UNIVERZA V LJUBLJANI
Fakulteta za elektrotehniko

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ZBIRKA REŠENIH VAJ PRI PREDMETU

OBDELAVA SIGNALOV

NA VISOKOŠOLSKEM STROKOVNEM ŠTUDIJSKEM PROGRAMU ELEKTROTEHNIKE

Ljubljana, 2012

Predgovor

Zbirka rešenih vaj je nastala v okviru avditornih vaj, ki smo jih izvajali pri predmetu *Teorija obdelave signalov* na univerzitetnem študiju elektrotehnike in pri predmetu *Obdelava signalov* na viskošolskem strokovnem študiju elektrotehnike na Fakulteti za elektrotehniko, Univerze v Ljubljani, v letih od 2003 do 2008.

Rešitve nalog so izvedene s programskim paketom *Wolfram Mathematica 8*. Zbirka je nastala s pomočjo naslednje literature:

- MIHELIČ, France. *Signali. 1. izd.* Ljubljana: Fakulteta za elektrotehniko, 2006.
- MIHELIČ, France, GYERGYÉK, Ludvik, EBENŠPANGER, Tomaž. *Signali : priročnik z zbirko rešenih nalog. 4. popravljena in dopolnjena izd.* Ljubljana: Založba FE in FRI, 2009.

Ljubljana, 2012 Janez Žibert

Kazalo

Izražava signalov
z Walshevimi temeljnimi funkcijami

Walsheve temeljne funkcije

Definicija prvih štirih funkcij

$$W_0[t\_] := \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{True} \end{cases}$$

$$W_1[t\_] := \begin{cases} 1 & 0 < t \le \frac{1}{2} \\ -1 & \frac{1}{2} < t \le 1 \\ 0 & \text{True} \end{cases}$$

$$W_2[t\_] := \begin{cases} 1 & 0 < t \le \frac{1}{4} \\ -1 & \frac{1}{4} < t \le \frac{1}{2} \\ 1 & \frac{1}{2} < t \le \frac{3}{4} \\ -1 & \frac{3}{4} < t \le 1 \\ 0 & \text{True} \end{cases}$$

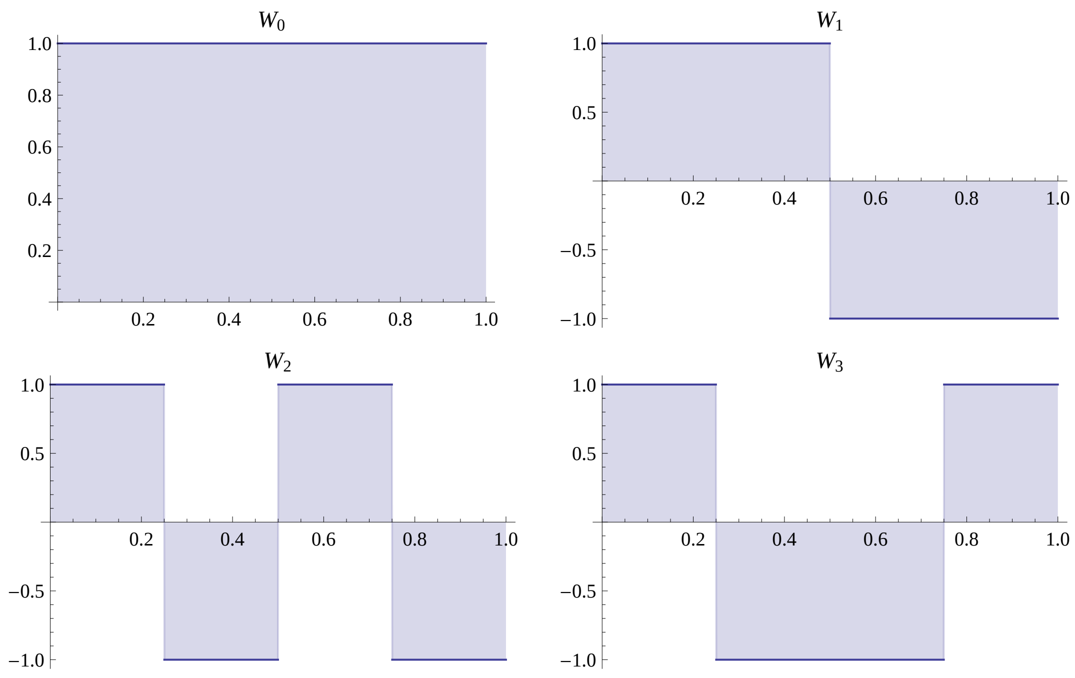
$$W_3[t\_] := \begin{cases} 1 & 0 < t \le \frac{1}{4} \\ -1 & \frac{1}{4} < t \le \frac{3}{4} \\ 1 & \frac{3}{4} < t \le 1 \\ 0 & \text{True} \end{cases}$$

Koeficienti:

$K_0 := 1;$
$K_1 := 1;$
$K_2 := 1;$
$K_3 := 1;$

Izris funkcij:

```
gw0 = Plot[W0[t], {t, 0, 1}, PlotRange → All, PlotLabel → "W0", Filling → Axis];
gw1 = Plot[W1[t], {t, 0, 1}, PlotRange → All, PlotLabel → "W1", Filling → Axis];
gw2 = Plot[W2[t], {t, 0, 1}, PlotRange → All, PlotLabel → "W2", Filling → Axis];
gw3 = Plot[W3[t], {t, 0, 1}, PlotRange → All, PlotLabel → "W3", Filling → Axis];
GraphicsGrid[{{gw0, gw1}, {gw2, gw3}}]
```



Aproksimacija signala (naloga)

Naloga:

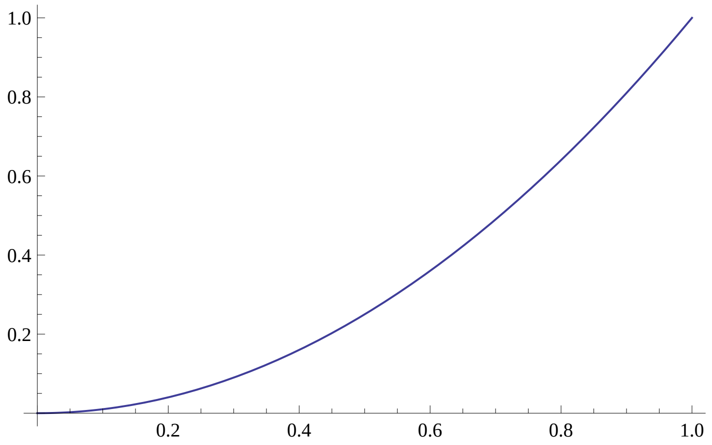
Signal $\mathbf{x(t) = t^2}$ na intervalu **[0,1]** izrazite s približkom *prvih štirih Walshevih* temeljnih funkcij.

Določite še napako aproksimacije in skicirajte približek.

Rešitev:

```
x[t_] := t^2;
```

```
Plot[x[t], {t, 0, 1}, PlotRange → All]
```

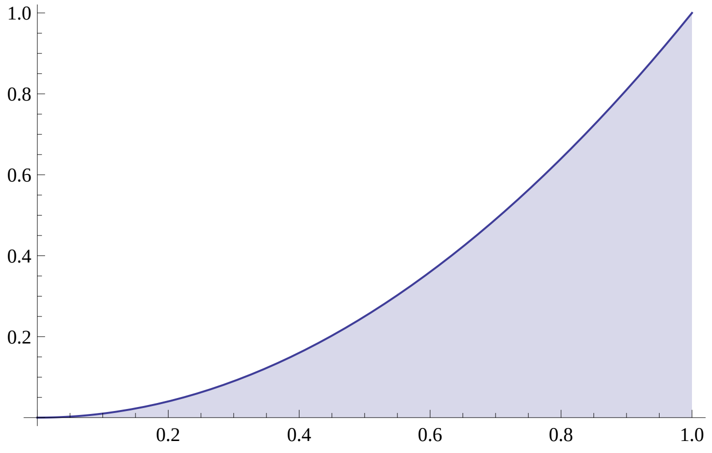


Izračun koeficientov:

$$c_0 = \frac{1}{1} * \int_0^1 t^2 * 1 \, dt$$

$$\frac{1}{3}$$

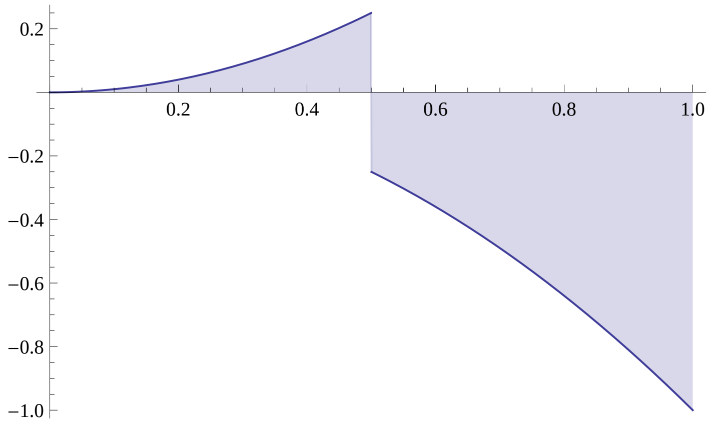
```
Plot[{x[t] * W0[t]}, {t, 0, 1}, Filling → Axis]
```



$$c_1 = \frac{1}{1} * \left(\int_0^{1/2} t^2 * 1 \, dt + \int_{1/2}^1 t^2 * (-1) \, dt \right)$$

$$-\frac{1}{4}$$

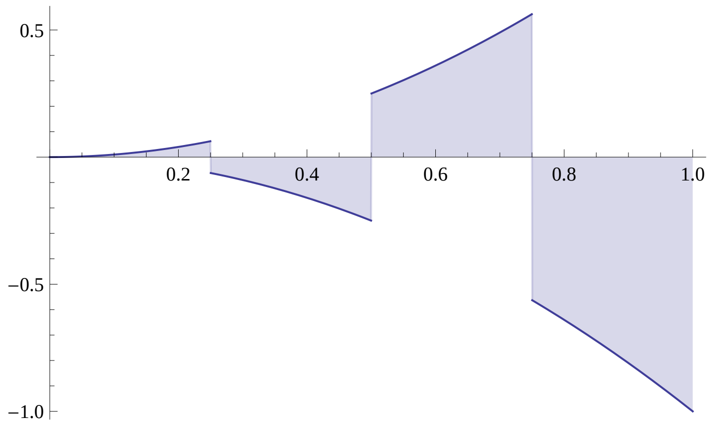
```
Plot[{x[t] * W1[t]}, {t, 0, 1}, Filling → Axis]
```



$$C_2 = \frac{1}{1} * \left(\int_0^{1/4} t^2 * 1\, dt + \int_{1/4}^{1/2} t^2 * (-1)\, dt + \int_{1/2}^{3/4} t^2 * 1\, dt + \int_{3/4}^{1} t^2 * (-1)\, dt\right)$$

$$-\frac{1}{8}$$

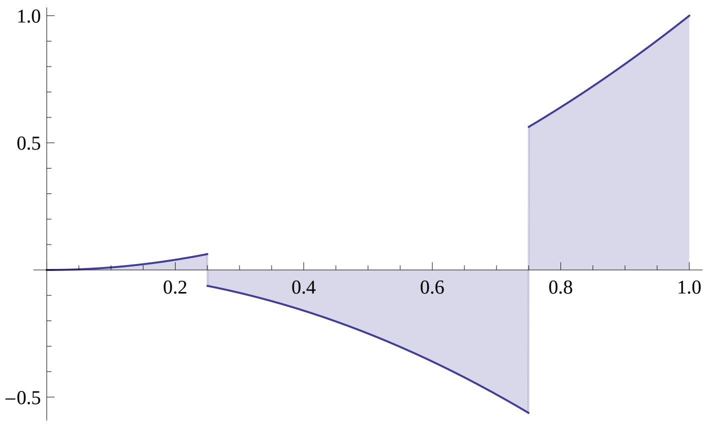
```
Plot[{x[t] * W2[t]}, {t, 0, 1}, Filling → Axis]
```



$$C_3 = \frac{1}{1} * \left(\int_0^{1/4} t^2 * 1\, dt + \int_{1/4}^{3/4} t^2 * (-1)\, dt + \int_{3/4}^{1} t^2 * 1\, dt\right)$$

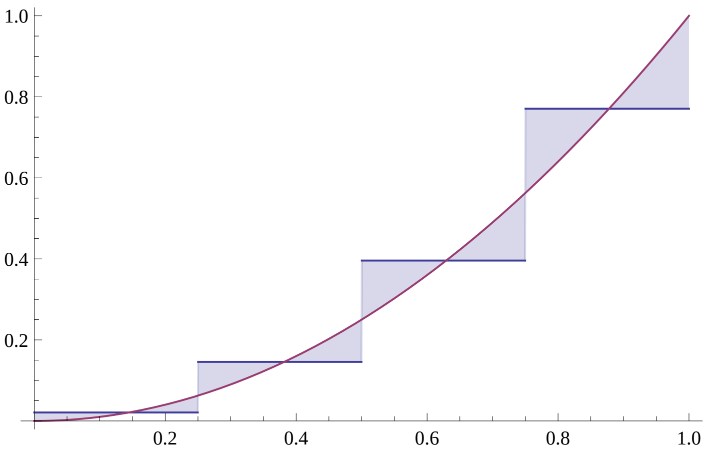
$$\frac{1}{16}$$

```
Plot[{x[t] * W3[t]}, {t, 0, 1}, Filling → Axis]
```



Izris aproksimiranega signala:

```
Plot[{C0 W0[t] + C1 W1[t] + C2 W2[t] + C3 W3[t], x[t]}, {t, 0, 1}, Filling → {1 → {2}}]
```



Izračun napake:

$t_1 = 0;$

$t_2 = 1;$

$$\epsilon = \frac{1}{t_2 - t_1} * \left(\int_0^1 t^2 * t^2 \, dt - \left(K_0 * C_0^2 + K_1 * C_1^2 + K_2 * C_2^2 + K_3 * C_3^2\right)\right)$$

$$\frac{79}{11\,520}$$

```
N[%]
```

0.00685764

Izražava signalov s Haarovimi temeljnimi funkcijami

Haarove temeljne funkcije

Definicija prvih štirih funkcij

$$H_0[t\_] := \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{True} \end{cases}$$

$$H_1[t\_] := \begin{cases} 1 & 0 < t \le \frac{1}{2} \\ -1 & \frac{1}{2} < t \le 1 \\ 0 & \text{True} \end{cases}$$

$$H_2[t\_] := \begin{cases} 1 & 0 < t \le \frac{1}{4} \\ -1 & \frac{1}{4} < t \le \frac{1}{2} \\ 0 & \text{True} \end{cases}$$

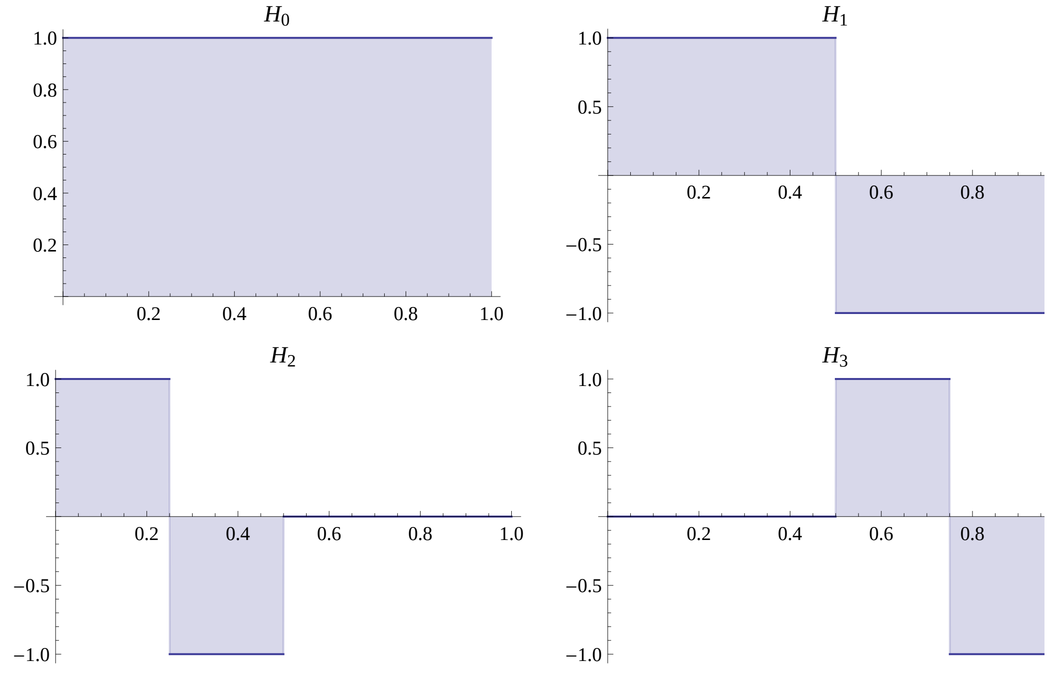
$$H_3[t\_] := \begin{cases} 1 & \frac{1}{2} < t \le \frac{3}{4} \\ -1 & \frac{3}{4} < t \le 1 \\ 0 & \text{True} \end{cases}$$

