Automorphism groups of Cayley digraphs of $\mathbb{Z}_p^3$

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Abstract

We calculate the full automorphism group of Cayley digraphs of $\mathbb{Z}_p^3$, p an odd prime, as well as determine the 2-closed subgroups of $S_m \wr S_p$ with the product action.

1 Introduction

In the last several decades, there has been considerable interest in vertex-transitive digraphs, that is, digraphs whose automorphism group acts transitively on the vertex set of the digraph. As vertex-transitive digraphs are studied for their symmetry, a natural and fundamental question which immediately arises is that, given a vertex-transitive digraph Γ, what are all symmetries of Γ? That is, what is $\mathrm{Aut}(\Gamma)$, the automorphism group of Γ? This problem is also named as the *König problem* [16], and it is well-known to be a quite difficult one (cf. [18]). As one would expect, only modest progress has been made towards a solution. In this paper, we will give a description of the automorphism group of a Cayley digraph of $\mathbb{Z}_p^3$, p an odd prime. The automorphism groups of Cayley digraphs have been determined for the groups $\mathbb{Z}_p$ [1], $\mathbb{Z}_p^2$ [13], $\mathbb{Z}_{p^2}$ [18] (see also [13] for a different,

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later proof), $\mathbb{Z}_n$ (for arbitrary n see [23, Theorem 2.3] which summarizes results proven in [14, 19, 20], and see [25] for a polynomial time algorithm to compute the automorphism group; for the special case $n = pq$, p and q are distinct primes, see also [18] or [9] for a different, later proof, and for the case n is square-free see [11] for an independent computation of the automorphism group). See also [9] for the automorphism groups of every vertex-transitive graph of order pq, where p and q are distinct primes.

A classical result of Sabadussi states that, a digraph is isomorphic to a Cayley digraph of a group G if and only if its automorphism group contains a regular subgroup isomorphic to G. A 2-closed permutation group is simply the automorphism group of a color digraph, and the automorphism group of a Cayley digraph is a 2-closed group (see also Section 2). Our main result below gives in fact all 2-closed groups which contain a regular elementary abelian subgroup of order p^3.

Theorem 1.1. *Let $G \leqslant S_{p^3}$ be a 2-closed group, p is an odd prime, such that G contains a regular elementary abelian subgroup. Then one of the following is true:*

(1) G is primitive, and permutation isomorphic to one of the following groups:

- *(a) S_{p^3};*
- *(b) a primitive subgroup of $\mathrm{AGL}(3,p)$;*
- *(c) $S_3 \wr S_p$ with the product action.*

(2) G is imprimitive, and permutation isomorphic to one of the following groups:

- *(a) an imprimitive subgroup of $\mathrm{AGL}(3,p)$;*
- *(b) $X \wr Y$, where $X \leqslant S_{p^i}$ and $Y \leqslant S_{p^j}$ are 2-closed groups, containing a regular elementary abelian subgroup, and $1 \leqslant i, j$, $i + j = 3$;*
- *(c) $S_p \times X$ or $S_{p^2} \times Y$, where $X \leqslant S_{p^2}$ and $Y \leqslant S_p$ are 2-closed groups, containing a regular elementary abelian subgroup;*
- *(d) $A((S_p \times S_p) \times Z)$ or $A((X \wr Y) \times Z)$, where $Z < \mathrm{AGL}(1,p)$, $X, Y \leqslant S_p$ are 2-closed groups, and $A \leqslant \mathrm{Aut}(\mathbb{Z}_p^3)$.*

The rest of this paper is organized as follows. In the next section, we gather most definitions and preliminary results needed later. In Section 3, we determine the primitive 2-closed groups that contain a regular subgroup isomorphic to $\mathbb{Z}_p^3$. In Section 4, we consider the 2-closed subgroups of $S_m \wr S_p$ with the product action. We remark that results in Section 4 are not needed for the proof of Theorem 1.1. Imprimitive 2-closed groups that contain a regular subgroup isomorphic to $\mathbb{Z}_p^3$ are computed in Section 5, where the work is broken down according to various possibilities for a Sylow p-subgroup - the various possibilities are listed in Theorem 5.4, and were determined explicitly in [28] and implicitly in [7].

2 Preliminaries

Notation is relatively standard. For permutation group theoretical terminology not defined here the reader is referred to [6].

Let Ω be a set and $G \leqslant S_\Omega$ be a transitive group. Let G act on $\Omega \times \Omega$ by $g(\omega_1, \omega_2) = (g(\omega_1), g(\omega_2))$ for every $g \in G$ and $\omega_1, \omega_2 \in \Omega$. We define the *2-closure of* G, denoted $G^{(2)}$, to be the largest subgroup of S_Ω whose orbits on $\Omega \times \Omega$ are the same as G's. Let $\mathcal{O}_1, \ldots, \mathcal{O}_r$ be the orbits of G acting on $\Omega \times \Omega$. Define digraphs $\Gamma_1, \ldots, \Gamma_r$ by $V(\Gamma_i) = \Omega$ and $E(\Gamma_i) = \mathcal{O}_i$. Each Γ_i, $1 \leqslant i \leqslant r$, is an *orbital digraph of* G, and it is straightforward to show that $G^{(2)} = \cap_{i=1}^r \mathrm{Aut}(\Gamma_i)$. Let $\{\Phi_1, \ldots, \Phi_s\}$ be an arbitrary partition of $\Omega \times \Omega$ such that $\Phi_1 := \{(\omega, \omega) \colon \omega \in \Omega\}$. The pair $\Phi := (\Omega, \{\Phi_1, \ldots, \Phi_s\})$ is called a color digraph, and its automorphism group is $\mathrm{Aut}(\Phi) := \cap_{i=1}^s \mathrm{Aut}((\Omega, \Phi_i))$. To the sets Φ_i, $1 \leqslant i \leqslant s$, we shall also refer as the color classes of Φ. Clearly, the automorphism group of a vertex-transitive graph or digraph is 2-closed, and the 2-closed subgroups of S_Ω coincide with the automorphism groups of color digraphs with vertex set Ω.

Let $S \subseteq G$. We define the *Cayley digraph of* G with *connection set* S, denoted $\mathrm{Cay}(G, S)$, to be the digraph with vertex set G and arc set $\{(g, gs) : g \in G, s \in S\}$. By a *Cayley color digraph of* H we mean a color digraph with vertex set H, each color class of which is an arc set of a Cayley digraph of H. For $g \in G$, define $g_L : G \to G$ by $g_L(h) = gh$. It is easy to see that $g_L \in \mathrm{Aut}(\mathrm{Cay}(G, S))$. We set $G_L := \{g_L : g \in G\}$, which is the *left-regular representation* of G, and thus $G_L \leqslant \mathrm{Aut}(\mathrm{Cay}(G, S))$.

The following classical result of Burnside [3] is quite useful for analyzing transitive groups of prime degree, especially now that, as a consequence of the Classification of Finite Simple Groups, all doubly transitive groups are known [4].

Theorem 2.1. *Let G be a transitive group of prime degree. Then either G is doubly transitive, or G contains a normal Sylow p-subgroup.*

Equivalently (see [6, Exercise 3.5.1]), we have

Theorem 2.2. *Let G be a transitive group of prime degree p. Then we may relabel the set upon which G acts so that $G \leqslant \mathrm{AGL}(1, p)$, or G is doubly transitive.*

As essentially observed by Alspach [1], this yields the following result giving all 2-closed groups of prime degree.

Theorem 2.3. *Let G be a 2-closed group of prime degree p. Then either G is permutation isomorphic to a proper subgroup of $\mathrm{AGL}(1, p)$, or $G = S_p$.*

The 2-closed subgroups of S_{p^2} that contain a regular elementary abelian subgroup were determined in [13, Theorem 14].

Theorem 2.4. *Let G be a 2-closed subgroup of S_{p^2} such that G contains the left regular representation of $\mathbb{Z}_p^2$.*

1. *If G is doubly transitive, then $G = S_{p^2}$.*

2. *If G is simply primitive and solvable, then $G \leqslant \mathrm{AGL}(2,p)$.*

3. *If G is simply primitive and nonsolvable, then $G \leqslant \mathrm{AGL}(2,p)$ or $G = S_2 \wr S_p$ in its product action.*

4. *If G is imprimitive, solvable, and has an elementary abelian Sylow p-subgroup, then either $G < \mathrm{AGL}(1,p) \times \mathrm{AGL}(1,p)$ or $G = S_3 \times S_3$ (and $p = 3$).*

5. *If G is imprimitive, nonsolvable, and has an elementary abelian Sylow p-subgroup, then either $G = S_p \times S_p$ or $G = S_p \times A$, where $A < \mathrm{AGL}(1,p)$.*

6. *If G is imprimitive with Sylow p-subgroup of order at least p^3, then $G = G_1 \wr G_2$, where G_1 and G_2 are 2-closed permutation groups of degree p.*

We shall have need of the following result of Kalužnin and Klin [17] (this result is also contained in the more easily accessible [5, Theorem 5.1]).

Lemma 2.5. *Let $G \leqslant S_X$ and $H \leqslant S_Y$ be transitive groups. Then in their coordinate-wise action on $X \times Y$, we have*

$$(G \times H)^{(2)} = G^{(2)} \times H^{(2)}, \text{ and } (G \wr H)^{(2)} = G^{(2)} \wr H^{(2)}.$$

Let A be a finite set of order n, and $\mathcal{R}el(A)$ to be the set of all relations on A. We define a *combinatorial object* X to be a subset of $\mathcal{R}el(A)$ following Muzychuk [24] (see this reference as well for various equivalent definitions of a combinatorial object). We define a *Cayley object of a group* G to be a combinatorial object X (e. g. digraph, graph, design, code) such that the left regular representation $G_L \leqslant \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the *automorphism group of* X (note that this implies that the vertex set of X is in fact G). If X is a Cayley object of G in some class $\mathcal{K}$ of combinatorial objects with the property that whenever Y is another Cayley object of G in $\mathcal{K}$, then X and Y are isomorphic if and only if they are isomorphic by a group automorphism of G, then we say that X is a *CI-object of G in $\mathcal{K}$*. If every Cayley object of G in $\mathcal{K}$ is a CI-object of G in $\mathcal{K}$, then we say that G is a *CI-group with respect to* $\mathcal{K}$. If G is a CI-group with respect to every class of combinatorial objects, then G is a *CI-group*.

Babai [2] characterized the CI-property in the following manner.

Lemma 2.6. *For a Cayley object X of G the following are equivalent:*

1. *X is a CI-object;*

2. *given a permutation $\varphi \in S_G$ such that $\varphi^{-1}G_L\varphi \leqslant \mathrm{Aut}(X)$, G_L and $\varphi^{-1}G_L\varphi$ are conjugate in $\mathrm{Aut}(X)$.*

The problem of determining which groups G are CI-groups with respect to digraphs has attracted considerable attention over the last 30 or so years. The interested reader is referred to [21]. The following result is due to the first author of this paper [7], and independently, by M.-Y. Xu [28].

