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ON A NEW APPROACH TO ESTIMATE THE SHAPE PARAMETER OF THE INVERSE GAUSSIAN DISTRIBUTION

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In this paper, we propose a new simple method for estimating the shape parameter of the inverse Gaussian distribution. This new method makes use of the reciprocal property of the distribution. Also, bias-reduced versions of this proposed estimator are introduced. Furthermore, a new test for the coefficient of variation in an inverse Gaussian distribution is derived. The performance of the estimators are evaluated via Monte Carlo simulations. In general, compared with the maximum likelihood estimator, the proposed method has smaller bias. Two real data sets are used to illustrate the proposed methodology.

Key words: Maximum likelihood estimator, Modified moment estimator, Hypothesis testing, Inverse Gaussian distribution.

1. Introduction

A first detailed study on the inverse Gaussian (IG) distribution was presented by Tweedie (1957a,b). Posteriorly, Folks and Chhikara (1978) presented its main properties and applications, and a complete study was provided by Chhikara and Folks (1989). Also, Johnson, Kotz and Balakrishnan (1994) summarized important statistical properties of this distribution. The IG model is an alternative failure time distribution, such as the Birnbaum–Saunders, gamma, lognormal and Weibull distributions. A random variable X follows an IG distribution with parameters $\mu > 0$ and $\lambda > 0$, denoted by $X \sim \mathrm{IG}(\mu, \lambda)$, if its cumulative distribution function (CDF) is given by

$$F(x; \mu, \lambda) = \Phi\left(\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} - 1\right)\right) + \exp\left(\frac{2\lambda}{\mu}\right)\Phi\left(-\sqrt{\frac{\lambda}{x}}\left(\frac{x}{\mu} + 1\right)\right), \quad x > 0, \mu > 0, \lambda > 0, \qquad (1)$$

where $\Phi(\cdot)$ is the standard normal CDF, μ is the mean and λ is a shape parameter.

The IG distribution has been applied in several areas including actuarial science, engineering, hydrology, meteorology, management, physiology, etc. For details, see Chhikara and Folks (1989) and Seshadri (1999).

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The maximum likelihood estimators (MLEs) of μ and λ have been discussed by Tweedie (1957a). Patel (1965) derived moment estimators of μ and λ and their asymptotic variances for the one-sided and two-sided truncated IG distributions. Padgett and Wei (1979) have discussed moment and maximum likelihood estimation and Cheng and Amin (1981) have discussed maximum likelihood estimation, both for the three-parameter IG distribution, which includes an unknown shifted origin parameter. In this context, the main aim of this paper is to propose a simple alternative estimator (hereafter NE) of the shape parameter of the IG distribution, based on the reciprocal property of this distribution. We also propose bias-reduced versions of the NE based on the theoretical bias and upon inspecting the pattern of the bias via Monte Carlo (MC) simulation studies. Additionally, we derive a test for the coefficient of variation of the IG distribution.

The rest of the paper proceeds as follows. Section 2 introduces the IG distribution and some of its basic properties. In Section 3, we describe the MLEs, the NE and its bias-reduced form. Also, some properties of the estimators are discussed. In Section 4, we introduce a test statistic based on the proposed estimator for testing hypotheses about the coefficient of variation of the IG distribution. Comparisons of the estimators and tests via MC simulation studies are shown in Section 5. In Section 6, we consider two empirical examples, and conclusions are outlined in Section 7.

2. The inverse Gaussian distribution and some of its properties

The probability density function (PDF) associated with (1) is given by

$$f(x;\mu,\lambda) = \sqrt{\frac{\lambda}{2\pi}}x^{-3/2}\exp\left(-\frac{\lambda(x-\mu)^2}{2\mu^2 x}\right), \quad x>0. \tag{2}$$

The coefficient of variation, skewness and kurtosis of the IG distribution are given by $\sqrt{\mu/\lambda}$, $3\sqrt{\mu/\lambda}$ and $15\sqrt{\mu/\lambda}$, respectively. Some PDFs are presented in Figure 1 for various values of $\lambda(\mu=1)$ and $\mu(\lambda=1)$. Figure 1(left) shows that the skewness of the IG distribution increases, for a fixed value of μ, as λ decreases.

The characteristic function of $X \sim \mathrm{IG}(\mu,\lambda)$ is given by

$$\varphi_X(t) := E(e^{itX}) = \int_0^\infty e^{itx} f(x;\mu,\lambda)\mathrm{d}x = \exp\left\{\frac{\lambda}{\mu}\left[1-\left(1-\frac{2i\mu^2 t}{\lambda}\right)^{1/2}\right]\right\}.$$

The kth moment of X is obtained by taking the kth derivative of $(-i)^k\varphi_X(t)$ and letting $t=0$,

$$\mathrm{E}(X^k) = \mu^k \sum_{l=0}^{k-1}\frac{(k-1+l)!}{l!(k-1-l)!}\left(2\frac{\lambda}{\mu}\right)^{-l}.$$

In particular, the mean and variance associated with (2) are given by

$$\mathrm{E}(X) = \mu \quad \text{and} \quad \mathrm{Var}(X) = \frac{\mu^3}{\lambda}.$$

Moreover, if $X \sim \mathrm{IG}(\mu,\lambda)$, then $W = X^{-1}$ has the PDF given by

$$f(w;\mu,\lambda) = \left(\frac{\lambda}{2\pi w}\right)^{1/2}\exp\left(-\frac{\lambda(1-\mu w)^2}{2\mu^2 w}\right), \quad w>0.$$