Koeficienti:

$$K_0 := 1;$$
$$K_1 := 1;$$
$$K_2 := \frac{1}{2};$$
$$K_3 := \frac{1}{2};$$

Izris funkcij:

```
gw0 = Plot[H0[t], {t, 0, 1}, PlotRange → All, PlotLabel → "H0",
   Filling → Axis];
gw1 = Plot[H1[t], {t, 0, 1}, PlotRange → All, PlotLabel → "H1",
   Filling → Axis];
gw2 = Plot[H2[t], {t, 0, 1}, PlotRange → All, PlotLabel → "H2",
   Filling → Axis];
gw3 = Plot[H3[t], {t, 0, 1}, PlotRange → All, PlotLabel → "H3",
   Filling → Axis];
GraphicsGrid[{{gw0, gw1}, {gw2, gw3}}]
```



Aproksimacija signala (naloga 1)

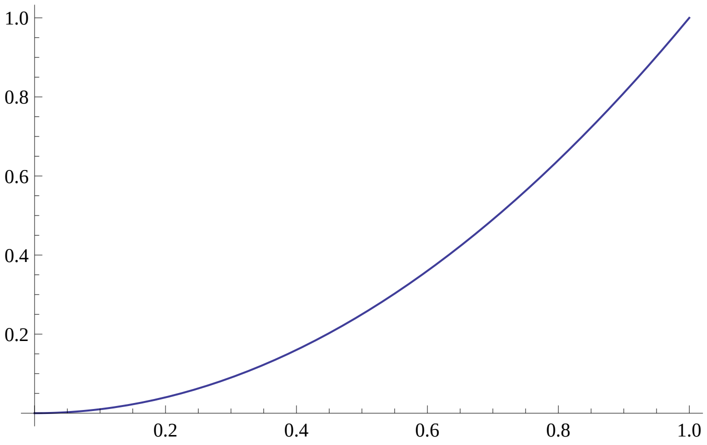
Naloga:

Signal $\mathbf{x(t) = t^2}$ na intervalu $\mathbf{[0,1]}$ izrazite s približkom *prvih štirih Haarovih* temeljnih funkcij.
Določite še napako aproksimacije in skicirajte približek.

Rešitev:

```
x[t_] := t^2;
```

```
Plot[x[t], {t, 0, 1}, PlotRange → All]
```

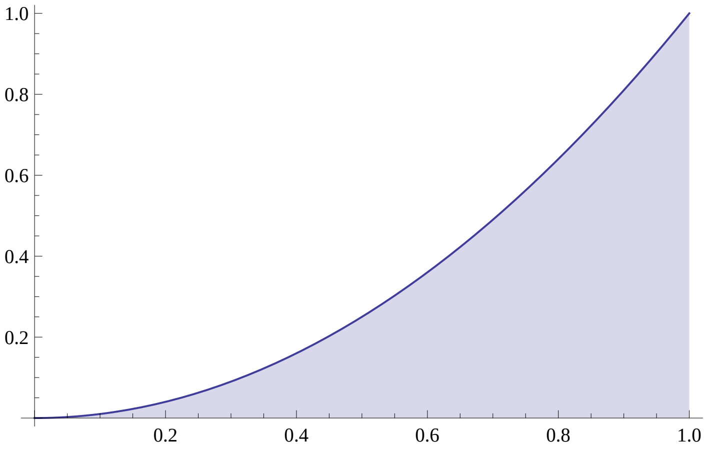


Izračun koeficientov:

$$C_0 = \frac{1}{1} * \int_0^1 t^2 * 1 \, dt$$

$$\frac{1}{3}$$

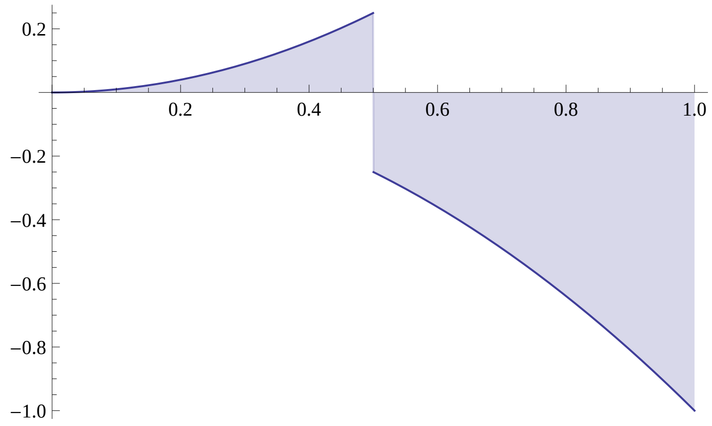
```
Plot[{x[t] * H0[t]}, {t, 0, 1}, Filling → Axis]
```



$$C_1 = \frac{1}{1} * \left(\int_0^{1/2} t^2 * 1 \, dt + \int_{1/2}^1 t^2 * (-1) \, dt \right)$$

$$-\frac{1}{4}$$

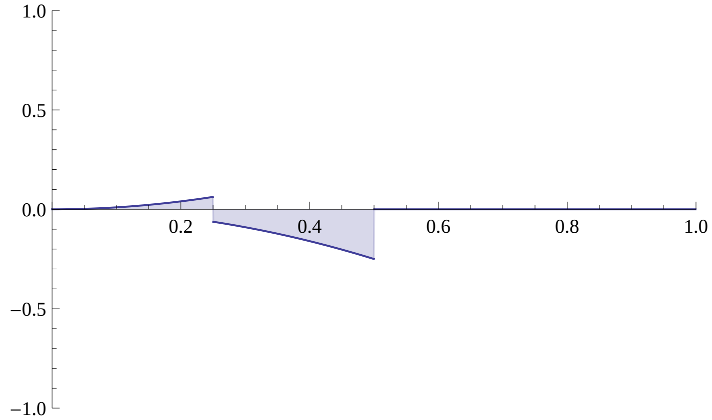
```
Plot[{x[t] * H1[t]}, {t, 0, 1}, Filling → Axis]
```



$$C_2 = \frac{1}{\frac{1}{2}} * \left(\int_0^{1/4} t^2 * 1\, dt + \int_{1/4}^{1/2} t^2 * (-1)\, dt\right)$$

$$-\frac{1}{16}$$

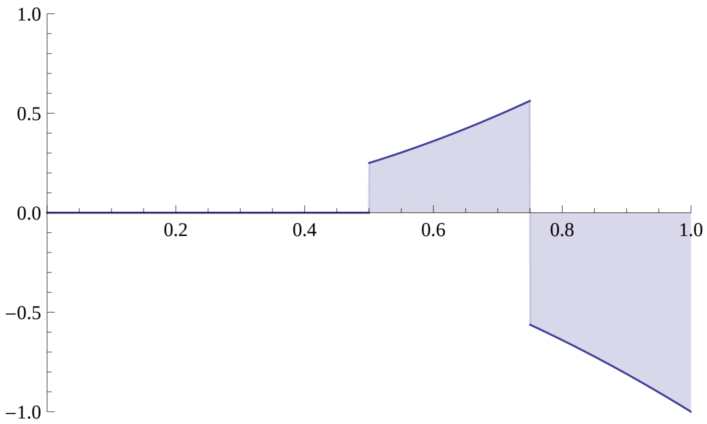
```
Plot[{x[t] * H2[t]}, {t, 0, 1}, Filling → Axis, PlotRange → {{0, 1}, {-1, 1}}]
```



$$C_3 = \frac{1}{\frac{1}{2}} * \left(\int_{\frac{1}{2}}^{\frac{3}{4}} t^2 * 1\, dt + \int_{\frac{3}{4}}^{1} t^2 * (-1)\, dt\right)$$

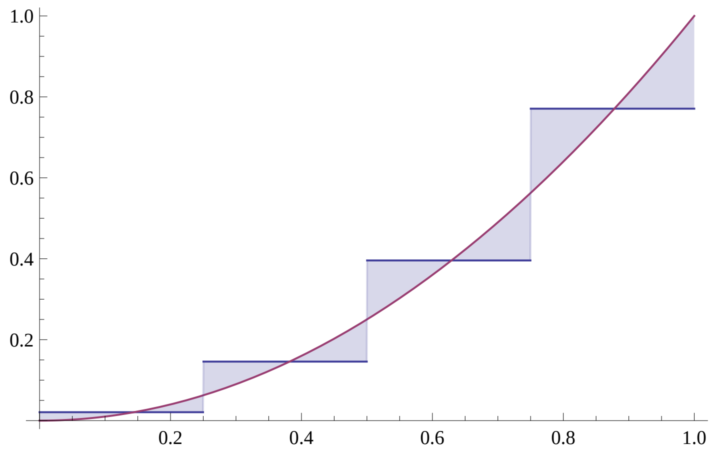
$$-\frac{3}{16}$$

```
Plot[{x[t] * H3[t]}, {t, 0, 1}, Filling → Axis, PlotRange → {{0, 1}, {-1, 1}}]
```



Izris aproksimiranega signala:

```
Plot[{C0 H0[t] + C1 H1[t] + C2 H2[t] + C3 H3[t], x[t]}, {t, 0, 1},
 Filling → {1 → {2}}]
```



Izrračun napake:

$t_1 = 0;$

$t_2 = 1;$

$$\epsilon = \frac{1}{t_2 - t_1} * \left(\int_0^1 t^2 * t^2 \, dt - \left(K_0 * C_0^2 + K_1 * C_1^2 + K_2 * C_2^2 + K_3 * C_3^2\right)\right)$$

$$\frac{79}{11520}$$

```
N[%]
```

```
0.00685764
```

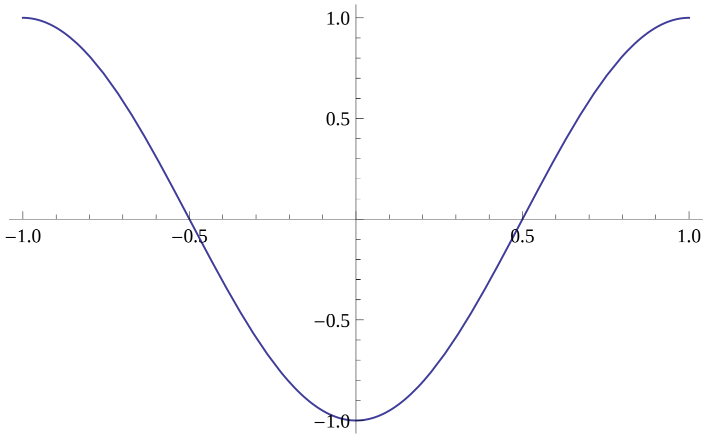
Aproksimacija signala na drugem intervalu (naloga 2)

Naloga:
Signal **x(t) = -**`Cos[πt]` na **intervalu [-1,1]** izrazite s približkom prvih štirih Haarovih temeljnih funkcij.
Določite še **razliko v napaki** aproksimacije, če aproksimiramo s prvimi štirimi Haarovimi funkcijami ali samo s prvimi tremi Haarovimi funkcijami.

Rešitev:

```
x[t_] := -Cos[π*t];
```

```
Plot[x[t], {t, -1, 1}, PlotRange → All]
```



Haarove funkcije imamo definirane na intervalu [0,1] zato jih moramo premakniti na interval [-1,1] in izvajati aproksimacijo s premaknjenimi Haarovimi t.f.

- ***Premaknjene Haarove. t.f.***

Poiščemo preslikavo **u: [-1,1] → [0,1]; u[t] = a*t+b**

```
u[t_] := 1/2 * t + 1/2;
```

```
Ĥ0[t_] := H0[u[t]];
Ĥ1[t_] := H1[u[t]];
Ĥ2[t_] := H2[u[t]];
Ĥ3[t_] := H3[u[t]];

a = 1/2;
K̂0 := K0/a
K̂1 := K1/a
K̂2 := K2/a
K̂3 := K3/a
```

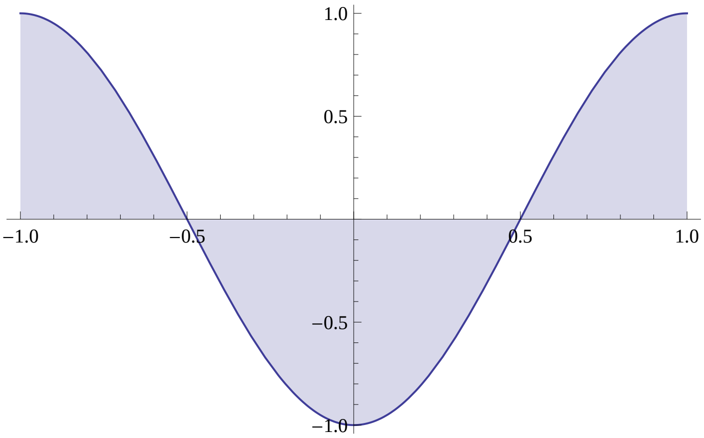
```
{K̂0, K̂1, K̂2, K̂3}
```

{2, 2, 1, 1}

$$C_0 = \frac{1}{\hat{K}_0} * \int_{-1}^{1} x[t] * \hat{H}_0[t] \, dt$$

0

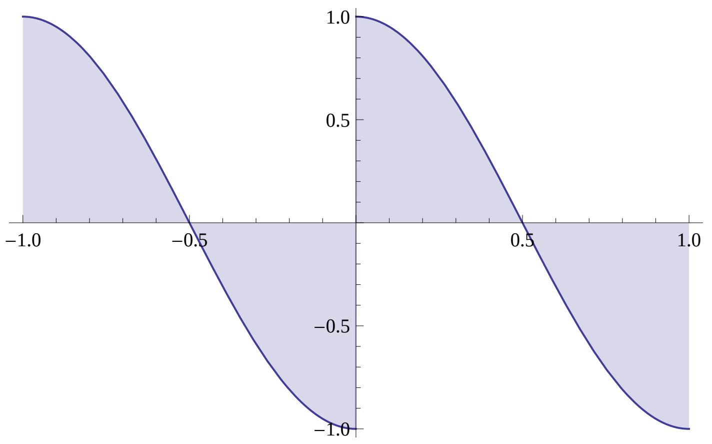
```
Plot[{x[t] * Ĥ0[t]}, {t, -1, 1}, Filling → Axis]
```



```
C1 = 1/K̂1 * ∫_{-1}^{1} x[t] * Ĥ1[t] dt
```

