INVERSE KIES DISTRIBUTION: PROPERTIES AND APPLICATIONS

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Abstract: In this paper, a new class of distribution namely “the inverse Kies distribution” is proposed and some of its important aspects are studied by deriving expressions for its percentile measures, raw moments, reliability measures etc. The maximum likelihood estimation of the parameters of the distribution is discussed and the distribution has been fitted to certain real life data sets. The asymptotic behaviour of maximum likelihood estimators of the parameters of the distribution are also studied by using simulated data sets.

1. Introduction

Keller, Kamath and Perera (1982) introduced and studied the inverse Weibull distribution (IWD) as a failure model for analysing wear, fatigue and corrosion occurring in mechanical components. They defined the IWD through its cumulative distribution function (c.d.f.)

$$G(u) = \begin{cases} 0, & \text{for } u < 0 \\ \exp\left[-\left(\frac{\alpha}{u}\right)^{\beta}\right], & \text{for } u \geq 0, \end{cases}$$

in which $\alpha > 0$ and $\beta > 0$. Keller, Goblin and Farnworth (1985) considered certain applications of the IWD and fitted the distribution to failure data of dynamic components while Erto (1989) provided some of its physical applications. Carter, Wampler and Stablein (1983) also used the IWD in certain survival data analysis studies. The IWD has been further studied by Calabria and Pulcini (1990), Khan, Pasha and Pasha (2008), Pawlas and Szynal (2000), Mahmoud, Sultan and Amer (2003) and Elshahat and Ismail (2014). Certain other re-parametric versions of the IWD are proposed in the

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literature recently by authors like de Gusmão, Ortega and Cordeiro (2011), Khan and King (2012), Khan (2010) and Baharith, Mousa, Atallah and Elgayar (2014). All these families of distributions possess decreasing/upside down bathtub-shaped hazard rate functions.

Kumar and Dharmaja (2014) considered a functional form of the Weibull distribution namely "the Kies distribution" (KD) which has the following probability density function (p.d.f.), in which $0 \leq a \leq \upsilon \leq b < \infty$, $\lambda > 0$ and $\beta > 0$.

$$q(\upsilon) = q(\upsilon; a, b, \lambda, \beta) = \lambda\beta\,(b-a)\,(\upsilon-a)^{\beta}\,(b-\upsilon)^{-\beta-1}\exp\left\{-\lambda\left(\frac{\upsilon-a}{b-\upsilon}\right)^{\beta}\right\}.$$

The c.d.f. of the KD is

$$Q(\upsilon) = \begin{cases} 0, & \text{for } \upsilon < a \\ 1-\exp\left[-\lambda\left(\frac{\upsilon-a}{b-\upsilon}\right)^{\beta}\right], & \text{for a } \leq \upsilon < b \\ 1, & \text{for } \upsilon \geq b. \end{cases} \tag{1}$$

In an analogous way of developing Kies distribution as a functional form of Weibull distribution, through this paper we develop a functional form of the IWD and named it as "the inverse Kies distribution" or in short "the IKD". The paper is organised as follows. In Section 2 it is shown that the IKD possess increasing or bathtub-shaped or reverse shaped hazard rate functions depending on its scale and shape parameters and thus the IKD is a useful distribution for modelling the complete lifetime of systems with bathtub-shaped or reverse S-shaped hazard rate functions, where both the Weibull and the inverse Weibull models are not suitable. This flexible nature of the hazard rate function of the IKD helps to give better fits in such situations compared to existing life time distributions in the literature like the Kies distribution (KD) (cf. Kumar and Dharmaja, 2014) the generalised inverse Weibull distribution (GIWD) (cf. de Gusmão et al., 2011), the exponentiated generalised inverse Weibull distribution (EGIWD) (cf. Elbatal and Muhammed, 2014), the modified inverse Weibull distribution (MIWD) (cf. Khan and King, 2012), the beta inverse Weibull distribution (BIWD) (cf. Hanook, Shahbaz, Mohsin and Kibria, 2013), the beta generalised inverse Weibull distribution (BGIWD) (cf. Baharith et al., 2014) etc. Above all, compared to these extended inverse Weibull models, the IKD possess some useful properties as discussed in Section 2 of the paper. In Section 3 we discuss the estimation of the parameters of the IKD and in Section 4 we consider two real life data sets for illustrating the usefulness of the proposed distribution.

We present the following integral/series representations which we need in the sequel. For details regarding these representations see Gradshteyn and Ryhzik (2007, p. 346). For $\mathrm{Re}(\nu) > 0$, $\mathrm{Re}(\mu) > 0$,

$$\int_0^u x^{\nu-1}\exp(-\mu x)\,dx = \mu^{-\nu}\,\gamma(\nu, \mu u), \tag{2}$$

$$\int_u^{\infty} x^{\nu-1}\exp(-\mu x)\,dx = \mu^{-\nu}\,\Gamma(\nu, \mu u),$$

and

$$\int_u^{\infty} x^{-\nu}\exp(-x)\,dx = u^{-\nu\mid 2}\exp(-u\mid 2)\,W_{-\frac{\nu}{2},\left(\frac{1-\nu}{2}\right)}(u), \tag{3}$$

in which

$$\gamma(\alpha, u) = \sum_{i=0}^{\infty}\frac{(-1)^i}{i!}\frac{u^{\alpha+u}}{\alpha+i}, \tag{4}$$

$$\Gamma(\alpha,u) = \{\Gamma(\alpha) - \gamma(\alpha,u)\}$$

and for $|\arg(-x)| < \frac{3\pi}{2}$

$$W_{k_1,k_2}(x) = \frac{\Gamma(-2k_2)}{\Gamma\left(\frac{1}{2}-k_2-k_1\right)} M_{k_1,k_2}(x) + \frac{\Gamma(2k_2)}{\Gamma\left(\frac{1}{2}+k_2-k_1\right)} M_{k_1,-k_2}(x), \tag{5}$$

where

$$M_{k_1,k_2}(x) = \exp\left(-\frac{x}{2}\right) x^{k_2+\frac{1}{2}} \sum_{n=0}^{\infty} \left\{ \frac{\left(\frac{1}{2}-k_1+k_2\right)_n}{(1+2k_2)_n} \frac{x^n}{n!} \right\},$$

with $(c)_n = c(c+1)\dots(c+n-1)$, for $n \geq 1$ and $(c)_0 = 1$.

2. The Inverse Kies Distribution

Here we present the definition of the inverse Kies distribution and discuss some of its important properties.

Definition 1 A continuous random variable X is said to follow the inverse Kies distribution if its c.d.f. $F(x)$ is of the following form, for $c \geq 0$, $c < d$, $\vartheta > 0$ and $\beta > 0$.

$$F(x) = \begin{cases} 0, & \text{for } x < c \\ \exp\left[-\vartheta\left(\frac{x-c}{d-x}\right)^{-\beta}\right], & \text{for } c \leq x < d \\ 1, & \text{for } x \geq d \end{cases} \tag{6}$$

A distribution with c.d.f. (6) is hereafter denoted by IKD(c, d, ϑ, β).

On differentiating (6) with respect to x we get the following p.d.f., $f(x)$, of the IKD(c, d, ϑ, β),

$$f(x) = f(x;c,d,\vartheta,\beta) = \vartheta\beta(d-c)(d-x)^{\beta-1}(x-c)^{-\beta-1}\exp\left[-\vartheta\left(\frac{x-c}{d-x}\right)^{-\beta}\right], \tag{7}$$

for $0 \leq c \leq x \leq d < \infty$, $\vartheta > 0$ and $\beta > 0$.

Now, we have the following results.

