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Some NP-complete edge packing and partitioning problems in planar graphs

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Abstract

Graph packing and partitioning problems have been studied in many contexts, including from the algorithmic complexity perspective. Consider the packing problem of determining whether a graph contains a spanning tree and a cycle that do not share edges. Bernáth and Király proved that this decision problem is NP-complete and asked if the same result holds when restricting to planar graphs. Similarly, they showed that the packing problem with a spanning tree and a path between two distinguished vertices is NP-complete. They also established the NP-completeness of the partitioning problem of determining whether the edge set of a graph can be partitioned into a spanning tree and a (not-necessarily spanning) tree. We prove that all three problems remain NP-complete even when restricted to planar graphs.

Keywords: Edge packing, spanning trees, NP-completeness, planar graphs, graph partitioning

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1 Introduction

A connected graph contains a spanning tree. Does it contain two spanning trees which do not share edges? In other words, can the graph stay connected after removing the edges of a spanning tree? This problem can be solved in polynomial time [5, 7]. In general, given two classes A and B of graphs, one could ask the **packing problem** of whether a graph G contains edge-disjoint subgraphs $A \in \mathsf{A}$ and $B \in \mathsf{B}$. Similarly, one could consider the **covering problem** where the union of A and B is G, or the **partitioning problem**, where A and B are edge-disjoint and whose union is G.

Bernáth and Király [2] considered seven classes, including paths, cycles, and trees. They noted that there are 44 natural[1] graph theoretic questions under this setup. Prior to their work, some of these problems were known to be in P, while some were NP-complete. They settled the status of each remaining problem in the sense of either giving a polynomial time algorithm or proving that the problem is NP-complete. Moreover, for the NP-complete problems, they noted that most of these remain NP-complete even when restricted to planar graphs.[2] However, five of these cases were left open. The goal of this paper is to settle the three remaining cases that involve spanning trees.

Theorem 1.1. *The following problems are* NP*-complete, even when restricted to planar graphs:*

(i) the packing problem with a spanning tree and a cycle;[3]

(ii) the packing problem with a spanning tree and a path between two distinguished vertices; and

(iii) the partitioning problem with a spanning tree and a tree.

The paper is organized as follows. In Section 2, we give definitions and some background. The NP-completeness of the three problems are presented in sections 3, 4, and 5, respectively. We conclude in Section 6 with some remarks and briefly discuss the remaining two problems (the partitioning problems with two trees and with a cut and a forest).

2 Definitions and background

Let G be a graph, and write $V(G)$ and $E(G)$ for its vertex and edge sets, respectively. We are concerned with undirected simple graphs, and follow standard terminology. Given a (connected) graph G, a subgraph H of G is **nonseparating** if $G - E(H)$ is connected.

Following the notation of [2], albeit with a different font, we write C for the class of all cycles, SpT the class of spanning trees,[4] T the class of (not-necessarily spanning) trees, and P_{st} the class of s–t paths (paths from s to t), where s and t are distinguished vertices. The

[1] For example, it makes no sense to pack paths as a trivial path consisting of a single vertex can be used.

[2] Most of these remain NP-complete even when restricted to planar graphs of maximum degree 3 or 4.

[3] Previously, this problem was (erroneously) claimed to be in P. See Section 6.2 for a discussion.

[4] Formally, SpT depends on (the number of vertices of) the underlying graph in question. By abuse of notation, we ignore these trivial technicalities.

packing problem with classes A and B is denoted $\mathsf{A} \wedge \mathsf{B}$, while the partitioning problem is denoted $\mathsf{A} + \mathsf{B}$.

The decision problem Planar $\mathsf{C} \wedge \mathsf{SpT}$ takes a *planar* graph G as input, and outputs whether there is a cycle $Q \in \mathsf{C}$ and a spanning tree $T \in \mathsf{SpT}$ such that Q and T are edge-disjoint subgraphs of G. As $G - E(Q)$ contains a spanning tree if and only if it is connected, we are equivalently asking to decide the existence of a *nonseparating* cycle in G.

