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COMMUNICATIONS ON NUMBER THEORY AND COMBINATORIAL THEORY

On Two-Player Pebbling

Garth Isaak[1], Matthew Prudente[2], Andrea Potylycki[1], William Fagley[2], and Joseph Marcinik[3]

[1]Department of Mathematics, Lehigh University
[2]Department of Science & Mathematics, Alvernia University
[3]Department of Physics, University of California Los Angeles

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Abstract

Graph pebbling can be extended to a two-player game on a graph G, called *Two-Player Graph Pebbling*, with players Mover and Defender. The players each use pebbling moves, the act of removing two pebbles from one vertex and placing one of the pebbles on an adjacent vertex, to win. Mover wins if they can place a pebble on a specified vertex. Defender wins if the specified vertex is pebble-free and there are no more pebbling moves on the vertices of G. The *Two-Player Pebbling Number* of a graph G, $\eta(G)$, is the minimum m such that for every arrangement of m pebbles and for any specified vertex, Mover can win. We specify the winning player for paths, cycles, and the join of certain graphs.

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1 Introduction

From this point, G will denote a simple connected graph on n vertices and P_n will denoted a path on n vertices labeled $v_1v_2\dots v_n$. The *complement* $\overline{G}$ is the graph such that $V\left(\overline{G}\right) = V(G)$ and $uv \in E\left(\overline{G}\right) \iff uv \notin E(G)$. For any graphs, G and H, the *join*, $J = G \vee H$, is the graph such that $V(J) = V(G) \cup V(H)$ and $E(J) = \{uv \mid uv \in E(G),\, uv \in E(H), \text{ or } u \in V(G) \text{ and } v \in V(H)\}$. The *diameter* of a graph G, $diam(G) = \max_{u,v}\{dist(u,v)\}$, is the maximum distance over every pair of vertices in G.

We now define two *graph-pebbling* specific terms.

Definition 1.1. Given a graph G, a *configuration* C is an arrangement of pebbles to the vertices of G with $C(v)$ as the number of pebbles on vertex v.

Definition 1.2. A *pebbling move* consists of removing two pebbles from one vertex and placing one pebble on an adjacent vertex.

So, for any pebbling move, we obtain a new configuration C' such that

- $|C'| = |C| - 1$,
- there exists an edge, uv, where $C'(u) = C(u) - 2$ and $C'(v) = C(v) + 1$, and
- $C'(x) = C(x),\ \forall\, x \neq u, v$.

We use pebbling moves to place one of the pebbles on a specified goal vertex r, called the *root*. Given a configuration C on a graph G and a root $r \in V(G)$, we say C is *r-solvable* provided one can use a sequence of pebbling moves to place a pebble on the root. For a vertex r, if there is an $m \in \mathbb{N}$ so that for any configuration C of m pebbles, C is r-solvable, then we set $\pi(G,r)$ to be the smallest m. The *pebbling number* of a graph, $\pi(G)$, is the maximum $\pi(G,r)$ over all $r \in V(G)$.

We can extend graph pebbling to a game with two players, Mover and Defender, as defined in [2]. Each player will use pebbling moves, starting with Mover, in an attempt to win. Mover wins provided they can place one of the pebbles on the root. Defender wins if the root is pebble-free and there are no more pebbling moves on the vertices of G. There are two rules.

(1) Each player must take their turn.

(2) If Mover pebbles from vertex u to vertex v, then Defender cannot pebble from v to u on their next turn.

By *turn*, we mean an individual player's pebbling move. A *round* consists of two turns, first made by Mover and then by Defender.

A variation for which Defender could forfeit their turn was considered in [2], however this lead to severely restricting the graphs for which Mover could win. While playing the game, each player will employ a *strategy*, a choice function from the set of possible configurations to the collection of legal pebbling moves. A strategy will be *winning* on a configuration for either player provided the other player can not win, regardless of the strategy they can apply. Some starting configurations allow Mover to win on their first turn.

Fact 1.3. *If a starting configuration has two pebbles on any vertex in the neighborhood of* r*, then Mover has a winning strategy.*

So we define the following:

Definition 1.4. A *nontrivial configuration* on the vertices of G will have 0 or 1 pebbles on vertices in the neighborhood of r.