Result 1 For any c, d, ϑ and $\beta \in R^+ = [0,\infty)$, if X follows the IKD(c, d, ϑ, β), then Z_1 = a + $\left[\frac{X-c}{d-X}\right]$ follows the IWD of Khan et al. (2008), with c.d.f.

$$F_1(z) = \exp\left[-\vartheta(z-a)^{-\beta}\right]. \tag{8}$$

Proof. The c.d.f. $F_1(z)$ of $Z_1 = a + \frac{X-c}{d-X}$ is the following, for any $z > 0$.

$$\begin{aligned} F_1(z) &= P(Z_1 \leq z) = P\left\{a + \frac{X-c}{(d-X)} \leq z\right\} = P\left\{X \leq \frac{c+d(z-a)}{1+(z-a)}\right\} \\ &= F\left\{\frac{c+d(z-a)}{1+(z-a)}\right\}, \end{aligned}$$

which gives (8), in the light of (6). ■

Result 2 For any a, c, d, ϑ and $\beta \in R^+$, if X follows the IKD(c, d, ϑ, β), then $Z_2 = a + \left[\frac{X-c}{d-X}\right]^\beta$ follows the inverse generalised exponential distribution of Khan (2009), with c.d.f.

$$F_2(z) = \exp\left[-\vartheta (z-a)^{-1}\right]. \tag{9}$$

Proof. The c.d.f. $F_1(z)$ of $Z_2 = a + \left[\frac{X-c}{d-X}\right]^\beta$ is the following, for any $z > 0$,

$$\begin{aligned} F_2(z) &= P(Z_2 \le z) = P\left\{a + \left[\frac{X-c}{d-X}\right]^\beta \le z\right\} \\ &= P\left\{X \le \frac{c + d(z-a)^{\frac{1}{\beta}}}{1+(z-a)^{\frac{1}{\beta}}}\right\} = F\left\{\frac{c + d(z-a)^{\frac{1}{\beta}}}{1+(z-a)^{\frac{1}{\beta}}}\right\}, \end{aligned}$$

which gives (9), in the light of (6). ■

As a consequence of Result 2, we have the following corollary.

Corollary 1 If X follows the IKD(c, d, ϑ, β), then $Z_2 = \left[\frac{X-c}{d-X}\right]^{\frac{\beta}{2}}$, follows the inverse Rayleigh distribution cited in Ahmad, Ahmad and Ahmed (2014).

Result 3 If X follows the IKD(c, d, ϑ, β), then $Z_3 = \frac{1}{X}$ follows the KD with c.d.f. (1) in which $a = \dfrac{1}{d}$, $b = \dfrac{1}{c}$ and $\lambda = \theta\left(\dfrac{d}{c}\right)^\beta$.

Proof. The c.d.f. $F_3(z)$ of $Z_3 = \frac{1}{X}$ is the following, for any z > 0,

$$F_3(z) = P(Z_3 \le z) = P\left\{X \ge \frac{1}{z}\right\} = 1 - F\left\{\frac{1}{z}\right\},$$

which implies (1) with $a = \dfrac{1}{d}$, $b = \dfrac{1}{c}$ and $\lambda = \theta\left(\dfrac{d}{c}\right)^\beta$, in the light of (6). ■

Result 4 The survival function $S(x)$ and the hazard rate function $h(x)$ of the IK(c, d, ϑ, β) are the following, for $0 \le c \le x \le d < \infty$

$$S(x) = 1 - \exp\left[-\vartheta\left(\frac{x-c}{d-x}\right)^{-\beta}\right]$$

and

$$h(x) = \vartheta\beta(d-c)(d-x)^{\beta-1}(x-c)^{-\beta-1}\left\{\exp\left[\vartheta\left(\frac{x-c}{d-x}\right)^{-\beta}\right]-1\right\}^{-1}.$$

The proof is straightforward and hence omitted.

We have plotted the c.d.f. of IKD(0.01, 5, 0.5, β) for particular values of its parameter β in Figure 1. We have considered such c.d.f. plots of IKD(c, d, ϑ, β) for different values of its parameters and observed that for any arbitrary but fixed values of c, d and ϑ, the curves of the c.d.f. $F(x)$ for various values of β intersect at $x = \{(c+d) \mid 2\}$ and the value of $F(x)$ at $x = \{(c+d) \mid 2\}$ is $\{\exp(-\vartheta)\}$.

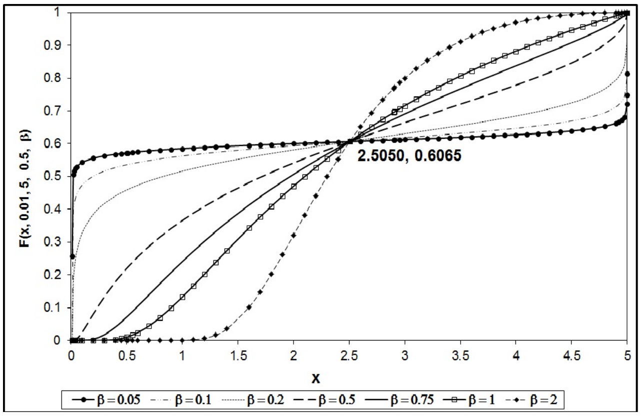

Figure 1: The c.d.f. plots of IKD(0.01, 5, 0.5, β) for particular values of β.

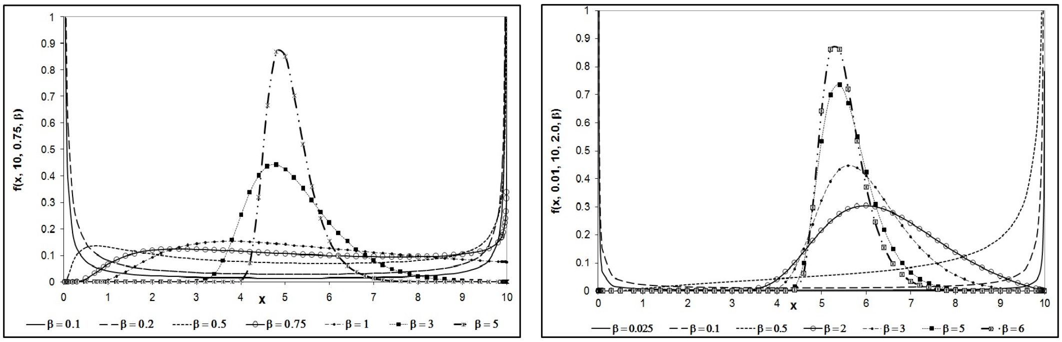

Figure 2: The p.d.f. plots of IKD(0,10, 0.75, β) (Left) and IKD(0.01, 10, 2, β) (Right) for particular values of β.

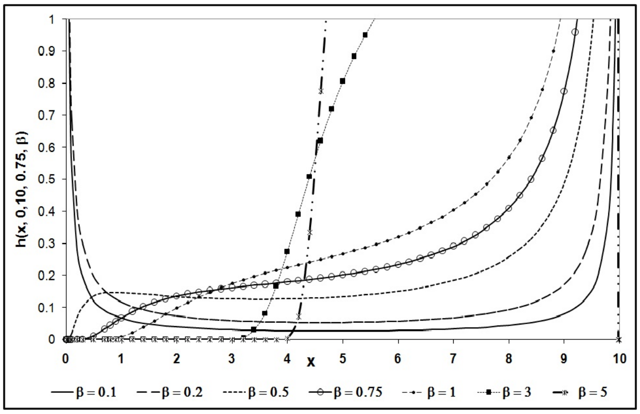

Figure 3: The hazard rate function plots of IK(0, 10, 0.75, β) for particular values of β.

Thus the inverse Kies characteristic life is given by $x = \frac{c+d}{2}$, the time at which $\exp(-\vartheta)$ percent units will fail. Here, it can be noted that the value of $F(x)$ at $x = 2.5050$ is $\exp(-0.5)$, for any value of β. Further, we have plotted the p.d.f. $f(x)$ for IKD(0.01, 5, 0.5, β) and IKD(0, 10, 2, β) for particular values of its parameter β in Figure 2 and hazard rate function $h(x)$ of IKD(0, 10, 0.75, β) for particular values of its parameter β in Figure 3.

Now we obtain the percentile function of IKD(c, d, ϑ, β) through the following result, by inverting its c.d.f. $F(x)$.

Result 5 For any $\beta > 0$, the percentile function x_P of IKD(c, d, ϑ, β) with c.d.f. (6) is the following.

$$x_P = \left(c\delta'_P + d\right)\left(1 + \delta'_P\right)^{-1}, \tag{10}$$

in which $\delta'_P = \left[-\vartheta^{-1}\ln(P)\right]^{\beta^{-1}}$ such that $P = F(x_P)$.