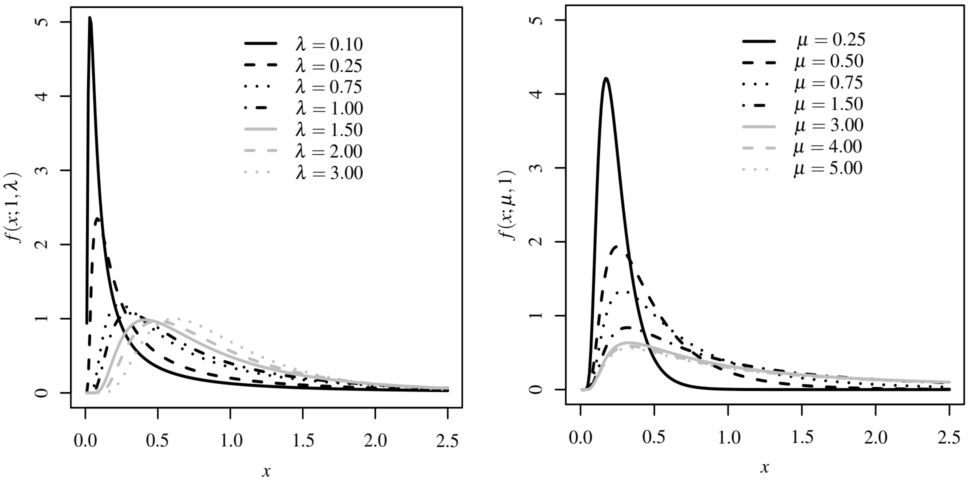

Figure 1. PDF of the IG($\mu = 1, \lambda$) (left) and IG($\mu, \lambda = 1$) (right) distributions for the indicated values of λ and μ.

The kth moment of W is given by

$$\mathrm{E}(W^k) = \frac{E(X^{k+1})}{\mu^{2k+1}}.$$

Therefore, we also readily have

$$\mathrm{E}(W) = \frac{1}{\mu} + \frac{1}{\lambda} \quad \text{and} \quad \mathrm{Var}(W) = \frac{1}{\lambda\mu} + \frac{2}{\lambda^2}.$$

3. Maximum likelihood and proposed estimators

In this section, we present the MLE and propose two new estimators for the shape parameter of the IG distribution.

3.1 Maximum likelihood estimators

Suppose $X_1, X_2, \ldots, X_n$ is a random sample of size n from a random variable X with the PDF in (2). Then, the log-likelihood function for the vector of parameters (μ, λ) can be written as

$$\ell(\mu, \lambda) = \frac{n}{2}\log\left(\frac{\lambda}{2\pi}\right) - \frac{3}{2}\sum_{i=1}^{n}\log(X_i) - \lambda\sum_{i=1}^{n}\frac{(X_i - \mu)^2}{2\mu^2 X_i},$$

and the MLEs of μ and λ are given by

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad \widehat{\lambda} = \frac{n}{\sum_{i=1}^{n}(1/X_i - 1/\widehat{\mu})}.$$

$\widehat{\mu}$ follows an inverse Gaussian distribution with parameters μ and $n\,\lambda$, whereas $n/\widehat{\lambda} \sim (1/\lambda)\chi^2_{n-1}$. Also, $\widehat{\mu}$ and $\widehat{\lambda}$ are independent; see Chhikara and Folks (1989).

Consider again the random sample $X_1, X_2, \ldots, X_n$ and $x_1, x_2, \ldots, x_n$ their observations. Denote the sample arithmetic and harmonic means by

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad r = \left[\frac{1}{n}\sum_{i=1}^{n} x_i^{-1}\right]^{-1},$$

respectively. Note that we can rewrite $\widehat{\mu}$ and $\widehat{\lambda}$ as

$$\widehat{\mu} = \overline{x} \quad \text{and} \quad \widehat{\lambda} = \frac{\overline{x}}{\overline{x}/r - 1} = \frac{\overline{x}\, r}{\overline{x} - r} = \frac{\overline{x}}{\overline{x}/r - 1}. \tag{3}$$

Property 3.1 Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the $IG(\mu, \lambda)$ distribution with the PDF in (2). Consider the random variables $\overline{X} = n^{-1}\sum_{i=1}^{n} X_i$ and $R = [n^{-1}\sum_{i=1}^{n} X_i^{-1}]^{-1}$. Then, $(\overline{X}, R^{-1})^\top$ is asymptotically bivariate normal distributed, that is, we have that

$$\sqrt{n}\begin{pmatrix} \overline{X} - E(X) \\ R^{-1} - E(X^{-1}) \end{pmatrix} \xrightarrow[n\to\infty]{} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{21} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}\right),$$

where

$$\sigma_{11} = \mathrm{Var}(X) = \frac{\mu^3}{\lambda}, \quad \sigma_{22} = \mathrm{Var}(X^{-1}) = \frac{1}{\lambda\mu} + \frac{2}{\lambda^2}, \quad \sigma_{12} = \sigma_{21} = \mathrm{Cov}(X, X^{-1}) = -\frac{\mu}{\lambda}.$$