Theorem 2.7. *The group $\mathbb{Z}_p^3$, p is a prime, is a CI-group with respect to color digraphs.*

The above theorem is of interest here because of the following lemma. Recall that, for a group G, and a subgroup $H \leqslant G$, the *normal closure* H^G is the group $\langle g^{-1}Hg \colon g \in G\rangle$.

Lemma 2.8. *Let H be a group, and $G \leqslant S_H$ be a 2-closed group such that $H_L \leqslant G$. If H is a CI-group with respect to color digraphs, then $G = A[(H_L)^G]^{(2)}$, where $A = \mathrm{Aut}(H) \cap G$.*

Proof. Let $g \in G$. Then $g^{-1}H_Lg \leqslant (H_L)^G$, and as H is a CI-group with respect to color digraphs (see Theorem 2.7), by Lemma 2.6, there exists $\delta_g \in [(H_L)^G]^{(2)}$ such that $\delta_g^{-1}g^{-1}H_Lg\delta_g = H_L$. Then $g\delta_g$ normalizes H_L, and so by [6, Corollary 4.2B], we have that $g\delta_g \in \mathrm{Aut}(H)H_L$. As $H_L \leqslant [(H_L)^G]^{(2)}$, by replacing δ_g with an appropriate $\delta_g h_L$, we get that $g\delta_g \in A = \mathrm{Aut}(H) \cap G$, and the result follows. □

Definition 2.9. Let $G \leqslant S_n$ be a transitive permutation group, admitting complete block systems $\mathcal{A}$ and $\mathcal{B}$ consisting of m blocks of size k and k blocks of size m, respectively, where $mk = n$. If, whenever $A \in \mathcal{A}$ and $B \in \mathcal{B}$, we have that $|A \cap B| = 1$, then we say that $\mathcal{A}$ and $\mathcal{B}$ are *orthogonal*, and write $\mathcal{A} \perp \mathcal{B}$.

The following result is [9, Lemma 2.2].

Lemma 2.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be orthogonal block systems of G. Then G is equivalent to a subgroup of $S_m \times S_k$ with the natural coordinate-wise action.*

Definition 2.11. Let G be a transitive permutation group admitting a complete block system $\mathcal{B}$. For $B \in \mathcal{B}$, we define $\mathrm{Stab}_G(B) := \{g \in G : g(B) = B\}$. Thus $\mathrm{Stab}_G(B)$ is the set-wise stabilizer of the block $B \in \mathcal{B}$. We define $\mathrm{fix}_G(\mathcal{B}) := \{g \in G : g(B) = B \text{ for all } B \in \mathcal{B}\}$. Thus $\mathrm{fix}_G(\mathcal{B})$ is the subgroup of G which fixes every block of $\mathcal{B}$ set-wise. For $g \in G$, we define $g/\mathcal{B}$ to be the permutation induced by g acting on the blocks in $\mathcal{B}$, and set $G/\mathcal{B} := \{g/\mathcal{B} \colon g \in G\}$.

Remark 2.12. While not in the statement of [9, Lemma 2.2], several useful facts can be extracted from the proof of that result. Namely, G is in fact contained in $G/\mathcal{A} \times G/\mathcal{B}$, $(G/\mathcal{A}) \cap G = \mathrm{fix}_G(\mathcal{B})$, $(G/\mathcal{B}) \cap G = \mathrm{fix}_G(\mathcal{A})$, and thus $(G/\mathcal{A})/((G/\mathcal{A}) \cap G) \cong (G/\mathcal{B})/((G/\mathcal{B}) \cap G)$.

3 The primitive groups

In this section, we will compute the full automorphism group of every primitive 2-closed group that contains a regular subgroup isomorphic to $\mathbb{Z}_p^3$. Throughout this section, for $0 \leqslant i \leqslant k$, we let T_i be the subset of $\mathbb{Z}_m^k$ that consists of those elements of $\mathbb{Z}_m^k$ with exactly i coordinates that are 0.

Lemma 3.1. *Let $K \leqslant S_k$ be a transitive group, and let $G = K \wr S_m$ with the product action, so that G is primitive. Let Γ be an orbital digraph of G, so that Γ is a Cayley digraph of $\mathbb{Z}_m^k$ with connection set T. Then there exists $0 \leqslant i \leqslant k$ such that $T \subseteq T_i$; and if $i = 0, 1,$ or $k-1$, then $T = T_i$.*

Proof. Let $t = (a_1, \ldots, a_k) \in T$, i. e., the identity $\bar{0} := (0, \ldots, 0)$ in $\mathbb{Z}_m^k$ is adjacent to t in Γ. Let $0 \leqslant i \leqslant k$ be such that $t \in T_i$. Let $1 \leqslant j_1, \ldots, j_i \leqslant k$ such that $a_{j_\ell} = 0$, $1 \leqslant \ell \leqslant i$. As $S_m^k \leqslant G$, and acts coordinate-wise, after fixing $\bar{0}$ and letting $\mathrm{Stab}_{S_m^k}(\bar{0})$ act on t, we see that $(b_1, \ldots, b_k) \in T$, where $b_{j_\ell} = a_{j_\ell} = 0$, $1 \leqslant \ell \leqslant i$, and if $n \neq j_\ell$, $1 \leqslant \ell \leqslant i$, then $b_n \in (\mathbb{Z}_m \setminus \{0\})$. Hence $(b_1, \ldots, b_k) \in T_i$. As each T_i is invariant under permutation of coordinates, and $K \wr 1_{S_m}$ permutes the coordinates, we have that $T \subseteq T_i$.

If, in addition, $i = 0$, then the action of $\mathrm{Stab}_{S_m^k}(\bar{0})$ on t produces every element of T_0, and so $T = T_0$.

If $i = 1$, then the action of $\mathrm{Stab}_{S_m^k}(\bar{0})$ on t produces every element of T_1 that is 0 is some fixed coordinate (given by t), and as $K \wr 1_{S_m}$ permutes the coordinates transitively and fixes $\bar{0}$ we obtain in T every element with 0 in exactly one fixed coordinate, and so $T = T_1$.

If $i = k-1$, then the action of $\mathrm{Stab}_{S_m^k}(\bar{0})$ on t produces every element of T_{k-1} that is not 0 in some fixed coordinate (given by t), and as $K \wr 1_{S_m}$ permutes the coordinates transitively and fixes $\bar{0}$, we have that $T = T_{k-1}$. □

Proof. (PART (1) OF THEOREM 1.1) As G is primitive, by [22, Theorem 1.1], G is permutation isomorphic to a subgroup of $\mathrm{AGL}(3,p)$, or a subgroup of $S_3 \wr U$ with the product action, where U is a primitive group of degree p with nonabelian simple socle T, or $A_{p^3} \leqslant G \leqslant S_{p^3}$. As $(A_{p^3})^{(2)} = S_{p^3}$, we need consider only the case when $G \leqslant S_3 \wr U$. Then G has socle $\mathrm{soc}(G) = T^3$. By Theorem 2.1, T is doubly transitive, and so by Lemma 2.5, we have that $(T^3)^{(2)} = S_p^3 \leqslant G$. Therefore, $G = K \wr S_p$, where $K \leqslant S_3$ is a transitive group. By Lemma 3.1, the orbital digraphs of G are the Cayley digraphs of $\mathbb{Z}_p^3$ with connections sets T_0, T_1, and T_2 (using the notation of Lemma 3.1). It is easy to see that $S_3 \wr S_p$ is contained in the automorphism groups of all of these orbital digraphs, and as the 2-closure is the intersection of the automorphism groups of all orbital digraphs, we have that $G^{(2)} = S_3 \wr S_p$. This completes the proof of part (1) of Theorem 1.1. □

Recall that, a 2-closed simply primitive group $G \leqslant S_{p^2}$ is permutation isomorphic to either $S_2 \wr S_p$ with product action, or a subgroup of $\mathrm{AGL}(2,p)$ (see Theorem 2.4). This result in conjunction with the above proof may lead one to suspect that this may be the case in a more general context. Our goal in the next section is to show that this is in general far from being true.

4 Primitive 2-closed subgroups of $\mathbf{S_m \wr S_p}$

In this section, we digress from the main goal of this paper, and consider primitive 2-closed subgroups of $S_m \wr S_p$, with the product action. According to the O'Nan-Scott Theorem [6, Theorem 4.1A], a primitive group of prime-power degree p^m is either a subgroup of $\mathrm{AGL}(m,p)$, has nonabelian simple socle, or is isomorphic to a subgroup of $S_n \wr U$ with the product action, where U is primitive of degree a power p^d and $nd = m$. Guralnick [15] has determined all primitive groups of prime-power degree with nonabelian simple socle, and with one exception of degree 27, all are doubly transitive. Ignoring this exception as well as all primitive groups with the product action that can be constructed with it, we see that every other primitive group of prime-power degree constructed using the product action must have socle T^r for some r, where T is a doubly transitive nonabelian simple group. Then $(T^r)^{(2)} = S_{p^i}^r$, where T has degree p^i. Thus every 2-closed, simply primitive group of prime-power degree not a power of 3 is either a subgroup of $\mathrm{AGL}(m,p)$ or of $S_n \wr S_{p^i}$ with the product action. We remark that the results in this section are not used in the proof of Theorem 1.1. We begin with a definition.

Definition 4.1. A *k-uniform hypergraph* X is an ordered pair (V,E), where V is a set, and E is a subset of the set of all subsets of V of size k. An *automorphism* of X is a bijection $g : V \to V$ such that $g(e) \in E$ for every $e \in E$. We denote by $\mathrm{Aut}(X)$ the automorphism group of X, which is the set (group) of all automorphisms of X. We say X is *vertex-transitive* if $\mathrm{Aut}(X)$ acts transitively on V.

Theorem 4.2. *Let G be the automorphism group of a vertex-transitive k-uniform hypergraph of order m. Then $G \wr S_p$ with the product action is a transitive, primitive, 2-closed subgroup of S_{p^m}.*

Proof. Let X be a vertex-transitive k-uniform hypergraph of order m with $\mathrm{Aut}(X) = G$. Let $S = \{(a_1,\ldots,a_k) : a_i \neq 0, 1 \leqslant i \leqslant k\}$, and $\Gamma = \mathrm{Cay}(\mathbb{Z}_p^k, S)$. Note that $S_p^k \leqslant \mathrm{Aut}(\Gamma)$. For each $\{a_1,a_2,\ldots,a_k\} = T \in E(X)$, let ι_T be the natural inclusion map from $\mathbb{Z}_p^k$ to $\mathbb{Z}_p^m$ that maps the j^{th} coordinate of $\mathbb{Z}_p^k$ to the a_j^{th}-coordinate of $\mathbb{Z}_p^m$ as the identity, and is 0 in every other coordinate of $\mathbb{Z}_p^m$. Set $S' = \{\iota_T(S) : T \in E(X)\}$, and $\Gamma' = \mathrm{Cay}(\mathbb{Z}_p^m, S')$. We will show that $\mathrm{Aut}(\Gamma') = G \wr S_p$ with the product action.