We say $\eta(G, r)$ is the minimum number $m \in \mathbb{N}$ such that given any configuration of m pebbles and a given root vertex, r, Mover has a winning strategy. From this, we say the *Two-Player Pebbling Number* is $\eta(G) = \max_{r \in V}\{\eta(G, r)\}$. However, if for a graph G, a root r, and arbitrarily large $t \in \mathbb{N}$, there exists a configuration of size at least t', for $t' > t$, for which Defender has a winning strategy, then $\eta(G, r) = \infty$. The following provides a sufficient condition for $\eta(G, r) = \infty$.

Theorem 1.5 ([2])**.** *For a graph* G*, let* $S \subset V(G)$ *be a cut set of* G *and let* $G_0, G_1, \ldots G_k$ *be the components of* $G - S$ *with* $r \in G_0$*. If for every* $v \in S$*,* $|N(v) - (V(G_0) \cup S)| \geq 2$ *and for every* $x \in N(S) - (V(G_0) \cup S)$*,* $|N(x) - S| \geq 2$*, then* $\eta(G, r) = \infty$*.*

We say a configuration is *Mover-win* if the Mover has a winning strategy and *Defender-win* otherwise. Below are a few basic results for $\pi(G)$ and $\eta(G)$.

Fact 1.6. *For any graph* G*,* $\eta(G) \geq \pi(G)$*.*

Mover cannot win with fewer than $\pi(G)$ pebbles.

Fact 1.7. *If* $\eta(G, r) = \infty$ *for some* $r \in V(G)$*, then* $\eta(G) = \infty$*.*

If Defender has a winning strategy for at least one choice of r, then Defender has a winning strategy for G, overall.

Proposition 1.8 ([2])**.** *If* $deg(r) = |V(G)| - 1$*, then* $\eta(G, r) = |V(G)|$*.*

The next two classical pebbling results are included as comparisons for bounds for Two-Player Pebbling we consider later.

Fact 1.9 ([1])**.** *For every positive integer* n*, we have* $\pi(P_n) = 2^{n-1}$*.*

Theorem 1.10 ([4])**.** *For every integer* $k \geq 2$*, we have* $\pi(C_{2k}) = 2^k$ *and for every integer* $k \geq 1$*, we have* $\pi(C_{2k+1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1$*.*

There are many significant differences between classical pebbling and Two-Player pebbling. For classical pebbling, the pebbling number for a graph, $\pi(G)$, is nonincreasing when edges are added to G. This is not the case for Two-Player pebbling, as stated in [2].

Additionally, for classical pebbling, solvability is preserved when adding pebbles to a configuration. This is not the case for Two-Player Pebbling; a configuration C for which Mover has a winning strategy does not necessarily imply that Mover has a winning strategy on the configuration C' where is the same configuration as C with one additional pebble added. A simple example (suggested by a referee) is the graph $K_{1,3}$, a central vertex with edges to 3 adjacent leaves. If one leaf is the root the configuration with 4 pebbles on another leaf is Mover-win and the configuration with 4 pebbles on another leaf and 1 pebble on the central vertex is Defender win as can easily be checked.

We briefly discuss a few more examples to illustrate the complexities of Two-Player pebbling. Consider paths $P_n = v_1, v_2, \ldots, v_n$ with the root v_1 and write configurations as $(C(v_1), C(v_2), \ldots, C(v_n))$. Given a path on 5 vertices, P_5, one can check that $(0, 0, 3, 2, 1)$ is Mover-win and $(0, 0, 3, 2, 2)$ is Defender win, providing an example that, even on paths, adding pebbles might not preserve wins for Mover.

A useful tool for certain configurations comes from classical pebbling [3]. Define the *weight* of a configuration on a path as $P(C) = \sum C(v_i)/2^{i-1}$. In [3], the authors define weight in the context of placing m pebbles on the root. For our purposes, we require $m = 1$. A move toward the root does not change the weight. If $P(C) \geq 1$ either some vertex has at least 2 pebbles and a move is possible or there is a pebble on the root. So the configuration is solvable if and only if $P(C) \geq 1$. So in the pebbling game Defender has a winning strategy on a path if $P(C) < 1$ by playing arbitrarily.