Now one can obtain expressions for the first quartile Q_1, the second quartile Q_2 and the third quartile Q_3 of IKD(c, d, ϑ, β) by putting $P = 0.25$, 0.50 and 0.75 respectively in (10). Further we have the following results, based on Result 5.

Result 6 Galton's percentile oriented measures for skewness g_a of IKD(c, d, ϑ, β) with c.d.f. (6) is

$$g_a = \frac{\left(\delta'_{0.8} - \delta'_{0.5}\right)\left(1 + \delta'_{0.2}\right)}{\left(\delta'_{0.5} - \delta'_{0.2}\right)\left(1 + \delta'_{0.8}\right)}.$$

Proof follows from the definition of g_a as $g_a = \frac{(x_{0.8} - x_{0.5})}{x_{0.5} - x_{0.2}}$.

Remark 1 If X follows IKD(c, d, ϑ, β) with c.d.f. (6), then the distribution is symmetric ($g_a = 1$) if $\vartheta = \omega_\beta$, positively skewed ($g_a > 1$) if $\vartheta < \omega_\beta$ and negatively skewed ($g_a < 1$) if $\vartheta > \omega_\beta$ where g_a is the Galton's percentile oriented measure of skewness and ω_β is given by the following equation.

$$\omega_\beta = \left\{\frac{[\ln(2)]^{\frac{1}{\beta}}[\ln(5)]^{\frac{1}{\beta}} + [\ln(2)]^{\frac{1}{\beta}}\left[\ln\left(\frac{5}{4}\right)\right]^{\frac{1}{\beta}} - 2[\ln(5)]^{\frac{1}{\beta}}\left[\ln\left(\frac{5}{4}\right)\right]^{\frac{1}{\beta}}}{[\ln(5)]^{\frac{1}{\beta}} + \left[\ln\left(\frac{5}{4}\right)\right]^{\frac{1}{\beta}} - 2[\ln(2)]^{\frac{1}{\beta}}}\right\}^{\beta}.$$

Result 7 The Schmid-Trede percentile oriented measure L (cf. Schmid and Trede, 2003) for kurtosis of IKD(c, d, ϑ, β) with c.d.f. (6) is the following.

$$L = \frac{\left\{\delta'_{0.975}\left(1 + \delta'_{0.025}\right) - \delta'_{0.025}\left(1 + \delta'_{0.975}\right)\right\}\left(1 + \delta'_{0.75}\right)\left(1 + \delta'_{0.25}\right)}{\left(\delta'_{0.75} - \delta'_{0.25}\right)\left(1 + \delta'_{0.975}\right)\left(1 + \delta'_{0.125}\right)}. \tag{11}$$

Proof. The Schmid-Trede percentile oriented measure L for kurtosis is the product of a measure T of tail and a measure P of peakedness as $T = \frac{x_{0.975} - x_{0.025}}{x_{0.875} - x_{0.125}}$, $P = \frac{x_{0.875} - x_{0.125}}{x_{0.75} - x_{0.25}}$. Thus, $L = TP = \frac{x_{0.975} - x_{0.025}}{x_{0.75} - x_{0.25}}$, which gives (11) in the light of Result 5. ■

Remark 2 If X follows IKD(c, d, ϑ, β) with c.d.f. (6), then the distribution is ($L = 2.9058$) for that value of β, for which $\vartheta \cong \left(\frac{-B' - \sqrt{B'^2 - 4A'C'}}{2A'}\right)^{\beta}$, while the distribution is leptokurtic ($L > 2.9058$)

when the quadratic form $A'X^2+B'X+C'$ for which $X=\vartheta^{\frac{1}{\beta}}$ is positive definite and platykurtic ($L<2.9058$) when the quadratic form is negative definite.

$$\begin{aligned} A' &= [\ln(4)]^{\frac{1}{\beta}}\left[\ln\left(\frac{4}{3}\right)\right]^{\frac{1}{\beta}}\left\{[\ln(40)]^{\frac{1}{\beta}}-\left[\ln\left(\frac{40}{39}\right)\right]^{\frac{1}{\beta}}\right\}-2.9058\,[\ln(40)]^{\frac{1}{\beta}} \\ &\quad\times\left[\ln\left(\frac{40}{39}\right)\right]^{\frac{1}{\beta}}\left\{[\ln(4)]^{\frac{1}{\beta}}-\left[\ln\left(\frac{4}{3}\right)\right]^{\frac{1}{\beta}}\right\}, \\ B' &= 3.9058\left\{[\ln(40)]^{\frac{1}{\beta}}\left[\ln\left(\frac{4}{3}\right)\right]^{\frac{1}{\beta}}-\left[\ln\left(\frac{40}{39}\right)\right]^{\frac{1}{\beta}}[\ln(4)]^{\frac{1}{\beta}}\right\} \\ &\quad+1.9058\left\{\left[\ln\left(\frac{40}{39}\right)\right]^{\frac{1}{\beta}}\left[\ln\left(\frac{4}{3}\right)\right]^{\frac{1}{\beta}}-[\ln(40)]^{\frac{1}{\beta}}[\ln(4)]^{\frac{1}{\beta}}\right\} \end{aligned}$$

and

$$C'=[\ln(40)]^{\frac{1}{\beta}}-\left[\ln\left(\frac{40}{39}\right)\right]^{\frac{1}{\beta}}-2.9058\left\{[\ln(4)]^{\frac{1}{\beta}}-\left[\ln\left(\frac{4}{3}\right)\right]^{\frac{1}{\beta}}\right\}.$$

We have computed values of Q_2 , g_a and L for particular values of the parameters of IKD(c, d, ϑ, β) and given in Table 1. From Table 1, it can be observed that the IKD(0, 10, 0.75, β) is symmetric when $\beta\cong0.4379$ and when $\beta>0.4379$ the curve is positively skewed. The IKD(0.01, 10, 2, β) is symmetric when $\beta\cong1.3142$ and when $\beta>1.3142$ the curve is positively skewed. Further it can be observed that the IKD(0, 10, 0.75, β) is mesokurtic for $\beta\cong2.6246$ and the IKD(0.01, 10, 2, β) is mesokurtic for $\beta\cong3.1853$.

Next we have the following results.

Result 8 If X follows IKD(c, d, ϑ, β), then the failure probability function $F(x)$ given in (6) satisfies the equation

$$F(\Delta_{xy})=F(x)\times F(y),$$

where

$$\Delta_{xy}=\frac{c\left\{\left(\frac{x-c}{d-x}\right)^{-\beta}+\left(\frac{y-c}{d-y}\right)^{-\beta}\right\}^{\frac{1}{\beta}}+d}{1+\left\{\left(\frac{x-c}{d-x}\right)^{-\beta}+\left(\frac{y-c}{d-y}\right)^{-\beta}\right\}^{\frac{1}{\beta}}}.$$

The proof is straightforward from (6) and hence omitted.

Result 9 The random numbers from IKD(c, d, ϑ, β) can be generated through the probability integral transformation for specified values of its parameters, by the formula

$$X=\left(c\delta'_Z+d\right)\left(1+\delta'_Z\right)^{-1}, \tag{12}$$

where Z is a random variable uniformly distributed over (0,1).

Table 1: The computed values of Q_2, g_a and L for IKD(c, d, ϑ, β) for particular values of the parameters.