Theorem 1.1 asserts that the following three decision problems are NP-complete.

Planar $\mathsf{C} \wedge \mathsf{SpT}$
Instance: Planar graph G.
Decide: Does G contain a nonseparating cycle?

Planar $\mathsf{P}_{st} \wedge \mathsf{SpT}$
Instance: Planar graph G and distinguished vertices $s, t \in V(G)$.
Decide: Does G contain a nonseparating s–t path?

Planar $\mathsf{T} + \mathsf{SpT}$
Instance: Planar graph G.
Decide: Can $E(G)$ be partitioned into a tree and a spanning tree?

These three problems are trivially in NP. We prove their NP-hardness by similar reductions from a planar version of boolean satisfiability, which we define below.

A **boolean variable** takes **boolean values** $+$ (true) and $-$ (false). We identify $+$ and $-$ with 1 and -1, respectively, allowing us to negate and multiply boolean values by inheriting the notations and operations from $\mathbb{Z}$. A **literal** is a variable x or its negation $-x$. A finite collection of literals is called a **clause**. A **boolean expression** φ (in conjunctive normal form) consists of a set $X = \{x_1, \dots, x_n\}$ of variables and a set $C = \{C^1, \dots, C^m\}$ of clauses. Let the **associated graph** G_φ be the graph with vertex set $X \sqcup C$ and edges

$$\{x_i C^j : x_i \text{ or } -x_i \text{ occurs in } C^j\} \cup \{x_i x_{i+1} : i \in \{1, \dots, n\}\},$$

where subscripts are hereafter read modulo n. A boolean expression φ is **planar** if its associated graph G_φ is. An assignment $f : X \to \{\pm\}$ of boolean values to the variables is **satisfying** if each clause contains a $+$ literal under such an assignment.

Planar SAT
Instance: Planar boolean expression φ.
Decide: Does φ admit a satisfying assignment?

Lichtenstein [3] proved that Planar SAT is NP-complete, even when each clause contains precisely three literals.

3 Planar $\mathsf{C} \wedge \mathsf{SpT}$ is NP-complete

We reduce Planar SAT to Planar $\mathsf{C} \wedge \mathsf{SpT}$. Given a boolean expression φ with variables $X = \{x_1, \dots, x_n\}$ and clauses $\{C^1, \dots, C^m\}$ such that the associated graph G_φ is planar, we form a new planar graph H_φ from G_φ such that H_φ contains a nonseparating cycle if and only if φ admits a satisfying assignment.

3.1 Reduction construction

Fix a proper plane drawing of G_φ. Color each edge x_iC^j by $+$ if $x_i \in C^j$ and $-$ if $-x_i \in C^j$. (Note that if a clause C^j contains both x_i and $-x_i$, then we may omit C^j. Therefore, without loss of generality, assume this does not happen.)

For each i, take a small neighborhood D_i about x_i containing only the vertex x_i and initial segments of edges leaving x_i. We locally modify the plane graph. First, subdivide edge x_ix_{i+1} to the path $x_it_is_{i+1}x_{i+1}$, where the new vertices s_i and t_i lie within D_i for each i.

From now on, we focus on a fixed i and perform local replacements. Each new vertex is added inside D_i, and is decorated with a subscript i, which may be suppressed for notational convenience. Similarly, new edges are to be drawn inside D_i, the reader is encouraged to check that each new edge can be drawn without introducing crossings.

