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Developing Critical Thinking Military Officers

Thor Martinsen
Naval Postgraduate School, thor@nps.edu

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Thor Martinsen
Department of Applied Mathematics
Naval Postgraduate School
Monterey, CA 93943
thor@nps.edu

The nation that will insist on drawing a broad line of demarcation between the fighting man and the thinking man is liable to find its fighting done by fools and its thinking done by cowards.
- *Sir William Francis Butler* [1]

I. Introduction

In May of 2020 the Joint Chiefs of Staff published guidance entitled *Developing Today's Joint Officers for Tomorrow's Ways of War.* In it they state, "To achieve intellectual overmatch against adversaries, we must produce the most professionally competent, strategic-minded, and critically thinking officers possible" [2]. What then is critical thinking? These days many different definitions exist. The word critical comes from the Greek word *kritikos*, which pertains to judging or discerning. In the broadest sense, critical thinking involves objective analysis of facts to form a judgment [3]. More specifically, critical thinking can be described as self-directed, self-disciplined, self-monitored, and self-corrective thinking [4].

The origins of critical thinking can be traced back to antiquity and the teachings of Socrates that were recorded by his pupil Plato. Interestingly, both of these men were not only philosophers but also warriors who fought in the Peloponnesian War [5], [6]. Plato founded the first institution of higher learning in the Western World, known as the Academy [7]. He also championed the dialectic form of discourse in which people with differing viewpoints seek to establish truth through reasoning and argumentation free from emotional and pejorative elements [8]. The famous philosopher Aristotle, who studied at the Academy and also tutored Alexander the Great, wrote a treatise entitled, *The Art of Rhetoric* [9]. In it he explained that there are three ways to persuade an audience: *ethos* (appeal to the speaker's character), *pathos* (appeal to emotion), and *logos* (appeal to logic) [10]. However, there is a distinct, albeit subtle, difference between that which is rhetorically persuasive and that which is verifiably true. While we do not discount the importance of teaching officers to recognize the role *ethos* and *pathos* can play in their thinking, the nature of war dictates that warfighters be well-versed in logic in order to make sound decisions based upon the realities present on the battlefield. In the words of Alfred Tarski, "There can be no doubt that the knowledge of logic is of considerable practical importance for everyone who desires to think and to infer correctly" [11].

Reasoning is the way in which thinking moves from one idea to another. Reasoning involves the recognition of relationships, the ability to create abstractions, and the ability to draw parallels. We distinguish between deductive and inductive reasoning. Deductive reasoning is inference carried out from the general to the specific (specialization). Here our information about the universe of discourse is complete and our conclusions are guaranteed, provided that we have applied the rules of inference correctly. Inductive reasoning on the other hand, is an attempt to reason about the general based upon the specific (generalization). In this case, our information about the universe of discourse is incomplete, and our conclusions are based upon evidentiary data from a sample population. The conclusions we arrive at are therefore not guaranteed, rather probable [12]. Observation and inductive reasoning together form the basis for the scientific method. While inductive reasoning is not part of formal logic, both forms of reasoning are important in military operations.

II. Deductive Reasoning

Logic is the study of inference [13]. Logic includes syllogistic and symbolic logic. Syllogistic logic can be found in Aristotle's book *Prior Analytics*, which is the first study of formal logic [14]. Modern symbolic logic consists of propositional and predicate logic and expands upon this early work. Predicate logic, also referred to as first-order logic, is an extension of proposition logic.

A. Propositional Logic

When introducing students to deductive reasoning, propositional logic is a natural place to start. Propositional logic consists of atoms and connectives. The atoms are simple propositions that can either be true or false. Simple propositions can in turn be modified and combined to form more complex, molecular, propositions. Common logical connectives are listed in Table I.
Given truth assignments of two propositions p and q we demonstrate the logic operations using the truth table found in Table II.
Of these operations, students typically struggle most with the implication and the principle of *ex falso quodlibet*, that is

TABLE I
LOGICAL CONNECTIVES

Symbol	Connective	Explanation
$\neg$	Negation	It is false that
$\wedge$	Conjunction	And
$\vee$	Disjunction	Or
$\oplus$	Exclusive Disjunction	Either, but not both
$\rightarrow$	Implication	If, then
$\leftrightarrow$	Biconditional	If and only if

TABLE II
TRUTH TABLE EXAMPLE

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \oplus q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	F	T	T
T	F	F	F	T	T	F	F
F	T	T	F	T	T	T	F
F	F	T	F	F	F	T	T

"from falsehood, anything" follows. It is worth mentioning that $\wedge$, $\vee$, $\oplus$, and $\leftrightarrow$ are commutative binary operations. If so inclined, one could discuss the material implication by showing, $p \rightarrow q$, is equivalent to the disjunction $\neg p \vee q$. From here one can add additional propositions and demonstrate how more complicated compound statements can be captured and analyzed. It is subsequently helpful to cover associativity and distributivity. Students usually take to those concepts easily due to their familiarity with analogous rules in arithmetic. Once the distributive laws are covered, De Morgan's laws make for a nice follow-on.