Proof. By the strong law of large numbers, it follows that $\overline{X}$ and R^{-1} converge almost surely to $E(X)$ and $E(X^{-1})$, respectively. Additionally, from the central limit theorem, $\overline{X}$ and R^{-1}, as well as any linear combination of these quantities, asymptotically follow a bivariate normal distribution. Therefore,

$$\sqrt{n}\begin{pmatrix} \overline{X} - E(X) \\ R^{-1} - E(X^{-1}) \end{pmatrix} \xrightarrow[n\to\infty]{} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{11} = \mathrm{Var}(X) & \sigma_{12} = 1 - \mathrm{E}(X)\mathrm{E}(X^{-1}) \\ \sigma_{21} = 1 - \mathrm{E}(X)\mathrm{E}(X^{-1}) & \sigma_{22} = \mathrm{Var}(X^{-1}) \end{pmatrix}\right). \quad \blacksquare$$

3.2 Proposed estimator

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n where $X \sim \mathrm{IG}(\mu, \lambda)$, then define

$$Z_{ij} = X_i\, X_j^{-1}, \quad \text{for } 1 \le i \ne j \le n,$$

where $Z_{ij} = 1/Z_{ji}$; see Balakrishnan and Zhu (2014) and Balakrishnan, Saulo, Bourguignon and Zhu (2017). Now, it is possible to show that

$$E(Z_{ij}) = E(X_i\, X_j^{-1}) = \mu\left(\frac{1}{\mu} + \frac{1}{\lambda}\right). \tag{4}$$

Note that the sample mean of Z_{ij} is given by

$$\overline{Z} = \frac{1}{2\binom{n}{2}} \sum_{1\le i\ne j\le n} Z_{ij}. \tag{5}$$

Now, if we equate (5) to (4) with $\widehat{\mu} = \bar{x}$, we get an estimator of λ as

$$\widetilde{\lambda}^* = \frac{\overline{x}}{\overline{Z} - 1}. \tag{6}$$

We are now going to show that the proposed estimator $\widetilde{\lambda}^*$ can be written as a function of the MLE $\widehat{\lambda}$. From (5), we have

$$\begin{aligned} \overline{Z} = \frac{1}{2\binom{n}{2}} \sum_{1 \leq i \neq j \leq n} \frac{X_i}{X_j} &= \frac{1}{n(n-1)} \left[\sum_{1 \leq i \neq j \leq n} \frac{X_i}{X_j} + n \right] - \frac{n}{n(n-1)} \\ &= \frac{n(\overline{x}/r)}{n-1} - \frac{1}{n-1} \\ &= \frac{n(\overline{x}/r) - 1}{n-1}. \end{aligned}$$

Therefore, based on (3) the estimator $\widetilde{\lambda}^*$ in (6) can be rewritten as

$$\widetilde{\lambda}^* = \frac{\overline{x}}{\frac{n(\overline{x}/r)-1}{n-1} - 1} = \frac{n-1}{n} \frac{\overline{x}}{\overline{x}/r - 1} = \frac{n-1}{n} \widehat{\lambda}.$$

Property 3.2 The estimator $\widetilde{\lambda}^*$ is consistent.

Proof. First, note that for $\overline{Z} \xrightarrow{p} 1 + \mu/\lambda$, $Var[\overline{Z}] \to 0$. Then, taking probability limits in (6), we obtain

$$\text{plim}_{n\to\infty} \widetilde{\lambda}^* = \text{plim}_{n\to\infty} \frac{\overline{x}}{\overline{Z} - 1} = \frac{\mu}{\mu/\lambda} = \lambda. \qquad \blacksquare$$

Property 3.3 Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the IG distribution with PDF as given in (2). Then, $\widetilde{\lambda}^* \leq \widehat{\lambda}$.

3.2.1 Asymptotic distribution of $\overline{Z}$

Here, the asymptotic behaviour of $\overline{Z}$ is considered in order to derive the asymptotic distribution of $\widetilde{\lambda}^*$. Note that

$$E(\overline{Z}) = E\left(\frac{X_i}{X_j}\right) = \mu\left(\frac{1}{\mu} + \frac{1}{\lambda}\right) = \frac{\lambda + \mu}{\lambda},$$

and

$$\begin{aligned} E(\overline{Z}^2) &= \frac{1}{n^2(n-1)^2} E\left[\sum_{1 \leq i \neq j \neq k \neq l \leq n} \frac{X_i X_j}{X_k X_l} + \sum_{1 \leq i \neq j \neq k \leq n} \frac{X_i^2}{X_j X_k} + \sum_{1 \leq i \neq j \neq k \leq n} \frac{X_j X_k}{X_i^2} \right. \\ &\quad \left. + 2 \sum_{1 \leq i \neq j \neq k \leq n} \frac{X_i X_j}{X_i X_k} + \sum_{1 \leq i \neq j \leq n} \frac{X_i^2}{X_j^2} + \sum_{1 \leq i \neq j \leq n} \frac{X_i X_j}{X_j X_i} \right] \\ &= \frac{(n-2)(n-3)}{n(n-1)} \frac{(\lambda+\mu)^2}{\lambda^2} + \frac{(n-2)}{n(n-1)} \frac{(\lambda+\mu)^3 + \mu\lambda(\lambda+2\mu) + \lambda(\lambda+\mu)^2 + 2\lambda^2(\lambda+\mu)}{\lambda^3} \\ &\quad + \frac{1}{n(n-1)} \frac{\mu(\lambda+2\mu)(\lambda+\mu) + (\lambda+\mu)^3 + \lambda^3}{\lambda^3}. \end{aligned}$$