For the configuration $(0, 0, 0, 0, 16)$ on P_5, Mover has a winning strategy. It is not hard to check that Mover can force the configuration $(0, 0, 2, 2, 4)$ after 4 rounds. If Mover pebbles to $(0, 1, 0, 2, 4)$ then Defender can pebble 'backwards' to $(0, 1, 0, 0, 5)$ which has weight less than 1 and Defender then has a winning strategy. If instead, Mover pebbles from $(0, 0, 2, 2, 4)$ to $(0, 0, 3, 0, 4)$ Defender must pebble to either $(0, 1, 1, 0, 4)$ or $(0, 0, 3, 1, 2)$ each of which can be easily checked to have a winning strategy for Mover. Hence, Mover has a winning strategy on $(0, 0, 0, 0, 16)$ that involves ensuring that Defender can never pebble away from the root. This is necessary since the initial configuration has weight exactly 1 and any move away from the root will make it smaller. This example also illustrates that Mover pebbling toward the root from the closest vertex possible is not always an optimal strategy.

As stated in Fact 1.9, for any path $\pi(P_n) = 2^{n-1}$. Checking that $\eta(P_5, v_1) = \pi(P_5) = 16$ of course involves checking that all configurations with 16 or more pebbles admit a winning strategy for Mover. While long and involving many cases, one can check that $\eta(P_4) = 2^3 = 8 = \pi(P_4)$ and $\eta(P_5) = 2^4 = 16 = \pi(P_5)$.

For P_6 and configuration $(0, 0, 0, 0, 0, 32)$ it can be checked that Mover is no longer able to prevent Defender from pebbling away from the root at some point. Thus Defender can take the weight below 1 and the configuration is Defender-win. Hence $\eta(P_6, v_1) > 32$. Similarly, the configuration with 2^{n-1} pebbles on v_{n-1} can be used to show that $\eta(P_n, v_1) > 2^{n-1} = \pi(P_n)$.

2 Paths & Cycles

In this section, we will show that Mover has a winning strategy for paths and cycles. That is we show $\eta(G)$ is finite for paths and cycles and provide bounds on these values.

2.1 General Upper Bound for Paths

The following general lemma will allow us to assume that path P_n is labeled as $v_1, v_2, \ldots, v_n$ with v_1 the root.

Lemma 2.1. *If r is a cut vertex of G and $G_1, G_2, \ldots G_k$ are the graphs induced by the components $G - r$ and r, then $\eta(G, r) = 1 + \sum_{i=1}^{k}(\eta(G_i, r) - 1)$.*

Proof. Let r be a cut vertex of G and $G_1, G_2, \ldots G_k$ be the graphs induced by the components of $G - r$ with r. Let C be a configuration with $\sum_{i=1}^{k}(\eta(G_i, r) - 1)$ pebbles arranged so that G_i receives $\eta(G_i, r) - 1$ with a configuration that is winning for Defender. Defender playing a winning strategy on each G_i is a winning strategy for Defender on G; Mover cannot reach the root from any G_i.

Now, suppose C' is a configuration with $\sum_{i=1}^{k}(\eta(G_i, r)-1)+1$ pebbles. By the Pigeonhole Principle, at least one component G_k will have at least $\eta(G_k, r)$ pebbles distributed on it. Thus Mover has a winning strategy on G by always playing in G_k and using a winning strategy for G_k. □

It is tempting to conclude that placing $2\eta(P_{n-1})$ pebbles on P_n will be enough to allow Mover to win. The *first* $\eta(P_{n-1})$ pebbles used to place a pebble on v_2 and then as $\eta(P_{n-1})$ pebbles remain Mover can place a second pebble on v_2. Since Defender cannot 'undo' this move, Mover wins on the next turn. This approach fails because, while Mover still wins with more than $\eta(P_{n-1})$ pebbles on P_{n-1}, more moves, and hence more pebbles, may be necessary as the presence of additional pebbles allows Defender to avoid making moves that 'help' Mover. In particular, we stated above that $\eta(P_6) > 32 = \pi(P_6)$, but $\pi(P_5) = 2^{5-1} = 16$. This will be further illustrated in a future paper. For now, we have an upper bound for $\eta(P_n, v_1)$.