We first show that $G \wr S_p \leqslant \mathrm{Aut}(\Gamma')$. By construction, Γ' is a Cayley graph of $\mathbb{Z}_p^m$. Thus $\mathrm{Aut}(\Gamma')$ contains the left-regular representation of $\mathbb{Z}_p^m$ as a transitive subgroup, as does $G \wr S_p$. Thus to settle $G \wr S_p \leqslant \mathrm{Aut}(\Gamma')$ it suffices to show that every element σ of $G \wr S_p$ that fixes the identity $\bar{0}$ in $\mathbb{Z}_p^m$ satisfies $\sigma(S') = S'$.

For $h \in S_m$, we denote by $\tilde{h}$ the element of $S_m \wr 1_{S_p}$ with the product action corresponding to h. Let $g \in G$. Then $\tilde{g}(\bar{0}) = \bar{0}$ and

$$\tilde{g}(S') = \tilde{g}\{\iota_T(S) : T \in E(X)\} = \{\iota_{g(T)}(S) : T \in E(X)\} = S',$$

as $g \in G = \mathrm{Aut}(X)$. Thus $G \wr 1_{S_p} \leqslant \mathrm{Aut}(\Gamma')$.

Let $\bar{t} \in S'$, and T be a set in $E(X)$ such that $\bar{t} \in \iota_T(S)$, and let $h \in \mathrm{Stab}_{S_p^m}(\bar{0})$. As $\mathrm{Aut}(\Gamma) = S_p^k$, $\iota_T^{-1}h(\iota_T(S)) = S$. Then

$$h(\bar{t}) = \iota_T\iota_T^{-1}h(\bar{t}) \in \iota_T\iota_T^{-1}h(\iota_T(S)) = \iota_T(S) \subseteq S',$$

and $h \in \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_p^m, S'))$. Thus $S_p^m \leqslant \mathrm{Aut}(\Gamma')$. Thus $G \wr S_p$ with the product action is contained in $\mathrm{Aut}(\Gamma')$.

It now follows by [6, Lemma 2.7A] that $G \wr S_p$ with the product action is primitive so that $\mathrm{Aut}(\Gamma')$ is primitive. As $\mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_p^m, S'))$ contains $(\mathbb{Z}_p^m)_L$, by [22, Theorem 1.1] we have that $\mathrm{Aut}(\Gamma') \leqslant H \wr S_p$ with the product action, for some $H \leqslant S_m$ with $G \leqslant H$.

In order to show that $\mathrm{Aut}(\Gamma') \leqslant G \wr S_p$, it suffices to show that $H \leqslant G$. Let $h \in H$, $T \in E(X)$, and $\bar{s} = (s_1, \dots, s_k) \in S$ such that $s_i \neq 0$ for all $1 \leqslant i \leqslant k$, and $\bar{t} = \iota_T(\bar{s})$. Then $\tilde{h}(\bar{0}\bar{t}) = \tilde{h}(\bar{0})\tilde{h}(\bar{t}) = \bar{0}\tilde{h}(\bar{t})$. Thus $\tilde{h}(\bar{t}) \in S'$. Also $\tilde{h}$ maps the set of nonzero coordinates T of $\bar{t}$ to the set of nonzero coordinates $h(T)$ of $\tilde{h}(\bar{t})$. As the set of nonzero coordinates of $\bar{t}$ are T, and the set of nonzero coordinates of $\tilde{h}(\bar{t})$ form an edge of X, we have that $h(T) \in E(X)$. Thus $H \leqslant \mathrm{Aut}(X) = G$, so that $\mathrm{Aut}(\Gamma') \leqslant G \wr S_p$ and the result follows. $\square$

Remark 4.3. It is known that every 2-closed group is also k-closed [27, Theorem 5.10]. It is then apparent that the problem of determining all transitive 2-closed groups is "easier" than determining all transitive k-closed groups for a fixed $k \geqslant 3$. The above result essentially says that this may not in fact be the case. The automorphism group of a k-uniform hypergraph is k-closed (although we remark that there are certainly k-closed groups which are not the automorphism group of a k-uniform hypergraph), and the previous result basically states that in order to determine the transitive 2-closed subgroups of S_{p^m}, we must already know many (those that are the automorphism groups of k-uniform hypergraphs) transitive k-closed groups of degree m.

Theorem 4.4. *Let Γ be an orbital digraph of a 2-closed primitive subgroup G of $S_m \wr S_p$, p a prime, with the product action, where G has nonabelian socle. Let $\bar{a} = (a_1, \dots, a_m)$ be a neighbor of $(0, \dots, 0)$ in Γ, and $U = \{i : a_i \neq 0\}$. Then*

(1) $G = H \wr S_p$ with product action, where $H \leqslant S_m$ is a transitive group.

(2) $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(X) \wr S_p$ with the product action, where X is the k-uniform hypergraph defined by $V(X) := \mathbb{Z}_m$, and $E(X) := \{h(U) : h \in H\}$, where $k = |U|$.

Proof. As $S_m \wr S_p$ with the product action has degree p^m and G has nonabelian socle, we have by [6, Theorem 4.1A] that $\mathrm{soc}(G) = T^m$ for some nonabelian simple group T of degree p. By Theorem 2.1, we have that T is doubly transitive. As the 2-closure of a doubly transitive group is a symmetric group, by Lemma 2.5 we have that $(T^m)^{(2)} = S_p^m$, so that $T = A_p$. Then $(T^m)^{(2)} \leqslant G^{(2)} \leqslant \mathrm{Aut}(\Gamma)$. We conclude that $G = H \wr S_p$ with the product action for some transitive group $H \leqslant S_p$, and $\mathrm{Aut}(\Gamma) = L \wr S_p$ with the product action for some $H \leqslant L \leqslant S_m$, in particular, (1) follows.

In part (2), as $\bar{a} = (a_1, \dots, a_m)$ is a neighbor of $\bar{0} = (0, \dots, 0)$ in Γ, we have that $\bar{0}(b_1, \dots, b_m) \in E(\Gamma)$, where $b_i = a_i$ if $a_i = 0$ and $b_i \in \mathbb{Z}_p^*$ if $a_i \neq 0$ as $S_p^m \leqslant \mathrm{Aut}(\Gamma)$. Then $(-a_1, \dots, -a_m)$ is a neighbor of $\bar{0}$, so that Γ is a graph.

Observe that Γ is a Cayley graph of $\mathbb{Z}_p^m$, and as Γ is an orbital digraph, Γ is arc-transitive. Let $\Gamma = \mathrm{Cay}(\mathbb{Z}_p^m, S)$. Thus $\mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(\bar{0})$ is transitive on S. Note that any element $\gamma \in \mathrm{Stab}_{S_p^m}(\bar{0})$ maps the nonzero coordinates of any element s of $\mathbb{Z}_p^m$ bijectively to the nonzero coordinates of $\gamma(s)$, as does any element $\gamma \in L \wr 1_{S_p}$. As $\mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(\bar{0}) = \langle \mathrm{Stab}_{S_p^m}(\bar{0}), L \wr 1_{S_p} \rangle$, we have that every element of S contains exactly the same number of nonzero coordinates, and an element $\gamma \in \mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(\bar{0})$ maps the nonzero coordinates of $\bar{s} \in S$ to the nonzero coordinates of $\gamma(s)$. We first show that $\mathrm{Aut}(X) \wr S_p \leqslant \mathrm{Aut}(\Gamma)$.

Let $x \in \mathrm{Aut}(X)$ and $e \in E(\Gamma)$. Denote by $\tilde{x}$ the element of $\mathrm{Aut}(X) \wr 1_{S_p}$ corresponding to x. As $1_{S_m} \wr S_p = S_p^m \leqslant G^{(2)} \leqslant \mathrm{Aut}(\Gamma)$ and is transitive, there exists $\delta \in S_p^m$ such that one endpoint of $\delta(e)$ is $\bar{0}$. As $\tilde{x} \in \mathrm{Aut}(\Gamma)$ if and only if $\tilde{x}\delta \in \mathrm{Aut}(\Gamma)$, we can and do assume that one endpoint of e is $\bar{0}$. Let $\bar{c} = (c_1, \ldots, c_m)$ denote the endpoint of e that is not $\bar{0}$ so that $c \in S$, and let $V = \{i : c_i \neq 0\}$. As G acts arc-transitively on Γ, there is some $g \in G$ that maps the arc from $\bar{0}$ to $\bar{a}$ to the arc from $\bar{0}$ to $\bar{c}$. Then g stabilizes $\bar{0}$ and maps $\bar{a}$ to $\bar{c}$. Let $g = \tilde{h}\delta$, where $h \in H$, $\delta \in S_p^m$. As g maps the nonzero coordinates of $\bar{a}$ to the nonzero coordinates of $\bar{c}$ bijectively, we have that $h(U) = V$, and so $V \in E(X)$.

As $x \in \mathrm{Aut}(X)$, $x(V) \in E(X)$, there is some element of S that is 0 in every coordinate not contained in $x(V)$ and is not 0 in every coordinate contained in $x(V)$. Hence $(d_1, \ldots, d_m) \in S$, where $d_i = 0$ if $i \notin x(V)$ and if $i \in x(V)$, then $d_i \in \mathbb{Z}_p^*$. Then $\tilde{x}(c_1, \ldots, c_m) = (d_1, \ldots, d_m)$ where $d_i = 0$ if $i \notin x(V)$ and $d_i \in \mathbb{Z}_p^*$ if $i \in x(V)$. Thus $\tilde{x}(e) \in E(\Gamma)$ and $\tilde{x} \in \mathrm{Aut}(\Gamma)$. Thus $\mathrm{Aut}(X) \wr S_p \leqslant \mathrm{Aut}(\Gamma)$.