The truth table, first introduced by Ludwig Wittgenstein in 1921 in *Tractatus Logico-Philosophicus*, is an excellent pedagogical tool for teaching logic [15]. It provides students the means with which to systematically reason about propositions, and is particularly helpful when trying to establish logical equivalence between different molecular propositions. Incidentally, Wittgenstein, who is considered among the most influential figures in 20th century philosophy, also had a military background, having served as a lieutenant in the Austrian 7th Army during World War I [16]. When introducing truth tables, it is helpful to mention that for n propositions, the size of the input set is 2^n. It is also important to demonstrate a method of imposing a lexicographic or reverse-lexicographic order on the truth assignments, so students do not inadvertently omit or repeat rows in the table.

Given the fact that we live and fight in the information age, we would be remiss in our duty if we at some point failed to replace the traditional true and false truth assignments, T and F, with 1 and 0, and demonstrate Boolean algebra and the close connection propositional logic shares with digital logic and computing. Possibilities abound; however, given limited time and a primary goal of teaching reasoning skills and not engineering, we recommend limiting this aspect of one's lecture to a few examples. One approach that works well is to introduce the AND, OR, and NOT (inverter) gates, and subsequently demonstrate how a simple circuit such as an XOR gate can be implemented in different ways. This invariably leads students to recognize the important role logical equivalence can play in circuit optimization, and how reductions in size, power consumption, heat, and cost can be achieved in modern computers. These days, computer programming skills are arguably as important to warfighters as fluency in foreign languages, and we therefore recommend reminding students that computer programs consist of three main control structures, namely sequential, selection, and iteration logic. Finally, it is also worth mentioning to students that George Boole's book, *The Laws of Thought*, published in 1854, along with Claude Shannon's 1937 thesis entitled, *A Symbolic Analysis of Relay and Switching Circuits*, which build upon Boole's work, together laid the foundation for digital computing and the information age [17], [18].

B. Rules of Inference

When teaching inference, we begin by mentioning the three fundamental axiomatic rules of rational discourse [19]:

1) The law of identity. (What is, is. $p = p$.)
2) The law of non-contradiction. (Nothing can both be and not be. $\neg(p \wedge \neg p)$.)
3) The law of excluded middle. (Everything must either be or not be. $p \vee \neg p$.)

An argument consists of a set of propositions, called premises, followed by a conclusion. For example:

All men are mortal.
Socrates is a man.
Therefore, Socrates is mortal.

We analyze arguments by abstracting away the specifics and focus instead on the underlying structure of the argument. Letting p represent the proposition, "is a man," q represent the proposition, "is mortal," and $p \rightarrow q$ represent the implication, "if man, then mortal," we can rewrite the above argument as:

$$\begin{array}{l} p \rightarrow q \\ \underline{p \qquad} \\ \therefore q \end{array}$$

This argument is part of a classical form of inference known as a syllogism. Syllogisms consist of a major and a minor premise followed by a conclusion. In this case, the implication $p \rightarrow q$ is the major premise, the proposition p is the minor premise, and q is the conclusion. This particular syllogism is called Modus Ponens; it was known to Aristotle and the ancient Greeks [14]. An argument is valid if and only if, in every instance where all the premises are true, the conclusion is also true [13]. Otherwise, the argument is invalid. Using a truth table one can quickly check the validity of an argument. We do so by identifying so-called critical rows of the table where all premises are true, and then verify that the corresponding conclusion also is true. Arguments that are valid and contain premises that are all true are called sound [13]. We compile, in Table III, a list of ten basic rules of inference we believe every critical thinker should know [20].