The above results yield

$$
\begin{aligned}
\mathrm{Var}(\overline{Z}) &= E(\overline{Z}^2) - \frac{(\lambda+\mu)^2}{\lambda^2} \\
&= \frac{(n-2)(n-3)}{n(n-1)}\frac{(\lambda+\mu)^2}{\lambda^2} + \frac{(n-2)}{n(n-1)}\frac{(\lambda+\mu)^3 + \mu\lambda(\lambda+2\mu) + \lambda(\lambda+\mu)^2 + 2\lambda^2(\lambda+\mu)}{\lambda^3} \\
&\quad + \frac{1}{n(n-1)}\frac{\mu(\lambda+2\mu)(\lambda+\mu) + (\lambda+\mu)^3 + \lambda^3}{\lambda^3} - \frac{(\lambda+\mu)^2}{\lambda^2} \\
&= \frac{6-4n}{n(n-1)}\frac{(\lambda+\mu)^2}{\lambda^2} + \frac{(n-2)}{n(n-1)}\frac{(\lambda+\mu)^3 + \mu\lambda(\lambda+2\mu) + \lambda(\lambda+\mu)^2 + 2\lambda^2(\lambda+\mu)}{\lambda^3} \\
&\quad + \frac{1}{n(n-1)}\frac{\mu(\lambda+2\mu)(\lambda+\mu) + (\lambda+\mu)^3 + \lambda^3}{\lambda^3} - \frac{(\lambda+\mu)^2}{\lambda^2} \\
&\approx -\frac{4}{n-1}\frac{(\lambda+\mu)^2}{\lambda^2} + \frac{1}{n-1}\frac{(\lambda+\mu)^3 + \mu\lambda(\lambda+2\mu) + \lambda(\lambda+\mu)^2 + 2\lambda^2(\lambda+\mu)}{\lambda^3} \\
&= \frac{(\lambda+\mu)^3 - 3\lambda(\lambda+\mu)^2 + \mu\lambda(\lambda+2\mu) + 2\lambda^2(\lambda+\mu)}{(n-1)\lambda^3} \\
&= \frac{\mu^2(2\lambda+\mu)}{\lambda^3(n-1)}.
\end{aligned}
$$

Now, note that it is possible to rewrite $\overline{Z}$ as

$$
\overline{Z} = \frac{n}{n-1}\left(\frac{\overline{X}}{R}\right) - \frac{1}{n-1}. \tag{7}
$$

By using Property 3.1, which says $\overline{X}$ and R^{-1} are asymptotically distributed as bivariate normal, and the delta method, we obtain from (7) that

$$
\sqrt{n}\left(\overline{Z} - \frac{\lambda+\mu}{\lambda}\right) \xrightarrow[n\to\infty]{} \mathrm{N}\left(0, \frac{\mu^2(2\lambda+\mu)}{\lambda^3}\right). \tag{8}
$$

3.2.2 Asymptotic distribution of $\widetilde{\lambda}^*$

In order to obtain the distribution of $\widetilde{\lambda}^*$ we use the result in (8) and the Taylor series expansion of $\widetilde{\lambda}^*$. Note that

$$
\begin{aligned}
\widetilde{\lambda}^* = \frac{\overline{x}}{\overline{Z}-1} &= g(\overline{x},\overline{Z}) = g(a,b) + (\overline{x}-a)\frac{\partial}{\partial s}g(a,b) + (\overline{Z}-b)\frac{\partial}{\partial \overline{Z}}g(a,b) \\
&\quad + (\overline{x}-a)(\overline{Z}-b)\frac{\partial^2}{\partial\overline{x}\partial\overline{Z}}g(a,b) + \frac{1}{2}(\overline{x}-a)^2\frac{\partial^2}{\partial\overline{x}^2}g(a,b) + \frac{1}{2}(\overline{Z}-b)^2\frac{\partial^2}{\partial\overline{Z}^2}g(a,b) + R_2 \\
&= \lambda + (\overline{x}-\mu)\frac{\lambda}{\mu} - \left(\overline{Z} - \frac{\lambda+\mu}{\lambda}\right)\frac{\lambda^2}{\mu} - (\overline{x}-\mu)\left(\overline{Z} - \frac{\lambda+\mu}{\lambda}\right)\frac{\lambda^2}{\mu^2} + \frac{1}{2}\left(\overline{Z} - \frac{\lambda+\mu}{\lambda}\right)^2\frac{2\lambda^3}{\mu^2} + R_2,
\end{aligned}
$$

where $a = \mu$, $b = (\lambda+\mu)/\lambda$ and

$$
|R_k| \leq \frac{1}{(k+1)!}\sum_{l=0}^{k+1}\binom{k+1}{l}M|\overline{x}-a|^l|\overline{Z}-b|^{k+1-l} = \frac{M}{(k+1)!}(|\overline{x}-a||\overline{Z}-b|)^{k+1}, \tag{9}
$$

with (9) holding if all partial derivatives of g of order $k+1$ are bounded in magnitude by M. Note that to find the order of convergence we compute

$$\frac{\left|\frac{M}{(k+2)!}(|\overline{x}-a||\overline{Z}-b|)^{k+2}\right|}{\left|\frac{M}{(k+1)!}(|\overline{x}-a||\overline{Z}-b|)^{k+1}\right|^{r}} = \left|M^{1-r}\frac{(k+1)!^{r}}{(k+2)!}(|\overline{x}-a||\overline{Z}-b|)^{((k+1)(1-r)+1)}\right|,$$

and find the largest value of r such that the following limit is finite

$$\lim_{k\to+\infty}\left|M^{1-r}\frac{(k+1)!^{r}}{(k+2)!}(|\overline{x}-a||\overline{Z}-b|)^{((k+1)(1-r)+1)}\right|.$$

When $r=1$ this limit is zero. Therefore, the order of convergence is 1.