Suppose now that $f \in \mathrm{Aut}(\Gamma)$. We write $f = \tilde{\ell}\delta$, where $\tilde{\ell} \in L \wr 1_{S_p}$ and $\delta \in S_p^m$. As $S_p^m \leqslant \mathrm{Aut}(X) \wr S_p$, it suffices to show that $\ell \in \mathrm{Aut}(X)$ (using the same notation as above backwards). Let $W \in E(X)$, so that $W = h(U)$ for some $h \in H$. As every element of S contains exactly the same number of nonzero coordinates, there exists $\bar{s} \in S$ such that $\bar{s}$ is nonzero precisely in the coordinates contained in W. As Γ is an orbital digraph, there exists $g \in \mathrm{Stab}_G(\bar{0})$ such that $g(\bar{s}) = \tilde{\ell}(\bar{s})$ (i.e. the image of the edge from $\bar{0}$ to $\bar{s}$ under g and $\tilde{\ell}$ are the same). Let $h' \in H$ such that $g = \tilde{h}'\delta'$, $\delta' \in S_p^m$. Then $\ell(W) = h'(W) = (h'h)(U) \in E(X)$ and so $\ell \in \mathrm{Aut}(X)$. Thus $\mathrm{Aut}(\Gamma) \leqslant \mathrm{Aut}(X) \wr S_p$ and so $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(X) \wr S_p$. □

5 The imprimitive groups

Before starting to derive the groups in part (2) of Theorem 1.1, we prove some more general results.

Definition 5.1. A complete block system $\mathcal{B}$ of a permutation group G is *genuine* if $\mathcal{B}$ is formed by the orbits of a normal subgroup of G.

Lemma 5.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be genuine orthogonal complete block systems of a 2-closed permutation group G, with $\mathcal{A}$ consisting of m blocks of size k. Then G contains a transitive normal subgroup $L = X \times Y$, where $\mathrm{fix}_G(\mathcal{B}) = X \leqslant S_m$ and $\mathrm{fix}_G(\mathcal{A}) = Y \leqslant S_k$ are 2-closed groups.*

Furthermore, if G contains a regular abelian CI-group H with respect to color digraphs, then $G = A(X \times Y)$, where $A = \mathrm{Aut}(H) \cap G$.

Proof. As both $\mathcal{A}$ and $\mathcal{B}$ are genuine, we have that $X := \mathrm{fix}_G(\mathcal{A}) \neq 1$, and $Y := \mathrm{fix}_G(\mathcal{B}) \neq 1$. As $\mathcal{A} \perp \mathcal{B}$, we have that $X \cap Y = 1$. Hence $\langle X, Y\rangle \cong X \times Y$ and $\langle X, Y\rangle \triangleleft G$ as $X, Y \triangleleft G$. Let G act on Ω, and let $\omega_1, \omega_2 \in \Omega$. Then there exists $A \in \mathcal{A}$ such that $\omega_1 \in A$, and $B \in \mathcal{B}$ such that $\omega_2 \in B$. As $\mathcal{A} \perp \mathcal{B}$, $A \cap B$ is a singleton, say $\{\omega_3\}$. Also, as $\mathcal{A}$ and $\mathcal{B}$ are genuine, $\mathrm{fix}_G(\mathcal{A})$ acts transitively on A and $\mathrm{fix}_G(\mathcal{B})$ acts transitively on B. Then there exists $\alpha \in \mathrm{fix}_G(\mathcal{A})$ such that $\alpha(\omega_1) = \omega_3$ and $\beta \in \mathrm{fix}_G(\mathcal{B})$ such that $\beta(\omega_3) = \omega_2$. Then $\beta\alpha(\omega_1) = \omega_2$ and $\langle X, Y\rangle$ is transitive. By Lemma 2.5 we have that $(X \times Y)^{(2)} = X^{(2)} \times Y^{(2)}$

and $X \leqslant X^{(2)} \leqslant \mathrm{fix}_{(X\times Y)^{(2)}}(\mathcal{A}) = X$, so that $X = X^{(2)}$. A similar argument then shows that $Y^{(2)} = Y$.

Now suppose G contains a regular abelian subgroup H which is a CI-group with respect to color digraphs. As a transitive abelian group is regular [26, Proposition 4.3], we must have that $\mathrm{fix}_H(\mathcal{A}) \neq 1 \neq \mathrm{fix}_H(\mathcal{B})$. As above, $\mathrm{fix}_H(\mathcal{A}) \cap \mathrm{fix}_H(\mathcal{B}) = 1$ and $\langle \mathrm{fix}_H(\mathcal{A}), \mathrm{fix}_H(\mathcal{B})\rangle$ is transitive, so that $\langle \mathrm{fix}_H(\mathcal{A}), \mathrm{fix}_H(\mathcal{B})\rangle = H$. Thus $H \leqslant X \times Y$, and as $(X\times Y) \triangleleft G$, we have that $H^G \leqslant (X \times Y)$. Hence $[H^G]^{(2)} \leqslant (X\times Y)^{(2)} = X \times Y$. By Lemma 2.8, $X \times Y = A_1[(H^G)^{(2)}]$, $A_1 = \mathrm{Aut}(H) \cap (X \times Y)$, and $G = A[(H^G)^{(2)}]$, $A = \mathrm{Aut}(H) \cap G$. Then $A_1 \leqslant A$, and

$$G = A[(H^G)^{(2)}] = AA_1[(H^G)^{(2)}] = A[A_1[(H^G)^{(2)}]] = A(X \times Y).$$

□

Corollary 5.3. *Let $G \leqslant S_n$ be a transitive 2-closed group, such that G contains a regular abelian CI-group H with respect to color digraphs. If $(H^G)^{(2)}$ admits orthogonal complete block systems $\mathcal{A}$ and $\mathcal{B}$, with $\mathcal{A}$ consisting of m blocks of size k. Then there exist 2-closed groups $X \leqslant S_m$ and $Y \leqslant S_k$, such that $G = A(X \times Y)$, where $A = \mathrm{Aut}(H) \cap G$.*

Proof. By Lemma 5.2, there exist 2-closed groups $X \leqslant S_m$ and $Y \leqslant S_k$ such that $(H^G)^{(2)} = A_1(X \times Y)$, where $A_1 = \mathrm{Aut}(H) \cap (H^G)^{(2)}$. By Lemma 2.8, $G = A[A_1(X\times Y)] = A(X \times Y)$, where $A = \mathrm{Aut}(H) \cap G$. □

Let $G \leqslant S_{p^3}$ be a 2-closed group, such that G contains a regular elementary abelian subgroup, and let P be a Sylow p-subgroup of G. Then $P^{(2)} \leqslant G$, and since $P^{(2)}$ is a p-group (see [27, Exercise 5.28]), we have that P is 2-closed. Thus P is described by the following result which is explicit in [28], and implicit in [7].

Theorem 5.4. *Let $P \leqslant S_{p^3}$ be a transitive 2-closed p-group, such that P contains a regular elementary abelian subgroup, where p is an odd prime. Then P is permutation isomorphic to one of the following groups:*

1. $\mathbb{Z}_p^3$,
2. $\mathbb{Z}_p \wr (\mathbb{Z}_p \wr \mathbb{Z}_p)$,
3. $\mathbb{Z}_p \wr (\mathbb{Z}_p \times \mathbb{Z}_p)$,
4. $(\mathbb{Z}_p \times \mathbb{Z}_p) \wr \mathbb{Z}_p$,
5. $(\mathbb{Z}_p \wr \mathbb{Z}_p) \times \mathbb{Z}_p$,
6. $\mathbb{Z}_p^3 \rtimes \langle\gamma\rangle$, *where* $\gamma((i,j,k)) = (i, j+i, k+j)$, $(i,j,k) \in \mathbb{Z}_p^3$.

Below we go through cases (1)-(6) separately. Part (2) of Theorem 1.1 will follow directly from Proposition 5.7, Lemma 5.8, and Propositions 5.12 and 5.13.

5.1 Case (1)

In this subsection, we deal with the most difficult case - when a Sylow p-subgroup P is a regular elementary abelian subgroup. We begin with verifying a special case of a conjecture of the first author [10, Conjecture 6.8].

Definition 5.5. Let $\mathcal{A}$ and $\mathcal{B}$ be two complete block systems of a permutation group G. We write $\mathcal{A} \preceq \mathcal{B}$ if any block in $\mathcal{B}$ is a union of blocks in $\mathcal{A}$ (note that, $\mathcal{A} \prec \mathcal{B}$ is used in the usual meaning, i. e., $\mathcal{A} \preceq \mathcal{B}$ but $\mathcal{A} \neq \mathcal{B}$).

Below we say that, a series $\mathcal{B}_1 \prec \cdots \prec \mathcal{B}_\ell$ of complete block systems of G is *maximal*, if there is no nontrivial complete block system $\mathcal{B}$ of G for which either $\mathcal{B} \prec \mathcal{B}_1$, $\mathcal{B}_i \prec \mathcal{B} \prec \mathcal{B}_{i+1}$, or $\mathcal{B}_\ell \prec \mathcal{B}$ for some $1 \leqslant i \leqslant \ell - 1$.

Theorem 5.6. *Let $G \leqslant S_{p^k}$ be a transitive group with an abelian Sylow p-subgroup P, and a maximal series $\mathcal{B}_1 \prec \cdots \prec \mathcal{B}_\ell$ of genuine complete block systems $\mathcal{B}_i$ of G, where if $B_i \in \mathcal{B}_i$ and $B_{i+1} \in \mathcal{B}_{i+1}$, then $|B_{i+1}|/|B_i| \leqslant p^2$. Then P^G is permutation isomorphic to a direct product $\Pi_{i=1}^r G_i$ with the coordinate-wise action, and each G_i is either cyclic of prime-power order, or a doubly transitive nonabelian simple group.*

Proof. Note that P is transitive by [26, Theorm 3.4]. We proceed by induction on k. If $k = 1$, then the result follows by Theorem 2.1. Let $k \geqslant 2$ and assume that the result is true for all $i < k$ and let $G \leqslant S_{p^k}$ satisfy the hypothesis. Then G admits a genuine complete block system $\mathcal{B}_1$ consisting of p^{k-m} blocks of size p^m, where $m = 1$ or $m = 2$. As P is abelian, P is regular [26, Proposition 4.4], and so a Sylow p-subgroup of $\mathrm{fix}_G(\mathcal{B}_1)$ is of order p^m and is abelian. Note that $\mathrm{fix}_G(\mathcal{B}_1)$ must act faithfully on each $B \in \mathcal{B}_1$. This follows as if $\mathrm{fix}_G(\mathcal{B}_1)$ does not act faithfully on some $B \in \mathcal{B}_1$, then the kernel K of this action is nontrivial on some block $B' \in \mathcal{B}_1$ and normal, and so has orbits of order p or p^m on B'. Thus p divides the order of K. We can then conclude that a Sylow p-subgroup of $\mathrm{fix}_G(\mathcal{B}_1)$ does not have order p^m. Thus $\mathrm{fix}_G(\mathcal{B}_1)$ acts faithfully on each $B \in \mathcal{B}_1$. The complete block system $\mathcal{B}_1$ is minimal, and so $\mathrm{Stab}_G(B)$ acts primitively on every block $B \in \mathcal{B}_1$. Applying Theorems 2.1 and 2.4 to the normal subgroup $\mathrm{fix}_G(\mathcal{B}_1)|_B \triangleleft \mathrm{Stab}_G(B)|_B$, results in the cases below.