TABLE III
TEN BASIC RULES OF INFERENCE

Name	Inference Rule
Modus Ponens	$p \to q,\, p \vdash q$
Modus Tollens	$p \to q, \neg q \vdash \neg p$
Disjunctive Syllogism	$p \vee q, \neg p \vdash q$
Material Implication	$p \to q \vdash \neg p \vee q$
Hypothetical Syllogism	$(p \to q) \wedge (q \to r) \vdash p \to r$
Composition	$(p \to q) \wedge (p \to r) \vdash p \to (q \wedge r)$
Resolution	$(p \vee q) \wedge (\neg p \vee r) \vdash (q \vee r)$
Disjunction Elimination	$p \vee q,\, p \to r,\, q \to r \vdash r$
Transposition	$p \to q \vdash \neg q \to \neg p$
Negation Introduction	$(p \to q) \wedge (p \to \neg q) \vdash \neg p$

* The turnstile logic symbol $\vdash$ represents ”proves”.

C. First-order Logic

Propositional logic can be extended to predicate logic. The logical connectives remain the same, however, we formalize the concept of propositions by introducing Boolean functions called predicates that are evaluated with respect to variables drawn from sets. Returning to our example of the mortality of man, the propositions p and q now become predicates P and Q, and the universal implication $p \to q$ is represented by $\forall x, P(x) \to Q(x)$. Predicate logic is a more nuanced and expressive ontology than that of propositional logic. Whereas we previously required separate symbols for each instance of a given attribute, we now can instantiate a predicate for multiple objects. Universal and existential quantification allow us to assert the truth (or falsity) of a sentence for all or for at least one element of a set. Used in concert, nested quantifiers allow us to create multi-variable predicates that capture complex relationships between variables and assert facts that propositional logic fails to adequately describe [21]. For example: $\forall x \in \mathbb{Z}^+ \exists y \in \mathbb{Q}, P(x,y)$, where $P(x,y)$ asserts $x \cdot y = 1$ and $\mathbb{Z}^+$ and $\mathbb{Q}$ represent the set of positive integers and set of rational numbers respectively. The German philosopher and mathematician Gottlob Frege first introduced quantified variables in his 1879 book entitled *Begriffsschrift* [21], [22]. Although the significance of his work was largely overlooked during his lifetime, Frege is today credited with inventing axiomatic predicate logic and is considered by many to be one of the greatest logicians since Aristotle [21]. Once students have been exposed to propositional logic, they typically take to predicate logic very easily. The idea of using a Boolean function to evaluate an input and determine whether it satisfies a particular attribute seems natural. To ease the transition to first-order logic, it is helpful to begin by introducing existential and universal quantification and set-builder notation. Subsequently, it is important to ensure that students understand the concepts of domain of discourse, the binding of variables, and the importance of specifying the scope of nested quantifiers.

First order logic is both complete and sound. Kurt Godel’s 1929 completeness theorem shows that if a formula in first-order logic is valid, then there exists a formal proof of it in the form of a finite deduction. Conversely, soundness says that only valid formulas are provable in this deductive system. Together the completeness theorem tells us that a first-order formula is logically valid if and only if it is the conclusion of a formal deduction [23].

D. Proof

The word proof comes from the Latin word *probare*, which means to test. A proof is a rigorous and exhaustive argument that uses deductive reasoning to establish, with logical certainty, the truth (or falsity) of a statement. As such, the proof process represents the epitome of critical thinking. The first known mathematical proof can be found in the work of the sixth century BCE mathematician Thales of Miletus [24]. The Greek mathematician Euclid of Alexandria later introduced the axiomatic method in his treatise the *Elements* circa 300 BCE. This method involves using, without proof, self-evident propositions called axioms as the starting point for logical derivations of theorems [25]. The *Elements* is perhaps the most influential work on logic and mathematical reasoning ever written. There are four fundamental proof techniques with which we believe all critical thinkers should be familiar:

1) Direct Proof.
2) Proof by Contrapositive.
3) Proof by Contradiction.
4) Proof by Induction.

When teaching proofs, we recommend proceeding in the above order and, in each case, beginning by sketching the proof technique’s logical structure. Relating the approaches back to the, now familiar, rules of inference such as implication, disjunction elimination, transposition, and negation introduction helps students understand the techniques. We find that it is best to use theorems involving elementary number-theoretic properties such as parity, primality, divisibility, and closure when demonstrating proof techniques. While it may be tempting to showcase theorems involving lemmas and more advanced mathematics, we recommend against doing so. In such cases, students tend to focus on the mathematics and lose sight of the proof technique being demonstrated.