Lemma 2.2. *For $n \geq 4$, we have $\eta(P_n, v_1) \leq \frac{3}{2} \cdot 2^{n-1} - n$ and for any configuration with at least $\eta(P_n, v_1)$ pebbles Mover has a winning strategy that uses at most $\frac{3}{2} \cdot 2^{n-2} - 1$ rounds.*

Proof. We will use induction on n. The basis, $n = 4$ involves checking multiple cases but is straightforward.

For a nontrivial configuration on P_n with a least $\frac{3}{2}\cdot 2^{n-1}-n$, consider the subpath P_{n-1} on $v_2,\ldots,v_n$. By induction Mover can place a pebble on v_2 using at most $\frac{3}{2}\cdot 2^{n-3}-1$ rounds (trivially if $C(v_2)=1$). This uses at most $2\cdot(\frac{3}{2}\cdot 2^{n-3}-1)$ pebbling moves, and counting 1 for the pebble on v_1 at least $(\frac{3}{2}\cdot 2^{n-1}-n)-(\frac{3}{2}\cdot 2^{n-2}-2)-1=\frac{3}{2}\cdot 2^{n-2}-(n-1)$ pebbles remain. Again by induction Mover has a strategy where a second pebble is placed on v_2 in at most $\frac{3}{2}\cdot 2^{n-3}-1$ rounds . If Mover places the second pebble on v_2, then Defender cannot undo this move and Mover can pebble from v_2 to the root v_1 in the next round. If Defender places the second pebble on v_2, then again Mover can pebble from v_2 to the root v_1 in the next round. Thus Mover wins and counting the last move to the root there are at most $2\cdot(\frac{3}{2}\cdot 2^{n-3}-1)+1=\frac{3}{2}\cdot 2^{n-2}-1$ rounds. □

Now, we can bound the Two Player Pebbling Number for paths.

Theorem 2.3. *For $n\geq 4$, we have $2^{n-1}\leq \eta(P_n)\leq \frac{3}{2}\cdot 2^{n-1}-n$.*

Proof. The lower bound follows by Facts 1.9 and 1.6. For the upper bound, by Lemma 2.1 it suffices to assume that the root is v_1. Then use Lemma 2.2. □

While we did not include the long details, the observations at the end of Section 1 show that the lower bound is tight for $n=4,5$ and strict for $n\geq 6$.

The bound on the number of rounds in Lemma 2.2 can be checked to be tight. That is, for any configuration with at least $4\left(\dfrac{3}{2}\cdot 2^{n-1}-1\right)$ pebbles on v_n, Defender can force at least $\frac{3}{2}\cdot 2^{n-2}-1$ rounds. There are enough pebbles that, after fewer than $\frac{3}{2}\cdot 2^{n-2}-1$ rounds at least 2 pebbles remain in v_n. Thus in each round, Defender could move from v_n to v_{n-1}. The only case where this is prevented is if Mover pebbled 'backwards' from v_n to v_{n-1}. Thus in each round Defender can force least one pebbling move between these two vertices. While Mover pebbling from v_{n-1} to v_n or Defender pebbling from v_n to v_{n-1} when, for example pebbling from v_{n-1} to v_n is an option, might not be part of optimal strategies, they are allowable moves to be considered. Assume Defender plays so that in each round at least one move is between v_n and v_{n-1}. At least 2^{n-2} pebbling moves and hence at least $\frac{1}{2}\cdot 2^{n-2}$ rounds are required to place 2^{n-2} pebbles on v_{n-1}. Then at least $2^{n-1}-1=2^{n-2}-1$ moves on $v_1,v_2,\ldots,v_{n-1}$ are required from the classical pebbling bound and for each of these moves, by Defender's strategy, the other move in the round is between v_n and v_{n-1}. Hence at least $2^{n-2}-1$ additional rounds are required for a total of $\frac{3}{2}\cdot 2^{n-2}-1$ rounds. It is likely that maximum number of rounds could be forced with configurations with fewer than $4\left(\dfrac{3}{2}\cdot 2^{n-1}-1\right)$ pebbles on v_n. However, this number is needed for the argument that is given.