β	$c=0, d=10, \vartheta=0.75$			$c=0.01, d=10, \vartheta=2$		
	Q_2	g_a	L	Q_2	g_a	L
0.2500	5.7818	0.7766	1.1093	9.8579	0.0500	4.9192
0.4000	5.4911	0.9639	1.3275	9.3402	0.2054	2.9578
0.4379	5.4489	**1.0000**	1.3876	9.1843	0.2528	2.7615
0.5000	5.3933	1.0510	1.4863	8.9287	0.3322	2.5395
1.0000	5.1970	1.2437	2.1480	7.4288	0.8328	2.2421
1.2500	5.1576	1.2771	2.3717	7.2406	0.8956	2.3408
1.3142	5.1499	1.2831	2.4193	6.9163	**1.0000**	2.3697
1.5000	5.1314	1.2967	2.5385	6.6994	1.0657	2.4534
2.0000	5.0985	1.3173	2.7565	6.2981	1.1745	2.6475
2.5000	5.0788	1.3272	2.8830	6.0480	1.2315	2.7849
2.6246	5.0751	1.3289	**2.9058**	5.9999	1.2414	2.8118
2.7500	5.0717	1.3303	2.9262	5.9556	1.2501	2.8363
3.0000	5.0657	1.3327	2.9607	5.8781	1.2645	2.8789
3.1853	5.0619	1.3341	2.9819	5.8283	1.2732	**2.9058**
3.5000	5.0563	1.3361	3.0112	5.7554	1.2852	2.9440
4.0000	5.0493	1.3382	3.0456	5.6628	1.2989	2.9903
4.5000	5.0438	1.3397	3.0699	5.5904	1.3084	3.0240
5.0000	5.0394	1.3408	3.0878	5.5323	1.3153	3.0492
10.0000	5.0197	1.3443	3.1472	5.2694	1.3378	3.1364

The proof follows by inverting the c.d.f. $F(x)$ of IKD(c, d, ϑ, β).

Result 10 If X follows IKD(c, d, ϑ, β) with c.d.f. (6), then the r^{th} raw moment μ_r' of IKD(c, d, ϑ, β) is the following.

$$\begin{aligned}\mu_r' &= \sum_{i=0}^{r}\left[\binom{r}{i}(d-c)^i c^{r-i}\sum_{j=0}^{\infty}\left\{\frac{(-1)^j (i)_j \gamma\left(\frac{j}{\beta}+1\right)}{j!\,\vartheta^{\frac{j+1}{\beta}}}\right\}\right] \\ &\quad+\sum_{i=0}^{r}\left[\binom{r}{i}(d-c)^i c^{r-i}\sum_{j=0}^{\infty}\left\{\frac{(-1)^j (i)_j W_{-\left(\frac{i+j}{2\beta}\right),\left(\frac{\beta-i-j}{2\beta}\right)}(\vartheta)}{j!\,\vartheta^{-\left(\frac{i+j}{2\beta}\right)}\exp\left(\frac{\vartheta}{2}\right)}\right\}\right], \end{aligned} \tag{13}$$

where $\gamma(\alpha,u)$is as given in (4) and $W_{k_1,k_2}(x)$ is the Whittaker function as defined in (5).

Proof. By definition, the r^{th} raw moment of IKD(c, d, ϑ, β) with p.d.f. (7) is

$$\mu_r' = \int_c^d x^r (d-c)\,\vartheta\beta\,(x-c)^{-\beta}(d-x)^{\beta}\exp\left[-\vartheta\left(\frac{x-c}{d-x}\right)^{-\beta}\right]dx. \tag{14}$$

If we put $u = \left(\frac{x-c}{d-x}\right)^{-\beta}$ in (14), we get

$$\mu_r' = \int_c^d \vartheta \left[\frac{(d-c)}{1+u^{\frac{1}{\beta}}} + c\right]^r \exp(-\vartheta u) du.$$

Now by applying the binomial theorem and rearranging the terms, we obtain

$$\mu_r' = \sum_{i=0}^{r} \left[\binom{r}{i} (d-c)^i c^{r-i} \int_0^\infty \vartheta \left\{\frac{\exp(-\vartheta u)}{\left(1+u^{\frac{1}{\beta}}\right)^i}\right\} du\right].$$

On splitting the integral and then expanding $\left(1+u^{\frac{1}{\beta}}\right)^{-i}$, we get the following

$$\begin{aligned}\mu_r' \quad = \quad & \sum_{i=0}^{r} \left[\binom{r}{i} (d-c)^i c^{r-i} \sum_{j=0}^{\infty} \left\{\frac{(-1)^j (i)_j}{j!} \int_0^1 \left(\vartheta u^{\frac{j}{\beta}} \exp(-\vartheta u)\right) du\right]\right\} \\ & + \sum_{i=0}^{r} \left[\binom{r}{i} (d-c)^i c^{r-i} \sum_{j=0}^{\infty} \left\{\frac{(-1)^j (i)_j}{j!} \int_1^\infty \left(\frac{\vartheta \exp(-\vartheta u)}{u^{\left(\frac{i+j}{\beta}\right)}}\right) du\right\}\right],\end{aligned}$$

which leads to (13) in the light of (2), (3), (4) and (5). ■

By using (13), one can compute the values of mean, variance, moment measure of skewness(γ_1) and moment measure of kurtosis (γ_2) of IKD(c, d, ϑ, β) for particular values of its parameters. We have obtained plots of the mean and variance of IKD(c, d, ϑ, β) in Figure 4, and that of the skewness and kurtosis in Figure 5.

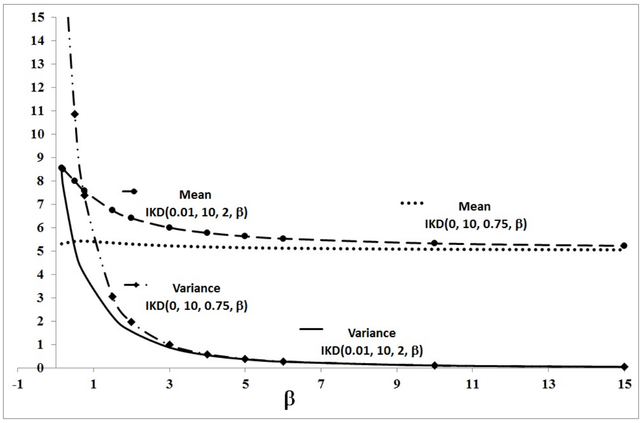

Figure 4: The plots of the mean and variance of IKD(c, d, ϑ, β) for particular values of the parameters.

From Figure 4 it can be observed that (i) the mean is an increasing function of β (ii) while the variance is an increasing function of β, for $d < \vartheta$ and the variance is a decreasing function of β, for $d \geq \vartheta$. From Figure 5 it can be observed that, the skewness of the curve moves from negatively

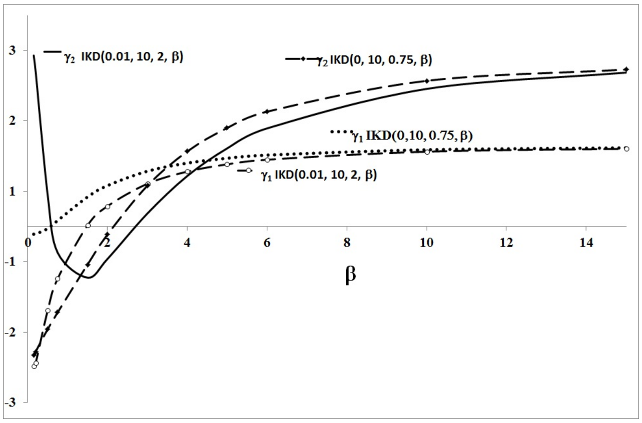

Figure 5: The plots of moment measure of skewness (γ_1) and moment measure of kurtosis (γ_2) of IKD(c, d, ϑ, β) for particular values of the parameters.

skewed to positively skewed in the case of IKD(0, 10, 0.75, β) and IKD(0, 10, 2, β) and peakedness of the curve moves from platykurtic to leptokurtic as β increases in the case of IKD(0, 10, 0.75, β). But for IKD(0, 10, 2, β) the peakedness of the curve moves from leptokurtic to platykurtic and then to leptokurtic.

Result 11 If X follows IKD(c, d, ϑ, β) with c.d.f. (6), then the mean residual life function of IKD is the following.