(1) $\mathrm{fix}_G(\mathcal{B}_1)|_B$ *is doubly transitive with nonabelian socle.*

By [10, Lemma 4.3], we have that P^G is permutation isomorphic to $G_1 \times K$, where $G_1 \leqslant S_{p^m}$ and $K \leqslant S_{p^{k-m}}$ with the canonical action such that $G_1 \cong \mathrm{fix}_{P^G}(\mathcal{B}_1)$ and $K \cong P^G/\mathcal{B}_1$ (we observe that not all of this information is contained in the statement of [10, Lemma 4.3], but can be extracted from the proof of that lemma). Then $\mathrm{soc}(G_1)$ is a doubly transitive nonabelian simple group, and as $\mathrm{soc}(G_1)$ char G_1, we obtain that $G_1 = \mathrm{soc}(G_1)$ is a doubly transitive nonabelian simple group. By the induction hypothesis, as $P^G/\mathcal{B}_1 = (P/\mathcal{B}_1)^{G/\mathcal{B}_1}$, we have that K is permutation isomorphic to $\Pi_{i=2}^r G_i$ with the canonical action, where each G_i is cyclic, or a doubly transitive nonabelian simple group. We conclude that P^G is permutation isomorphic to $\Pi_{i=1}^r G_i$ with the canonical action where each G_i is either cyclic, or a doubly transitive nonabelian simple group.

(2) $\mathrm{fix}_G(\mathcal{B}_1)|_B$ *is permutation isomorphic to a subgroup of* $\mathrm{AGL}(m,p)$.

As $\mathrm{fix}_G(\mathcal{B}_1)$ acts faithfully on $B \in \mathcal{B}_1$, we have that $\mathrm{fix}_G(\mathcal{B}_1)$ contains a normal Sylow p-subgroup. Then $\mathrm{fix}_{P^G}(\mathcal{B}_1) = \mathrm{fix}_P(\mathcal{B}_1)$ is semiregular of order p^m by [10, Lemma 5.1] and the fact that a Sylow p-subgroup of $\mathrm{fix}_P(\mathcal{B}_1)$ is semiregular. By the induction hypothesis, we have that $P^G/\mathcal{B}_1 = \Pi_{i=2}^r G_i$ with the canonical action and each G_i is either cyclic, or a doubly transitive nonabelian simple group. Then P^G is permutation isomorphic to a direct product of abelian groups of prime power order and nonabelian simple groups with the canonical action by [10, Lemma 5.5].

(3) $m = 2$, *and* $\mathrm{fix}_G(\mathcal{B}_1)|_B$ *is permutation isomorphic to a subgroup of* $S_2 \wr H$, *where* H *is a doubly transitive group of degree p with nonabelian socle.*

Thus $p \geqslant 5$, and a Sylow p-subgroup of $\mathrm{fix}_G(\mathcal{B}_1)$ is elementary abelian, so P has at least 2 elementary divisors. It then follows by [10, Lemma 3.6] that P^G admits a complete block system $\mathcal{B}$ whose blocks are strictly contained in blocks of $\mathcal{B}_1$. This case then reduces to those considered above (replacing G with P^G and observing that as P is a Sylow p-subgroup, P^G is generated by all Sylow p-subgroups of G, all of which are contained in P^G, and P^{P^G} is generated by all Sylow p-subgroups of P^G, all of which are contained in P^{P^G}, so that $P^G = P^{P^G}$), and the result follows by induction. □

The groups appearing in case (1) are given by the following proposition.

Proposition 5.7. *Let* $G \leqslant S_{p^3}$ *be* 2*-closed and imprimitive with a Sylow p-subgroup* $P \cong \mathbb{Z}_p^3$. *Then* G *is permutation isomorphic to one of the following groups:*

(1) a subgroup of $\mathrm{AGL}(3,p)$*;*

(2) $S_p \times X$, *where* $X \leqslant S_{p^2}$ *is a* 2*-closed group;*

(3) $A((S_p \times S_p) \times X)$, *where* $X < \mathrm{AGL}(1,p)$, *and* $A \leqslant \mathrm{Aut}(\mathbb{Z}_p^3)$.

Proof. By Theorem 5.6, we have that P^G is permutation isomorphic to $K_1 \times H$, or $K_1 \times K_2 \times K_3$ with the coordinate-wise action, where $H \leqslant S_{p^2}$ is a primitive nonabelian simple group, and $K_i \leqslant S_p$ is a transitive simple group, and $i = 1,2,3$. As by Lemma 2.5 for transitive permutation groups M and N, $(M \times N)^{(2)} = M^{(2)} \times N^{(2)}$, we have that $(P^G)^{(2)} = H \times K \times L$ or $H \times S_{p^2}$, where $H, K, L = \mathbb{Z}_p$ or S_p. Note, however, that a Sylow p-subgroup of $H \times S_{p^2}$ is not elementary abelian, so the only possibility is that $(P^G)^{(2)} = H \times K \times L$.

If $H = K = L = \mathbb{Z}_p$, then $(P^G)^{(2)} = P^G \triangleleft G$ so that $G \leqslant \mathrm{AGL}(3,p)$ and (1) follows. We may thus assume that $p \geqslant 5$. If $H = K = L = S_p$, then as $P \leqslant A_{p^3}$, it must be the case that $P^G = A_p^3$. Notice that as $p \geqslant 5$, we have that A_p is simple. Also, we must have that any nontrivial normal subgroup of A_p^3 is either A_p or A_p^2 (as factors). As G is imprimitive, G admits a complete block system $\mathcal{B}$ consisting of p^i blocks of size p^{3-i}, where $i = 1$ or $i = 2$. Then $\mathrm{fix}_G(\mathcal{B}) \cap P^G$ is a normal subgroup of G, and so of $P^G = A_p^3$, and has Sylow p-subgroup of order p^{3-i}. We conclude that $\mathrm{fix}_G(\mathcal{B}) \cap P^G = A_p^{3-i}$, and so the centralizer of $\mathrm{fix}_G(\mathcal{B}) \cap P^G$ in G is S_p^i. As the centralizer of a normal subgroup is normal, we have that $S_p^i \triangleleft G$, and so G admits a complete block system $\mathcal{C}$ consisting of p^{3-i} blocks of size p^i. It is also not difficult to see that $\mathcal{B} \perp \mathcal{C}$, and so $G \leqslant S_p \times S_{p^2}$, see Lemma 2.10. As $S_p^3 \leqslant G$, we conclude that $G = S_p \times X$, $X \leqslant S_{p^2}$, and as by Lemma 2.5, $S_p \times X = G^{(2)} = S_p^{(2)} \times X^{(2)}$, we have that X is 2-closed, and so (2) follows.

The only remaining possibility is that exactly two of H, K, and L are S_p or $\mathbb{Z}_p$ and the remaining group is either $\mathbb{Z}_p$ or S_p. Let k be the number of the groups H, K, and L that are S_p and j the number that are $\mathbb{Z}_p$. Then $k + j = 3$, and $P^G = A_p^k \times \mathbb{Z}_p^j$. Then the center of P^G, $C(P^G)$, is nontrivial and $C(P^G) = \mathbb{Z}_p^j$. As the center of a group is characteristic, $\mathbb{Z}_p^j \triangleleft G$. Similarly, the commutator subgroup of P^G, $(P^G)'$, is also nontrivial and $(P^G)' = A_p^k$. As the commutator subgroup of a group is also characteristic, we have that $A_p^k \triangleleft G$. We conclude that G admits orthogonal complete block systems formed by the orbits of $\mathbb{Z}_p^j$ and A_p^k, respectively, and we may assume that $\mathcal{B}$ is formed by the orbits of $\mathbb{Z}_p^j$. We denote the orbits of A_{p^k} by $\mathcal{C}$, so $\mathcal{B} \perp \mathcal{C}$. By Lemma 2.10, we have that $G \leqslant S_{p^k} \times S_{p^j}$.

Now if $k = 1$, then as $(S_p \times \mathbb{Z}_p \times \mathbb{Z}_p) \leqslant G$ and $(S_p \times \mathbb{Z}_p \times \mathbb{Z}_p)/\mathcal{B} = S_p$, we must have that $G = S_p \times X$, $X \leqslant S_{p^2}$, and we conclude X is a 2-closed group by repeating the above argument, and so (2) follows.

Thus let $k = 2$, and so $S_p \times S_p \leqslant G/\mathcal{B}$, and $G/\mathcal{C} < \mathrm{AGL}(1, p)$. If $G/\mathcal{B} = S_p \times S_p$, then $G \leqslant S_p \times S_p \times S_p$, and thus G admits a complete block system $\mathcal{D}$ consisting of p blocks of size p^2 such that $G/\mathcal{D} = S_p$. Then (2) follows again by arguments at the end of the preceding paragraph. Otherwise, $G/\mathcal{B}$ must be primitive by [13, Theorem 4], and also by [13, Theorem 4], we have that $G/\mathcal{B} = S_2 \wr S_p$ with the product action. Then G is a subgroup of $(S_2 \wr S_p) \times G/\mathcal{C}$. By Lemma 5.2, $G = A(\mathrm{fix}_G(\mathcal{C}) \times \mathrm{fix}_G(\mathcal{B}))$. We know that $\mathrm{fix}_G(\mathcal{C}) \leqslant S_2 \wr S_p$ and $S_p \times S_p \leqslant \mathrm{fix}_G(\mathcal{C})$. If $\mathrm{fix}_G(\mathcal{C}) = S_2 \wr S_p$ then the result follows with $X = \mathrm{fix}_G(\mathcal{B})$. Otherwise, $\mathrm{fix}_G(\mathcal{C}) = S_p \times S_p$ and $\mathrm{fix}_G(\mathcal{B}) = X$, where X has index two in $G/\mathcal{C}$, and $G = A(S_p \times S_p \times X)$. □

5.2 Cases (2)-(4)

In this subsection, we dispose of the cases where the Sylow p-subgroup P can be written as a nontrivial wreath product. Specifically, we handle cases (2), (3) and (4) of Theorem 5.4. The required groups are the wreath products given in (2) (b) of Theorem 1.1. This follows directly from the next lemma.