Subdivide each edge x_iC^j with a new vertex $\beta^j = \beta_i^j$, and color the vertex with the color of x_iC^j. Delete vertex x_i (and all incident edges). Let k be a sufficiently large number, say, two times the number of the β vertices introduced. Add paths $P_i^+ = sv^0v^1 \dots v^{4k}t$ and $P_i^- = su^0u^1 \dots u^{4k}t$, where $u^j = v^j$ for $j \equiv 2 \pmod 4$. Draw them in such a way that they "cross" each other at these common points (see Figure 1), and color the edges in P^+ and P^- by $+$ and $-$, respectively. Add path $u^j\omega^jv^j$ for each $j \equiv 0 \pmod 4$, and paths $u^{j-1}\sigma^jv^{j+1}$ and $v^{j-1}\tau^ju^{j+1}$ for each $j \equiv 2 \pmod 4$.

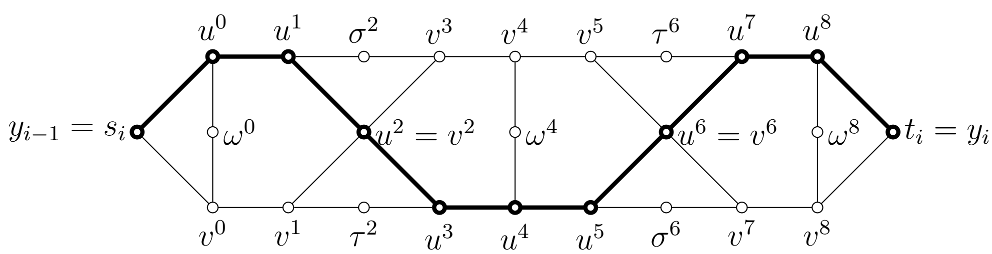

Figure 1: Replacement within D_i; $k = 2$; P_i^- darkened.

Finally, for each β^j, pick an edge of P^+ or P^- of the same color, subdivide that edge with a vertex a^j, and join $a^j\beta^j$ by an edge. The edges from the subdivisions stay in P^+ or P^- in the obvious way. As k is large, we may pick these edges distinct, such that the new edges $a^j\beta^j$ can be properly drawn within D_i without introducing crossings.

After doing such local replacements for all i, contract t_is_{i+1} to a new vertex y_i for each i. We consider P_i^+ and P_i^- to be y_{i-1}–y_i paths. This concludes the construction of H_φ.

3.2 Correctness of reduction

First, note that the construction produces a *planar* graph $H := H_\varphi$ in polynomial time.

Suppose $f : X \to \{+,-\}$ is a satisfying assignment for φ. For $r \in \{+,-\}$, let $Q^r = \bigcup_i P_i^{rf(x_i)}$. (Recall that the multiplication of $\{+,-\}$ is afforded by the identification with $\pm 1 \in \mathbb{Z}$.) Note that Q^+ and Q^- are edge-disjoint cycles. We claim that $H - E(Q^-)$ is connected, which means there is a spanning tree of H that is edge-disjoint from Q^-, as desired.

Indeed, $H - E(Q^-)$ contains the cycle Q^+, and the σ, τ, and ω vertices are clearly connected to Q^+. Moreover, each vertex in $V(Q^-)\setminus V(Q^+)$ is connected to Q^+ through a σ, τ, or ω vertex. As a_i^j and β_i^j are connected to C^j for each j, it remains to show that each C^j is connected to Q^+. By definition, Q^+ contains $P_i^{f(x_i)}$. As f is satisfying, C^j contains a literal x_i such that $f(x_i) = +$, or a literal $-x_i$ such that $f(x_i) = -$. In either case, β_i^j is colored with $f(x_i)$, and hence a_i^j lies on $P_i^{f(x_i)}$. That is, $C^j\beta_i^j a_i^j$ is a path from C^j to Q^+, as desired.

Conversely, suppose H contains a cycle Q such that $H - E(Q)$ is connected. Note that Q cannot contain a vertex of degree 2 (in H), lest the vertex be isolated in $H - E(Q)$. Therefore, Q is contained in the subgraph H' of H where the vertices of degree 2, namely, the β, σ, τ, and ω vertices, are deleted. Moreover, the (now) isolated C vertices may also be deleted from H'. Let Y consists of the y vertices and the u^j vertices for $j \equiv 2 \pmod 4$. There are two **large** faces that contain Y on its boundary. Each of the remaining **small** faces is bounded by precisely two vertices in Y and two paths between them.

