$  for each of the rest of vertices and edges.

After applying the above modification on the total labeling  $\lambda$  to have  $\lambda^*$ , we can obtain that there is no two vertex of the same weight.

Thus, the total labeling  $\lambda^*$  above is a totally irregular total  $(mn + 1)$ -labeling and the exact value of the total irregularity strength of  $m$  copies of the friendship graph is  $mn + 1$ . ■

### 3. Conclusion

By Theorem 1, we have showed that  $m$  copies of the friendship graph  $mF_n$  is a totally irregular total graph and the total irregularity strength of  $mF_n$  is equal to its total edge irregularity strength.

### References

- [1] M. Baca, S. Jendrol, M. Miller, and J. Ryan, (2007), On irregular total labeling, *Discrete Math.*, 307, 1378–1388.
- [2] Chartrand, G, Jacobson, M.S., Lehel, J., Oellermann, O.R., Ruiz, S., Saba, F. (1988), Irregular networks. *Congr. Numer.* 64, 187-192.
- [3] Nurdin, Baskoro, E.T., Salman, A.N.M., and Gaos, N.N. (2010), On the Total Vertex Irregularity Strength of Trees, *Discrete Mathematics*, 310, 3043-3048.
- [4] Rinovia Simanjuntak, Susilawati Susilawati, Edy Tri Baskoro, (2020), Total vertex irregularity strength for trees with many vertices of degree two, *Electron. J. Graph Theory Appl.* 8 (2), 415–421.
- [5] Susilawati, Edy Tri Baskoro, Rinovia Simanjuntak, (2018), Total vertex irregularity strength of trees with maximum degree five, *Electron. J. Graph Theory Appl.* 6 (2), 250–257.
- [6] A. Ahmad, M. Baca, M. Numan, (2013), On Irregularity Strength of Disjoint Union of Friendship Graphs,

Electron. J. Graph Theory Appl. 1 (2), 100–108.

- [7] J. Ivančco and S. Jendrořl, (2006), Total edge irregularity strength of trees, *Discuss. Math. Graph Theory*, 26, 449–456.
- [8] C. C. Marzuki, A. N. M. Salman, and M. Miller, (2013), On the total irregularity strength of cycles and paths, *Far East J. Math. Sci. (FJMS)*, 82 (1), 1–21.
- [9] R. Ramdani and A.N.M. Salman, (2013), On the total irregularity strength of some cartesian product graphs, *AKCE Int. J. Graphs Comb.*, 10 (2), 199–209.
- [10] R. Ramdani, A.N.M. Salman, H. Assiyatun, A. Semanicova-Fenovcikova, M. Baca, (2015), Total irregularity strength of three family of graphs, *Math. Comput. Sci.*, 9, 229–237.
- [11] M. I. Tilukay, A. N.M. Salman, and E. R. Persulesy, (2015), On the total irregularity strength of fan, wheel, triangular book, and friendship graphs, *Procedia Computer Science*, 74, 124–131.
- [12] M. I. Tilukay, B.P. Tomasouw, F. Y. Rumlawang, A. N. M. Salman, (2017), The total irregularity strength of complete graphs and complete bipartite graphs, *Far East J. Math. Sci* 102 (2), 317-327
- [13] M. I. Tilukay, Pranaya D. M. Taihuttu, A. N. M. Salman, F. Y. Rumlawang, Zeth A. Leleury (2021). Complete bipartite graph is a totally irregular total graph, *Electronic Journal of Graph Theory and Applications* 9(2), 387-396
- [14] Galian, J A. (2019) A dynamic survey of graph labeling. *Electronic Journal of Combinatorics*. 22, 1-535 Retrieved Feb. 8<sup>th</sup>, 2020, from <https://www.combinatorics.org/files/Surveys/ds6/ds6v22-2019.pdf>