Case (i): For $x < \frac{c+d}{2}$,

$$\mu(x) = \frac{d-x}{1-\exp[-\eta(x)]} - \frac{(d-c)}{\beta}\sum_{j=0}^{\infty}\left\{\frac{(-1)^j(2)_j\gamma\left(\frac{j+1}{\beta},\vartheta\right)}{j!\vartheta^{\frac{j+1}{\beta}}\{1-\exp[-\eta(x)]\}} - \beta\frac{\vartheta^{\frac{j-2\beta}{\beta}}[\Delta_j(\vartheta)-\Delta_j(\vartheta\eta(x))]}{\{1-\exp[-\eta(x)]\}}\right\}. \tag{15}$$

Case(ii): For $x \geq \frac{c+d}{2}$

$$\mu(x) = \frac{d-x}{1-\exp[-\eta(x)]} - \frac{(d-c)}{\beta}\sum_{j=0}^{\infty}\frac{(-1)^j(2)_j\gamma\left(\frac{j+1}{\beta},\vartheta\eta(x)\right)}{j![\vartheta\eta(x)]^{\frac{j+1}{\beta}}\{1-\exp[-\eta(x)]\}}, \tag{16}$$

where $\eta(x) = (d-x)^{\beta}(x-c)^{-\beta}$ and for $j = 0, 1, 2, \ldots$

$$\Delta_j(\vartheta\tau) = (-1)^j(2)_j(j!)^{-1}\exp(-\vartheta\tau)W_{-\left(\frac{j+1+\beta}{2\beta}\right),-\left(\frac{j+1}{2\beta}\right)}(\vartheta\tau),$$

in which $\gamma(\alpha,u)$ is the incomplete gamma function defined in (4) and $W_{k_1,k_2}(x)$ is the Whittaker function as defined in (5).

Proof. By definition, the mean residual life function of IKD(c, d, ϑ, β) with p.d.f. (7) is the following, for $x \in [c,d]$

$$\begin{aligned}
\mu(x) &= E[X-x|X>x] \\
&= \frac{1}{S(x)}\int_x^d S(t)dt \\
&= \frac{1}{\{1-\exp[\vartheta\eta(x)]\}}\int_x^d \{\exp(-\vartheta\eta(t))\}dt.
\end{aligned}$$

If we put $u=\eta(t)$, we get

$$\mu(x) = \frac{d-x}{\{1-\exp[\vartheta\eta(x)]\}} - \frac{(d-c)}{\beta\{1-\exp[\vartheta\eta(x)]\}}\int_0^{\eta(x)}\left\{\frac{u^{\frac{1-\beta}{\beta}}\exp(-\vartheta u)}{\left(1+u^{\frac{1}{\beta}}\right)^2}\right\}du. \tag{17}$$

If we assume that $x<2^{-1}(c+d)$, then we obtain the following from (17) by splitting the integral and expanding $\left(1+u^{\frac{1}{\beta}}\right)^{-2}$.

$$\begin{aligned}
\mu(x) &= \frac{d-x}{\{1-\exp[\vartheta\eta(x)]\}} - \frac{(d-c)}{\beta\{1-\exp[\vartheta\eta(x)]\}}\times \\
&\sum_{j=0}^{\infty}\frac{(-1)^j(2)_j}{j!}\int_0^1\left[u^{\frac{j+1}{\beta}-1}\exp(-\vartheta u)\right]du - \int_1^{\eta(x)}\left[u^{-\frac{(j+\beta+1)}{\beta}}\exp(-\vartheta u)\right]du.
\end{aligned}$$

On rearranging the terms, we get the following from (18)

$$\begin{aligned}
\mu(x) =& \frac{d-x}{\{1-\exp[\vartheta\eta(x)]\}} - \frac{(d-c)}{\beta\{1-\exp[\vartheta\eta(x)]\}}\sum_{j=0}^{\infty}\frac{(-1)^j(2)_j}{j!}\int_0^1 u^{\frac{j+1}{\beta}-1}\exp(-\vartheta u)du- \\
&\frac{(d-c)}{\beta\{1-\exp[\vartheta\eta(x)]\}}\sum_{j=0}^{\infty}\frac{(-1)^j(2)_j}{j!}\left\{\int_1^{\infty}u^{-\frac{(j+\beta+1)}{\beta}}\exp(-\vartheta u)du-\right. \\
&\left.\int_{\eta(x)}^{\infty}u^{-\frac{(j+\beta+1)}{\beta}}\exp(-\vartheta u)du\right\},
\end{aligned}$$

which gives (15) by using (2), (3), (4) and (5).

If we assume that $x \geq 2^{-1}(c+d)$, then we have the following from (17) by expanding $\left(1+u^{\frac{1}{\beta}}\right)^{-2}$.

$$\begin{aligned}
\mu(x) &= \frac{d-x}{\{1-\exp[\vartheta\eta(x)]\}} - \frac{(d-c)}{\beta\{1-\exp[\vartheta\eta(x)]\}}\sum_{j=0}^{\infty}\left\{\frac{(-1)^j(2)_j}{j!}\times\right. \\
&\left.\int_0^{\eta(x)}u^{\left(\frac{j+1}{\beta}-1\right)}\exp(-\vartheta u)du.\right\},
\end{aligned}$$

which leads to (16) in the light of (3) and (5). ■

Result 12 Let X be the strength of a system which is subjected to a stress Y, and if X follows IKD(c, d, ϑ_1, β) and Y follows IKD(c, d, ϑ_2, β), then for known values of c and d, $R = P(Y < X)$, the measure of system performance is

$$R = \frac{\vartheta_1}{\vartheta_1 + \vartheta_2}. \tag{18}$$

Proof: Let $f_1(x)$ denote the p.d.f. of X and $f_2(x)$ denote the p.d.f. of Y, then

$$R = \int_c^d \left[\left(\int_c^x f_2(y)\, dy \right) f_1(x) \right] dx \tag{19}$$

By using (6) we obtain the following from (19),

$$\begin{aligned} R &= \int_c^d \left[\left(\exp\left\{ -\vartheta_2 \left(\frac{x-c}{d-x} \right)^{-\beta} \right\} \right) f_1(x) \right] dx \\ &= \vartheta_1 \int_c^d \left[\beta (d-c) \frac{(d-x)^{\beta-1}}{(x-c)^{\beta+1}} \exp\left(-(\vartheta_1+\vartheta_2) \left(\frac{x-c}{d-x} \right)^{-\beta} \right) \right] dx. \end{aligned} \tag{20}$$

On substituting $u = \left(\frac{x-c}{d-x}\right)^{-\beta}$ in (20), we get (18).

In order to establish the following theorem, we need the following lemmas in Kumar and Dharmaja (2014), which are from Rinne (2008).

Theorem 1 If X follows IKD(c, d, ϑ, β) with c.d.f. (6), then for any $y \in [c,d)$, and for every $0 \le c \le x \le d < \infty$, $\vartheta > 0$ and $\beta > 0$,

$$\begin{aligned} E&\left\{ -\log\left[1 - \exp\left\{ -\vartheta \left(\frac{X-c}{d-X} \right)^{-\beta} \right\} \right] \,\middle|\, X \ge y \right\} \\ &= 1 - \log\left[1 - \exp\left\{ -\vartheta \left(\frac{y-c}{d-y} \right)^{-\beta} \right\} \right], \end{aligned} \tag{21}$$

$$E\left[1 - \exp\left\{ -\vartheta \left(\frac{X-c}{d-X} \right)^{-\beta} \right\} \,\middle|\, X > y \right] = \frac{1 - \exp\left[-\left(\frac{y-c}{d-y} \right)^{-\beta} \right]}{2} \tag{22}$$

and

$$E\left(\left(\frac{X-c}{d-X} \right)^{-\beta} \,\middle|\, X \le y \right) = \left(\frac{y-c}{d-y} \right)^{-\beta} + \frac{1}{\vartheta}. \tag{23}$$

Proof. Since the c.d.f. $F(x)$ of IKD(c, d, ϑ, β) given in (6) has the form

$$F(x) = \begin{cases} 0, & \text{for } x < c \\ 1 - \exp\left(\log\left\{ 1 - \exp\left[-\vartheta \left(\frac{x-c}{d-x} \right)^{-\beta} \right] \right\} \right), & \text{for } x \in [c,d) \\ 1, & \text{for } x \ge d, \end{cases}$$

and $\phi_1(X) = -\log\left[1 - \exp\left\{ -\vartheta \left(\frac{X-c}{d-X} \right)^{-\beta} \right\} \right]$ is a strictly increasing differentiable function from $[c,d)$ onto $[0,\infty)$, by Lemma 1 of Kumar and Dharmaja (2014), we get (21).