0

```
Plot[{x[t] * Ĥ1[t]}, {t, -1, 1}, Filling → Axis]
```



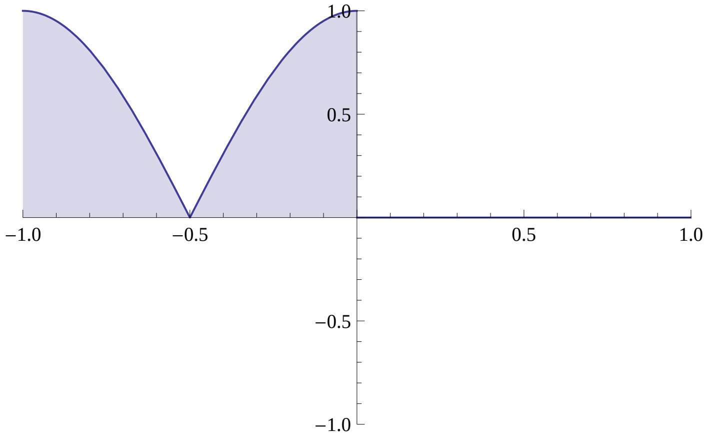
Pomoč pri izračunu koeficientov C_2 in C_3 :

$$\int \mathrm{Cos}[\pi * t]\, dt = \frac{\mathrm{Sin}[\pi t]}{\pi}$$

```
C2 = 1/K̂2 * ∫_{-1}^{1} x[t] * Ĥ2[t] dt
```

$$\frac{2}{\pi}$$

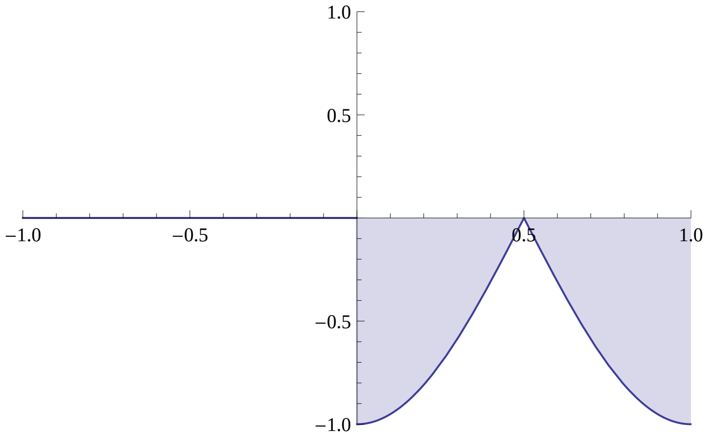
```
Plot[{x[t] * Ĥ2[t]}, {t, -1, 1}, Filling → Axis,
 PlotRange → {{-1, 1}, {-1, 1}}]
```




```
C3 = 1/K̂3 * ∫_{-1}^{1} x[t] * Ĥ3[t] dt
```

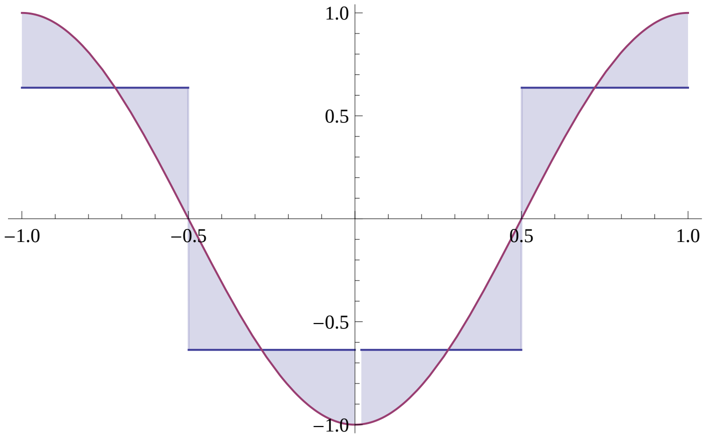
$$-\frac{2}{\pi}$$

```
Plot[{x[t] * Ĥ3[t]}, {t, -1, 1}, Filling → Axis,
 PlotRange → {{-1, 1}, {-1, 1}}]
```



- *Izris aproksimiranega signala:*

```
Plot[{C0 Ĥ0[t] + C1 Ĥ1[t] + C2 Ĥ2[t] + C3 Ĥ3[t], x[t]}, {t, -1, 1},
 Filling → {1 → {2}}]
```



- *Izračun razlike napake, če aproksimiramo s prvimi 4-imi H.t.f. ali pa samo s 3-mi H.t.f.*

$$\Delta\epsilon = \epsilon 4 - \epsilon 3 = \frac{1}{t_2 - t_1} * K_3 * C_3 * C_3;$$

V našem primeru:

$$\Delta\epsilon = \frac{1}{2} * \left(-\frac{2}{\pi}\right) * \left(-\frac{2}{\pi}\right)$$

$$\frac{2}{\pi^2}$$

```
N[%]
```

```
0.202642
```

Izražava signalov z Walshevimi temeljnimi funkcijami na premaknjenem intervalu

Walsheve temeljne funkcije

Definicija prvih štirih funkcij

$$W_0[t\_] := \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{True} \end{cases}$$

$$W_1[t\_] := \begin{cases} 1 & 0 < t \le \frac{1}{2} \\ -1 & \frac{1}{2} < t \le 1 \\ 0 & \text{True} \end{cases}$$

$$W_2[t\_] := \begin{cases} 1 & 0 < t \le \frac{1}{4} \\ -1 & \frac{1}{4} < t \le \frac{3}{4} \\ 1 & \frac{3}{4} < t \le 1 \\ 0 & \text{True} \end{cases}$$

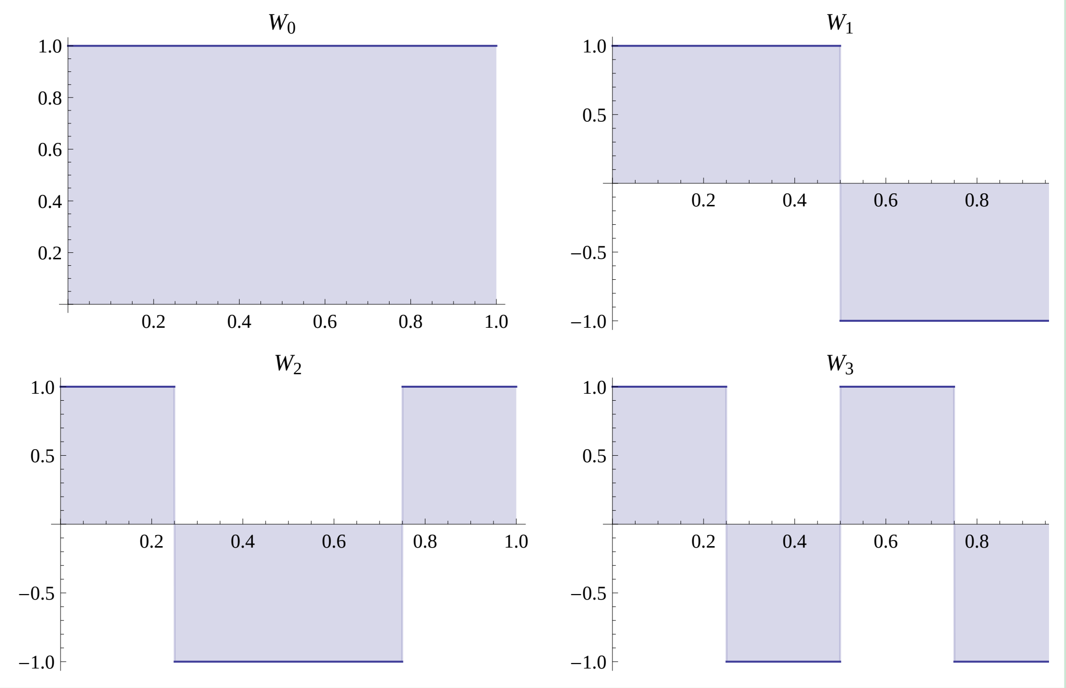
$$W_3[t\_] := \begin{cases} 1 & 0 < t \le \frac{1}{4} \\ -1 & \frac{1}{4} < t \le \frac{1}{2} \\ 1 & \frac{1}{2} < t \le \frac{3}{4} \\ -1 & \frac{3}{4} < t \le 1 \\ 0 & \text{True} \end{cases}$$

Koeficienti:

$K_0 := 1;$
$K_1 := 1;$
$K_2 := 1;$
$K_3 := 1;$

Izris funkcij:

```
gw0 = Plot[W0[t], {t, 0, 1}, PlotRange → All, PlotLabel → "W0",
   Filling → Axis];
gw1 = Plot[W1[t], {t, 0, 1}, PlotRange → All, PlotLabel → "W1",
   Filling → Axis];
gw2 = Plot[W2[t], {t, 0, 1}, PlotRange → All, PlotLabel → "W2",
   Filling → Axis];
gw3 = Plot[W3[t], {t, 0, 1}, PlotRange → All, PlotLabel → "W3",
   Filling → Axis];
GraphicsGrid[{{gw0, gw1}, {gw2, gw3}}]
```

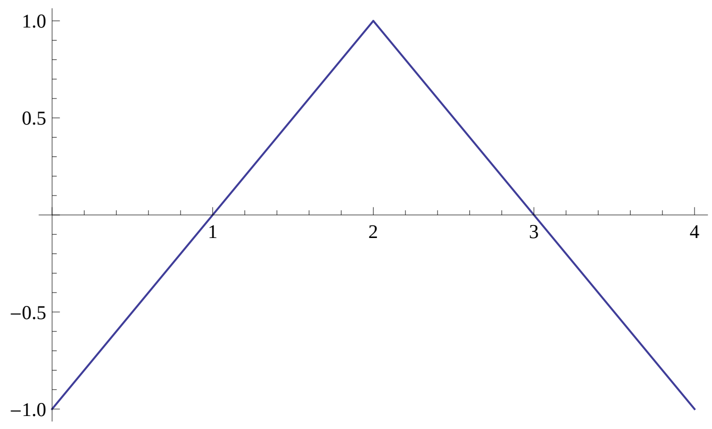


Aproksimacija signala na premaknjenem intervalu (naloga 1)

Naloga 1:
Signal na sliki izrazite s približkom prvih štirih Walshevih temeljnih funkcij.
Aproksimacijo signala skiciraj.
Kakšna je razlika v napaki, če signal aproksimiramo samo s prvimi tremi Walshevimi funkcijami?

```
            ⎧  t - 1   0 < t ≤ 2
x[t_] :=    ⎨ -t + 3   2 < t ≤ 4;
            ⎩   0       True
```

```
Plot[x[t], {t, 0, 4}, PlotRange → All]
```



Rešitev:

- *Premaknjene Walsheve. t.f.*

Poiščemo preslikavo **u: [0,4] → [0,1]; u[t] = a*t+b**

```
u[t_] := 1/4 * t;
```

```
Ŵ0[t_] := W0[u[t]];
Ŵ1[t_] := W1[u[t]];
Ŵ2[t_] := W2[u[t]];
Ŵ3[t_] := W3[u[t]];

a = 1/4;
K̂0 := K0/a
K̂1 := K1/a
K̂2 := K2/a
K̂3 := K3/a
```

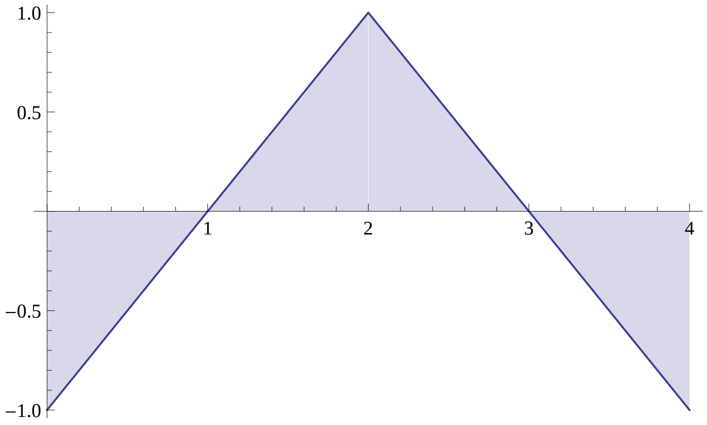
```
{K̂0, K̂1, K̂2, K̂3}
```

```
{4, 4, 4, 4}
```

```
C₀ = 1/K̂₀ * ∫₀⁴ x[t] * Ŵ₀[t] ⅆt
```

```
0
```

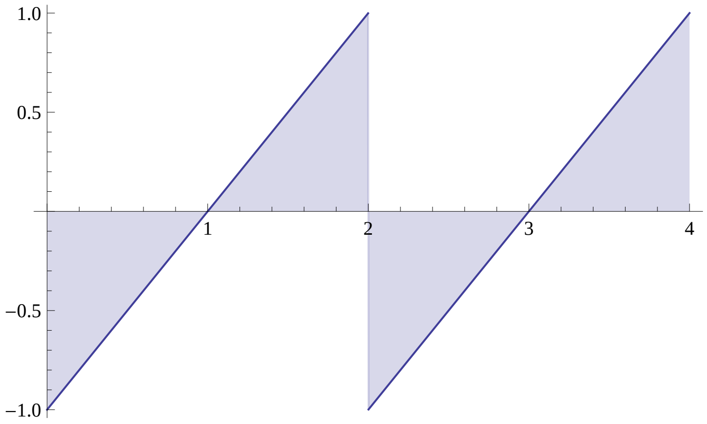
```
Plot[{x[t] * Ŵ₀[t]}, {t, 0, 4}, Filling → Axis]
```




```
C₁ = 1/K̂₁ * ∫₀⁴ x[t] * Ŵ₁[t] ⅆt
```

```
0
```

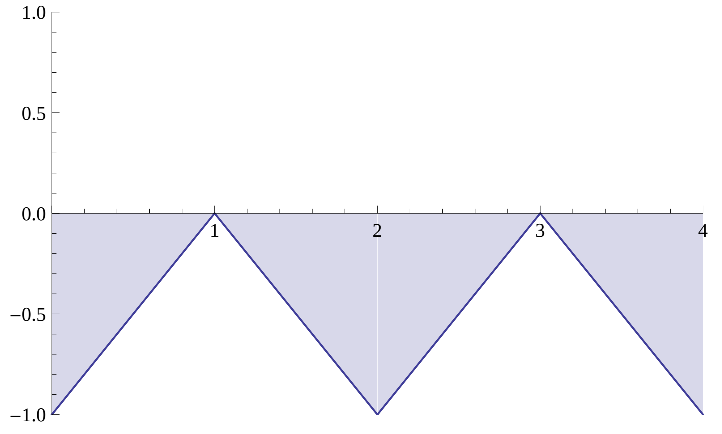
```
Plot[{x[t] * Ŵ₁[t]}, {t, 0, 4}, Filling → Axis]
```



```
C2 = 1/K̂2 * ∫_0^4 x[t] * Ŵ2[t] ⅆt
```

$-\frac{1}{2}$

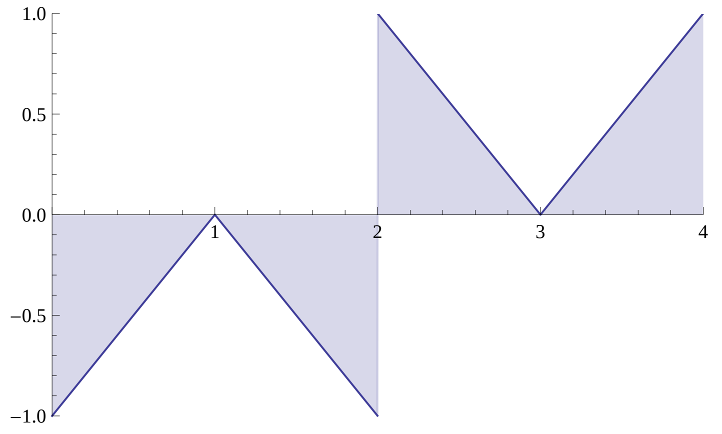
```
Plot[{x[t] * Ŵ2[t]}, {t, 0, 4}, Filling → Axis, PlotRange → {{0, 4}, {-1, 1}}]
```




```
C3 = 1/K̂3 * ∫_0^4 x[t] * Ŵ3[t] ⅆt
```

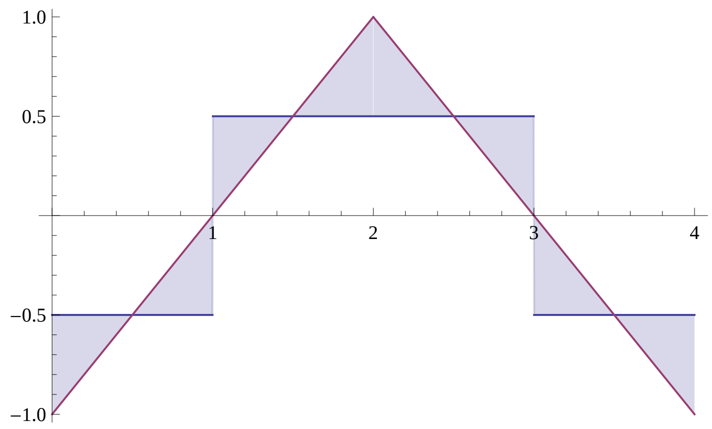
0

```
Plot[{x[t] * Ŵ3[t]}, {t, 0, 4}, Filling → Axis, PlotRange → {{0, 4}, {-1, 1}}]
```