Note that the pebbling bound in Lemma 2.2 is smaller than the number of pebbles, $3\cdot 2^{n-1}-2$ needed to play $\frac{3}{2}\cdot 2^{n-1}-1$ rounds. The slightly better pebbling bound is possible, since with a smaller number of pebbles Defender can be forced to make moves that *help* Mover. As an example, the configuration with 2 pebbles on v_n and 1 pebble on every other non-root vertex is Mover-win. The example showing that many rounds can be forced suggests that $\eta(P_n)$ might be closer to the upper bound. However, as the argument requires a large number of pebbles on v_n, it does carry over to a lower bound on η.

2.2 Bounds for Cycles

Now, we move on to cycles. We will consider even cycles and odd cycles in different cases. For even cycles, we can label the vertices of C_{2n} as r, a_1, a_2, a_3, $\ldots$, a_{n-1}, x, b_{n-1}, $\ldots$, b_3, b_2, b_1. Thus, we can consider the vertices of the cycles as two internally disjoint paths with the same starting and ending vertices, x and r, respectively. As Mover can choose to pebble on only one of these paths as a strategy, we can consider a variation of Two-Player Pebbling for paths.

We define a new pebbling game on paths $v_1, v_2, \ldots, v_n$. The rules for Mover are the same as the regular Two-Player Pebbling game. In this variation, Defender can pass, remove 2 pebbles from vertex v_n or add a pebble to vertex v_n, in addition to making pebbling moves.

Observe that this will capture the moves on a subpath from the root of a cycle. In particular, each move that Defender makes on the cycle corresponds to a move in the new game. It may be that no moves off the subpath are available in which case the pass option on the subpath is not available. Mover could also make moves on the cycle corresponding to the extra options in the new game. However, a winning strategy for Mover in the new game translates to a winning strategy on the cycle since the options for Mover are a superset of the those in the new game and the options for Defender are a subset. For the same reasons, any lower bounds corresponding to strategies for Defender in the new game do not translate to a strategy for Defender on the cycle.

Lemma 2.4. *For the new pebbling game on paths P_n, $n \geq 4$ and any configuration with at least $2^n + 2^{n-1} - 2n$ pebbles, Mover has a winning strategy that uses at most $2^{n-1} - 1$ rounds.*

Proof. The proof is similar to the proof of Lemma 2.2. We will use induction on n. The basis, $n = 4$ involves checking multiple cases but is straightforward.

For a nontrivial configuration on P_n with at least $2^n + 2^{n-1} - 2n$ pebbles, consider the subpath P_{t-1} on $v_2, \ldots, v_t$. By induction, Mover has a strategy that will place a pebble on v_2 using at most $2^{n-2} - 1$ rounds, trivially if $C(v_2) = 1$. This uses at most $3 \cdot (2^{n-2} - 1)$ pebbles, as on each round Mover uses 2 pebbles to place a pebble on another vertex and Defender could possibly remove 2 pebbles from v_n. If Defender places a new pebble on v_n, passes, or uses 2 pebbles to place a pebble on another vertex then fewer pebbles are used. Counting 1 for the pebble on v_1, at least $(2^n + 2^{n-1} - 2n) - 3(2^{n-2} - 1) - 1 = 2^{n-1} + 2^{n-2} - 2(n-1)$ pebbles remain. By induction Mover has a strategy where a second pebble is placed on v_2 in at most $2^{n-2} - 1$ rounds. If Mover places the second pebble on v_2, then Defender cannot undo this move and Mover can pebble from v_2 to the root v_1 in the next round. If Defender places the second pebble on v_2, then Mover can pebble from v_2 to the root v_1 in the next round. Thus Mover wins and counting the last move to the root there are at most $2 \cdot (2^{n-2} - 1) + 1 = 2^{n-1} - 1$ rounds. □

Now, we consider an upper bound for even cycles C_{2n}.

Theorem 2.5. *If $n \geq 2$, then we have $\eta(C_{2n}) \leq 2^{n+2} + 2^{n+1} - 4n - 3$.*

Proof. Let C_{2n} be an even cycle with $2n$ vertices labeled r, a_1, a_2, a_3, $\dots$, a_{n-1}, x, b_{n-1}, $\dots$, b_3, b_2, b_1. Let d be the number of pebbles on $r, a_1, \dots, a_{n-1}$ and e the number on $r, b_1, \dots, b_{n-1}$ and f the number of pebbles on x. If $d \geq 2^n + 2^{n-1} - 2n$ then Mover can play a winning strategy for the new game described in Lemma 2.4 on this path as a winning strategy on the cycle as noted in the paragraph before that lemma. Thus, if Defender has a winning strategy $d < 2^n + 2^{n-1} - 2n$. That is, $d \leq 2^n + 2^{n-1} - 2n - 1$.