Since the c.d.f. $F(x)$ of IKD(c, d, ϑ_2, β) given in (6) has the form

$$F(x)=\begin{cases} 0, & \text{for } x<c \\ 1-\left\{1-\exp\left[-\vartheta\left(\frac{x-c}{d-x}\right)^{-\beta}\right]\right\}^{1}, & \text{for } x\in[c,d) \\ 1, & \text{for } x\geq d, \end{cases}$$

and $\phi_2(X)=1-\exp\left\{-\left(\frac{X-c}{d-X}\right)^{-\beta}\right\}$ is real valued, continuous and differentiable function on $[a,b)$ with $E[\phi_2(X)]=1\mid 2$ and for any $k\in[0,1)$, $g(k)=0$, and $\psi(k)=1\mid 2$ by Lemma 2 of Kumar and Dharmaja (2014), we obtain (22) ■

Further, since the c.d.f. $F(x)$ of IKD(c, d, ϑ, β) takes the following form

$$F(x)=\begin{cases} 0, & \text{for } x<c \\ \exp\left[-\vartheta\left(\frac{x-c}{d-x}\right)^{-\beta}\right], & \text{for } x\in[c,d) \\ 1, & \text{for } x\geq d, \end{cases}$$

and $\phi_3(X)=\{(X-c)\mid(d-X)\}^{-\beta}$, for $0<c\leq x\leq d<\infty$ is a real-valued monotone function continuously differentiable on $(c,d]$ with $\lim_{x\downarrow c}\phi_3(X)=\infty$ with $E[\phi_3(X)]=\frac{1}{\vartheta}$ and $d=\frac{1}{\vartheta}$, by Lemma 3 of Kumar and Dharmaja (2014), we obtain (23).

Theorem 2 Let $X_1, X_2, \ldots, X_n$ be n independent and identically distributed (i.i.d.) random variables following IKD(c, d, ϑ, β) with c.d.f. (6) and let $Y=\max(X_1,X_2,\ldots,X_n)$. Then Y follows the IKD(c, d, $n\,\vartheta$, β). Conversely, if Y follows IKD(α, σ, δ, θ), then each X_i, $i=1,2,\ldots,n$ follows IKD(α, σ, $n^{-1}\delta$, θ).

Proof. If X_1, X_2, ..., X_n are i.i.d. IKD(c, d, ϑ, β) variates each with p.d.f. (7), the p.d.f. $f_n(y)$ of $Y=\max(X_1,X_2,\ldots,X_n)$ is the following, for any $0\leq c\leq x\leq d<\infty$, $\vartheta>0$ and $\beta>0$.

$$f_n(y)=n\vartheta\beta(d)(y-c)^{-\beta-1}(d-y)^{\beta-1}\exp\left\{-n\vartheta\left[\frac{y-c}{d-y}\right]^{-\beta}\right\},$$

since the p.d.f. $g_n(z)$ of maximum of n i.i.d. random variates each with p.d.f. $g(z)$ and its c.d.f. $G(z)$ is

$$g_n(z)=ng(z)(G(z))^{n-1}.$$

Thus Y follows IKD(c, d, nϑ, β).

Conversely, assume that $Y=\max(X_1,X_2,\ldots,X_n)$ follows IKD(α, σ, δ, θ), then the c.d.f. $F_n(y)$ of Y is

$$F_n(y)=\exp\left(-\delta\left(\frac{y-\alpha}{\sigma-y}\right)^{-\theta}\right), \tag{24}$$

in the light of (6). For any i.i.d. random variates $Z_1, Z_2, \ldots, Z_n$ each with c.d.f. $G(z)$, the c.d.f. $G_n(z)$ of Z = max(Z_1, Z_2, ..., Z_n) is

$$G_n(z)=[G(z)]^n. \tag{25}$$

Now we obtain the following from (24) in the light of (25),

$$[F(y)]^n=\exp\left\{-\delta\left(\frac{y-\alpha}{\sigma-y}\right)^{-\theta}\right\}.$$

Thus the p.d.f. of X_1 is

$$f(x) = F'(x) = \theta \frac{\delta}{n} (\sigma - \alpha)(x - \alpha)^{-\theta - 1} (\sigma - x)^{\theta - 1} \exp\left[-\frac{\delta}{n} \left(\frac{x - \alpha}{\sigma - x} \right)^{-\theta} \right].$$

■

3. Estimation

Here we discuss the maximum likelihood estimation of the parameters of the IKD(c, d, ϑ, β). Consider the following log-likelihood function ℓ of a random sample $X_1, X_2, \ldots, X_n$ from a population following IKD(c, d, ϑ, β) with p.d.f. (7),

$$\begin{aligned} \ell &= n\log(\beta) + n\log(\vartheta) + n\log(d-c) - (\beta+1)\sum_{i=1}^{n}\log(x_i - c) \\ &\quad + (\beta-1)\sum_{i=1}^{n}\log(d-x_i) - \vartheta\sum_{i=1}^{n}\left(\frac{x_i-c}{d-x_i}\right)^{-\beta}. \end{aligned} \tag{26}$$

On differentiating (26) with respect to the parameters c, d, ϑ, β of IKD(c, d, ϑ, β), we get the following likelihood equations:

$$-\frac{n}{(d-c)} - (\beta+1)\sum_{i=1}^{n}\frac{1}{(x_i-c)} - \vartheta\beta\sum_{i=1}^{n}\frac{(b-x_i)^{\beta}}{(x_i-c)^{\beta+1}} = 0, \tag{27}$$

$$\frac{n}{(d-c)} + (\beta-1)\sum_{i=1}^{n}\frac{1}{(d-x_i)} - \vartheta\beta\sum_{i=1}^{n}\frac{(d-x_i)^{\beta-1}}{(x_i-c)^{\beta}} = 0, \tag{28}$$

$$\frac{n}{\vartheta} - \sum_{i=1}^{n}\left(\frac{x_i-c}{d-x_i}\right)^{-\beta} = 0 \tag{29}$$

and

$$\frac{n}{\beta} - \sum_{i=1}^{n}\log(x_i-c) + \sum_{i=1}^{n}\log(d-x_i) - \vartheta\sum_{i=1}^{n}\left(\frac{x_i-c}{d-x_i}\right)^{-\beta}\log\left(\frac{d-x_i}{x_i-c}\right) = 0. \tag{30}$$

When these likelihood equations do not always have a solution, the maximum of the likelihood function is attained at the border of the domain of the parameters. Since the MLE of the unknown parameters c, d, ϑ and β are not obtained in closed forms, it is not possible to derive the exact distributions of the MLE. So we obtain the second order partial derivatives of log-likelihood function with respect to the parameters c, d, ϑ, β and we observe that they give negative values for $c > 0$, $d > c$, $\vartheta > 0$ and $\beta > 0$. Therefore, the MLEs $\hat{c}$, $\hat{d}$, $\hat{\vartheta}$ and $\hat{\beta}$ of the parameters c, d, ϑ and β of IKD(c, d, ϑ, β) with p.d.f. (7) can be obtained by solving the likelihood equations (27), (28), (29) and (30), with the help of mathematical software such as MATHCAD, MATLAB, R etc. Further we obtain the Fisher information matrix $I(\theta)$ as

$$I(\theta) = \begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{bmatrix},$$

in which the expressions for the elements of $I(\theta)$ are as given in Appendix A.

4. Data Analysis

For numerical illustration, we have considered the following two uncensored data sets. The first data set consists of 57 breaking strengths of carbon fibers of length 1 taken from Lawless (2003) and second data set is on the survival times (in days) of 72 guinea pigs injected with different doses of *tubercle bacilli* taken from Kundu and Howlader (2010).

In case of both the data sets, since the values are non-negative, we have fitted the three parameter IKD(d, ϑ, β) with $c = 0$ (which we denoted as IKD(d, ϑ, β)) to both the data sets with the help of the R package. The estimated values of the parameters of the IKD(d, ϑ, β) and corresponding values of standard error (SE), t-values and p-values are obtained by using the R's `maxLik` package (cf. Henningsen and Toomet, 2011) and presented in Table 2.

Table 2: Fitted values of IKD(d, ϑ, β) to the data sets.