Note that the asymptotic distribution of $\widetilde{\lambda}^{*}$ is

$$\sqrt{n}(\widetilde{\lambda}^{*}-\lambda) \xrightarrow[n\to\infty]{} \mathrm{N}(0,\Omega),$$

where $\Omega = E([(\overline{X}-\mu)\lambda/\mu - (\overline{Z}-(\lambda+\mu)/\lambda)\lambda^{2}/\mu]^{2})$, which can be solved numerically.

3.2.3 Bias-reduced versions of the proposed estimator $\widetilde{\lambda}^{*}$

We here present two bias-reduced estimators.

1. Based on the Taylor series expansion of $\widetilde{\lambda}^{*}$, we have

   $$\mathrm{Bias}_{1}(\widetilde{\lambda}^{*}) \approx \frac{2\lambda+\mu}{n-1}.$$

   Note that

   $$E(\widetilde{\lambda}^{*}) = \lambda + \mathrm{Bias}_{1}(\widetilde{\lambda}^{*}) = \frac{\lambda(n+1)+\mu}{n-1}.$$

   Then, solving the above equation for λ and denoting it by $\widetilde{\lambda}^{*}_{u1}$, we get

   $$\widetilde{\lambda}^{*}_{u1} = \left(\widetilde{\lambda}^{*} - \frac{\widehat{\mu}}{n-1}\right)\frac{n-1}{n+1}.$$

2. Now, by inspecting the pattern of the bias of the NE through MC simulation studies (based on the simulation results, we study the possible relationship between the bias, n and λ by means of regression), we verify that for small and moderate sample sizes,

   $$\mathrm{Bias}_{2}(\widetilde{\lambda}^{*}) \approx \frac{3\lambda}{n}.$$

   Thus, by implementing a standard bias reduction method, we can obtain a bias-reduced version of the NE, which is given by

   $$\widetilde{\lambda}^{*}_{u2} = \frac{n}{n+3}\widetilde{\lambda}^{*}. \tag{10}$$

Note that $\widetilde{\lambda}^{*}_{u1}$ is the bias-reduced estimator of λ based on the second order bias and $\widetilde{\lambda}^{*}_{u2}$ (alternative bias-reduced estimator for $\widetilde{\lambda}^{*}$) is the bias-reduced estimator of λ based on the MC simulation. For an account of the bias-reduced estimator based on the MC simulation, see Ng, Kundu and Balakrishnan (2003).

4. Hypothesis testing

In this section, we propose a new test on the coefficient of variation defined as $\phi = \sqrt{\mu/\lambda}$ for the IG(μ, λ) population. The testing problem may, equivalently, be considered in terms of the parameter $\upsilon = \phi^2 = \mu/\lambda$. In particular, the interest lies in testing the null hypothesis

$$H_0 : \upsilon \leq \upsilon_0 \text{ against the alternative hypothesis } H_a : \upsilon > \upsilon_0,$$

for a given υ_0. Seshadri (1988) showed that the estimator of the square of the coefficient of variation in the IG distribution is a U-statistic, which is given by

$$U = \frac{\overline{X}V}{n-1} = \widehat{\mu}/\widehat{\lambda}^* = \widehat{\upsilon}^*, \tag{11}$$

where $V = \sum_{i=1}^{n}(X_i^{-1} - \overline{X}^{-1})$. For testing H_0 against H_a, the new statistic, based on (10), is

$$U^* = \frac{n+3}{n}(\overline{Z} - 1) = \widehat{\mu}/\widehat{\lambda}_u^* = \widehat{\upsilon}_u^*. \tag{12}$$

We opt to use (10) because it provides the best performance; see Section 5. The limiting distributions (from (8)) of the statistics $\sqrt{n-1}(\widehat{\upsilon}_u^* - \upsilon_0)/(\upsilon_0\sqrt{\upsilon_0 + 2})$ and $\sqrt{n-1}(\widehat{\upsilon}^* - \upsilon_0)/(\upsilon_0\sqrt{\upsilon_0 + 2})$ are both normal with mean zero and variance 1, under H_0. The null hypothesis is rejected for a given nominal level, γ say, if the test statistic exceeds the upper $100(1-\gamma)\%$ quantile of the N(0,1) distribution.

5. Numerical evaluation

In this section we carry out two MC simulation studies, the first one to evaluate the performance of the proposed estimators and the second one to evaluate the performance of the new test statistic described in Section 4. The numerical evaluations were implemented using the R software (R Core Team, 2016) by means of some packages already available at `http://cran.r-project.org` and new routines added to these packages.

5.1 Numerical results: estimation

Here we conduct an extensive MC simulation study in order to evaluate the performances of the estimators presented anteriorly. The scenario of this simulation considers the sample sizes $n \in \{10, 20, 50, 100, 200, 500\}$, the values of the true shape parameter as $\lambda \in \{0.10, 0.25, 0.75, 1.00, 1.50, 2.00, 3.00\}$, 10 000 MC replications, and without loss of generality the value of μ is set at 1.00. Note that the values of λ cover high, moderate and low skewness.

Table 1 presents the empirical values of the bias and MSE of the NE and its bias-reduced versions, as well as the corresponding values for the MLEs. From these results, we have the following findings:

1. As n increases, the bias and MSE of all the estimators decrease, as expected;
2. The NE of λ displays biases and MSEs that are, in most cases, smaller than or equal to those of the corresponding MLE for all sample sizes considered in this study;
3. As λ increases, the finite sample performances of the estimators of λ, the shape parameter, deteriorate;

4. The bias-reduced estimator we propose, $\widetilde{\lambda}^*_{u2}$, outperforms, in most cases, all other estimators by delivering an adjusted estimator with smaller bias and MSE.