Lemma 5.8. *Let $G \leqslant S_n$ be a transitive group such that G contains a transitive subgroup H of the form $H = H_1 \wr H_2$, where $H_1 \leqslant S_m$, $H_2 \leqslant S_k$, and $mk = n$. Then $G^{(2)} = G_1 \wr G_2$, where $G_1 \leqslant S_{m_1}$ and $G_2 \leqslant S_{k_1}$ are 2-closed groups, $m_1 k_1 = n$.*

Proof. Let $\Gamma_1, \ldots, \Gamma_r$ be the orbital digraphs of G. Let $\mathcal{B}$ be the complete block system of $H_1 \wr H_2$ formed by the orbits of $1_{H_1} \wr H_2$. Then in Γ_i, if there is a directed edge from B to B', $B, B' \in \mathcal{B}$, then there is a directed edge from every vertex of B to every vertex of B'. We conclude that $\Gamma_i = D_{i,1} \wr D_{i,2}$, where $D_{i,1}$ is a digraph of order m and $D_{i,2}$ is a digraph of order k. Coloring the edges of $D_{i,j}$, $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant 2$, with color i, we have that $D_1 = \cup_{i=1}^r D_{i,1}$ and $D_2 = \cup_{i=1}^r D_{i,2}$ are color digraphs of order m and k, respectively.

Further, setting $D = D_1 \wr D_2$, it is apparent that $\mathrm{Aut}(D) = G^{(2)}$. The result then follows by [12, Theorem 5.7]. $\square$

For the rest of the paper we fix the following notation: let $\tau_1, \tau_2, \tau_3 : \mathbb{Z}_p^3 \to \mathbb{Z}_p^3$ be given by

$$\tau_1(i,j,k) := (i+1,j,k),\ \tau_2(i,j,k) := (i,j+1,k),\ \tau_3(i,j,k) := (i,j,k+1).$$

Hence $\langle \tau_1, \tau_2, \tau_3 \rangle$ is the left (and right) regular representation of $\mathbb{Z}_p^3$. Further, for $1 \leqslant i, j \leqslant 3$, we denote by $\mathcal{B}_i$ the partition of $\mathbb{Z}_p^3$ into the orbits of τ_i, and by $\mathcal{B}_{i,j}$ the partition consisting of the orbits of $\langle \tau_i, \tau_j \rangle$.

5.3 Case (5)

In this subsection, we let $G \leqslant S_{\mathbb{Z}_p^3}$ be a 2-closed imprimitive group, such that G has a Sylow p-subgroup

$$P := \langle \tau_1, \tau_2, \tau_3|_B : B \in \mathcal{B}_{2,3} \rangle,$$

where $\tau_3|_B((i,j,k)) := (i,j,k+1)$ if $(i,j,k) \in B$, and $\tau_3|_B((i,j,k)) := (i,j,k)$ otherwise. In the next three preparatory lemmas we show that G admits complete block systems with block size both p and p^2.

Lemma 5.9. *G admits a complete block system of p^2 blocks of size p.*

Proof. To the contrary assume that G does not admit a complete block system consisting of p^2 blocks of size p. Then G admits a complete block system $\mathcal{B}$ consisting of p blocks of size p^2. Note that as a Sylow p-subgroup of $G/\mathcal{B}$ has order p, we must have that $\langle \tau_3|_B : B \in \mathcal{B}_{2,3} \rangle \leqslant \mathrm{fix}_G(\mathcal{B})$.

We claim that $\mathcal{B} \neq \mathcal{B}_{2,3}$. To the contrary assume that $\mathcal{B} = \mathcal{B}_{2,3}$, and pick an orbit T of $\mathrm{Stab}_G(\bar{0})$, $T \not\subset \langle (0,1,0), (0,0,1) \rangle$. Now the Cayley digraph $\mathrm{Cay}(\mathbb{Z}_p^3, T)$ is an orbital digraph of G. Let $H \leqslant \mathbb{Z}_p^3$ such that $T + H = T$, i. e., H is largest subgroup in $\mathbb{Z}_p^3$ such that T is a union of cosets of H. Then $|H| \neq 1$, as $\langle (0,0,1) \rangle \leqslant H$. It can be proved using [26, Proposition 23.5] and following the proof of [26, Theorem 24.12] that, the cosets of H form a complete block system of G. It follows that $|H| = p^2$, and we readily deduce that $H = \langle (0,1,0), (0,0,1) \rangle$. As the orbit T was arbitrarily chosen not in $\langle (0,1,0), (0,0,1) \rangle$, it is then not difficult to see that any orbital digraph of G is isomorphic to a wreath product of a circulant digraph of order p and a vertex-transitive graph of order p^2. As G is 2-closed, we conclude that $(\mathbb{Z}_p \wr \mathbb{Z}_p^2) \leqslant G$, and so $p \cdot (p^2)^p$ divides $|G|$. However, a Sylow p-subgroup of G has order $|P| = p^2 \cdot p^p$, we must have that $p + 2 \geqslant 2p + 1$ so that $p \leqslant 1$, a contradiction.

As $\mathcal{B} \neq \mathcal{B}_{2,3}$, we find that $\mathrm{fix}_P(\mathcal{B})$ is faithful on every block $B \in \mathcal{B}$. From this we deduce that $\mathrm{fix}_G(\mathcal{B})$ is also faithful on every block $B \in \mathcal{B}$. For otherwise, there exists $K \triangleleft \mathrm{fix}_G(\mathcal{B})$ such that $K|_B$ is nontrivial while $K|_{B'}$ is trivial, $B, B' \in \mathcal{B}$. By the previous observation p does not divide $|K|$. But, as $K|_B \lhd \mathrm{fix}_G(\mathcal{B})|_B$, it has orbits of the same size p^m, $m \geqslant 1$, and hence $|K|$ is divisible by p, a contradiction.

As $\mathrm{fix}_G(\mathcal{B})$ acts faithfully on $B \in \mathcal{B}$ and a Sylow p-subgroup of $\mathrm{fix}_G(\mathcal{B})$ has order p^{p+1} (as $|P| = p^{p+2}$ and a Sylow p-subgroup of $G/\mathcal{B}$ has order p), and as a Sylow p-subgroup of S_{p^2} has order p^{p+1} and is isomorphic to $\mathbb{Z}_p \wr \mathbb{Z}_p$, we see that a Sylow p-subgroup of $\mathrm{fix}_G(\mathcal{B})|_B$ is isomorphic to $\mathbb{Z}_p \wr \mathbb{Z}_p$. Also observe that $\mathrm{Stab}_G(B)|_B$ is primitive for every $B \in \mathcal{B}$ by [6, Exercise 1.5.10]. By [13, Theorem 6] and [8, Lemma 17], the only primitive groups with Sylow p-subgroup isomorphic to $\mathbb{Z}_p \wr \mathbb{Z}_p$ are A_{p^2} and S_{p^2} (if $p = 2$, then only S_{p^2}). Whence $\mathrm{soc}(\mathrm{Stab}_G(B)|_B) = A_{p^2}$, and as $\mathrm{fix}_G(\mathcal{B})|_B \lhd \mathrm{Stab}_G(B)|_B$, if $p \geqslant 3$, we have that $\mathrm{fix}_G(\mathcal{B})|_B = A_{p^2}$ or S_{p^2}. Furthermore, as p is prime, $p^2 \neq 6$, and so by [4, Table], A_{p^2} has a unique representation of degree p^2. Applying [10, Lemma 4.1], we have that G is permutation isomorphic to a subgroup of $S_p \times S_{p^2}$, and G admits a complete block system consisting of p^2 blocks of size p, a contradiction. ☐

Lemma 5.10. *Let $A, B, C \leqslant \mathbb{Z}_p^3$, $|A| = |B| = p$, $|C| = p^2$, $\langle A, B\rangle \cap C = B$, and let $\Gamma = \mathrm{Cay}(\mathbb{Z}_p^3, S)$, where $S = (A \setminus \{\bar{0}\}) \cup (C \setminus B)$. Then the C-orbits form a complete block system of* $\mathrm{Aut}(\Gamma)$.

Proof. Let V and V' be two orbits of C. For the subgraph $\Gamma[V]$ of Γ induced by V, $\Gamma[V] \cong K_{p^2} - pK_p$, the complete graph K_{p^2} minus p disjoint complete graphs K_p. Let $\Gamma[V, V']$ be the bipartite graph with bipartition sets V and V', and with $E(\Gamma[V, V']) := (V \times V') \cap E(\Gamma)$. It can be seen that $\Gamma[V, V'] = p^2 K_2$. Let $g \in \mathrm{Aut}(\Gamma)$. Then $\Gamma[V^g] \cong K_{p^2} - pK_p$. If now V^g is not an orbit of C, then $|V^g \cap V'| \leqslant p$ for any orbit V' of C. Thus a vertex in V^g has at most $2p-2$ neighbors in $\Gamma[V^g]$, implying that $p^2 - p \leqslant 2p-2$, which contradicts $p > 2$. We obtain that V^g is also an orbit of C, and the lemma follows. ☐

For a subgroup $K \leqslant G$, we write $\bar{K}_L$ for the subgroup $\{k_L : k \in K\}$ of G_L.

Lemma 5.11. *G admits a complete block system of p blocks of size p^2.*

Proof. Observe that, the negative statement implies that G admits a unique nontrivial complete block system $\mathcal{B}$ consisting of p^2 blocks of size p. Then $\mathcal{B}$ consists of the orbits of a group $\bar{K}_L$, where $K \leqslant \mathbb{Z}_p^3$, $|K| = p$.

First, let $\bar{K}_L = \langle \tau_3 \rangle$, i. e., $\mathcal{B} = \mathcal{B}_3$. We claim that, for every orbit T of $\mathrm{Stab}_G(\bar{0})$,

$$(\mathbb{Z}_{p^2} \wr \mathbb{Z}_p) \leqslant \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_p^3, T)).$$

This is trivial if $T \subseteq K$, so let $T \nsubseteq K$. Next $T \setminus \langle (0,1,0), (0,0,1)\rangle \neq \emptyset$. For otherwise, $\langle T\rangle \leqslant \langle (0,1,0), (0,0,1)\rangle$, but $\langle T\rangle \neq K$. We get another nontrivial complete block system of G given by the $\overline{\langle T\rangle}_L$-orbits, which is not the case. Thus there exists $x \in T \setminus \langle (0,1,0), (0,0,1)\rangle$, and we see that the coset $\langle (0,0,1)\rangle + x \subseteq T$. As the cosets of $\langle (0,0,1)\rangle$ form $\mathcal{B}$, we conclude that $T + \langle (0,0,1)\rangle = T$, and from this the above inequality follows. Let $\mathcal{O}$ be the set of $\mathrm{Stab}_G(\bar{0})$-orbits. Since the Cayley digraphs $\mathrm{Cay}(\mathbb{Z}_p^3, T)$, $T \in \mathcal{O}$, comprise the orbital digraphs of G, we find that

$$(\mathbb{Z}_{p^2} \wr \mathbb{Z}_p) \leqslant \bigcap_{T \in \mathcal{O}} \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_p^3, T)) = G^{(2)} = G.$$

Thus $p^2 \cdot p^{p^2}$ divides $|G|$, contradicting that $|P| = p^{p+2}$.