1) Direct Proof $(p \to q)$: When teaching direct proofs, we suggest first showing students how to analyze the statement to be proved. Begin by identifying the antecedent and conclusion, and then define relevant sets and predicates so the claim can be expressed using predicate logic. Once this has been accomplished, emphasize the importance of using the applicable definitions to start the mathematical journey from antecedent to the conclusion. Students often make the mistake of arguing from examples when attempting to prove a universal statement. It therefore is important to teach them the method of generalizing from the generic particular.

It is recommended that students also be exposed to proof by exhaustion. Notice that this proof technique in essence is disjunctive elimination. It is perhaps worth mentioning that proving, for example, the logical equivalence of two propositional statements using a truth table is a form of proof by exhaustion. However, we also suggest showing students another proof where the universe of discourse can be divided into a manageable number of cases, each of which we in turn demonstrate lead to the desired conclusion. One such example that we find works well is using the quotient-remainder theorem and proving that the product of any three consecutive integers is divisible by three. When introducing existential proofs, it is instructive to highlight the difference between constructive and non-constructive proofs. There are plenty of examples of constructive proofs one can choose. When it comes to non-constructive proofs, we suggest using Dov Jarden’s proof that an irrational number raised to an irrational exponent may be rational:

> $(\sqrt{2}\,)^{\sqrt{2}}$ is either rational or irrational. If it is rational, our statement is proved. If it is irrational, then $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = (\sqrt{2}\,)^{\sqrt{2}\cdot\sqrt{2}} = (\sqrt{2}\,)^2 = 2$. Thus, the result is proven. [26]

2) Proof by Contrapositive $(\neg q \to \neg p)$: Transposition was among the ten rules of inference that we suggested every critical thinker should know. While students at this point are aware that the contrapositive is logically equivalent to the implication, many frequently wonder why one would resort to using the contrapositive over a direct proof. To motivate this, we recommend selecting an example that illustrates the fact that it sometimes is easier to prove a result using the contrapositive than proceeding directly. One such instructive example is proving that for all integers n, if n^3 is odd, then n is an odd integer. Proceeding directly, students frequently suggest setting $n^3 = 2k+1$, for some integer k. Doing so, and subsequently taking the cube root on both sides of the equation, yields $n = \sqrt[3]{2k+1}$. From here, students typically struggle with trying to reason about whether $\sqrt[3]{2k+1}$ is even or odd. Having unscrupulously forced them into this predicament, we instead suggest starting over using the contrapositive, which involves proving that for all integers n, if n is even, then n^3 is an even integer. Proceeding in a similar fashion as before, we now set $n = 2k$, for some integer k. Cubing both sides of the equation produces: $n^3 = 8k^3 = 2(4k^3)$. In this case, $2(4k^3)$ is clearly even and the desired result is easily established.

3) Proof by Contradiction $((\neg p \to q) \land (\neg p \to \neg q) \vdash p)$*:* Proof by contradiction, also referred to as *reductio ad absurdum*, makes use of negation introduction, the last of our ten highlighted rules of inference. Here we establish the truth of a proposition by assuming the opposite, and then show that this leads to a contradiction. Notice that this proof technique relies upon both the law of non-contradiction and the law of excluded middle. In keeping with our number-theoretic theme, we suggest demonstrating Euclid's classical proof of the irrationality of $\sqrt{2}$. Experience shows that some students have a negative reaction to this proof. The crux of their objections usually centers around the contradiction. These students typically view the fact that the derived fraction is not in its lowest form as a technicality rather than a legitimate contradiction. This is unfortunate. To avoid students in these cases forming the opinion that proof by contradiction is a sleight of hand, we suggest backing up and asking them to reconsider the initial set-up of the problem. Letting $\sqrt{2} = p/q$, where $p, q \in \mathbb{Z}$, $q \neq 0$, means that $2q^2 = p^2$. Observe that both p^2 and q^2 must necessarily contain zero or an even number of 2's. This in turn means that the respective prime factorizations of the left and right side of the equation, $2q^2 = p^2$, contain unequal numbers of 2's, thereby violating the Fundamental Theorem of Arithmetic. Faculty often also demonstrate the infinitude of primes using proof by contradiction. Their proofs frequently use Euclid's construction of a new primorial prime, $p = p_1 p_2 ... p_n + 1$, involving the product of a purported finite collection of all n primes. While this also serves as a suitable example, it is worth mentioning that Euclid's original proof was not, in fact, a proof by contradiction. Like nonconstructive proofs, proofs by contradiction provide little insight into the underlying forces at work in a theorem. Consequently, while indirect proofs are powerful tools, we recommend stressing to students that they be used sparingly, and only in cases where direct approaches are unavailable.