5.2 Numerical results: hypothesis testing

We shall now turn to the evaluation of the finite-sample behavior of the U-statistic test (Seshadri, 1988) (the test statistic for this test is given in (11)) and the new statistic (see (12)) for the parameter $\upsilon = \phi^2 = \mu/\lambda$ of the IG distribution. The MC simulation was carried out based on the following cases:

Case 1. $H_0 : \upsilon \leq 2$ versus $H_a : \upsilon > 2$;

Case 2. $H_0 : \upsilon \leq 2.5$ versus $H_a : \upsilon > 2.5$.

The simulation scenario considers: sample size $n \in \{10, 20, 30, 40, 50, 60, 100, 200, 500\}$, vector of true parameters $[\mu, \lambda] = [2.0, 1.0]$, nominal levels $\gamma \in \{10\%, 5\%, 1\%\}$, and 10,000 MC replications for each sample size. Note that in Case 2 the null hypothesis is not true.

Table 2 presents the null rejection rates of the two tests for Case 1. We note that in both tests, U and U^*, the size distortion decreases as the sample size increases. It is noteworthy that the U test is considerably liberal in small samples. For instance, when $n = 100$ and $\gamma = 5\%$, the null rejection rate of the U test is 0.1064, i.e., nearly twice the nominal level of the test. The new test (U^*) is much less size distorted. For example, its null rejection rate in the same situation is 0.0813; see Table 2.

Table 3 displays the the null rejection rates for Case 2. In this case, we observe that in both tests, U and U^*, the size distortion increases as the sample size increases. For instance, when $n = 100$ and $\gamma = 5\%$, the null rejection rate of the U^* test is 0.3660, and when $n = 500$ and $\gamma = 5\%$, the null rejection rate of the U^* test is 0.8020. In the simulation results for Case 2, presented in Table 3, we observe that the powers associated with both tests increase as a function of the sample size. The results presented in this table indicate that, although the test based on U outperforms the test based on U^* in terms of power, the discrepancy between the powers decreases as a function of the sample size.

6. Illustrative examples

We illustrate our methodology by using two real data sets. The first data set (Set I) gives the maximum flood levels in millions of cubic feet per second for the Susquehanna River at Harrisburg, Pennsylvania (1890–1969); see Seshadri (1999, p. 34). The second data set (Set II) corresponds to active repair times (hours) for an airborne communication transceiver; see von Alven (1964, p. 156). Table 5 provides some descriptive statistics of the corresponding data sets presented in Table 4, including central tendency statistics, as well as the standard deviation (SD), skewness (CS) and kurtosis (CK). From these statistics, we note that the IG model can be a good candidate for modeling these data mainly due to their asymmetric nature. In order to know whether the IG distribution fits these data or not, we computed Kolmogorov-Smirnov (KS) distances between the empirical and fitted distribution functions. We found the KS distances and the corresponding p-values (reported within brackets) for Set I and Set II to be 0.1528 (0.7384) and 0.0635 (0.9933), respectively. These results suggest that the IG distribution is indeed a good model for the Set I and Set II data sets.

Table 1. Simulated values of relative biases and MSEs of the NE and its bias-reduced version in comparison with those of MLEs ($\mu = 1.0$). The best result (the smallest bias and the smallest MSE) in each row is in bold.