- ***Izris aproksimiranega signala:***

```
Plot[{C0 Ŵ0[t] + C1 Ŵ1[t] + C2 Ŵ2[t] + C3 Ŵ3[t], x[t]}, {t, 0, 4},
 Filling → {1 → {2}}]
```



- ***Izračun razlike napake, če aproksimiramo s prvimi 4-imi W.t.f. ali pa samo s 3-mi W.t.f.***

$$\Delta\epsilon = \epsilon 4 - \epsilon 3 = \frac{1}{t_2 - t_1} * K_3 * C_3 * C_3;$$

V našem primeru:

$$\Delta\epsilon = \frac{1}{4} * 4 * 0 * 0$$

```
0
```

Izražava signalov s trigonometričnim temeljnimi funkcijami

Trigonometrične temeljne funkcije

Definicija prvih nekaj funkcij

```
T0[t_] := 1;
T1[t_] := Sin[ω*t];
T2[t_] := Cos[ω*t];
T3[t_] := Sin[2*ω*t];
T4[t_] := Cos[2*ω*t];
T5[t_] := Sin[3*ω*t];
T6[t_] := Cos[4*ω*t];
```

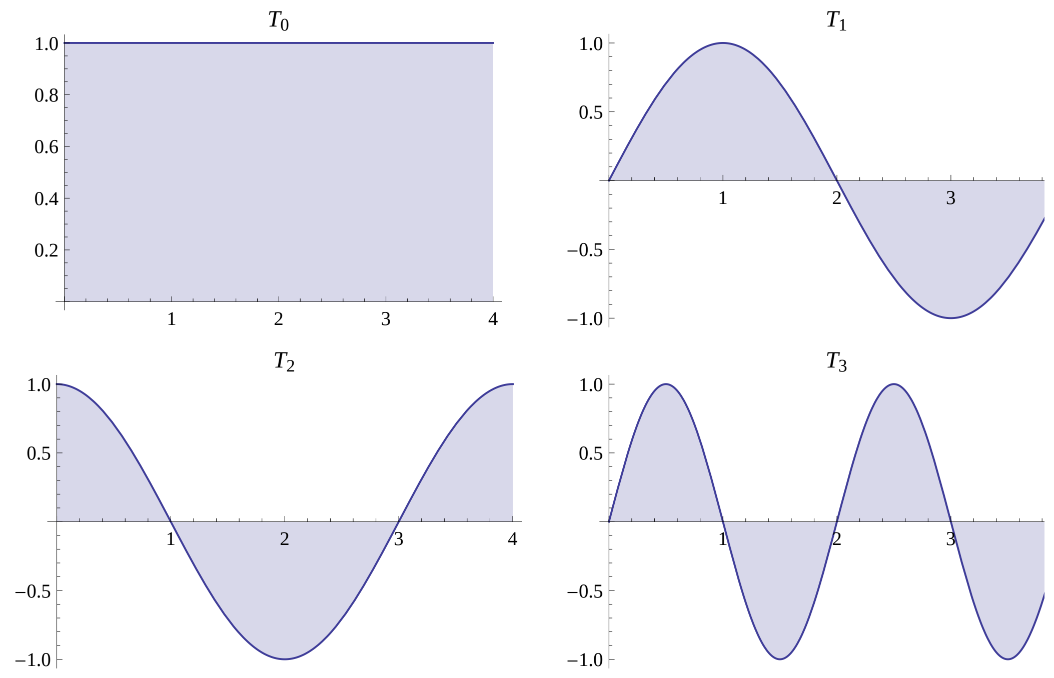
```
ω := 2*π/T;
```

Koeficienti:

```
K0 := T;
K1 := T/2;
K2 := T/2;
K3 := T/2;
K4 := T/2;
```

Izris funkcij:

```
T = 4;
gw0 = Plot[T0[t], {t, 0, T}, PlotRange → All, PlotLabel → "T0",
   Filling → Axis];
gw1 = Plot[T1[t], {t, 0, T}, PlotRange → All, PlotLabel → "T1",
   Filling → Axis];
gw2 = Plot[T2[t], {t, 0, T}, PlotRange → All, PlotLabel → "T2",
   Filling → Axis];
gw3 = Plot[T3[t], {t, 0, T}, PlotRange → All, PlotLabel → "T3",
   Filling → Axis];
GraphicsGrid[{{gw0, gw1}, {gw2, gw3}}]
```

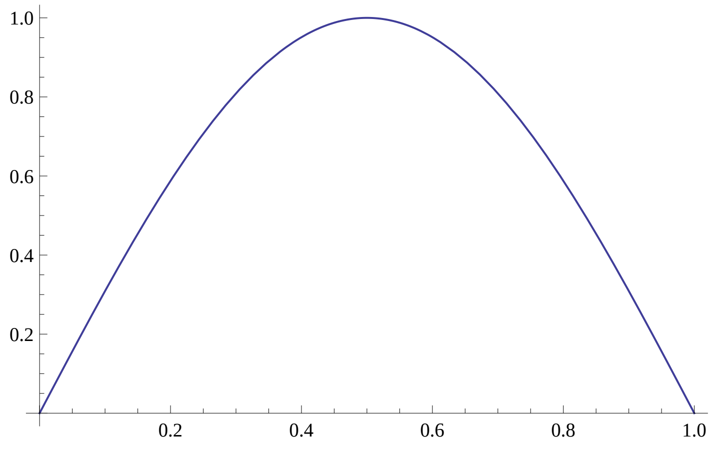


Aproksimacija signala (naloga 2)

Naloga:
Signal x(t) = Sin[πt] na intervalu [0,1] izrazite s približkom prvih štirih trigonometričnih temeljnih funkcij.
Določite še razliko v napako aproksimacije, če aproksimiramo samo s tremi trig. t.f. in skicirajte približek.

```
x[t_] := Sin[π * t];
```

```
Plot[x[t], {t, 0, 1}, PlotRange → All]
```



Rešitev:

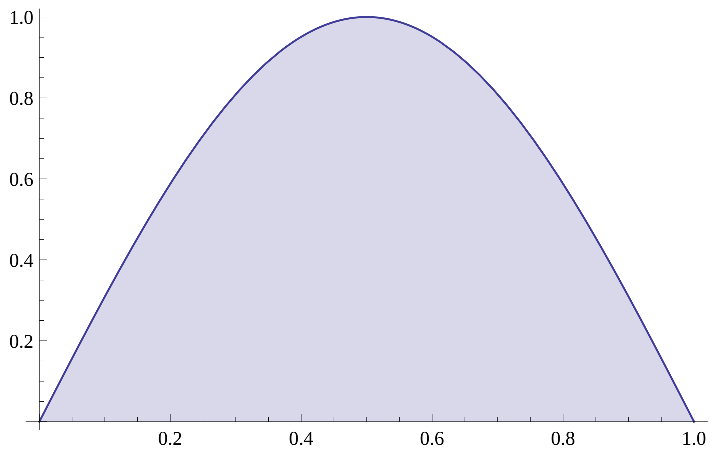
```
T = 1; ω = 2 * π;
```

```
K0 := 1;
K1 := 1 / 2;
K2 := 1 / 2;
K3 := 1 / 2;
```

$$C_0 = \frac{1}{1} * \int_0^1 \mathrm{Sin}[\pi * t] * 1\, dt$$

$$\frac{2}{\pi}$$

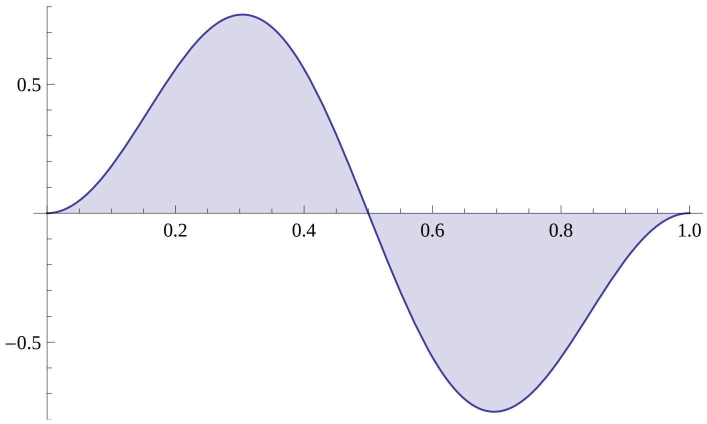
```
Plot[{x[t] * T0[t]}, {t, 0, 1}, Filling → Axis]
```



```
C₁ = 1/(1/2) * ∫₀¹ Sin[π*t]*Sin[2*π*t] ⅆt
```

0

```
Plot[{x[t]*T₁[t]}, {t, 0, 1}, Filling → Axis]
```



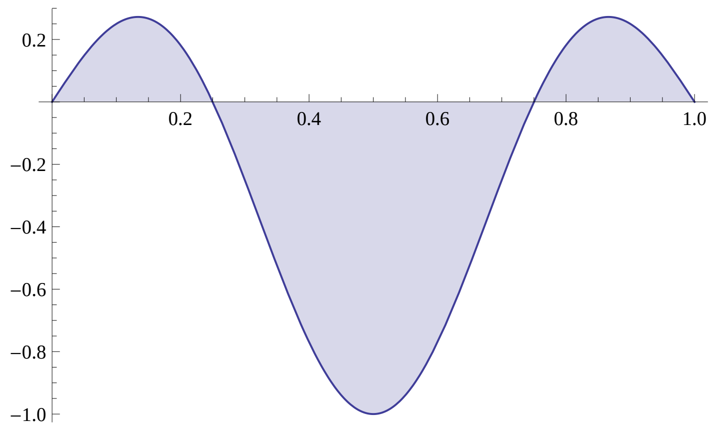

```
C₂ = 1/(1/2) * ∫₀¹ Sin[π*t]*Cos[2*π*t] ⅆt
```

$-\frac{4}{3\pi}$

Pomoč pri izračunu koeficienta C_2 :

```
Sin[α]*Cos[β] = 1/2 *Sin[α + β] + Sin[α − β]
```

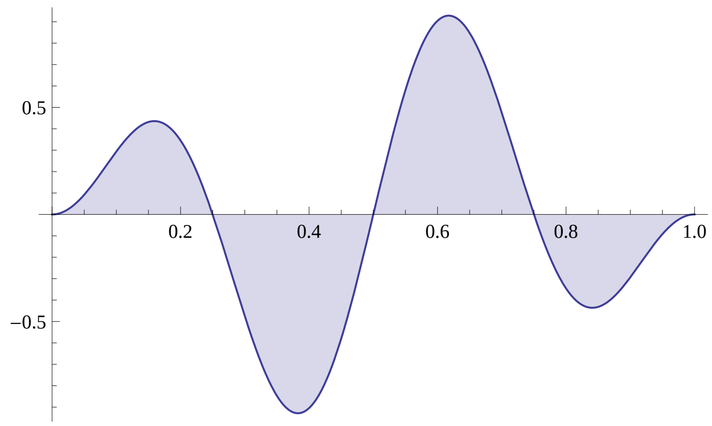
```
Plot[{x[t] * T2[t]}, {t, 0, 1}, Filling → Axis]
```



$$C_3 = \frac{1}{\frac{1}{2}} * \int_0^1 \text{Sin}[\pi * t] * \text{Sin}[3 * \pi * t] \, dt$$

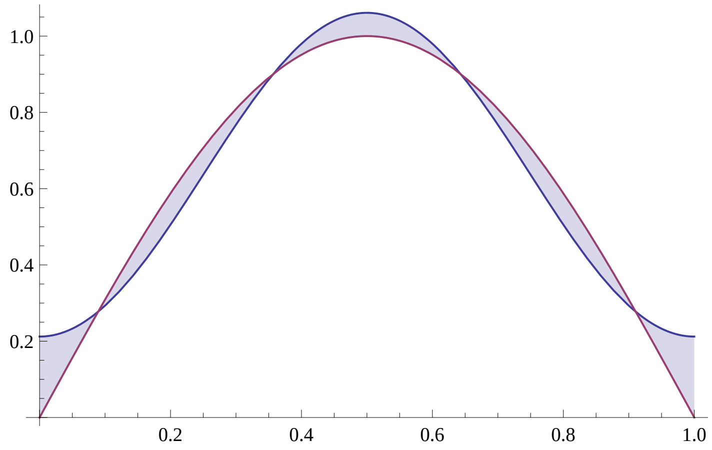
0

```
Plot[{x[t] * T3[t]}, {t, 0, 1}, Filling → Axis]
```



- *Izris aproksimiranega signala:*

```
Plot[{C0 T0[t] + C1 T1[t] + C2 T2[t] + C3 T3[t], x[t]}, {t, 0, 1},
 Filling → {1 → {2}}]
```



- *Izračun razlike napake, če aproksimiramo s prvimi 4-imi trig .t.f. ali pa samo s 3-mi trig.t.f.*

$$\Delta\epsilon = \epsilon 4 - \epsilon 3 = \frac{1}{t_2 - t_1} * K_3 * C_3 * C_3;$$

V našem primeru:

$$\Delta\epsilon = \frac{1}{1} * \frac{1}{2} * 0 * 0$$

```
0
```

Fourierjeva vrsta

Kompleksna Fourierjeva vrsta

```
f[t_] := Sum[F_n * Exp[i * n * ω * t], {n, -∞, ∞}];
ω := (2 * π)/T;
F_n := 1/T * Integrate[f[t] * Exp[-i * n * ω * t], {t, t_0, t_0 + T}];
```

$$f[t\_] := \sum_{n=-\infty}^{\infty} F_n * \mathrm{Exp}[i * n * \omega * t];$$

$$\omega := \frac{2 * \pi}{T};$$

$$F_n := \frac{1}{T} * \int_{t_0}^{t_0+T} f[t] * \mathrm{Exp}[-i * n * \omega * t]\, dt;$$

Realna Fourierjeva vrsta

$$f[t\_] := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n * \mathrm{Cos}[n * \omega * t] + b_n * \mathrm{Sin}[n * \omega * t];$$

$$\omega := \frac{2 * \pi}{T};$$

$$a_n := \frac{2}{T} * \int_{t_0}^{t_0+T} f[t] * \mathrm{Cos}[n * \omega * t]\, dt;$$

$$b_n := \frac{2}{T} * \int_{t_0}^{t_0+T} f[t] * \mathrm{Sin}[n * \omega * t]\, dt;$$

Amplitudni in fazni spekter

$$F_n = P_n + jQ_n = |F_n| * e^{j\Theta_n}$$

Amplitudni spekter

$$|F_n| = \sqrt{F_n * \overline{F_n}} = \sqrt{P_n{}^2 + Q_n{}^2} = \frac{1}{2}\sqrt{a_n{}^2 + b_n{}^2}$$

Fazni spekter

$$\phi_n = \begin{cases} \arctan\frac{Q_n}{P_n} & P_n > 0 \\ \arctan\frac{Q_n}{P_n} \pm \pi & P_n < 0 \end{cases}$$

$$= \begin{cases} -\arctan\frac{b_n}{a_n} & a_n > 0 \\ -\arctan\frac{b_n}{a_n} \pm \pi & a_n < 0 \end{cases}$$

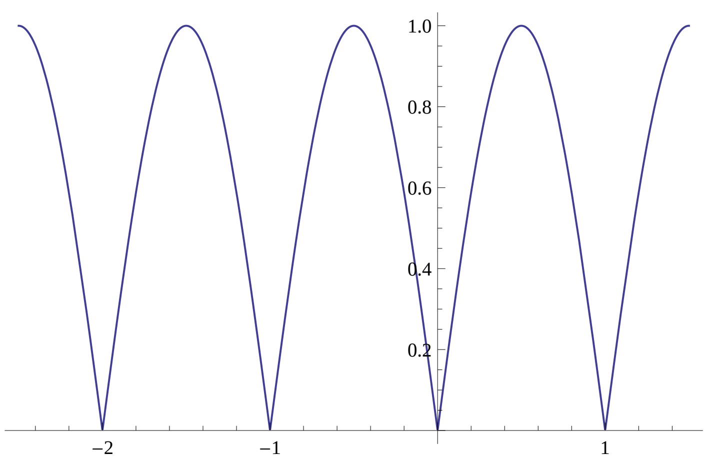
Naloga 1:

Naloga:
Izrazi periodično funkcijo f(t) s kompleksno F. v.
Določi tudi koeficiente realne F.v. ter zapiši realno vrsto.
Na koncu določi in nariši tudi amplitudni in fazni spekter.