Data set 1				
Parameter	Estimate	SE	t value	p value
d	6.1997	0.1858	33.367	$< 2e-16$
ϑ	1.6211	0.2222	7.296	$2.97e-13$
β	2.3755	0.3849	6.172	$6.73e-10$
Data set 2				
Parameter	Estimate	SE	t value	p value
d	0.3761611	0.0006827	550.9934	$< 2.2e-16$
ϑ	1.0661988	0.1047273	10.1807	$< 2.2e-16$
β	0.1661427	0.0409492	4.0573	$4.964e-05$

We have fitted the following models to the two data sets for comparison (a) the BGIWD(α, τ, σ, μ, ρ), in which α, τ, σ, μ and $\rho > 0$. (cf. Baharith et al., 2014), (b) the BIWD(α, σ, μ,ρ), in which α, σ, μ and $\rho > 0$ (cf. Khan, 2010), (c) the GIWD(α, σ, τ), in which α, σ and $\tau > 0$. (cf. de Gusmão et al., 2011), (d) the IWD(σ, ρ), in which σ and $\rho > 0$. (cf. Keller et al., 1982) and (e) the KD(a, b, λ, β), in which $0 \leq a < b < \infty$, $\lambda > 0$ and $\beta > 0$ (cf. Kumar and Dharmaja, 2014). Then we compared the fitted model the IKD(d, ϑ, β) to that of other fitted models, namely, the BGIWD(α, τ, σ, μ, ρ), the BIWD(α, σ, μ, ρ), the GIWD(α, σ, τ), the IWD (α, σ) and the KD(a, b, λ, β). For model comparison, we have used the Akaike information criterion (AIC), Bayesian information criterion (BIC) and the second order Akaike information criterion (AICc), which are as defined in Kumar and Dharmaja (2014). We have computed AIC, BIC and AICc in case of each of the fitted models for both the data sets and presented in Table 3. From the table, it can be observed that the IKD(d, ϑ, β) gives better fits to both the data sets compared to other competitive models.

Substituting the MLEs of the unknown parameters, the variance-covariance matrix Σ_1 and Σ_2 of the first and second data sets are respectively

$$\Sigma_1 = \begin{pmatrix} 0.03452245 & 0.02767106 & -0.03612933 \\ 0.02767106 & 0.04937312 & -0.01140359 \\ -0.03612933 & -0.01140359 & 0.14811577 \end{pmatrix}$$

Table 3: Fitting of various models to data set 1 and data set 2.

Data set 1					
Model	Estimates of the parameters	*Log*− likelihood	AIC	BIC	AICc
IKD(d, ϑ, β)	$d = 6.1997$ $\vartheta = 2.3755$ $\beta = 1.6211$	-69.8786	145.7572	151.8864	146.2100
BGIWD(α, τ, σ, μ, ρ)	$\alpha = 3.6546$ $\tau = 2.693$ $\sigma = 2.7080$ $\mu = 2.9870$ $\rho = 3.00101$	-75.3495	160.6990	170.9143	161.8755
BIWD(α, σ, μ,ρ)	$\alpha = 3.0010$ $\sigma = 2.9687$ $\mu = 3.988$ $\rho = 2.232$	-76.6914	161.3827	169.5549	162.1519
GIWD(α, σ, τ)	$\alpha = 68.1706$ $\tau = 2.1552$ $\sigma = 2.9770$	-87.6250	181.2500	187.3792	181.7028
IWD(α, σ)	$\alpha = 2.9890$ $\sigma = 3.0011$	-100.8710	205.7420	209.8281	205.9642
KD(a, b, λ, β)	$a = 1.9183$ $b = 16.0564$ $\lambda = 40.5395$ $\beta = 2.5003$	-70.1278	148.2555	156.4277	149.0247
Data set 2					
Model	Estimates of the parameters	*Log*− likelihood	AIC	BIC	AICc
IKD(d, ϑ, β)	$d = 0.3762$ $\vartheta = 0.1661$ $\beta = 1.0662$	248.255	-490.51	-490.938	-490.157
BGIWD(α, τ, σ, μ, ρ)	$\alpha = 0.5085$ $\tau = 0.8933$ $\sigma = 0.4634$ $\mu = 2.0447$ $\rho = 13.4915927$	107.134	-204.269	-204.982	-203.360
BIWD(α, σ, μ, ρ)	$\alpha = 0.3725$ $\sigma = 0.4947$ $\mu = 1.7751$ $\rho = 11.9740$	107.134	-206.268	-206.839	-205.671
GIWD(α, σ, τ)	$\alpha = 0.0041$ $\tau = 38.8135$ $\sigma = 1.4146$	101.7093	-197.419	-197.847	-197.066
IWD(α, σ)	$\alpha = 0.0542$ $\sigma = 1.4148$	101.7093	-199.419	-199.704	-199.245
KD(a, b, λ, β)	$a = 1.1749e-02$ $b = 5.5095$ $\lambda = 1.0234e+02$ $\beta = 1.1422$	104.2741	-200.548	-201.119	-199.951

and

$$\Sigma_2 = \begin{pmatrix} 4.660741\text{e}-07 & 2.749125\text{e}-05 & -9.661033\text{e}-06 \\ 2.749125\text{e}-05 & 1.096781\text{e}-02 & -3.766299\text{e}-03 \\ -9.661033\text{e}-06 & -3.766299\text{e}-03 & 1.676834\text{e}-03 \end{pmatrix}.$$

In addition, we have plotted the distribution functions of the fitted models against their empirical distributions in Figure 6 and corresponding WPP plots are obtained in in Figure 7. These figures supports the above conclusion that the IKD(d, ϑ, β) as a better model to the data sets compared to the existing models.

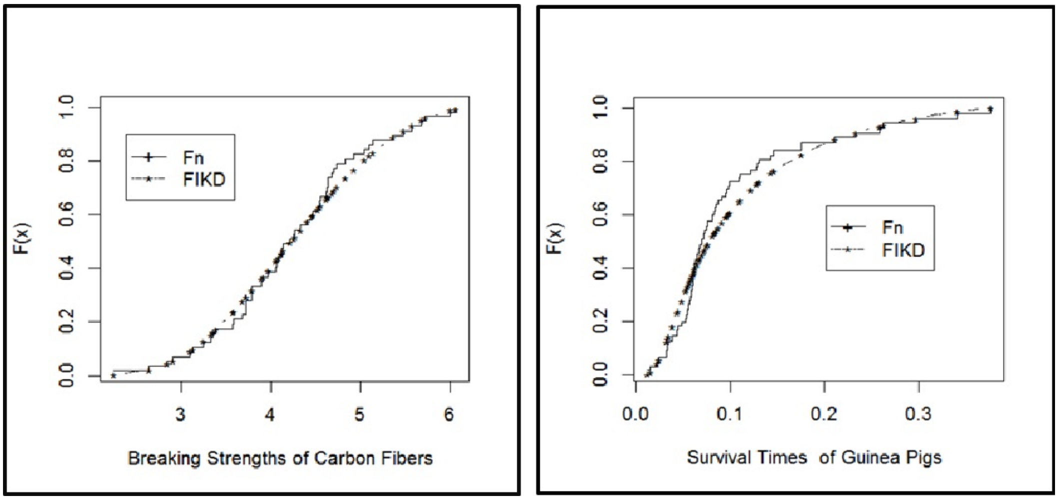

Figure 6: Empirical and fitted distribution function plots of IKD (d, ϑ, β) for data set 1 (Left) and data set 2 (Right).