Second, let $\bar{K}_L \neq \langle \tau_3 \rangle$, i. e., $\mathcal{B} \neq \mathcal{B}_3$. As $|P \cap \mathrm{fix}_G(\mathcal{B})| = p$, we find that $\mathrm{fix}_G(\mathcal{B})$ is faithful on every block $B \in \mathcal{B}$. It follows that $\mathrm{fix}_G(\mathcal{B}) = \bar{K}_L$. Put $N = C_G(\bar{K}_L)$, the centralizer of $\bar{K}_L$ in G. Then $N \lhd G$, and the group $G/\mathcal{B}$ is primitive. As $G/\mathcal{B}$ has a Sylow p-subgroup of order p^{p+1}, we have by Theorem 2.4, that $A_{p^2} \leqslant G/\mathcal{B}$. Since $N/\mathcal{B} \lhd G/\mathcal{B}$, we see that $N/\mathcal{B}$ is doubly transitive. Let us consider the orbits of $\mathrm{Stab}_N(\bar{0})$. As N centralizes $\bar{K}_L$, $\{x\}$ is such an orbit for every $x \in K$. Let U be an orbit of $\mathrm{Stab}_N(\bar{0})$, $U \not\subseteq K$. Since $N/\mathcal{B}$ is doubly transitive,

$$|U \cap (K + x)| = k > 0 \text{ for all } x \notin K.$$

Let $y \in K$. Then $y_L g = g y_L$ for all $g \in N$, and we see that if $U = \{g(u) \colon g \in \mathrm{Stab}_N(\bar{0})\}$, then

$$U + y = \{g y_L(u) \colon g \in \mathrm{Stab}_N(\bar{0})\} = \{g(u + y) \colon g \in \mathrm{Stab}_N(\bar{0})\},$$

i. e., $U + y$ is also an orbit of $\mathrm{Stab}_N(\bar{0})$. Note that if $U + y = U$, then $U + \langle y \rangle = U + K$, so $U = \mathbb{Z}_p^3 \setminus K$, a contradiction. Thus $\{U + y | y \in K\}$ are all distinct orbits of $\mathrm{Stab}_N(\bar{0})$, and $\cup_{y \in K}(U + y) = \mathbb{Z}_p^3 \setminus K$. We conclude that each $U + y$ contains $p^2 - 1$ elements, and so $k = 1$, and $|U| = p^2 - 1$. Then, besides the sets $\{x\}$, $x \in K$, there are p orbits of $\mathrm{Stab}_N(\bar{0})$.

As $\mathcal{B}$ is a complete block system of G, $\mathcal{B}$ is a complete block system of $P \cong (\mathbb{Z}_p \wr \mathbb{Z}_p) \times \mathbb{Z}_p$, so it cannot be the case that $\bar{K}_L = \langle \tau_1 \gamma \rangle$, where $\gamma \in \langle \tau_2, \tau_3 \rangle$, as P admits no complete block system formed by the orbits of $\langle \tau_1 \gamma \rangle$. Let $\bar{U} = U \cap \langle K, (0,0,1) \rangle = U \cap \langle (0,1,0), (0,0,1) \rangle$. Then $|\bar{U}| = p - 1$. Fix $u \in \bar{U}$. Among the $p+1$ subgroups $J < \langle (0,1,0), (0,0,1) \rangle$ of order $|J| = p$, there must be at least 3 with $\bar{U} \cap (J + u) = \{u\}$. In particular, choose a subgroup $L < \langle (0,1,0), (0,0,1) \rangle$, $|L| = p$, $L \neq K$, and $|\bar{U} \cap (L + x)| = 1$ for some $x \in K$. As we saw above, the set $U - x := \{u - x \colon u \in U\}$ is also an orbit of $\mathrm{Stab}_N(\bar{0})$, and furthermore, $|(U - x) \cap L| = 1$. Using the fact that the orbits of $\mathrm{Stab}_N(\bar{0})$ are the basic sets of a Schur-ring over $\mathbb{Z}_p^3$ [26, Theorem 24.1], we can apply a result of Schur and Wielandt [26, part (a) of Theorem 23.9] stating: if U' is any orbit of $\mathrm{Stab}_N(\bar{0})$, and $\alpha \in \mathbb{Z}_p^*$, then the set

$$\alpha \cdot U' := \{(\alpha i, \alpha j, \alpha k) \colon (i,j,k) \in U'\}$$

is also an orbit of $\mathrm{Stab}_N(\bar{0})$. As $|(U - x) \cap L| = 1$, we find that the sets $\alpha \cdot (U - x)$, $\alpha \in \mathbb{Z}_p^*$, form $p - 1$ distinct orbits of $\mathrm{Stab}_N(\bar{0})$. As there are p orbits of $\mathrm{Stab}_N(\bar{0})$ of size 1, and p orbits of $\mathrm{Stab}_N(\bar{0})$ of size $p^2 - 1$, there exists an orbit U_0 of order $p^2 - 1$ of $\mathrm{Stab}_N(\bar{0})$ satisfying that $\alpha \cdot U_0 = U_0$ for all $\alpha \in \mathbb{Z}_p^*$. Using the fact that $U_0 \setminus \langle (0,1,0), (0,0,1) \rangle$ is a union of $\langle \tau_3 \rangle$-orbits, we obtain U_0 in the form

$$U_0 = (L' \setminus \{\bar{0}\}) \cup (M \setminus \langle (0,0,1) \rangle),$$

where $L' < \langle (0,1,0), (0,0,1) \rangle$, $L' \neq K, L$, and $M < \mathbb{Z}_p^3$, $|M| = p^2$, $M \cap \langle (0,1,0)(0,0,1) \rangle = \langle (0,0,1) \rangle$. By Lemma 5.10, the $\bar{M}_L$-orbits form a complete block system of the group

$A := \mathrm{Aut}(\mathrm{Cay}(\mathbb{Z}_p^3, U'))$. Since $N \leqslant A$, we see that $L' = \langle(0,0,1)\rangle$, so that $U_0 = M \setminus \{\bar{0}\}$, and the p orbits of $\mathrm{Stab}_N(\bar{0})$ not in K are the sets $(M + x) \setminus K$.

Let U be the orbit of $\mathrm{Stab}_G(\bar{0})$ for which $U_0 = (M \setminus \{\bar{0}\}) \subseteq U$. Then $U \neq (M \setminus \{\bar{0}\})$, since otherwise the cosets of $M = \langle U \rangle$ form a complete block system of G. Using that $\alpha \cdot U = U$ for all $\alpha \in \mathbb{Z}_p^*$, we get $U = \mathbb{Z}_p^3 \setminus K$. This implies $(\mathbb{Z}_p \wr \mathbb{Z}_p \wr \mathbb{Z}_p) \leqslant G$. Thus $p \cdot p^p \cdot p^{p^2}$ divides $|G|$, a contradiction to $|P| = p^{p+2}$. □

Everything is prepared to determine G.

Proposition 5.12. *G is permutation isomorphic to one of the following groups:*

(1) $S_{p^2} \times X$, $X \leqslant S_p$ is a 2-closed group;

(2) $S_p \times (X \wr Y)$, $X, Y \leqslant S_p$ are 2-closed groups;

(3) $A((X \wr Y) \times Z)$, where $X, Y \leqslant S_p$ are 2-closed groups, $Z < \mathrm{AGL}(1,p)$, and $A \leqslant \mathrm{Aut}(\mathbb{Z}_p^3)$.

Proof. Let $\mathcal{A}$ be a complete block system of G consisting of p^2 blocks of size p, and $\mathcal{B}$ be a complete block system of G consisting of p blocks of size p^2, which are guaranteed by Lemmas 5.9 and 5.11, respectively. Note that, $\mathcal{A} \preceq \mathcal{B}_{2,3}$ holds.

First, we assume that $\mathcal{B}_3$ is not a complete block system of G. For the moment let $\mathcal{B} = \mathcal{B}_{2,3}$. Then $\langle \tau_3|_B \rangle^{\mathrm{fix}_G(\mathcal{B})}$, $B \in \mathcal{B}$, must be transitive on $B \in \mathcal{B}$, in which case a Sylow p-subgroup of G contains $\mathbb{Z}_p \wr \mathbb{Z}_p^2$, a contradiction. Let $\mathcal{B} \neq \mathcal{B}_{2,3}$. It is not hard to see that then $\mathcal{A} \perp \mathcal{B}$. By Lemma 2.10, $G \leqslant S_p \times S_{p^2}$. Further, as $\mathcal{B}_3$ is not a complete block system of G, we have that $G/\mathcal{A}$ is primitive with a Sylow p-subgroup $\mathbb{Z}_p \wr \mathbb{Z}_p$, and thus $A_{p^2} \leqslant (G/\mathcal{A})^{(2)}$. From this we obtain that $G = S_{p^2} \times X$, where $X \leqslant S_p$ is a 2-closed group, and so (1) follows.

Second, we assume that $\mathcal{B}_3$ is a complete block system of G. By [7, Lemma 2] $\mathcal{B}_{2,3}$ form a complete block system of G as well. Thus $G/\mathcal{B}_3$ is imprimitive. Using that $G/\mathcal{B}_3$ has a Sylow p-subgroup of order p^2, we find $G/\mathcal{B}_3 \leqslant S_p \times S_p$, and hence we may assume that the complete block system $\mathcal{B} \neq \mathcal{B}_{2,3}$. Observe that the complete block system that consists of the intersections of blocks in $\mathcal{B}_{2,3}$ with blocks in $\mathcal{B}$ is equal to $\mathcal{B}_3$. Also observe that the group $\mathrm{Stab}_G(\langle(0,1,0),(0,0,1)\rangle|_{\langle(0,1,0),(0,0,1)\rangle}$ is imprimitive, and it has a Sylow p-sugroup of order p^2. Thus it admits a complete block system consisting of p blocks of size p, which are distinct of the complete block system induced by $\mathcal{B}_3$. This extends to a complete block system of G, and this allows us to assume that the complete block system $\mathcal{A} \neq \mathcal{B}_3$. We obtain that $\mathcal{A} \perp \mathcal{B}$. Then G is a subgroup of $G/\mathcal{A} \times G/\mathcal{B} = (X \wr Y) \times G/\mathcal{B}$. If $\mathrm{fix}_G(\mathcal{A}) = S_p$, then G is in fact a direct product, $G = (X \wr Y) \times S_p$ and (2) follows. Otherwise, $G = A(\mathrm{fix}_G(\mathcal{B}) \times Z = A((X_1 \wr Y_1) \times Z)$, and (3) follows. □

5.4 Case (6)

The groups appearing in this case are given in the following proposition.