Signal:

$$x[t\_] := \begin{cases} A * \mathrm{Sin}[\pi * t] & 0 < t < 1 \\ 0 & \mathrm{True} \end{cases}$$

```
A = 1; Plot[x[t - 1] + x[t] + x[t + 1] + x[t + 2] + x[t + 3], {t, -2.5, 1.5},
 PlotRange → All]
```



Rešitev:

```
T = 1; ω = 2 * π;
```

Funkcija je soda: f(t)=f(-t). $F_n=P_n+i*Q_n$ => $Q_n = 0$. => $F_n=P_n$.

$$F_n = \frac{1}{T} * \int_{t_0}^{t_0+T} x[t] * \text{Cos}[n*2*\pi*t]\,dt$$

Izberemo $t_0 = 0$.

$$F_n = \frac{1}{1} * \int_{0}^{1} A * \text{Sin}[\pi*t] * \text{Cos}[n*2*\pi*t]\,dt$$

$$= \frac{2\,A\,\text{Cos}[n\,\pi]^2}{\pi - 4\,n^2\,\pi} = \frac{2\,A}{\pi\left(1 - 4\,n^2\right)}$$

Pri izračunu integrala si pomagamo z:

$$\int \text{Sin}[a*t] * \text{Cos}[b*t]\,dt$$

$$-\frac{\text{Cos}[(a-b)\,t]}{2\,(a-b)} - \frac{\text{Cos}[(a+b)\,t]}{2\,(a+b)}$$

Kompleksna F. v.

$$f[t] = \frac{2\,A}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{\left(1 - 4\,n^2\right)} * \text{Exp}[i*n*2*\pi*t];$$

Realna F. v.

$$a_n = F_n + \overline{F_n} = \frac{4\,A}{\left(1 - 4\,n^2\right)\pi};$$

$$b_n = i * \left(F_n - \overline{F_n}\right) = 0;$$

$$f[t] = \frac{2*A}{\pi} + \frac{4*A}{\pi} \sum_{n=1}^{\infty} \frac{1}{1 - 4\,n^2} * \text{Cos}[2*n*\pi*t];$$

Amplitudni spekter:

$$|F_n| = \begin{cases} \frac{2\,A}{(4\,n^2-1)\,\pi} & n \neq 0 \\ \frac{2*A}{\pi} & n = 0 \end{cases}$$

Fazni spekter:

$$\phi_n = \begin{cases} \pm\pi & n \neq 0 \\ 0 & n = 0 \end{cases}$$

Zaradi realnega signala mora biti fazni spekter liha funkcija. Zato :

$$= \begin{cases} +\pi & n > 0 \\ -\pi & n < 0 \\ 0 & n = 0 \end{cases}$$

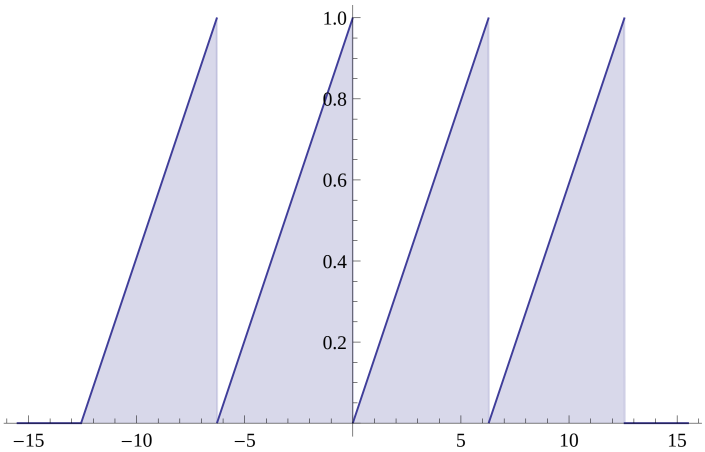
Naloga 2:

Naloga:
Razvij periodični signal f(t) v realno in kompleksno F.v.
Določi in nariši tudi amplitudni in fazni spekter.

Signal:

$$x[t\_] := \begin{cases} \frac{A}{2*\pi} * t & 0 < t \leq 2\pi \\ 0 & \text{True} \end{cases} \quad A > 0$$

```
A = 1; Plot[x[t + 4 * π] + x[t + 2 * π] + x[t] + x[t - 2 * π], {t, -15.5, 15.5},
 PlotRange → All, Filling → Axis]
```



Rešitev:

```
T = 2 * π; ω = 1;
```

Izberemo $t_0 = 0$.

$$a_n = \frac{2}{2*\pi} * \int_0^{2*\pi} \frac{A}{2*\pi} * t * \text{Cos}[n * \omega * t] \, dt$$

Pomagamo si z:

$$\int t * \mathrm{Cos}[a * t]\, dt$$

$$\frac{\mathrm{Cos}[a\, t]}{a^2} + \frac{t\, \mathrm{Sin}[a\, t]}{a}$$

$$a_0 = A;$$
$$a_n = 0;$$

$$b_n = \frac{2}{2 * \pi} * \int_0^{2*\pi} \frac{A}{2 * \pi} * t * \mathrm{Sin}[n * \omega * t]\, dt$$

Pomagamo si z:

$$\int t * \mathrm{Sin}[a * t]\, dt$$

$$-\frac{t\, \mathrm{Cos}[a\, t]}{a} + \frac{\mathrm{Sin}[a\, t]}{a^2}$$

$$b_0 = 0;$$
$$b_n = \frac{A}{-\pi * n};$$

Realna F. v.

$$f[t] = \frac{A}{2} - \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} * \mathrm{Sin}[n * t];$$

Preverimo, kako je v točkah nezveznosti :

$$f[2 * k * \pi] = \frac{A}{2} - \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} * \mathrm{Sin}[n * 2 * k * \pi] = \frac{A}{2}$$

Kompleksna F. v.

Pomagamo si z :

$$F_n = \frac{a_n - i * b_n}{2}, \quad \text{in} \quad F_{-n} = \overline{F_n}, \ \text{ker je vhodni signal realen.}$$

$$F_n = \begin{cases} \frac{i * A}{2 * \pi * n} & n \neq 0 \\ \frac{A}{2} & n = 0 \end{cases}$$

Amplitudni spekter:

$$|F_n| = \begin{cases} \frac{A}{2*\pi*|n|} & n \neq 0 \\ \frac{A}{2} & n = 0 \end{cases}$$

Fazni spekter:

$$\phi_n = \begin{cases} \pm\frac{\pi}{2} & n \neq 0 \\ 0 & n = 0 \end{cases}$$

Zaradi realnega signala mora biti fazni spekter liha funkcija. Zato :

$$= \begin{cases} +\frac{\pi}{2} & n > 0 \\ -\frac{\pi}{2} & n < 0 \\ 0 & n = 0 \end{cases}$$

Naloga 3:

Naloga:
Izrazi periodično funkcijo f(t) s kompleksno F. v.
Določi tudi koeficiente realne F.v. ter zapiši realno vrsto.
Na koncu določi in nariši tudi amplitudni in fazni spekter.

Signal:

$$f[t\_] := \mathrm{Sin}\left[\frac{\pi * t}{4} + \frac{\pi}{4}\right];$$

Rešitev:

Adicijski izrek:

$$f[t] = \mathrm{Sin}\left[\frac{\pi * t}{4}\right] * \mathrm{Cos}\left[\frac{\pi}{4}\right] + \mathrm{Cos}\left[\frac{\pi * t}{4}\right] * \mathrm{Sin}\left[\frac{\pi}{4}\right]$$

$$\omega = \frac{\pi}{4} = \frac{2 * \pi}{T} \; => \; T = 8$$

Realna F. v.

$$a_0 = 0$$

$$a_1 = \mathrm{Sin}\left[\frac{\pi}{4}\right] = \frac{\sqrt{2}}{2}$$

$$b_1 = \mathrm{Cos}\left[\frac{\pi}{4}\right] = \frac{\sqrt{2}}{2}$$

$$a_n = b_n = 0, \quad n > 1$$

Kompleksna F. v.

Ker velja:

$$F_n = \frac{a_n - i * b_n}{2}, \quad \text{in} \quad F_{-n} = \overline{F_n}, \text{ ker je vhodni signal realen.}$$

$$F_1 = \frac{a_1 - j * b_1}{2} =$$

$$\frac{\text{Sin}\left[\frac{\pi}{4}\right] - j * \text{Cos}\left[\frac{\pi}{4}\right]}{2} =$$

$$-\frac{j}{2}\left(\text{Cos}\left[\frac{\pi}{4}\right] + j * \text{Sin}\left[\frac{\pi}{4}\right]\right) = -\frac{j}{2}\,\text{Exp}\left[j * \frac{\pi}{4}\right]$$

$$F_{-1} = \overline{F_1} = \frac{j}{2}\,\text{Exp}\left[j * \frac{\pi}{4}\right]$$

Amplitudni spekter:

$$|F_n| = \begin{cases} \frac{1}{2} & n = 1 \\ \frac{1}{2} & n = -1 \\ 0 & n \neq \pm 1 \end{cases}$$

Fazni spekter:

$$\phi_n = \begin{cases} +\frac{\pi}{4} & n = -1 \\ -\frac{\pi}{4} & n = 1 \\ 0 & n \neq \pm 1 \end{cases}$$

Vpliv premika osnovnega signala na Fourierjevo vrsto

Če so $F[n]$ koeficienti F.v. signala *f[t]*, kako je s premaknjenim signalom *f[t - t_1]*?

Označimo s $F_1[n]$ koeficiente F. v. signala *f[t - t_1]*.

Velja :

$F_1[n] = e^{-jn\omega t_1} * F[n]$

Sledi :

$| F_1[n] | = | F[n] |$

$\phi_1[n] = \phi[n] - n\omega t_1$

Naloga 1:

Naloga:
Izrazi periodično funkcijo f(t) s kompleksno F. v.
Določi tudi koeficiente realne F.v. ter zapiši realno vrsto.
Na koncu določi in nariši tudi amplitudni in fazni spekter.

Signal:

$$f[t\_] := \text{Sin}\left[\frac{\pi * t}{4} + \frac{\pi}{4}\right];$$

Rešitev:

Zgornji signal lahko razumemo kot premaknjen osnovni signal:

$$f_1[t] = \text{Sin}\left[\frac{\pi * t}{4}\right]$$

$f[t] = f_1[t - t_1]$,

pri tem je $\frac{\pi * t}{4} + \frac{\pi}{4} = \frac{\pi * (t - t_1)}{4}$,

sledi $t_1 = -1$

$\omega = \frac{\pi}{4} = \frac{2 * \pi}{T} => T = 8$

Realna F. v. f_1

$$a_0 = 0$$
$$a_1 = 0$$
$$b_1 = 1$$
$$a_n = b_n = 0, \quad n > 1$$

Kompleksna F. v. f_1

Ker velja:

$$F_1[n] = \frac{a_n - j * b_n}{2}, \quad \text{in} \quad F_{-n} = \overline{F_n},$$

ker je vhodni signal realen.

$$F_1[1] = \frac{a_1 - j * b_1}{2} = \frac{-j}{2}$$

$$F_1[-1] = \overline{F_1[1]} = \frac{j}{2}$$

Kompleksna F. v. f

Upoštevamo premik $f[t] = f_1[t - t_1]$, $t_1 = -1$

$$F[1] = e^{-jn\omega t_1} * F_1[1] = \frac{-j}{2} * e^{-j\frac{\pi}{4}(-1)} = \frac{-j}{2} * e^{j\frac{\pi}{4}}$$
$$F[-1] = e^{-jn\omega t_1} * F_1[-1] = \frac{j}{2} * e^{-j\frac{\pi}{4}(-1)} = \frac{j}{2} * e^{j\frac{\pi}{4}}$$

Realna F. v. f

$$F[1] = \frac{-j}{2} * e^{j\frac{\pi}{4}} = \frac{-j}{2} * \left(\mathrm{Cos}\left[\frac{\pi}{4}\right] + j * \mathrm{Sin}\left[\frac{\pi}{4}\right]\right) = \frac{\sqrt{2}}{4} - j * \frac{\sqrt{2}}{4}$$
$$F[-1] = \frac{\sqrt{2}}{4} + j * \frac{\sqrt{2}}{4}$$

$$a_0 = 0$$
$$a_1 = F[1] + \overline{F[1]} = \frac{\sqrt{2}}{4} - j * \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{4} + j * \frac{\sqrt{2}}{4} = \frac{\sqrt{2}}{2}$$
$$b_1 = j\left(F[1] - \overline{F[1]}\right) = j\left(\frac{\sqrt{2}}{4} - j * \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4} - j * \frac{\sqrt{2}}{4}\right) = j\left(-j\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$
$$a_n = b_n = 0, \quad n > 0$$

Amplitudni spekter:

$$|F_n| = \begin{cases} \frac{1}{2} & n = 1 \\ \frac{1}{2} & n = -1 \\ 0 & n \neq \pm 1 \end{cases}$$

Fazni spekter:

$$\phi_n = \begin{cases} +\frac{\pi}{4} & n = -1 \\ -\frac{\pi}{4} & n = 1 \\ 0 & n \neq \pm 1 \end{cases}$$

Avtokorelacija periodičnih signalov

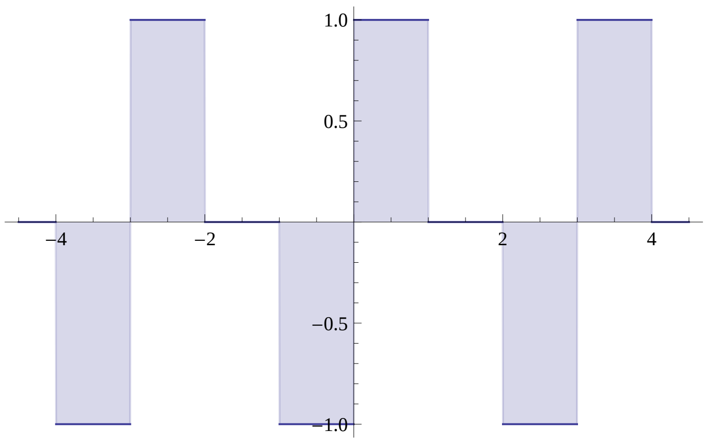
$$\varphi_{ii}[\tau] = \frac{1}{T} \int_{t_0}^{t_0+T} f_i[t] * f_i[t+\tau] \, dt$$

Naloga 1:

Naloga:
Za periodični signal f_1(t) določi avtokorelacijo in skiciraj nejen potek.