Similarly, Mover has a winning strategy unless $e \leq 2^n + 2^{n-1} - 2n - 1$ and $d + f \leq 2^{n+1} + 2^n - 2(n+1) - 1$ and $e + f \leq 2^{n+1} + 2^n - 2(n+1) - 1$. Combining these implies $2(d + e + f) \leq 2(2^n + 2^{n-1} - 2n - 1 + 2^{n+1} + 2^n - 2(n+1) - 1)$. So the number of pebbles $d + e + f \leq 2^{n+2} + 2^{n-1} - 4n - 4$. Hence, Mover has a winning strategy if number of pebbles is at least $2^{n+2} + 2^{n-1} - 4n - 3$. □

Now, we can move on to odd cycles. We can label the vertices of odd cycles, C_{2n+1} as r, a_1, a_2, $\dots$, a_{n-1}, a_n, b_n, b_{n-1}, $\dots$, b_2, b_1. Odd cycles can be thought of as two internally disjoint points with different starting vertices, a_n and b_n, but the same ending vertex, r. The following is an upper bound for odd cycles:

Theorem 2.6. *If $n \geq 2$, then $\eta(C_{2n+1}) \leq 2^{n+2} + 2^{n+1} - 4n - 5$.*

Proof. Let C_{2n+1} be an odd cycle with $2n+1$ vertices. By Lemma 2.4, Mover can win unless the number of pebbles on $r, a_1, \dots, a_n$ and the number on $r, b_1, \dots, b_n$ are both less than $2^{n+1} + 2^n - 2(n+1)$. Hence, since the case with a pebble on r is trivial the total number of pebbles is at most twice this, $2^{n+2} + 2^{n+1} - 4(n+1) - 2$. So Mover has a winning strategy if the number of pebbles is at least $2^{n+2} + 2^{n+1} - 4n - 5$. □

With these results, we have a nice bound for the Two-Player Pebbling Number for cycles,

$$\eta(C_{2n}), \eta(C_{2n+1}) \in \mathcal{O}(2^{n+2})$$

Since these are upper bounds, it is not unexpected that their bounds would match. However, we do conjecture that the exact value of $\eta(C_{2n})$ will differ from the exact value of $\eta(C_{2n+1})$, as do their classical pebbling numbers as noted in Theorem 1.10.

3 Joins of Certain Graphs

Previous results in [2] examined Two-Player Pebbling for $\overline{K}_m \vee H$ with the root in $\overline{K}_m$.

Theorem 3.1 ([2])**.** *Let $m \geq 3$ and H with $|V(H)| = n$ is any graph. Then if $r \in \overline{K}_m$, we have*

$$\eta(\overline{K}_m \vee H, r) = \begin{cases} m + 2n + 3, & n \textit{ is even} \\ m + 2n + 2, & n \textit{ is odd.} \end{cases}$$

Proposition 1.8 covers the case $m = 1$. The method used in Theorem 3.1 can be used in the case $m = 2$, producing a bound that is stronger than just using $m = 2$ in the bounds of Theorem 3.1. We will do this in Theorem 3.6.

We will also examine $\overline{K}_m \vee H$ with the root in H. In this case, if $m \geq 2$ and H has a component that does not contain the root and has minimum degree at least 2, then $\eta(\overline{K}_m \vee H, r) = \infty$ by Theorem 1.5. When $\eta(H)$ is finite we can get a bound on $\eta(\overline{K}_m \vee H, r)$ that is closely related to $\eta(H, r)$ and will do so below.

To motivate these results, we prove the following.