4) Proof by Induction $\forall P\,[P(0) \land \forall k\,(P(k) \to P(k+1)) \vdash \forall n\, P(n)]$*:* Unlike the three preceding proof techniques, all of which appeared in antiquity, mathematical induction is a more recent form of reasoning. The first rigorous use of mathematical induction can be found in the work of the 14th century mathematician and philosopher Gersonides [27]. The French mathematician Blaise Pascal later explicitly described the technique in *Traité du triangle arithmétique* in 1665 [28]. Mathematical induction is a form of deductive reasoning and, despite its name, should not be confused with inductive reasoning. It relies upon the Peano axioms. Notice that, as formulated above, proof by induction belongs to second-order logic since the first quantifier ranges over the predicates rather than the predicate variables. It is, however, possible to reduce mathematical induction to first-order logic via the introduction of an axiom schema. When carrying out proof by induction, one must prove that $\forall k\, P(k) \to P(k+1)$. Once this has been accomplished, invoking the induction principle carries out n applications of this step thereby allowing us to get from $P(0)$ to $P(n)$. It can be helpful to use either the domino or ladder analogies when explaining this process to students. When highlighting the difference between weak and strong induction, we recommend proving a theorem involving a sequence expressed as a multi-term recurrence relation such as a Lucas sequence. In these cases, it becomes easy for students to recognize that they will need the induction hypothesis to cover more than just the k^{th} term if they hope to prove $P(k+1)$. If teaching a class that includes computer science students, we also recommend including an example of mutual induction, which they will later use when studying finite state machines.

III. Inductive Reasoning

Information in war, as in many human endeavors, is incomplete. As tempting as it may be to remain within the safe confines of deductive reasoning, the pragmatist recognizes, despite its challenges, the need for critical thinking to venture into the unknown. In the words of Carl von Clausewitz, "War is the realm of uncertainty" [29]. Inductive reasoning provides a means with which to cut through the fog of war. It involves using evidence-based premises, synthesized from experience and observation, to draw probable conclusions about the larger world around us. Inductive reasoning was considered as far back as antiquity. Aristotle used the Greek word *epagoge* to describe the method. The Roman statesman and scholar Cicero later translated this into the Latin word *inductio* [30]. In addition to Statistical Syllogisms, there are three principal types of inductive reasoning: Generalization, Analogy, and Causation. Inductive arguments make use of enumerative or eliminative induction. A conclusion arrived at using enumerative induction is based upon the number of instances that support it, whereas a conclusion using eliminative induction is based upon the variety of instances that support it.

A. Statistical Syllogism

When teaching students how to reason inductively, we suggest beginning with the statistical syllogism. This form of inductive inference most closely resembles deductive logic. Statistical syllogisms use a generalization about a population to infer something about an individual or group of individuals [13]. Let A be a reference class, B be an attribute class, $x \in \mathbb{R}$, and i be an element of A. Then the general form of a statistical syllogism is as follows:

$$\begin{array}{l} x\ \%\ \text{of } A\text{'s are } B. \\ i \text{ is } A. \\ \hline \text{Therefore, } i \text{ is } B. \end{array}$$

It is worth pointing out to students that this argument is structurally similar to Modus Ponens. The difference is that the major premise no longer is a universally applicable implication, but rather a quantifiable generalization. The conclusion therefore is not guaranteed. The strength of the statistical syllogism depends upon the proportion of the reference class that also are members of the attribute class. Interestingly, Aristotle would likely have accepted such reasoning. In his book *Prior Analytics* he wrote, "That which people know to happen or not to happen, or to be or not to be, mostly in a particular way, is likely" [14], [31].

B. Inductive Generalization

Inductive generalization works by creating premises from a sample, and subsequently attempts to apply these premises and draw meaningful conclusions about a larger population. There are both anecdotal and quantitative forms of inductive generalization. We focus here on the latter, because they often lead to stronger conclusions. In a statistical generalization, the conclusion is inferred using a statistical sample of the population. The success of this form of inference hinges upon our ability to achieve a statistically-representative sample of the target population. To do so one must first know something about general population characteristics such as size and distribution. Provided that a randomly chosen sample of sufficient size can be obtained, the conclusion will be reliable within a quantifiable margin of error. When teaching statistical inference, it is important to ensure that students understand hypothesis testing and the proper construction of confidence intervals. Inferential statistics is a complex topic, and critical thinkers must understand these complexities if they hope to reason about the strength or weakness of a given statistical generalization. As Admiral Hyman Rickover once said, "The devil is in the details, but so is salvation" [32].