Signal:

```
Plot[f1[t] + f1[t + 3] + f1[t - 3], {t, -4.5, 4.5}, PlotRange → All,
 Filling → Axis]
```



$$f_1[t\_] := \begin{cases} -1 & -1 \le t \le 0 \\ 1 & 0 \le t \le 1 \\ 0 & 1 \le t < 2 \\ 0 & \text{True} \end{cases}$$

Rešitev

Zaradi periodičnosti avtokorelacije računamo za premike samo ene periode.

```
T = 3;
```

$$0 \le \tau \le 1$$

$$\varphi_{11} = \frac{1}{3}\left(\int_{-1}^{-\tau}(-1)*(-1)\,dt + \int_{-\tau}^{0}(-1)*(1)\,dt + \int_{0}^{1-\tau}(1)*(1)\,dt\right)$$

$$\frac{1}{3}(2-3\tau)$$

$1 \le \tau \le 2$

$$\varphi_{11} = \frac{1}{3}\left(\int_{-1-\tau}^{-2}(1)*(-1)\,dt + \int_{-2}^{-\tau}(0)*(-1)\,dt + \int_{-\tau}^{-1}(0)*(1)\,dt + \int_{-1}^{1-\tau}(-1)*(1)\,dt\right)$$

$$-\frac{1}{3}$$

$2 \le \tau \le 3$

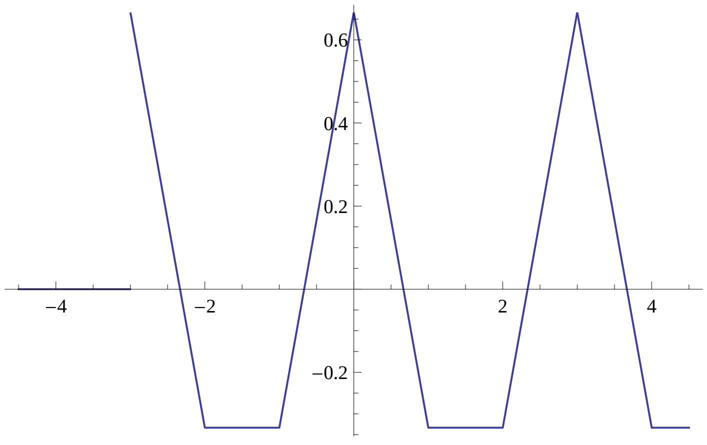
$$\varphi_{11} = \frac{1}{3}\left(\int_{-1-\tau}^{-3}(-1)*(-1)\,dt + \int_{-3}^{-\tau}(1)*(-1)\,dt + \int_{-\tau}^{-2}(1)*(1)\,dt + \int_{-2}^{1-\tau}(0)*(1)\,dt\right)$$

$$\frac{1}{3}(-7+3\tau)$$

Narišemo. *Avtokorelacija mora biti zvezna in periodična funkcija!!!*

$$\varphi_{11}[\tau\_] := \begin{cases} \frac{1}{3}*(2-3\tau) & 0 \le \tau \le 1 \\ -\frac{1}{3} & 1 \le \tau \le 2 \\ \frac{1}{3}*(-7+3\tau) & 2 \le \tau \le 3 \\ 0 & \text{True} \end{cases}$$

```
Plot[φ11[τ] + φ11[τ + 3] + φ11[τ - 3], {τ, -4.5, 4.5}]
```

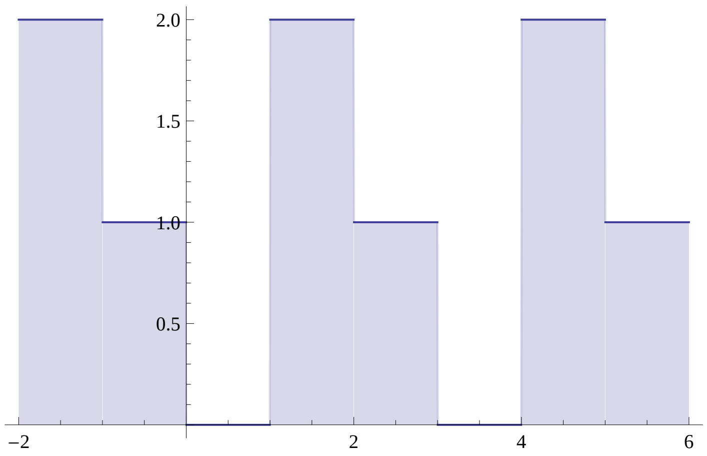


Naloga 2:

Naloga:
Za periodični signal $f_1(t)$ določi avtokorelacijo in skiciraj nejen potek.

Signal:

```
Plot[f1[t] + f1[t + 3] + f1[t + 6] + f1[t - 3] + f1[t - 6], {t, -2, 6},
 PlotRange → All, Filling → Axis]
```



$$f_1[t\_] := \begin{cases} 2 & 1 < t \leq 2 \\ 1 & 2 < t \leq 3 \\ 0 & \text{True} \end{cases}$$

Rešitev

Zaradi periodičnosti avtokorelacije računamo za premike samo ene periode.

```
T = 3;
```

$0 \leq \tau \leq 1$

$$\varphi_{11} = \frac{1}{3} * \left(\int_1^{2-\tau} 2*2\,dt + \int_{2-\tau}^{2} 2*1\,dt + \int_2^{3-\tau} 1*1\,dt\right)$$

$$\frac{1}{3}(1 + 4(1 - \tau) + \tau)$$

$$\frac{1}{3}(5 - 3\tau)$$

$1 \le \tau \le 2$

$$\varphi_{11} = \frac{1}{3} * \left(\int_{1-\tau}^{0} 1*2\, dt + \int_{1}^{3-\tau} 2*1\, dt \right)$$

$$\frac{1}{3} (2 (2 - \tau) + 2 (-1 + \tau))$$

$$\frac{2}{3}$$

$2 \le \tau \le 3$

$$\varphi_{11} = \frac{1}{3} * \left(\int_{1-\tau}^{-1} 2*2\, dt + \int_{-1}^{2-\tau} 1*2\, dt + \int_{2-\tau}^{0} 1*1\, dt \right)$$

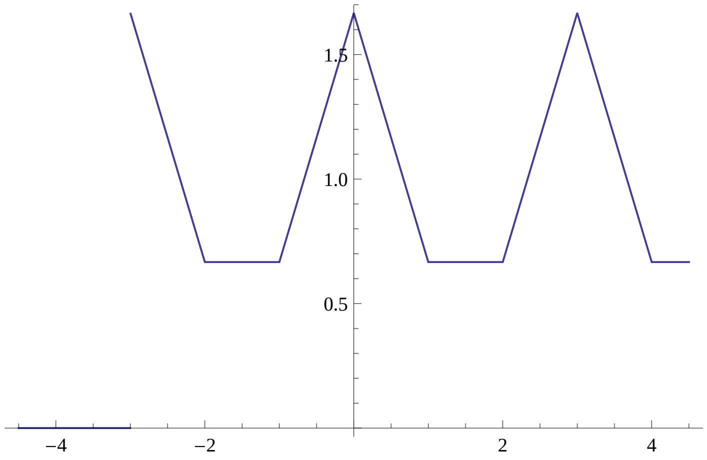
$$\frac{1}{3} (-2 + 2 (3 - \tau) + 4 (-2 + \tau) + \tau)$$

$$\frac{1}{3} (-4 + 3 \tau)$$

Narišemo. *Avtokorelacija mora biti zvezna in periodična funkcija!!!*

$$\varphi_{11}[\tau\_] := \begin{cases} \frac{1}{3} (5 - 3 \tau) & 0 \le \tau \le 1 \\ \frac{2}{3} & 1 \le \tau \le 2 \\ \frac{1}{3} (-4 + 3 \tau) & 2 \le \tau \le 3 \\ 0 & \text{True} \end{cases}$$

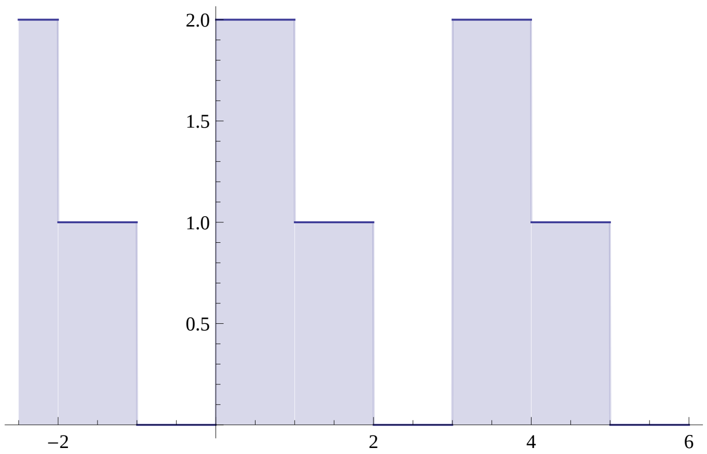
```
Plot[φ11[τ] + φ11[τ + 3] + φ11[τ - 3], {τ, -4.5, 4.5}]
```



Izračun avtokorelacije premaknjenega signala

```
f2[t_] := f1[t + 1]
```

```
Plot[f2[t] + f2[t + 3] + f2[t + 6] + f2[t - 3] + f2[t - 6], {t, -2.5, 6},
 PlotRange → All, Filling → Axis]
```




```
0 ≤ τ ≤ 1
```

$$\varphi_{11} = \frac{1}{3} * \left(\int_0^{1-\tau} 2*2\, dt + \int_{1-\tau}^{1} 2*1\, dt + \int_1^{2-\tau} 1*1\, dt\right)$$

$$\frac{1}{3}\,(1 + 4\,(1-\tau) + \tau)$$

$$\frac{1}{3}\,(5 - 3\,\tau)$$

$1 \le \tau \le 2$

$$\varphi_{11} = \frac{1}{3} * \left(\int_{-\tau}^{-1} 1*2\, dt + \int_0^{2-\tau} 2*1\, dt\right)$$

$$\frac{1}{3}\,(2\,(2-\tau) + 2\,(-1+\tau))$$

$$\frac{2}{3}$$

$2 \le \tau \le 3$

$$\varphi_{11} = \frac{1}{3} * \left(\int_{-\tau}^{-2} 2*2\, dt + \int_{-2}^{1-\tau} 1*2\, dt + \int_{1-\tau}^{-1} 1*1\, dt\right)$$

$$\frac{1}{3}\,(-2 + 2\,(3-\tau) + 4\,(-2+\tau) + \tau)$$

$$\frac{1}{3}\,(-4 + 3\,\tau)$$

Avtokorelacija je enaka kot zgoraj.

Fourierjeva transformacija

```
F[ω] = ∫_{-∞}^{∞} f[t] * e^(-iωt) dt
```

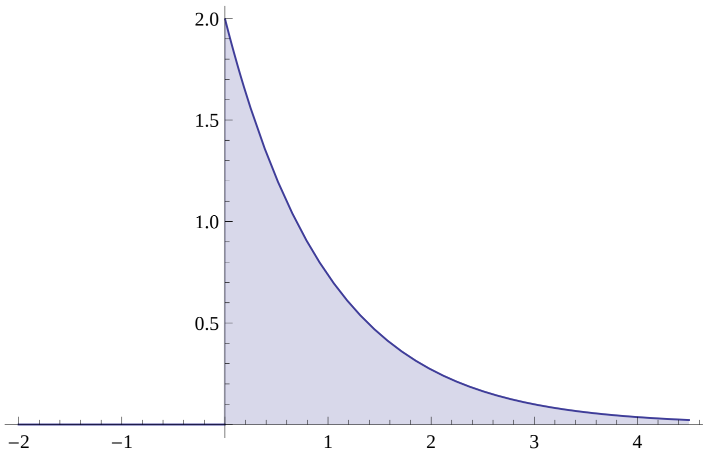
Naloga 1:

Naloga:
Izračunaj kompleksni spekter ter spekter amplitudne in fazne gostote za signal:

Signal:

```
f[t_] := { B * e^(-a*t)   t ≥ 0
         { 0              True
```

```
B = 2; a = 1; Plot[f[t], {t, -2.0, 4.5}, PlotRange → All, Filling → Axis]
```



Fourierjeva transformacija

$$F(\omega) = \int_{-\infty}^{\infty} f(t) * e^{-i\omega t}\, dt = \int_{0}^{\infty} B * e^{-a*t} * e^{-i\omega t}\, dt$$

$$= B * \int_{0}^{\infty} e^{-(i\omega + a)\, t}\, dt = B * \frac{e^{-(i\omega+a)\, t}}{-(i\omega + a)} \Bigg|_{0}^{\infty} =$$

$$= \frac{B}{i\omega + a} = \frac{B\,(a - i\omega)}{a^2 + \omega^2}$$

Spekter amplitudne gostote

$$|F(\omega)| = \sqrt{\left(\frac{Ba}{a^2+\omega^2}\right)^2 + \left(\frac{B\omega}{a^2+\omega^2}\right)^2} =$$

$$= \sqrt{\frac{B^2\left(a^2+\omega^2\right)}{\left(a^2+\omega^2\right)^2}} = \frac{B}{\sqrt{a^2+\omega^2}}$$

Spekter fazne gostote

$$\Theta(\omega) = \mathrm{ATan}\left[\frac{\frac{-B\omega}{a^2+\omega^2}}{\frac{Ba}{a^2+\omega^2}}\right] = -\mathrm{ATan}\left[\frac{\omega}{a}\right]$$

Izračun Fourierjeve transformacije delta funkcije

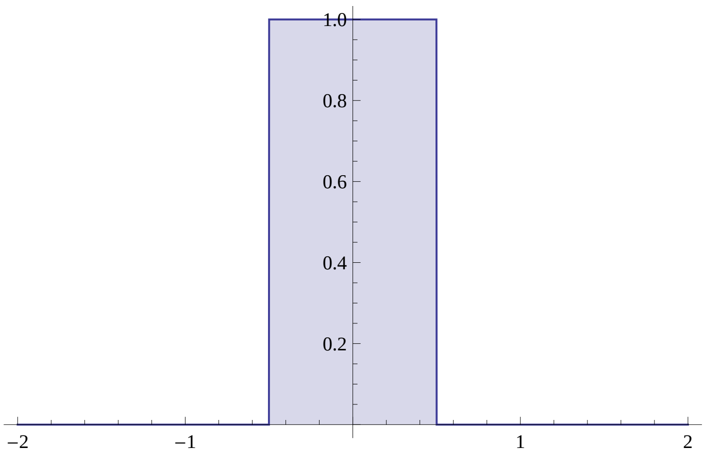
Naloga:
Izračunaj kompleksni spekter ter spekter amplitudne in fazne gostote za signal:

Signal:

$$b[t\_] := \begin{cases} E & \frac{-b}{2} \le t \le \frac{b}{2} \\ 0 & \text{True} \end{cases}$$

Pri čemer velja : **E, b > 0 in E*b = 1.**

```
E = 1; b = 1; Plot[b[t], {t, -2.0, 2.0}, PlotRange → All, Filling → Axis]
```



Fourierjeva transformacija *(funkcija je soda)*

$$F(\omega) = 2 * \int_0^{b/2} E * \text{Cos}[\omega * t]\, dt$$

$$= \frac{2 * E}{\omega} \text{Sin}[\omega t] \Big|_0^{b/2} =$$

$$= \frac{2 * E}{\omega} * \text{Sin}\left[\frac{\omega b}{2}\right] = \text{Lepše zapišemo} =$$

$$= E * b * \frac{\text{Sin}\left[\frac{\omega b}{2}\right]}{\frac{\omega b}{2}} = \frac{\text{Sin}\left[\frac{\omega b}{2}\right]}{\frac{\omega b}{2}} = P(\omega)$$

Spekter amplitudne gostote

$$|F(\omega)| = E*b*\frac{\left|\text{Sin}\left[\frac{\omega b}{2}\right]\right|}{\frac{|\omega b|}{2}}$$

Spekter fazne gostote

Kjer je P (ω) > 0, je Θ (ω) = 0;
Sicer pa je ± π.

Da bo fazni spekter liha funkcija se odločimo, da bo
pri ω < 0 in P (ω) < 0 , Θ (ω) = −π
pri ω > 0 in P (ω) < 0 , Θ (ω) = +π
pri P (ω) > 0, Θ (ω) = 0

Izpeljava tipalne funkcije δ(t)

$$\delta(t) := \lim_{b\to 0,\ Eb=1} b(t)$$

$$\lim_{b\to 0,\ Eb=1} F(\omega) = 1$$

Iz tega sledi, da je Fourierjeva transformacija delta funkcije δ (t) enaka 1.