Theorem 3.2. *Let G be a graph for which $\eta(G) = m$ and H be any graph with $|V(H)| = n$. If $r \in G$, then $\eta(G \vee H, r) \leq m + 3n$.*

Proof. Suppose C is a configuration with at least $m + 3n$ pebbles on the vertices of $G \vee H$. If initially any vertex of H has at least two pebbles then Mover will pebble to r and win. The strategy for Mover is to first place a single pebble on each vertex of H. This takes at most n rounds, the moves to H removing 2 pebbles from G and moves within G reducing the number of pebbles by 1. Thus during this phase the number of pebbles on G remains at least $m = \eta(G)$ and hence there are vertices with 2 pebbles to pebble to H. If at any point, Defender places a second pebble on a vertex of H, then Mover will pebble from that vertex to r and win.

Once a pebble is placed on each vertex of H, on Mover's next turn, at least m pebbles remain on G. Then Mover plays a winning strategy on G. Again if at any point, Defender places a second pebble on a vertex of H, then Mover will pebble from that vertex to r and win. Otherwise Mover wins from playing the strategy on G. □

A corollary to Theorem 3.2 is the case of $\eta(\overline{K}_m \vee G)$, where $m = 1$:

Corollary 3.3. *If $\eta(G)$ is finite, then $\eta(K_1 \vee G) \leq \eta(G) + 3$.*

This follows by applying Theorem 3.2 when $r \in G$ and Proposition 1.8 when $r = K_1$.

Recall that a wheel graph is is a join of a cycle to one additional vertex. A W_6, or a wheel with five vertices on the outer cycle, is shown in Figure 1. We then get the following:

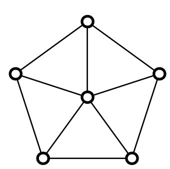

Figure 1: A Wheel Graph W_6 on six vertices

Corollary 3.4. *For any positive integer n, we have $\eta(W_n) \leq \eta(C_{n-1}) + 3$*

It may be tempting to consider Mover just playing a winning strategy on G and allowing Defender up to n moves to vertices of H, which would indicate a bound of $m+2n$. However this fails since this would be like allowing Defender up to n chances to forfeit their turn, changing the Two-Player pebbling number. This can be checked, for example with $P_4 \vee K_1$ which attains the upper bound.

An immediate Corollary is the following:

Corollary 3.5. *Let G and H be graphs for which Mover has winning strategies such that $|V(G)| = n_1$, $|V(H)| = n_2$, $\eta(G) = m_1$ and $\eta(H) = m_2$. Then $\eta(G \vee H) \leq max\{m_1 + 3n_2, m_2 + 3n_1]$.*

Another class of graphs that we can find an upper bound for the Two-Player Pebbling Number of is fan graphs. Recall a fan graph, $F_{m,n} = \overline{K_m} \vee P_n$, is the join of a independent set and a path. The graph in Figure 2 is an example of a fan graph.

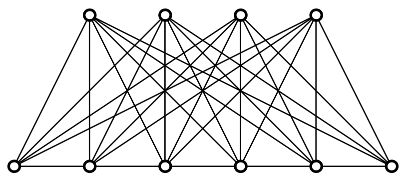

Figure 2: A Fan Graph $F_{4,6}$

In determining $\eta(F_{m,n})$, Theorem 3.2 covers the case that the root is on P_n. For the root in $\overline{K}_m$, Proposition 1.8 covers the case $m = 1$ and Theorem 3.1 covers the cases $m \geq 3$. To complete showing $\eta(F_{m,n}) < \infty$ we need a bound for $\eta(\overline{K}_2 \vee H, r)$ with $r \in \overline{K}_2$. We use the same ideas as in the proof of Theorem 3.1 given in [2].

Theorem 3.6. *Suppose H is any graph with $|V(H)| = n$ and minimum degree δ. If $r \in \overline{K}_2$ then $\eta(\overline{K}_2 \vee H, r) \leq n + 2\delta + 4$*

Proof. Call x the non root vertex in $\overline{K}_2$. Let v be a vertex with minimum degree δ in H, N be the closed neighborhood of v in H and W be the remaining vertices in H. Note that $|N| = \delta + 1$ and $|W| = n - \delta - 1$.

Suppose that C is a configuration of $n + 2\delta + 4$ pebbles. We may assume that C is nontrivial. Then, there is at most 1 pebble is on each vertex of H as they are adjacent to the root. So, we can assume that the only moves are from x.