The ω^j vertices prevent Q from containing (both paths of) a small face, lest $u^j\omega^j v^j$ form a connected component of $H - E(Q)$. Therefore, Q contains precisely one of the two paths for each small face. Similarly, the σ and τ vertices force Q to "oscillate" and contain either P_i^+ or P_i^- between y_{i-1} and y_i for each i. As such, $Q = \bigcup_i P_i^{-f(x_i)}$ for some $f : X \to \{\pm\}$.

It remains to show that this f is a satisfying assignment. Suppose not, and there is some clause C^j such that every literal evaluates to $-$ by f. This means that $a_i^j \in Q$ for each i, and hence C^j, together with the a_i^j and the β_i^j for all i, form a connected component in $H - E(Q)$, a contradiction.

4 Planar $\mathsf{P}_{st} \wedge \mathsf{SpT}$ is NP-complete

A similar reduction from Planar SAT to Planar $\mathsf{P}_{st} \wedge \mathsf{SpT}$ exhibits its NP-completeness.

4.1 Reduction construction

We follow the construction in Section 3.1. Given a boolean expression φ with planar G_φ, we form H_φ in the same way, except that when contracting $t_i s_{i+1}$ to a new vertex y_i for each i, we do not contract $t_n s_1$. Instead, delete the edge $t_n s_1$, and let $s = s_1$ and $t = t_n$. Call this graph $H := H'_\varphi$.

4.2 Correctness of reduction

Of course, H is a planar graph that is constructed in polynomial time. From a satisfying assignment, define Q^+ and Q^- the same way but note that they are edge-disjoint s–t paths. As before, $H - E(Q^-)$ is connected. Conversely, if there is an s–t path Q such that $H - E(Q)$ is connected, then $Q = \bigcup_i P_i^{-f(x_i)}$, yielding a satisfying assignment f. We omit the easy details.

5 Planar $\mathsf{T}+\mathsf{SpT}$ is NP-complete

We show that Planar $\mathsf{T}+\mathsf{SpT}$ is NP-complete by a reduction from Planar SAT.

5.1 Reduction construction

We continue with the construction in Section 4.1. Given a boolean expression φ with planar G_φ, we form H'_φ as above, but additionally insist that when subdividing edges of P^+ and P^- to create the a vertices, edges must belong to different small faces (as defined in Section 3.2). Add new cycles $ss's''s$ and $tt't''t$, where s', s'', t', t'' are new vertices. Call this new graph $H := H''_\varphi$.

5.2 Correctness of reduction

As before, H is a planar graph constructed in polynomial time. Suppose there is a tree T such that $H - E(T)$ is a spanning tree. Certainly T must contain an edge of $ss's''s$ and an edge of $tt't''t$. As such, T contains an s–t path Q. As $H - E(T)$ is a spanning tree, $H - E(Q)$ is connected. By the same argument as the previous two correctness proofs, we extract a satisfying assignment f from Q.

Conversely, take a satisfying assignment f and define Q^+ and Q^- as before. We alter Q^+ and Q^- such that they are a spanning tree and a tree, respectively, and their edge sets partition that of H. Take the edges incident to the σ, τ, and ω vertices, and add them (along with all incident vertices) to Q^+. Note that Q^+ is still a tree and contains all u and v vertices.