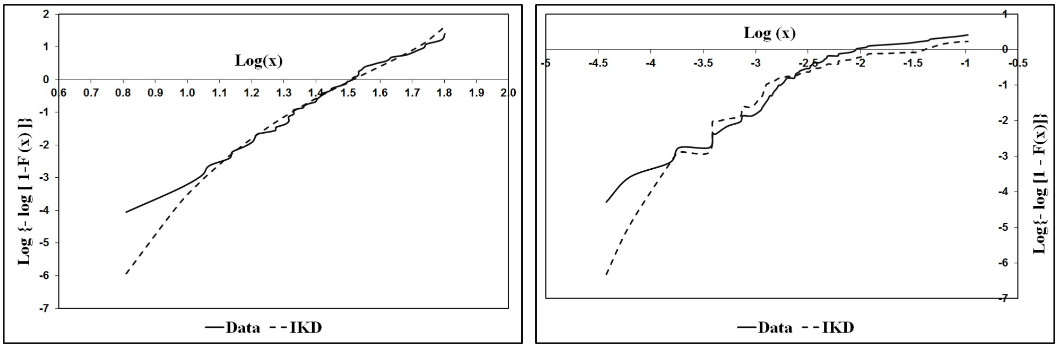

Figure 7: Weibull probability plots of the empirical and IKD (d, ϑ, β) for data set 1 (Left) and data set 2 (Right).

Table 4: Average bias and mean squared errors (in brackets) of the MLEs of the IKD(c, d, ϑ, β) based on simulated data sets for the parameter sets (i) $c = 0.01$, $d = 10$, $\vartheta = 2$ and $\beta = 1.1$ (negatively skewed) and (ii) $c = 0.0$, $d = 6.1997$, $\vartheta = 2.3755$ and $\beta = 1.6211$ (positively skewed).

		$n = 10$	$n = 25$	$n = 50$	$n = 100$
Parameter set (i)	c	2.8964	1.3132	0.1207	0.0422
		(9.8466)	(4.2184)	(0.1524)	(0.0047)
	d	1.3219	−0.2599	−0.0757	−0.0419
		(34.7285)	(0.1094)	(0.0057)	(0.0048)
	ϑ	−1.6504	−0.5807	0.0458	−0.0271
		(2.9214)	(0.6400)	(0.0798)	(0.0157)
	β	0.7209	−0.2079	−0.2027	−0.0467
		(2.8919)	(0.0536)	(0.0452)	(0.0330)
Parameter set (ii)	d	0.7193	0.6711	−0.3641	0.0610
		(3.5566)	(1.1843)	(0.1392)	(0.0153)
	ϑ	0.7045	0.5289	−0.3352	−0.1064
		(2.9837)	(0.8031)	(0.1347)	(0.0521)
	β	0.1957	−0.2852	0.2182	0.1049
		(8.5321)	(0.8465)	(0.2697)	(0.0540)

5. Simulation

In order to assess the properties of likelihood estimators of the parameters of the IKD(c, d, ϑ, β), we carry out a simulation study by generating n observations based on the sets of parameters (i) $c = 0.01$, $d = 10$, $\vartheta = 2$ and $\beta = 1.1$ (negatively skewed) and (ii) $c = 0.0$, $d = 6.1997$, $\vartheta = 2.3755$ and $\beta = 1.6211$ (positively skewed). As the c.d.f. (18) of IKD (c, d, ϑ, β) is in a closed explicit form, it is easy to generate pseudo-random numbers through the probability integral transformation. Therefore, if one has a uniform random number generator, then the random numbers from the IKD(c, d, ϑ, β) can be generated through the probability integral transformation for specified values of its parameters, by the formula (12). We considered 200 samples of sizes $n = 10$, 25, 50 and 100 to compare the performances of the different MLEs of the parameters of the distribution mainly with respect to their mean values and mean squared errors (MSEs). The results obtained are summarised in Table 4. From Table 4, it can be observed that as sample size increases mean value of the estimators approaches the original value of the respective parameters and MSEs of the estimators are also in decreasing order.

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Appendix A

The elements of Fisher Information matrix are the following,

$$I_{11}=E\left(\frac{\partial^2\ell}{\partial c^2}\right)=-\frac{n\beta}{(d-c)^2}+\frac{n(\beta+1)}{(d-c)^2}\left[\frac{2\Gamma\left(\frac{1}{\beta}+1\right)}{\vartheta^{\frac{1}{\beta}}}+\frac{\Gamma\left(\frac{1}{\beta}+2\right)}{\vartheta^{\frac{2}{\beta}}}\right]$$
$$-\frac{n\beta(\beta+1)}{(d-c)^2}\left[1+\frac{2\Gamma\left(\frac{1}{\beta}+2\right)}{\vartheta^{\frac{1}{\beta}}}+\frac{\Gamma\left(\frac{2}{\beta}+2\right)}{\vartheta^{\frac{2}{\beta}}}\right],$$

$$I_{12}=I_{21}=E\left(\frac{\partial^2\ell}{\partial d\partial c}\right)=\frac{n}{(d-c)^2}+\frac{n\beta^2}{(d-c)^2}\left[\frac{\Gamma\left(2-\frac{1}{\beta}\right)}{\vartheta^{-\frac{1}{\beta}}}+2+\frac{\Gamma\left(\frac{1}{\beta}+2\right)}{\vartheta^{\frac{1}{\beta}}}\right],$$

$$I_{13}=I_{31}=E\left(\frac{\partial^2\ell}{\partial\vartheta\partial c}\right)=\frac{-n\beta}{\vartheta(d-c)}+\frac{n\beta\Gamma\left(2+\frac{1}{\beta}\right)}{\vartheta^{1+\frac{1}{\beta}}(d-c)},$$

$$I_{14}=I_{41}=E\left(\frac{\partial^2\ell}{\partial\beta\partial c}\right)$$
$$=-\frac{n}{\beta(d-c)}\left\{1+[\Psi(2)-\ln(\vartheta)]+\frac{\Gamma\left(2+\frac{1}{\beta}\right)}{\vartheta^{\frac{1}{\beta}}}\left[\Psi\left(2+\frac{1}{\beta}\right)-\ln(\vartheta)\right]\right\},$$

$$I_{22}=E\left(\frac{\partial^2\ell}{\partial d^2}\right)=\frac{-n\beta}{(d-c)^2}+\frac{n(\beta-1)}{(d-c)^2}\left[\frac{\Gamma\left(1-\frac{2}{\beta}\right)}{\vartheta^{-\frac{2}{\beta}}}+\frac{2\Gamma\left(1-\frac{1}{\beta}\right)}{\vartheta^{-\frac{1}{\beta}}}\right]$$
$$-\frac{n\beta(\beta-1)}{(d-c)^2}\left[1+\frac{2\Gamma\left(\frac{1}{\beta}+2\right)}{\vartheta^{\frac{1}{\beta}}}+\frac{\Gamma\left(\frac{2}{\beta}+2\right)}{\vartheta^{\frac{2}{\beta}}}\right],$$

$$I_{23}=I_{32}=E\left(\frac{\partial^2\ell}{\partial\vartheta\partial d}\right)=\frac{-\beta n}{(d-c)\,\vartheta}-\frac{n\beta\Gamma\left(2-\frac{1}{\beta}\right)}{(d-c)\,\vartheta^{\frac{\beta-1}{\beta}}},$$

$$I_{24}=I_{42}=E\left(\frac{\partial^2\ell}{\partial\beta\partial d}\right)=\frac{n\Gamma\left(1-\frac{1}{\beta}\right)}{(d-c)\,\vartheta^{\frac{1}{\beta}}}-\frac{n\Gamma\left(2-\frac{1}{\beta}\right)}{(d-c)\,\vartheta^{\frac{-1}{\beta}}}$$

$$-\frac{n\beta}{(d-c)^2}\left[2+\frac{2\Gamma\left(2-\frac{1}{\beta}\right)}{\vartheta^{\frac{-1}{\beta}}}+\frac{\Gamma\left(\frac{1}{\beta}+2\right)}{\vartheta^{\frac{1}{\beta}}}\right],$$

$$I_{33}=E\left(\frac{\partial^2\ell}{\partial\vartheta^2}\right)=-\frac{n}{\vartheta^2},$$

$$I_{34}=I_{43}=E\left(\frac{\partial^2\ell}{\partial\beta\partial\vartheta}\right)=-\frac{n}{\beta\vartheta}\left[\psi(2)-\ln(\vartheta)\right]$$

and

$$I_{44}=E\left(\frac{\partial^2\ell}{\partial\beta^2}\right)=-\frac{n}{\beta^2}-\frac{n\left\{\left[\psi(2)-\ln(\vartheta)\right]^2+\zeta(2,2)\right\}}{\beta^2}.$$

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