Suppose that there are k pebble-free vertices in W and l pebble-free vertices in N. Thus there are $n + 2\delta + 4 - (n - k - l) = 2\delta + k + l + 4$ pebbles on x. If at any point Defender places a second pebble on a vertex of H, then Mover can win.

If $l \leq k$, the strategy for Mover will be to place a pebble on each pebble-free vertex of N and then pebble to v. If Defender does not pebble from v, Mover then wins. However, Defender can only pebble to another vertex with a pebble on it, as Defender cannot pebble back to x and hence Mover can still win. Mover can fill N within l rounds and then if at least 2 pebbles remain on x Mover can win as described. This requires $4l+2$ pebbles on x. This is possible because $2\delta+k+l+4 \geq 2(l-1)+l+l+4 = 4l+2$.

On the other hand, if $k<l$, then within $k+l$ pebbling moves all vertices of H will have a pebble. The next move, by either player places a second pebble on a vertex of H. If Mover pebbles next they win. If Defender pebbles next it must be to another vertex with a pebble on it, since Defender cannot pebble back to x in this case. This requires $2k+2l+2$ pebbles on x. This is possible because $2\delta+k+l+4 > 2(l-1)+k+k+4 \geq 2k+2l+2$. □

With this, we state the upper bound for fan graphs.

Corollary 3.7. *For $m \geq 1$ and $n \geq 2$, we have $\eta(F_{m,n}) \leq \eta(P_n)+3m$.*

Proof. When $m=1$, Theorem 3.3 applies.

For $m \geq 2$ and $r \in P_n$ Theorem 3.2 applies and $\eta(F_{m,n}, r) \leq \eta(P_n)+3m$.

For $m=2$ and $r \in \overline{K}_m$, $\eta(F_{2,n}, r) \leq n+2\cdot 1+4 = n+6 \leq m+2n+3$ by Theorem 3.6.

For $m \geq 3$ and $r \in \overline{K}_m$, $\eta(F_{m,n}, r) \leq m+2n+3$ by Theorem 3.1.

To complete the proof we note that $m+2n+3 \leq 2^{n-1}+3m \leq \eta(P_n)+3m$ when $m \geq 2$ and $n \geq 2$. □

4 Conclusion

Two-Player Graph Pebbling has many interesting questions. Determining finiteness or exact values of η for certain highly structured graphs can be challenging, as seen with paths and cycles in this paper. While grids, $P_n \square P_m$, have infinite η when $m,n \geq 4$ by Theorem 1.5, the cases when $n=2$ or 3 remain challenging. We have also considered *powers* of a certain graphs.

Definition 4.1. The k^{th} *power* of a graph, G^k is the graph with vertex set $V(G^k)=V(G)$ and edge set $E(G^k)=\{uv \mid d_G(u,v) \leq k\}$.

We are investigating powers of paths P_n^k. We can show for path powers P_n^k, if $2 \leq k \leq n-5$, then $\eta(P_n^k)=\infty$. However, the Two-Player Pebbling Number for P_n^{n-2} (the complete graph minus an edge), P_n^{n-3} and P_n^{n-4} are finite. For $K_n - e = P_n^{n-2} = \overline{K}_2 \vee K_{n-2}$ the bound from Theorem 3.6 is $4n-6$. However it is easy to check that $\eta(K_n-e)=2n-2$. It remains to determine values $\eta(P_n^{n-3})$ and $\eta(P_n^{n-4})$ exactly and to improve the bounds we have given for paths and cycles in this paper.

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References

[1] F.R.K. Chung, Pebbling in hypercubes, *SIAM J. Discrete Math.* **2** (1989), 467–472. http://dx.doi.org/10.1137/0402041

[2] G. Isaak, M. Prudente, Two-Player Pebbling on Diameter 2 Graphs, *International Journal of Game Theory* **50** (2021), 581–596. https://doi.org/10.1007/s00182-021-00766-0.

[3] D.P. Bunde, E.W.. Chambers, D. Cranston, K. Milans, and D.B. West, Pebbling and optimal pebbling in graphs, *Journal of Graph Theory* **57** (2008), 215–238.

[4] L. Pachter, H.S. Snevily, and B. Voxman, On pebbling graphs, *Congressus Numerantium* **107** (1995), 65–80.