Consider a clause C^j. Let i be minimal such that $a_i^j \in V(Q^+)$. As f is satisfying, such an i exists. Add the path $a_i^j\beta_i^jC^j$ to Q^+. Note that Q^+ is still a tree, since we added two new edges and two new vertices. For each $a_\ell^j \in V(Q^+)$, $\ell \neq i$, add $a_\ell^j\beta_\ell^jC^j$ to Q^+. Since both a_ℓ^j and C^j were already in Q^+, we added one new vertex and two edges, and hence created a single cycle. The cycle contains an a_ℓ^j–C^j path P avoiding β_ℓ^j. Let e be the first edge in P that intersects Q^-, which exists by the extra stipulation in Section 5.1 above. Delete e from Q^+ to destroy the only cycle. Add e to Q^-, which does not create cycles as they share precisely one vertex. Finally, for each $a_\ell^j \in V(Q^-)$, add $a_\ell^j\beta_\ell^jC^j$ to Q^+, which grows the tree Q^+ by two vertices and two edges.

After performing the procedure for all clauses, Q^+ is a spanning tree of H'_φ. For the remaining 6 edges of $H - E(H'_\varphi)$, add the paths $ss's''$ and $tt't''$ to Q^+ and the remaining two edges ss'' and tt'' to Q^-. The edge sets of Q^+ and Q^- partition that of H, while Q^+ is a spanning tree and Q^- is a tree, as desired.

6 Final remarks

6.1

The use of Planar SAT can be replaced by the NP-complete problem Planar 3SAT, where each clause has exactly three literals. If so, the planar graphs constructed here have maximum degree 4. Vertices of degree 4 are critically used to allow paths to cross each other.

It would be interesting to see if this can be circumvented by using some other "crossing gadget" to lower the maximum degree to 3.

Similarly, vertices of degree 2 are used, as in [2], to forbid paths or cycles from meandering into the wrong places and moreover control the way they turn. If one succeeds in lowering the maximum degree to 3, it would then be reasonable to ask each of these questions when restricted to planar *cubic* graphs, where every vertex is of degree 3.

6.2

In [2], $\mathsf{C} \wedge \mathsf{SpT}$ is listed among problems whose planar restrictions were not known to be NP-complete. In its draft arXiv version, [1] is referenced, which (erroneously) claims that $\mathsf{C} \wedge \mathsf{SpT}$ is in P when restricted to planar graphs. Indeed, [1] outlines an algorithm $\mathsf{FindNonSeparatingCycle}(G)$ that answers the $\mathsf{C} \wedge \mathsf{SpT}$ problem for a planar graph G. However, the algorithm fails on the graph shown in Figure 2, which contains a nonseparating cycle *abca*. If the face bounded by *abdea* is chosen in Step 1.2 of the algorithm, the recursive algorithm fails to identify a nonseparating cycle. This counterexample was communicated to Bernáth, leading to the correct list appearing in [2].

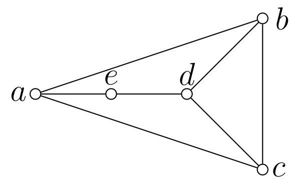

Figure 2: A graph on which the FindNonSeparatingCycle algorithm fails.

6.3

Partition the vertex set of a graph into two non-empty parts. The set of all edges intersecting both parts form a **cut**, which is **acyclic** if it contains no cycles. By planar duality, a nonseparating cycle determines an acyclic cut in the dual graph and, similarly, the existence of an acyclic cut guarantees a nonseparating cycle in the dual graph. As such, the problem of determining the existence of an acyclic cut in a (planar) graph (possibly with parallel edges) is NP-complete.

6.4

Pálvölgyi [6] showed that the problem $\mathsf{T}+\mathsf{T}$ of partitioning a graph into two (not necessarily spanning) trees is NP-complete. The reduction is from NAE-SAT, where an assignment is satisfying if each clause contains a + and a − ("not all equal"). The naïve approach of simply using the planar version does not work, since, somewhat surprisingly, PLANAR NAE-SAT can be solved in polynomial time [4].

The problem $\mathsf{Cut} + \mathsf{F}$ of partitioning a graph into a cut and a forest, equivalent to coloring the vertex set with two colors such that there are no monochromatic cycles, is NP-complete [2]. It is unknown whether these two problems remain NP-complete when restricted to planar graphs.

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