Proposition 5.13. *Let $G \leqslant S_{p^3}$ be a 2-closed imprimitive group, with a Sylow p-subgroup P permutation isomorphic to $\mathbb{Z}_p^3 \rtimes \langle\gamma\rangle$, where $\gamma((i,j,k)) = (i, i+j, k+j)$. Then G is permutation isomorphic to a subgroup of* $\mathrm{AGL}(3,p)$.

Proof. We may assume that $G \leqslant S_{\mathbb{Z}_p^3}$ such that $\langle\tau_1,\tau_2,\tau_3\rangle \leqslant G$, and $P := \langle\tau_1,\tau_2,\tau_3,\gamma\rangle$. Let $\mathcal{B}$ be a complete block system of G. First, we assume that $\mathcal{B}$ consists of p^2 blocks of size p. As G contains $P_1 := \langle\tau_1,\tau_2,\tau_3\rangle$, we have that $\mathcal{B}$ is genuine. Also note that as $\mathrm{fix}_P(\mathcal{B}) \neq 1$, we have that $\mathrm{fix}_P(\mathcal{B}) \cap C(P) \neq 1$. A straightforward computation will then show that $C(P) = \langle\tau_3\rangle$, so that $\mathcal{B}$ is formed by the orbits of $\langle\tau_3\rangle$.

As a Sylow p-subgroup of $\mathrm{fix}_G(\mathcal{B})$ has order p (as $\gamma \notin \mathrm{fix}_G(\mathcal{B})$), we have that $\mathrm{fix}_G(\mathcal{B})$ acts faithfully on $B \in \mathcal{B}$. By [10, Lemma 4.2], we have that either $\mathrm{fix}_G(\mathcal{B}) \cong \mathbb{Z}_p$, or G is permutation isomorphic to a subgroup of $S_{p^2} \times S_p$ with the coordinate-wise action. As $\gamma \in G$, G is not permutation isomorphic to a subgroup of $S_{p^2} \times S_p$ with the coordinate-wise action. Whence $\mathrm{fix}_G(\mathcal{B}) \cong \mathbb{Z}_p$. By [22, Theorem 1.1], we have that if $G/\mathcal{B}$ is doubly transitive, then $G/\mathcal{B} \leqslant \mathrm{AGL}(2,p)$ and so $G \leqslant \mathrm{AGL}(3,p)$ and the result follows. We thus assume without loss of generality that $G/\mathcal{B}$ is imprimitive. Then a Sylow p-subgroup of $G/\mathcal{B}$ has order p^3, and so by [13, Theorem 4], either $p = 3$ or $P/\mathcal{B} \triangleleft G/\mathcal{B}$. If $P/\mathcal{B} \triangleleft G/\mathcal{B}$, then it is straightforward to check, using [13, Lemma 6], and the fact that a Sylow p-subgroup of $G/\mathcal{B}$ has order p^3, that $G/\mathcal{B} \leqslant \mathrm{AGL}(2,p)$. Thus $G \leqslant \mathrm{AGL}(3,p)$ and the result follows as well. If $p = 3$, then by [13, Theorem 4 (5)], $G/\mathcal{B} = L(P/\mathcal{B})$, where $\mathbb{Z}_3 \times \mathbb{Z}_3 \leqslant L \leqslant S_3 \times \mathrm{AGL}(1,3) = \mathrm{AGL}(1,3) \times \mathrm{AGL}(1,3) \leqslant \mathrm{AGL}(2,3)$. Thus $G/\mathcal{B} \leqslant \mathrm{AGL}(2,3)$ and so $G \leqslant \mathrm{AGL}(3,3)$.

Second, we assume that $\mathcal{B}$ consisting of p blocks of size p^2. Then, as above, we have that $\mathcal{B}$ is genuine, formed by the orbits of some subgroup of P_1 of order p^2. We may assume without loss of generality that $\mathrm{Stab}_G(B)|_B$ is primitive, as otherwise by [6, Exercise 1.5.10], we have that G admits a complete block system with blocks of size p, and the result follows by arguments above. As $G/\mathcal{B} \leqslant S_p$, $P/\mathcal{B}$ has order p, and so $|\mathrm{fix}_P(\mathcal{B})| = p^3$. As γ has a fixed point, we conclude that $\gamma \in \mathrm{fix}_G(\mathcal{B})$. By [22, Theorem 1.1], there is no doubly transitive group of degree p^2 with nonabelian socle that contains a regular elementary abelian subgroup and contains a Sylow p-subgroup of order p^3, so we have that $\mathrm{Stab}_G(B)|_B$ is permutation isomorphic to a subgroup of $\mathrm{AGL}(2,p)$. By [10, Lemma 5.1], we have that P_1^G, the normal closure of P_1 in G, is contained in $(P_1^G/\mathcal{B}) \wr \mathbb{Z}_p^2$. As $\mathrm{fix}_P(\mathcal{B}) = \langle\tau_2,\tau_3,\gamma\rangle$, we conclude that $\mathrm{fix}_{P_1^G}(\mathcal{B}) = \langle\tau_2,\tau_3\rangle$. If $G/\mathcal{B} \leqslant \mathrm{AGL}(1,p)$, then P_1 is a normal Sylow p-subgroup of P_1^G, and so is characteristic in P_1^G. Thus $P_1 \triangleleft G$ and so $G \leqslant \mathrm{AGL}(3,p)$ as required. Note then that this implies that $p \geqslant 5$ as $S_3 = \mathrm{AGL}(1,3)$. Otherwise, by Theorem 2.2, $G/\mathcal{B}$ is doubly transitive with nonabelian socle. Then by Theorem 5.6, we have that $P_1^G = A_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. Then P_1^G admits a complete block system $\mathcal{B}_3$ consisting of p^2 blocks of size p formed by the orbits of $\langle\tau_3\rangle$, as does P. We conclude that $H := \langle P_1^G, P\rangle$ admits $\mathcal{B}_3$ as a complete block system. Then $H/\mathcal{B}_3$ is nonsolvable, has a regular elementary abelian subgroup, and has Sylow p-subgroup of order p^3. This, however, contradicts [13, Theorem 4]. $\square$

References

[1] Alspach, B., Point-symmetric graphs and digraphs of prime order and transitive permutation groups of prime degree, *J. Combin. Theory* **15** (1973), 12–17.

[2] Babai, L., Isomorphism problem for a class of point-symmetric structures, *Acta Math. Sci. Acad. Hung.* **29** (1977), 329-336.

[3] Burnside, W., On some properties of groups of odd order, *J. London Math. Soc.* 33 (1901), 162–185.

[4] Cameron, P. J., Finite permutation groups and finite simple groups, *Bull. London Math. Soc.* 13 (1981), 1–22.

[5] Cameron, P. J., Giudici, M., Kantor, W. M., Jones, G. A., Klin, M. H., Marušič, D., Nowitz, L. A., Transitive permutation groups without semiregular subgroups, *J. London Math. Soc.* **66** (2002), 325–333.

[6] Dixon, J. D., and Mortimer, B., "Permutation Groups", Springer-Verlag New York, Berlin, Heidelberg, Graduate Texts in Mathematics **163**, 1996.

[7] Dobson, E., Isomorphism problem for Cayley graphs of $\mathbb{Z}_p^3$, *Discrete Math.* **147** (1995), 87–94.

[8] Dobson, E., On groups of odd prime-power degree that contain a full cycle, *Discrete Math.* **299** (2005), 65–78.

[9] Dobson, E., Automorphism groups of metacirculant graphs of order a product of two distinct primes, *Combin. Prob. Comput.* **15** (2006), 105–130.

[10] Dobson, E., On overgroups of regular abelian p-groups, *Ars Math. Contemp.* **2** (2009), 59–76.

[11] Dobson, E., and Morris, J., On automorphism groups of circulant digraphs of square-free order, *Discrete Math.*, (**299**) 2005, 79-98.

[12] Dobson, E., and Morris, J., Automorphism groups of wreath product digraphs, *Elec. J. Combin.*, (**16**) 2009, #R17.

[13] Dobson, E., and Witte, D., Transitive permutation groups of prime-squared degree, *J. Algebraic Combin.* **16** (2002), 43–69.

[14] Evdokimov, S. A., and Ponomarenko, I. N., Characterization of cyclotomic schemes and normal Shcur rings over a cyclic group (Russian), *Algebra i Analiz* (2) **14** (2002), 11–55.

[15] Guralnick, R. M., Subgroups of prime power index in a simple group, *J. Algebra* **81** (1983), 304–311.

[16] Holton, D. A., The König question. Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975), pp. 323–342. *Congressus Numerantium*, No. XV, Utilitas Math., Winnipeg, Man., 1976.

[17] Kalužnin, L. A., and Klin, M. H., On some numerical invariants of permutation groups, *Latviĭsk. Mat. Ežegodnik* **18** (1976), 81–99.

[18] Klin, M. H., and Pöschel, R., The König problem, the isomorphism problem for cyclic graphs and the method of Schur, Proceedings of the Inter. Coll. on Algebraic methods in graph theory, Szeged 1978, *Coll. Mat. Soc. János Bolyai* **27**.

[19] Leung, K. H., and Man, S. H., On Schur rings over cyclic groups II, *J. Algebra* **183** (1996), 173–185.

[20] Leung, K. H., and Man, S. H., On Schur rings over cyclic groups, *Israel J. Math.* **106** (1998), 251–267.

[21] Li, C. H., On isomorphisms of finite Cayley graphs - a survey, *Discrete Math.* **246** (2002), 301–334.

[22] Li, C. H., The finite primitive permutation groups containing an abelian regular subgroup, *Proc. London Math. Soc.* (3) **87** (2003), 725–747.

[23] Li, C. H., Permutation groups with a cyclic regular subgroup and arc transitive circulants, *J. Algebraic Combin.* **21** (2005), 131–136.

[24] Muzychuk, M., On the isomorphism problem for cyclic combinatorial objects, *Disc. Math.* **197/198** (1999), 589-606.

[25] Ponomarenko, I., Determination of the automorphism group of a circulant association scheme in polynomial time (Russian), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **321** (2005), Vopr. Teor. Predst. Algebr. i Grupp. **12**, 251–267, 301; translation in J. Math. Sci. (N. Y.) **136** (2006), no. 3, 3972–3979.

[26] Wielandt, H. “Finite Permutation Groups” (trans. by R. Bercov), Academic Press, New York, 1964.

[27] Wielandt, H., “Permutation groups through invariant relations and invariant functions”,

[28] Xu, M. Y., On isomorphism of Cayley digraphs and graphs of groups of order p^3, preprint.