		Estimator of λ								Estimator of μ	
		Bias				MSE				Bias	MSE
n	λ	$\widehat{\lambda}$	$\widetilde{\lambda}^*$	$\widetilde{\lambda}^*_{u1}$	$\widetilde{\lambda}^*_{u2}$	$\widehat{\lambda}$	$\widetilde{\lambda}^*$	$\widetilde{\lambda}^*_{u1}$	$\widetilde{\lambda}^*_{u2}$	$\widehat{\mu}$	$\widehat{\mu}$
10	0.10	0.0422	0.0280	−0.0871	**−0.0015**	0.0096	0.0071	0.0200	**0.0037**	0.0096	1.0383
	0.25	0.1092	0.0733	−0.0783	**−0.0013**	0.0632	0.0469	0.0369	**0.0246**	0.0061	0.4149
	0.75	0.3307	0.2226	−0.0552	**−0.0018**	0.5876	0.4370	0.2352	**0.2292**	−0.0034	0.1299
	1.00	0.4348	0.2913	−0.0265	**−0.0067**	0.9935	0.7365	0.4913	**0.3856**	−0.0015	0.1019
	1.50	0.6411	0.4270	**−0.0062**	−0.0177	2.2863	1.7013	1.0566	**0.8991**	0.0014	0.0703
	2.00	0.8483	0.5635	**−0.0051**	−0.0281	3.9629	2.9446	1.6800	**1.5552**	−0.0014	0.0494
	3.00	1.2592	0.8333	**−0.0491**	−0.0513	8.7029	6.4594	4.1381	**3.4139**	−0.0033	0.0334
20	0.10	0.0178	0.0119	−0.0469	**−0.0027**	0.0021	0.0018	0.0047	**0.0012**	−0.0089	0.4972
	0.25	0.0421	0.0275	−0.0451	**−0.0087**	0.0129	0.0108	0.0113	**0.0076**	−0.0062	0.1962
	0.75	0.1339	0.0897	−0.0369	**−0.0198**	0.1203	0.1005	0.0795	**0.0703**	−0.0021	0.0655
	1.00	0.1721	0.1135	**−0.0288**	−0.0317	0.2031	0.1695	0.1423	**0.1194**	−0.0024	0.0502
	1.50	0.2637	0.1755	**−0.0370**	−0.0430	0.4931	0.4131	0.3024	**0.2909**	−0.0028	0.0327
	2.00	0.3495	0.2320	**−0.0255**	−0.0591	0.8296	0.6923	0.5374	**0.4863**	−0.0019	0.0250
	3.00	0.5289	0.3525	**−0.0222**	−0.0848	1.9295	1.6131	1.2276	**1.1330**	0.0004	0.0166
50	0.10	0.0060	0.0039	−0.0198	**−0.0020**	0.0005	0.0005	0.0009	**0.0004**	0.0060	0.2034
	0.25	0.0154	0.0101	−0.0194	**−0.0046**	0.0033	0.0030	0.0031	**0.0026**	0.0027	0.0825
	0.75	0.0489	0.0329	−0.0172	**−0.0114**	0.0311	0.0286	0.0257	**0.0246**	−0.0002	0.0270
	1.00	0.0656	0.0443	−0.0188	**−0.0148**	0.0532	0.0489	0.0440	**0.0420**	0.0007	0.0197
	1.50	0.0929	0.0611	**−0.0195**	−0.0273	0.1220	0.1126	0.0990	**0.0976**	−0.0003	0.0137
	2.00	0.1228	0.0803	**−0.0184**	−0.0374	0.2140	0.1975	0.1760	**0.1714**	0.0006	0.0102
	3.00	0.1796	0.1160	**−0.0193**	−0.0603	0.4680	0.4319	0.4065	**0.3761**	−0.0005	0.0066
100	0.10	0.0032	0.0022	−0.0099	**−0.0008**	0.0002	0.0002	0.0003	0.0002	−0.0036	0.0971
	0.25	0.0075	0.0049	−0.0102	**−0.0025**	0.0015	0.0014	0.0014	**0.0013**	−0.0028	0.0401
	0.75	0.0243	0.0166	−0.0104	**−0.0058**	0.0135	0.0129	0.0121	**0.0119**	0.0033	0.0133
	1.00	0.0306	0.0202	**−0.0086**	−0.0095	0.0226	0.0216	0.0213	**0.0201**	0.0008	0.0100
	1.50	0.0455	0.0300	**−0.0063**	−0.0145	0.0515	0.0493	0.0479	**0.0459**	−0.0003	0.0067
	2.00	0.0655	0.0449	**−0.0064**	−0.0147	0.0933	0.0892	0.0844	**0.0824**	0.0019	0.0051
	3.00	0.0953	0.0644	**−0.0066**	−0.0249	0.2132	0.2042	**0.1882**	0.1892	0.0002	0.0033
200	0.10	0.0014	0.0009	−0.0051	**−0.0006**	0.0001	0.0001	0.0001	0.0001	0.0003	0.0503
	0.25	0.0041	0.0028	−0.0048	**−0.0009**	0.0007	0.0007	0.0007	0.0007	−0.0001	0.0194
	0.75	0.0110	0.0072	−0.0055	**−0.0040**	0.0060	0.0059	0.0058	**0.0057**	0.0006	0.0069
	1.00	0.0132	0.0081	**−0.0055**	−0.0068	0.0106	0.0104	0.0104	**0.0100**	−0.0003	0.0049
	1.50	0.0249	0.0173	**−0.0045**	−0.0051	0.0248	0.0242	**0.0226**	0.0232	−0.0009	0.0033
	2.00	0.0245	0.0144	**−0.0052**	−0.0154	0.0423	0.0415	0.0420	**0.0403**	0.0002	0.0025
	3.00	0.0452	0.0300	**−0.0125**	−0.0148	0.0977	0.0956	**0.0910**	0.0922	0.0000	0.0016
500	0.10	0.0007	0.0005	−0.0019	**−0.0001**	0.0000	0.0000	0.0000	0.0000	0.0000	0.0200
	0.25	0.0016	0.0011	−0.0019	**−0.0004**	0.0003	0.0003	0.0003	0.0003	−0.0014	0.0082
	0.75	0.0042	0.0026	−0.0024	**−0.0018**	0.0023	0.0023	0.0023	**0.0022**	0.0004	0.0027
	1.00	0.0067	0.0047	**−0.0013**	**−0.0013**	0.0041	0.0041	**0.0040**	**0.0040**	−0.0004	0.0020
	1.50	0.0097	0.0067	**−0.0013**	−0.0023	0.0094	0.0093	0.0092	**0.0091**	0.0003	0.0013
	2.00	0.0135	0.0095	**−0.0005**	−0.0025	0.0164	0.0162	**0.0160**	**0.0160**	−0.0004	0.0010
	3.00	0.0168	0.0108	**−0.0032**	−0.0072	0.0367	0.0364	0.0360	**0.0359**	0.0002	0.0007

Table 2. Null rejection rates of the tests U^* and U for Case 1.