Lastnosti Fourierjeve transformacije

Linearnost

$\mathcal{F}\,(\alpha\, f(t) + \beta\, g(t)\,) = \alpha\, F(\omega) + \beta\, G(\omega)$

Premik

$\mathcal{F}\,(f(t - t_0)) = F(\omega)e^{-j\omega t_0}$

$\mathcal{F}^{\,-1}(F(\omega - \omega_0)) = f(t)e^{j\omega_0 t}$

Modulacija

$f(t) = g(t) * \cos(\omega_0\, t) = g(t) *[\,\frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right)]$

$F(\omega) = \frac{1}{2}[G(\omega + \omega_0) + G(\omega - \omega_0)]$

$f(t) = g(t) * \sin(\omega_0\, t) = g(t) *[\,-\frac{j}{2}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)]$

$F(\omega) = \frac{j}{2}[G(\omega + \omega_0) - G(\omega - \omega_0)]$

Lastnosti odvoda

$\mathcal{F}\,(\frac{d^n f(t)}{dt^n}) = (j\omega)^n\, F(\omega)$

Odvodi v točkah nezveznosti

$\frac{df(t)}{dt}\,|_{t=t_0} = \delta(t - t_0)\ [f^+(t_0) - f^-(t_0)]$

Ker velja $\mathcal{F}\,(a\,\delta(t - t_0)) = a\, e^{-j\omega t_0}$, sledi:

$\mathcal{F}\,(\frac{df(t)}{dt}\,|_{t=t_0}) = e^{-j\omega t_0}[f^+(t_0) - f^-(t_0)]$

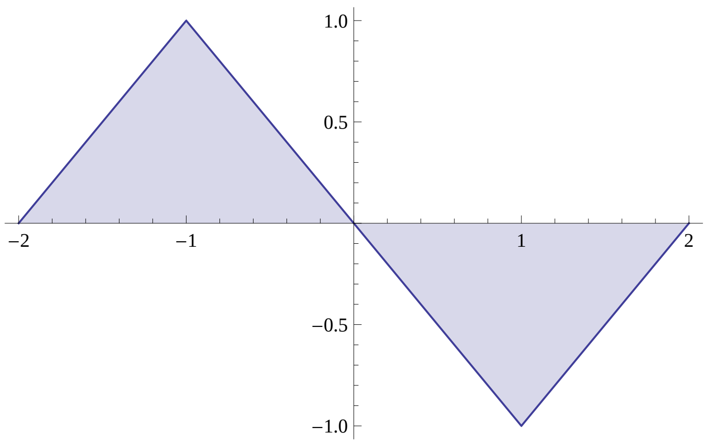
Naloga 1:

Naloga:

Izračunaj kompleksni spekter ter spekter amplitudne in močnostne gostote za signal podan na sliki:

- **Signal:**

```
Plot[f[t], {t, -2.0, 2.0}, PlotRange → All, Filling → Axis]
```



$$f[t\_] := \begin{cases} t+2 & -2 \le t \le -1 \\ -t & -1 \le t \le 1 \\ t-2 & 1 \le t \le 2 \\ 0 & \text{True} \end{cases}$$

- **Fourierjeva transformacija *(z odvajanjem)***

$$\frac{d^2 f(t)}{dt^2} = \delta\,(t+2)\,[1-0] + \delta\,(t+1)\,[-1-1] + \delta\,(t-1)\,[1+1] + \delta\,(t-2)\,[0-1]$$

$$\frac{d^2 f\,(t)}{dt^2} = \delta(t+2) - 2\,\delta(t+1) + 2\,\delta(t-1) - \delta(t-2)$$

Izračunamo F.t. leve in desne strani:

$$(j\omega)^2\,F\,(\omega) = \left(e^{j2\omega} - 2\,e^{j\omega} + 2\,e^{-j\omega} - e^{-j2\omega}\right)$$

$$F\,(\omega) = -\frac{1}{\omega^2}\left(e^{j2\omega} - 2\,e^{j\omega} + 2\,e^{-j\omega} - e^{-j2\omega}\right)$$

- **Spekter amplitudne gostote**

Pomagamo si s pravilom :

$$-\frac{2}{j}\sin(z) = e^{jz} - e^{-jz} \quad ALI \quad 2\cos(z) = e^{jz} + e^{-jz}$$

$$F\,(\omega) = -\frac{1}{\omega^2}\left(-\frac{2}{j}\sin(2\,\omega) + \frac{4}{j}\sin\,(\omega)\right)$$

$$F(\omega) = -\frac{2}{\omega^2}(\sin(2\omega) - 2\sin(\omega))$$

$$|F(\omega)| = \frac{2}{\omega^2}\left|\sin(2\omega) - 2\sin(\omega)\right|$$

- ### Spekter močnostne gostote

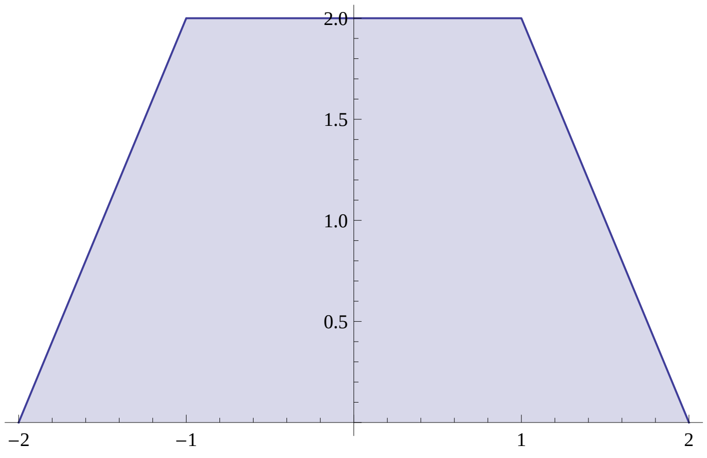
$$|F(\omega)|^2 = \frac{4}{\omega^4}(\sin(2\omega) - 2\sin(\omega))^2$$

Naloga 2:

Naloga:
Izračunaj kompleksni spekter ter spekter amplitudne in močnostne gostote za signal podan na sliki:

- ### Signal:

```
Plot[f[t], {t, -2.0, 2.0}, PlotRange → All, Filling → Axis]
```



$$f[t\_] := \begin{cases} 2*t+4 & -2 \le t \le -1 \\ 2 & -1 \le t \le 1 \\ -2*t+4 & 1 \le t \le 2 \\ 0 & \text{True} \end{cases}$$

- ### Fourierjeva transformacija *(z odvajanjem)*

$$\frac{d^2 f(t)}{dt^2} = \delta(t+2)[2-0] + \delta(t+1)[0-2] + \delta(t-1)[-2-0] + \delta(t-2)[0-(-2)]$$

$$\frac{d^2 f(t)}{dt^2} = 2\,(\delta(t+2) - \delta(t+1) - \delta(t-1) + \delta(t-2))$$

Izračunamo F.t. leve in desne strani:

$$(j\omega)^2 F(\omega) = 2\left(e^{j2\omega} - e^{j\omega} - e^{-j\omega} + e^{-j2\omega}\right)$$

$$F(\omega) = -\frac{2}{\omega^2}\left(e^{j2\omega} + e^{-j2\omega} - e^{j\omega} - e^{-j\omega}\right)$$

Spekter amplitudne gostote

Pomagamo si s pravilom :

$$-\frac{2}{j}\sin(z) = e^{jz} - e^{-jz} \quad \textit{ALI} \quad 2\cos(z) = e^{jz} + e^{-jz}$$

$$F(\omega) = -\frac{2}{\omega^2}\,(2\cos(2\omega) - 2\cos(\omega))$$

$$F(\omega) = -\frac{4}{\omega^2}\,(\cos(2\omega) - \cos(\omega))$$

$$|F(\omega)| = \frac{4}{\omega^2}\,\left|\cos(2\omega) - \cos(\omega)\right|$$

Spekter močnostne gostote

$$|F(\omega)|^2 = \frac{16}{\omega^4}\,(\cos(2\omega) - \cos(\omega))^2$$

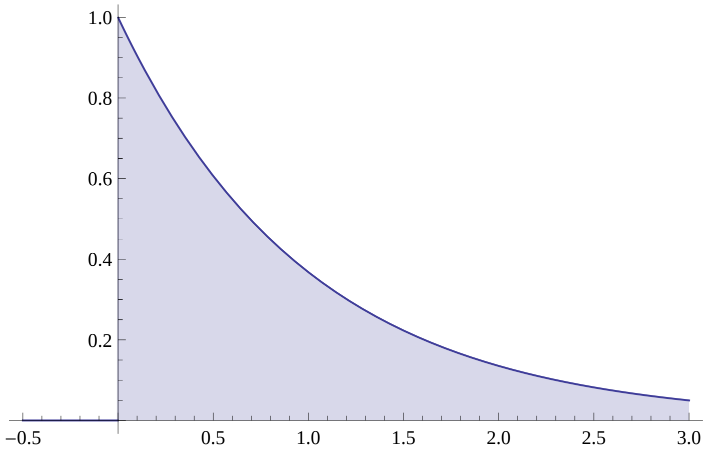
Linearni stacionarni sistemi

Naloga 1:

Naloga:
Izračunaj izhod LSS sistema s prevajalno funkcijo h(t) in vhodnim signalom u(t):

Prevajalna funkcija LSS:

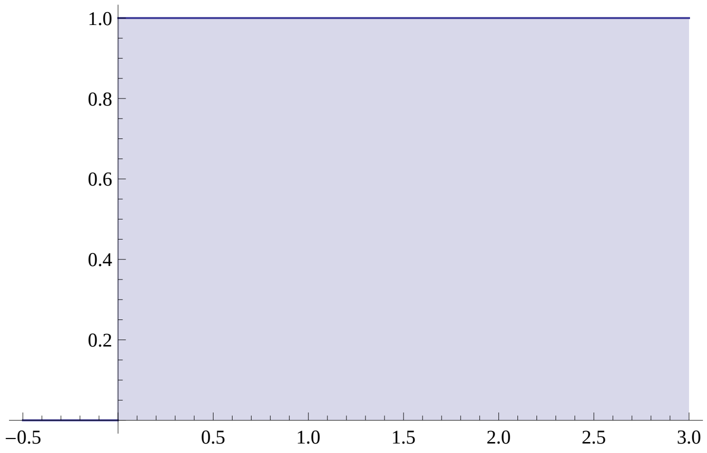
```
Plot[h[t], {t, -0.5, 3.0}, PlotRange → All, Filling → Axis]
```




```
h[t_] := { Exp[-t]  t ≥ 0
         {    0     True
```

- **Vhodni signal:**

```
Plot[u[t], {t, -0.5, 3.0}, PlotRange → All, Filling → Axis]
```



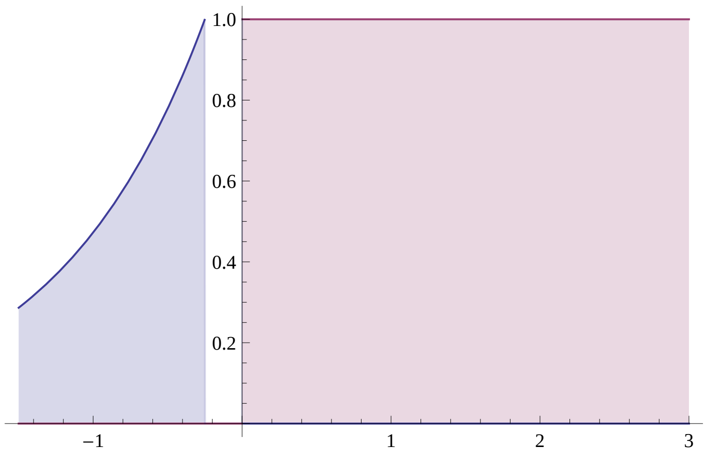
$$u[t\_] := \begin{cases} 1 & t \geq 0 \\ 0 & \text{True} \end{cases}$$

- **Konvolucija:**

$$y(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

- **Za interval t < 0:**

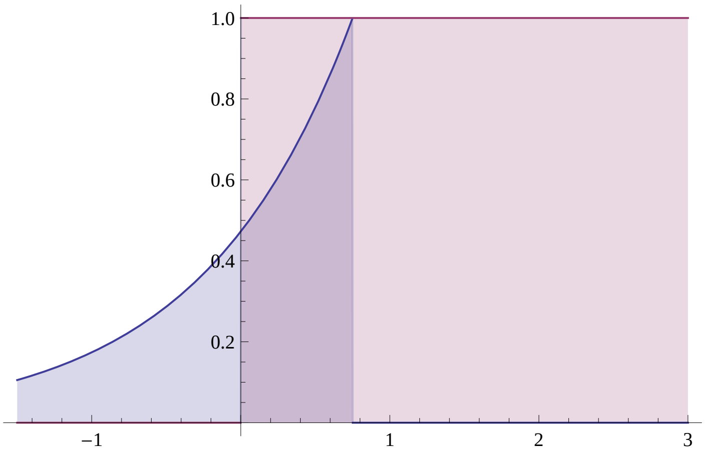
```
t = -1/4; Plot[{h[t - τ], u[τ]}, {τ, -1.5, 3.0}, PlotRange → All,
 Filling → Axis]
```




```
y = 0;
```

▫ Za interval t ≥ 0:

```
t = 3/4; Plot[{h[t - τ], u[τ]}, {τ, -1.5, 3.0}, PlotRange → All,
 Filling → Axis]
```



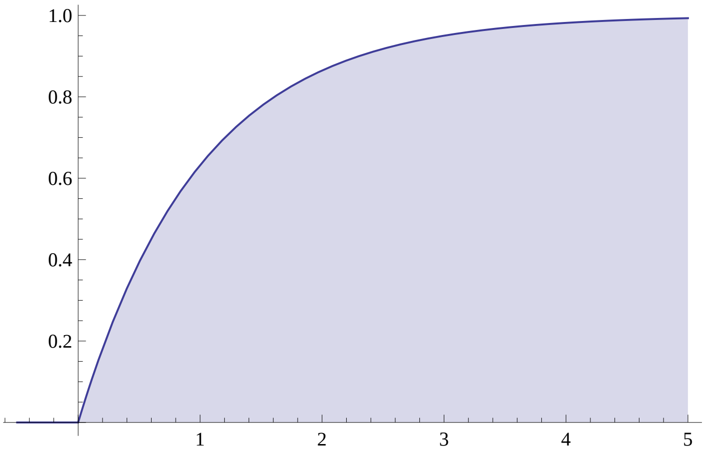

```
y = ∫_0^t 1 * (Exp[-(t - τ)]) dτ
```

$1 - e^{-t}$

▫ Skica izhodnega signala y(t)

```
y[t_] := { 1 - e^-t   t ≥ 0
         { 0          True
```

```
Plot[y[t], {t, -0.5, 5.0}, PlotRange → All, Filling → Axis]
```

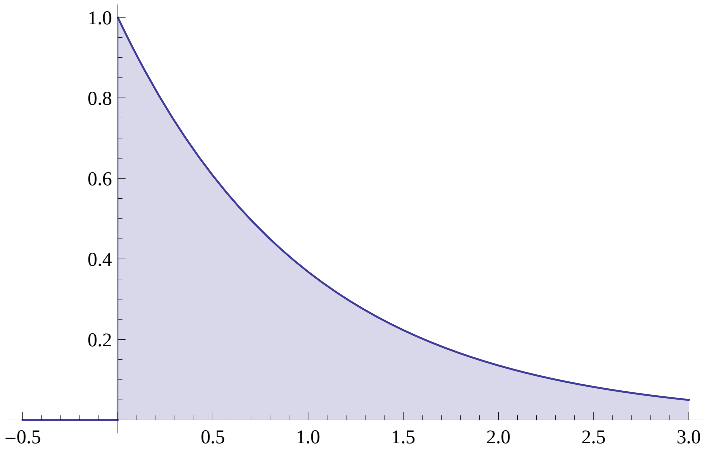


Naloga 2:

Naloga:
Izračunaj izhod LSS sistema s prevajalno funkcijo h(t) in vhodnim signalom u(t):

Prevajalna funkcija LSS:

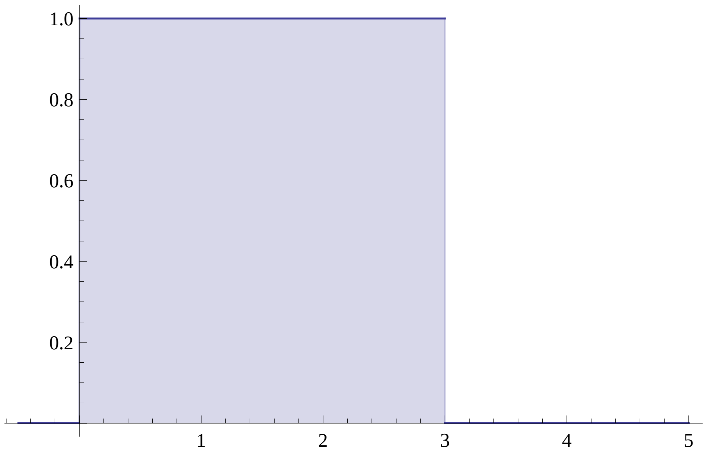
```
Plot[h[t], {t, -0.5, 3.0}, PlotRange → All, Filling → Axis]
```




```
h[t_] := { Exp[-t]  t ≥ 0
         {    0     True
```

Vhodni signal:

```
Plot[u[t], {t, -0.5, 5.0}, PlotRange → All, Filling → Axis]
```



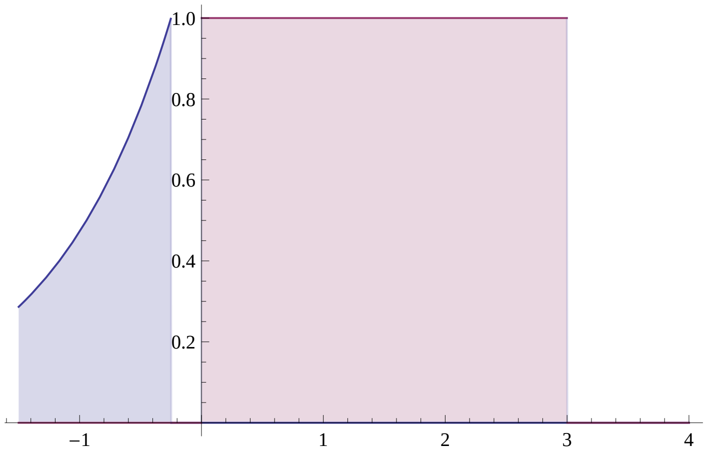
```
u[t_] := { 1   0 ≤ t ≤ 3
           0   True
```