C. Analogical Inference

Analogical inference works by recognizing shared properties among similar objects and infers that if an additional property is present in one of the objects, then that property probably applies to all of the objects [33], [34]. To highlight this form of reasoning, consider the following: Let P, Q, R, and S be predicates related to the existence of specified properties. Let x and y be elements from a set of similar objects U, and let $\Box$ be the unitary operator (similar in usage to C.L. Hamblin's semantics) that modifies a predicate from true to probable [35], [36]. Then the following represents an analogical argument:

$$\begin{array}{l} P(x) \wedge P(y) \\ Q(x) \wedge Q(y) \\ R(x) \wedge R(y) \\ \underline{S(x) \qquad\qquad} \\ \therefore \Box S(y) \end{array}$$

The strength of this form of argument depends upon the degree of similarity between the objects, as well as how relevant the known similarities are to the inferred similarity in the conclusion. After having been exposed to deductive logic, students may find such reasoning dissatisfying. However, it is worth mentioning that mathematics could not have developed were it not for our ability to abstract and use analogies. Somewhere in the distant past our ancestors must have recognized the shared property of quantity among sets of different objects, thereby giving rise to the concept of number. For example, two humans, two birds, and two deer, all share the property of "twoness".

D. Causal Inference

A commander's ability to predict enemy actions on the battlefield is critical to the outcomes in war. This complex and difficult undertaking involves knowing the enemy's force size, capabilities, and location, as well as understanding their psychology, tactics, and strategy. Inductive prediction attempts to draw conclusion about future events based upon a sample of specific past instances. Sun Tzu reminds us that, "The art of war is the art of deception," so in military matters such an endeavor must necessarily involve a great deal of risk [37]. We typically mitigate risk associated with inductive prediction by defining cones of uncertainty and using post-facto observation to either confirm a prediction or update our predictive model. On the battlefield, persistent Intelligence, Surveillance, and Reconnaissance (ISR) is employed to test and, if necessary, correct decisions based upon faulty inductive conclusions. Human beings are predisposed from an early age to reasoning based upon cause and effect. Causal inference works by building an implication $p \rightarrow q$ based upon an observed effect q, and an assumed causal connection to a preceding event p. When teaching inferential statistics, we often emphasize the fact that correlation does not imply causation. When teaching causal inference, it is important to impress upon students that the strength of the conclusion depends in large part upon our understanding of the underlying mechanisms governing the causal model about which we are trying to reason [34].

In his book, *On War*, Carl von Clausewitz wrote, "In the whole range of human activities, war most closely resembles a game of cards" [29]. The mathematical theory of probability was born out of attempts to analyze games of chance. Initial contributions from the 16th and 17th century mathematicians Gerolamo Cardano, Pierre de Fermat, Blaise Pascal, and Christiaan Huygens all stemmed from a desire to understand games of chance [38]. The 19th century mathematician Pierre Laplace, who incidentally taught at the French École Militaire, later developed probability theory into what we recognize today, including Bayesian inference [38], [39]. Bayesian inference computes the probability of a consequent based upon two antecedents, one in the form of a prior probability and the other in the form of a likelihood function of a statistical model built upon observational data. Ensuring that students have a good grasp on conditional probability is important to their ability to reason inductively. It is important that students understand that conditional probability allows us to update the probability of an event occurring based upon new information. One example we suggest using that highlights this concept is the Monty Hall problem. Such results can often be counter-intuitive. It is therefore important to give students adequate time to explore and absorb these concepts.

IV. Logical Fallacies

Logic is the study of both sound and faulty reasoning. When teaching critical thinking, it is important that we not only expose students to valid forms of inference, but also demonstrate what can go wrong with our thinking. We classify fallacies as either formal or informal. Formal fallacies are structural in nature, whereas informal fallacies are related to content [40]. Informal fallacies greatly outnumber formal fallacies. The use of fallacies in rhetoric is regrettably common. Here the goal of using sound logic is often outweighed by a desire to gain support for an issue. As previously mentioned, this exposition is focused on *logos* over *ethos* and *pathos*. We therefore forego a discussion on informal fallacies such as *argumentum ad hominem* (personal attack), *argumentum ad verecundiam* (appeal to authority) and *argumentum ad misericordiam* (appeal to pity) and examine instead fallacies common in deductive and inductive reasoning. Our treatment of such fallacies will not be complete, rather highlight a few common mistakes students make. Among formal fallacies, there are three that frequently occur in deductive reasoning.