	$\gamma = 10\%$		$\gamma = 5\%$		$\gamma = 1\%$	
n	U^*	U	U^*	U	U^*	U
10	0.1713	0.2830	0.1384	0.2399	0.0921	0.1674
20	0.1501	0.2318	0.1147	0.1784	0.0622	0.1080
30	0.1482	0.2137	0.1007	0.1566	0.0533	0.0843
40	0.1410	0.1954	0.0946	0.1409	0.0521	0.0785
50	0.1412	0.1888	0.0921	0.1311	0.0456	0.0666
60	0.1336	0.1768	0.0884	0.1267	0.0408	0.0607
100	0.1306	0.1656	0.0813	0.1064	0.0352	0.0476
200	0.1211	0.1432	0.0763	0.0911	0.0258	0.0332
500	0.1148	0.1281	0.0617	0.0701	0.0226	0.0261

Table 3. Null rejection rates of the tests U^* and U for Case 2.

	$\gamma = 10\%$		$\gamma = 5\%$		$\gamma = 1\%$	
n	U^*	U	U^*	U	U^*	U
10	0.2430	0.3950	0.1880	0.3130	0.1030	0.2080
20	0.2400	0.3500	0.1720	0.2500	0.1070	0.1620
30	0.2840	0.3810	0.2070	0.2860	0.1100	0.1580
40	0.3210	0.4100	0.2360	0.3120	0.1350	0.1820
50	0.3610	0.4410	0.2650	0.3420	0.1550	0.2010
60	0.3510	0.4150	0.2640	0.3320	0.1450	0.1820
100	0.4700	0.5150	0.3660	0.4220	0.2200	0.2610
200	0.6530	0.6800	0.5250	0.5720	0.3240	0.3600
500	0.8690	0.8830	0.8020	0.8180	0.6130	0.6450

Table 6 presents the ML estimates and the estimates obtained by using the proposed method, as well as the values of the KS, Cramér-von Mises (CM) and Anderson-Darling (AD) test statistics and their corresponding p-values. In general, the smaller the values of these statistics, the better the fit to the data. We observe in this case that the estimates from MLE and NE are quite similar. Also, the KS, CM and AD tests presented in Table 6 supports the IG model assumption made in our analysis.

Now, for the first application, suppose that we are interested in testing $H_0 : \upsilon \leq 1.16$ versus $H_a : \upsilon > 1.16$. Since the observed $U^* = 2.3382$ and $U = 2.1921$, the respective p-values are 0.9457 and 0.9364. These values are quite large for a 1% level of significance and therefore a squared CV of less than or equal to 1.16 is accepted. For the second application, suppose that we are interested in testing $H_0 : \upsilon \leq 0.54$ versus $H_a : \upsilon > 0.54$. Since the observed $U^* = 0.0863$ and $U = 0.0750$, the respective p-values are close to 0. Thus, we reject H_0 at 1% level.

7. Concluding remarks

In this paper, we have introduced a new method of estimating the shape parameter of the IG distribution based on a random sample. We have provided empirical evidence that suggests that the NE of λ has bias that is smaller than or equal to that of the corresponding MLE for all sample sizes

Table 4. Observations from the indicated data set.

Set I											
0.654	0.613	0.315	0.449	0.297	0.402	0.379	0.423	0.379	0.3235	0.296	0.740
0.418	0.412	0.494	0.416	0.338	0.392	0.484	0.265				
Set II											
0.2	0.3	0.5	0.5	0.5	0.6	0.6	0.7	0.7	0.7	0.8	0.8
1.0	1.0	1.0	1.0	1.1	1.3	1.5	1.5	1.5	1.5	2.0	2.0
2.2	2.5	2.7	3.0	3.0	3.3	3.3	4.0	4.0	4.5	4.7	5.0
5.4	5.4	7.0	7.5	8.8	9.0	10.3	22.0	24.5			

Table 5. Descriptive statistics.

	n	Min.	Median	Mean	Max.	SD	CS	CK
Set I	20	0.265	0.407	0.424	0.74	1.12	1.03	0.31
Set II	45	0.2	2	3.67	24.5	4.97	2.76	8.08

Table 6. Estimates of the parameters, KS, CM and AD statistics (p-values in parentheses) for the indicated date set.

		Parameter				
	Estimator	μ	λ	KS	CM	AD
Set I	$(\widehat{\mu}, \widehat{\lambda})$	0.4244	5.9547	0.1528 (0.7383)	0.0610 (0.3499)	0.3783 (0.3730)
	$(\widehat{\mu}, \widetilde{\lambda}^{*})$	0.4244	5.6569	0.1514 (0.7489)	0.0610 (0.3499)	0.3783 (0.3730)
	$(\widehat{\mu}, \widetilde{\lambda}^{*}_{u2})$	0.4244	4.9191	0.1473 (0.7781)	0.0610 (0.3499)	0.3781 (0.3734)
Set II	$(\widehat{\mu}, \widehat{\lambda})$	3.6755	1.7148	0.0626 (0.9944)	0.0270 (0.8800)	0.1891 (0.8950)
	$(\widehat{\mu}, \widetilde{\lambda}^{*})$	3.6755	1.6767	0.0626 (0.9944)	0.0270 (0.8800)	0.1893 (0.8949)
	$(\widehat{\mu}, \widetilde{\lambda}^{*}_{u2})$	3.6755	1.5719	0.0700 (0.9800)	0.0272 (0.8787)	0.1896 (0.8946)

considered in the simulation study. Also, we have introduced a bias-reduced version of the NE which outperformed all other estimators in our simulation study. The numerical results showed that the new test outperforms the U-test for the coefficient of variation. As part of future work, an extension of the proposed method of estimation to the bivariate IG (Al-Hussaini and Nagi, 1981) distribution will be of great interest; see Saulo, Balakrishnan, Zhu, Gonzales and Leão (2017).

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