A. Formal Fallacies

1) Affirming a disjunct: is a fallacy caused by students confusing the OR and XOR operations. The fallacy takes the following form:

$$\begin{array}{l} p \vee q \\ \underline{p \qquad} \\ \therefore \neg q \end{array}$$

In this case the offender treats the OR operator as an exclusive disjunction and fails to recognize that p and q could simultaneously both be true.

2) Affirming a consequent: sometimes also referred to as the converse error, is perhaps one of the most frequently occurring fallacies among students. Given an implication, the student fails to recognize that a different antecedent could also lead to the conclusion. The fallacious argument is as follows:

$$\begin{array}{l} p \rightarrow q \\ \underline{q \qquad} \\ \therefore p \end{array}$$

When given the implication $p \rightarrow q$, it is relatively common for students to mistakenly assume symmetry and attempt to invoke a biconditional relationship $p \leftrightarrow q$. To help students avoid this fallacy, we suggest demonstrating that $p \rightarrow q$ is not logically equivalent to $q \rightarrow p$. This can be done directly using a truth table, or perhaps more effectively by using the material implication and showing that $p \rightarrow q \equiv \neg p \vee q$ and $q \rightarrow p \equiv \neg q \vee p$. From here students can easily confirm that $\neg p \vee q \not\equiv \neg q \vee p$.

3) Denying the antecedent: sometimes referred to as the inverse error, is also a common student fallacy. The faulty argument takes the form:

$$\begin{array}{rl} & p \rightarrow q \\ & \underline{\neg\, p} \\ \therefore & \neg\, q \end{array}$$

Given the implication $p \rightarrow q$, the misguided student tries to invoke an inverse relationship. However, $p \rightarrow q$ is not logically equivalent to $\neg p \rightarrow \neg q$. In some instances, this error can be brought about by students confusing $\neg p \rightarrow \neg q$ with the contrapositive statement $\neg q \rightarrow \neg p$, which is logically equivalent to the original implication $p \rightarrow q$. It is worth pointing out to students that although neither $\neg p \rightarrow \neg q$ nor $q \rightarrow p$ are logically equivalent to $p \rightarrow q$, they are however equivalent to each other.

B. Informal Fallacies

Fallacies in inductive reasoning are typically related to the content and not the structure of the argument [13], [40]. The problematic content usually takes the form of a faulty generalization that is either built upon a statistically flawed sample or the result of errors in analysis of the available data. There are two common informal fallacies that occur in statistical syllogisms. The **accident** takes place when an exception to a general rule is ignored, whereas the **converse accident** happens when a rule applicable in an exceptional case is applied in general. When it comes to inductive generalizations, several mistakes are common. In a **hasty generalization**, one or a handful of examples are used to develop a generalization. In this case, the generalization fails to apply to the larger population. The **unrepresentative samples fallacy** occurs when a conclusion is drawn using samples of a population that are unrepresentative or biased. In a **slothful induction**, skeptics deny a correct conclusion despite strong supporting evidence, and assume instead that it is a coincidence. Conversely, in the **fallacy of exclusion**, proponents of an inductive argument exclude important evidence that calls into question the conclusion. Finally, in analogical inference, the **false analogy** is a common mistake. In this case, the objects being compared are altogether dissimilar or their similarity is not relevant with respect to the conclusion being inferred.

V. Context, Theory, Praxis, and Critique

When teaching the art of solving problems, it is important to prepare the pedagogical battlefield so to speak. Whenever possible, we recommend adopting a systematic approach focused on context, theory, praxis, and critique. First, set the stage by discussing why we care about the problem. Secondly, present the relevant theory related to how we solve the problem. Third, and perhaps most importantly, give students time to practice and internalize the techniques presented. Finally, demonstrate limitations to the methods and errors that can occur when attempting to solve the problem. Praxis informs critique. We argue that achieving basic proficiency in the mechanics of problem solving is not the desired end-state, rather the point from which students can begin to think critically about the problem and evaluate their solutions. What is the acceptable input domain? What is the solution codomain? Does the computed solution make sense? What are limitations of the method? Are there workarounds for situations where the algorithm fails? Perhaps some of the most interesting and insight-producing mathematics occurs in the latter situation. Take for example the use of least squares approximations when trying to solve overdetermined systems of linear equations. Inherent to the practice of critical thinking is the act of asking questions. Using the Socratic method in the classroom provides students with a critical-thinking exemplar. When it comes to critical thinking, we are reminded of Paul Halmos' famous quote:

> *Don't just read it; fight it! Ask your own question, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?* [41]

It is important to not lose sight of the fact that our mission is to educate officers, not train technicians. Avoid the temptation of introducing technology for technology's sake. Before giving students a computational tool, we must first equip them with the means with which to critically judge the tool's output. For example, when teaching trigonometry, students must first be taught the unit circle. Allowing students the use of a calculator without first ensuring that they can compute trigonometric function values by hand, undermines their understanding and confidence, forces their reliance upon a machine, and increases the likelihood of mistakes being made. Borrowing from an old adage, it is important that educators act as angling coaches and not charitable fishmongers. Technologies progress and systems change. The tools of tomorrow will almost certainly differ from those used in today's classrooms. We must therefore avoid teaching "buttonology" and mastery of specific tools, rather focus instead on the underlying algorithms. Computational tools, like weapon systems, have operating envelopes and performance parameters. Showing students when and how computational tools fail, is part and parcel of educating them on their use.

VI. IA Versus AI

Modern warfare is a marriage of man and machine. We believe that what military commanders need on the battlefield is machine augmented intelligence, sometimes referred to as Intelligence Amplification (IA), not merely Artificial Intelligence (AI). Perhaps the closest analog we currently have to AI employed in battle, is that of chess. Computer chess engines have been getting consistently better since IBM's Deep Blue computer first defeated the world chess champion, Garry Kasparov, in 1997 [42]. Today chess engines regularly beat chess grandmasters [43]. However, players such as the reigning world champion, Magnus Carlsen, frequently use chess engines to aid in their analysis and game preparation [44]. In fact, following his loss to Deep Blue, Kasparov subsequently invented advanced chess. This form of the game, sometimes referred to as centaur chess or cyborg chess, involves a human armed with a chess engine. Here the hybrid human-machine player, called a centaur, leverages computer computation and search capabilities, human creativity and intuition, along with anti-AI tactics to defeat a hybrid or machine adversary [45], [46], [47]. Much of the current focus within the Department of Defense is on AI, while arguably less attention is being given to how we should train the combined Human-Machine Learning system. Inductive reasoning, as previously discussed, is far more error-prone than deductive reasoning. Data science shows great promise in helping us reason inductively, however, it also brings with it great risk. Machine Learning systems can be vulnerable to enemy Machine Learning countermeasures in the form of deceptive data that is introduced in an effort to subvert or alter system behavior. It is therefore paramount that we equip warfighters with the necessary skills to reason critically about the solutions automated systems provide, and in so doing "cognitively fine-tune" our military human-machine reasoning systems.

VII. Conclusion

Teachers of critical thinking are in many respects like tour guides. We take students on excursions through monuments of some of man's greatest intellectual achievements over the past two and a half millennia. The routes we choose and time we dwell on certain features effects how our students experience and interact with their surroundings. We should strive to contextualize and connect the sites on the tour in such a way as to tell a compelling story. In order for our pupils to become proficient in the art of reasoning, we must allow them time to explore and internalize the ideas, interceding only when necessary to point out obstacles and pitfalls in their way. As Carl von Clausewitz wrote:

> *All thinking is indeed Art. Where the logician draws the line, where the premises stop which are the result of cognition - where judgment begins, there Art begins.* [29]

Our adversaries have steadily been closing the technological gap we have long enjoyed. On the battlefields of tomorrow, we may no longer be able to rely upon a technical advantage. We must therefore create a 21st century "cognitive phalanx" capable of delivering a warfighting advantage in spite of inferiority in numbers or technology that is not superior to that of our enemies. Instructing officers in critical thinking is important and must begin early. Exposing cadets, midshipmen, and junior officers to logic and reasoning introduces them to a systematic approach to problem solving and teaches rigor. It helps them exercise intellectual curiosity and judgement, and it reinforces confidence, discipline, and grit. This, in turn, makes them better warfighters and leaders. It also creates a solid foundation upon which they later can exercise more nuanced complex and strategic thinking. Sun Tzu tells us in *The Art of War* that, "Every battle is won before it's ever fought" [37]. The critical thinking skills we develop in our students today could quite possibly prove decisive in the outcomes of battles of the future.

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