- Konvolucija:

$$y(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

- Za interval t < 0:

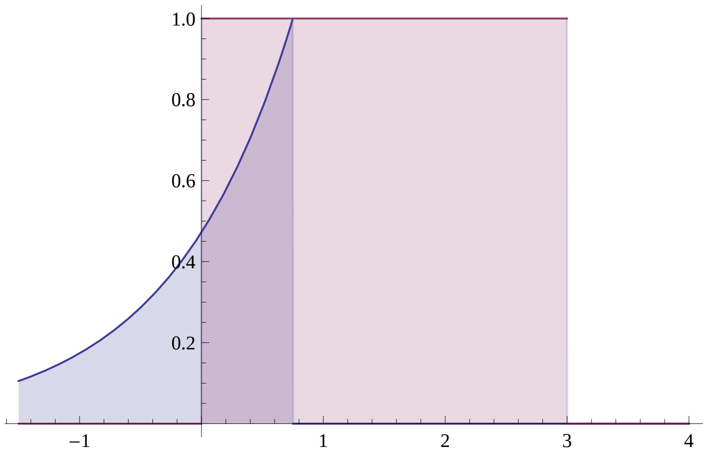
```
t = -1/4; Plot[{h[t - τ], u[τ]}, {τ, -1.5, 4.0}, PlotRange → All,
 Filling → Axis]
```




```
y = 0;
```

- **Za interval $0 \le t \le 3$:**

```
t = 3/4; Plot[{h[t - τ], u[τ]}, {τ, -1.5, 4.0}, PlotRange → All,
 Filling → Axis]
```

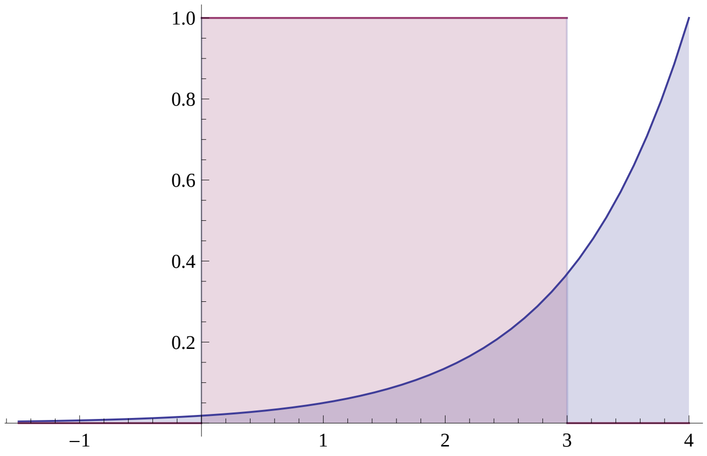


$$y = \int_0^t 1 * (\mathrm{Exp}[-(t - \tau)])\, d\tau$$

$$1 - e^{-t}$$

- **Za interval $t \ge 3$:**

```
t = 4; Plot[{h[t - τ], u[τ]}, {τ, -1.5, 4.0}, PlotRange → All,
 Filling → Axis]
```



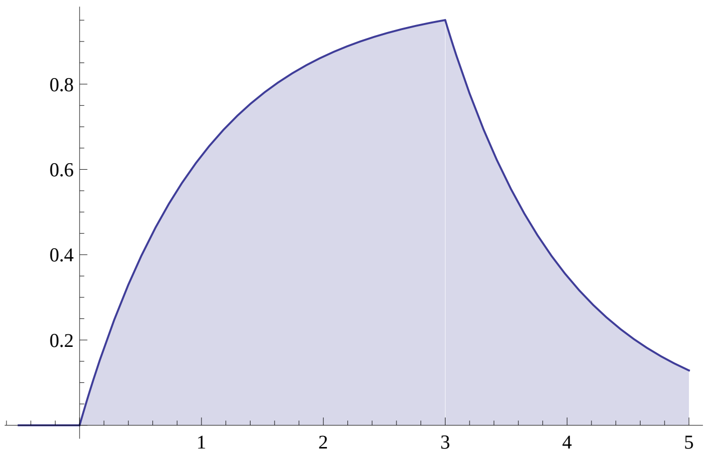
```
y = ∫_0^3 1 * (Exp[-(t - τ)]) dτ
```

$e^{-t}\left(-1 + e^{3}\right)$

Skica izhodnega signala y(t)

$$y[t\_] := \begin{cases} 1 - e^{-t} & 0 \le t \le 3 \\ e^{-t}\left(-1 + e^{3}\right) & t \ge 3 \\ 0 & \text{True} \end{cases}$$

```
Plot[y[t], {t, -0.5, 5.0}, PlotRange → All, Filling → Axis]
```

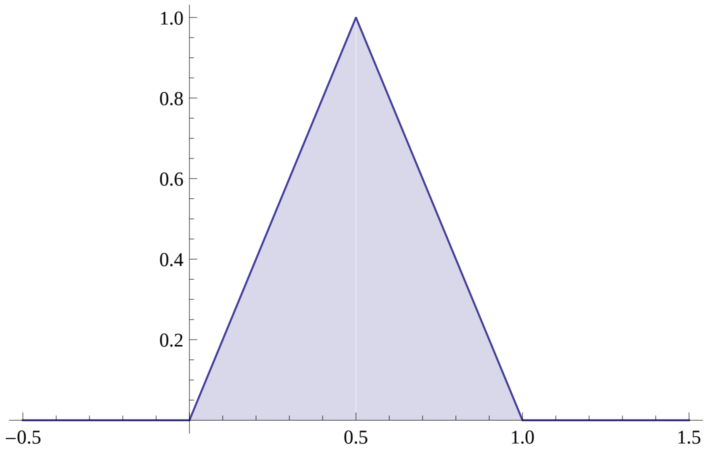


Naloga 2:

Naloga:
Izračunaj izhod LSS sistema s prevajalno funkcijo h(t) in vhodnim signalom u(t):

- **Prevajalna funkcija LSS:**

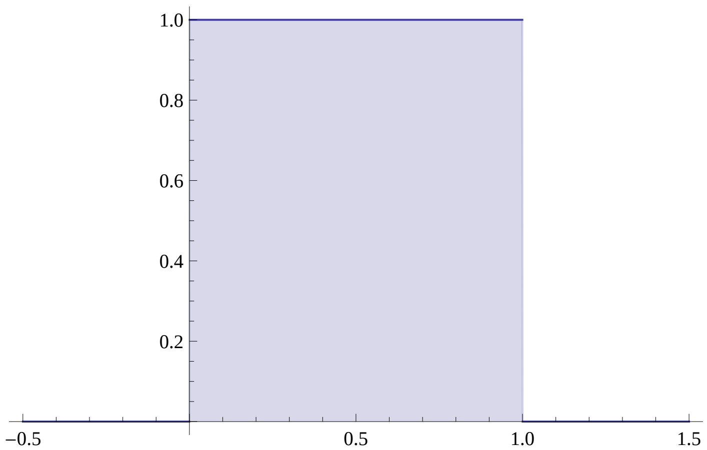
```
Plot[h[t], {t, -0.5, 1.5}, PlotRange → All, Filling → Axis]
```




```
h[t_] := { 2 t        0 ≤ t ≤ 1/2
           -2 t + 2   1/2 ≤ t ≤ 1
           0          True
```

- **Vhodni signal:**

```
Plot[u[t], {t, -0.5, 1.5}, PlotRange → All, Filling → Axis]
```



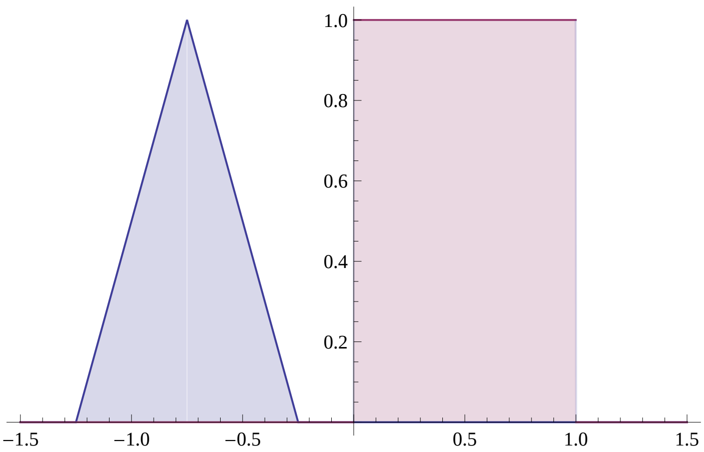
```
u[t_] := { 1  0 ≤ t ≤ 1
           0    True
```

- **Konvolucija:**

$$y(t) = \int_{-\infty}^{\infty} u(\tau)\, h(t-\tau)\, d\tau$$

- **Za interval t < 0:**

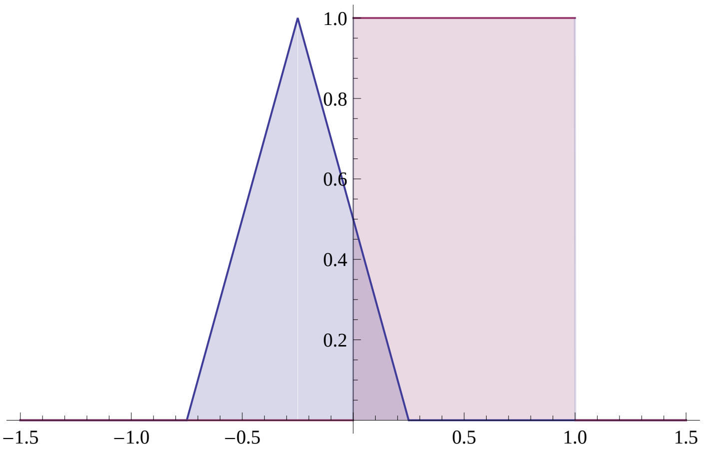
```
t = -1/4; Plot[{h[t - τ], u[τ]}, {τ, -1.5, 1.5}, PlotRange → All,
 Filling → Axis]
```




```
y = 0;
```

▫ Za interval $0 \le t \le \frac{1}{2}$:

```
t = 1/4; Plot[{h[t - τ], u[τ]}, {τ, -1.5, 1.5}, PlotRange → All,
 Filling → Axis]
```

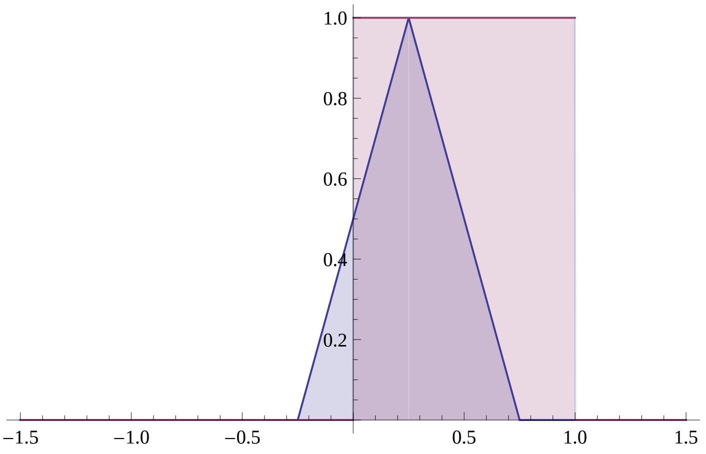



```
y = ∫_0^t 1 * (2 * (t - τ)) dτ
```

t^2

▫ Za interval $\frac{1}{2} \le t \le 1$:

```
t = 3/4; Plot[{h[t - τ], u[τ]}, {τ, -1.5, 1.5}, PlotRange → All,
 Filling → Axis]
```

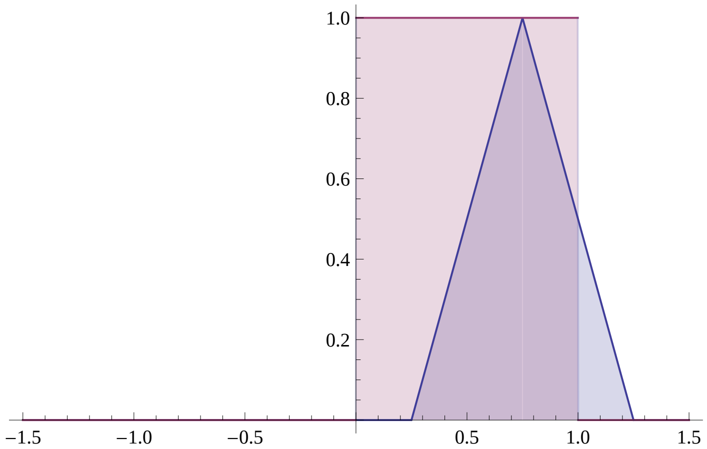


```
y = Integrate[1*(-2*(t - τ) + 2), {τ, 0, t - 1/2}] + 1/4
```

$$-\frac{1}{2} + 2t - t^2$$

- Za interval $1 \le t \le \frac{3}{2}$:

```
t = 5/4; Plot[{h[t - τ], u[τ]}, {τ, -1.5, 1.5}, PlotRange → All,
 Filling → Axis]
```

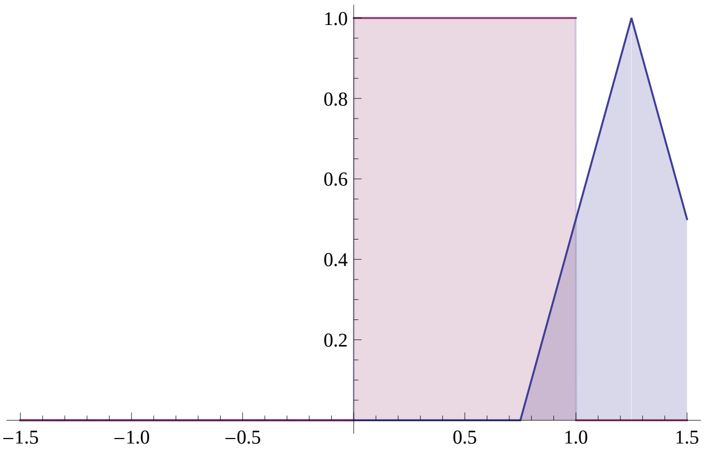



```
y = 1/4 + Integrate[1*(2*(t - τ)), {τ, t - 1/2, 1}]
```

$$-\frac{1}{2} + 2t - t^2$$

- Za interval $\frac{3}{2} \leq t \leq 2$:

```
t = 7/4; Plot[{h[t - τ], u[τ]}, {τ, -1.5, 1.5}, PlotRange → All,
 Filling → Axis]
```

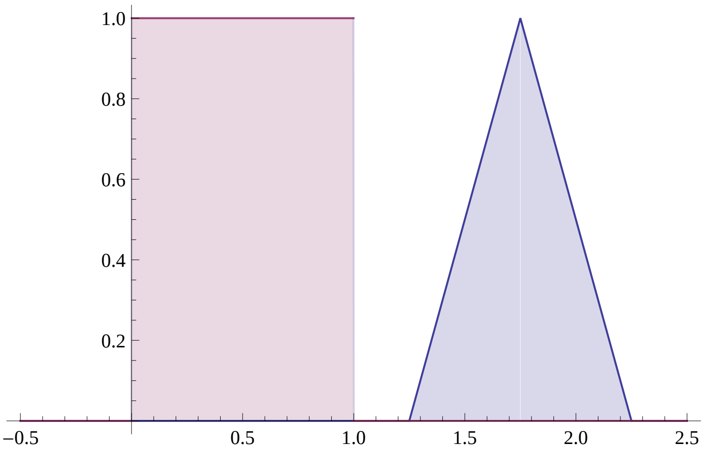



```
y = ∫_{t-1}^{1} 1 * (-2 * (t - τ) + 2) dτ
```

$4 - 4t + t^2$

- Za interval $t \geq 2$:

```
t = 9/4; Plot[{h[t - τ], u[τ]}, {τ, -0.5, 2.5}, PlotRange → All,
 Filling → Axis]
```

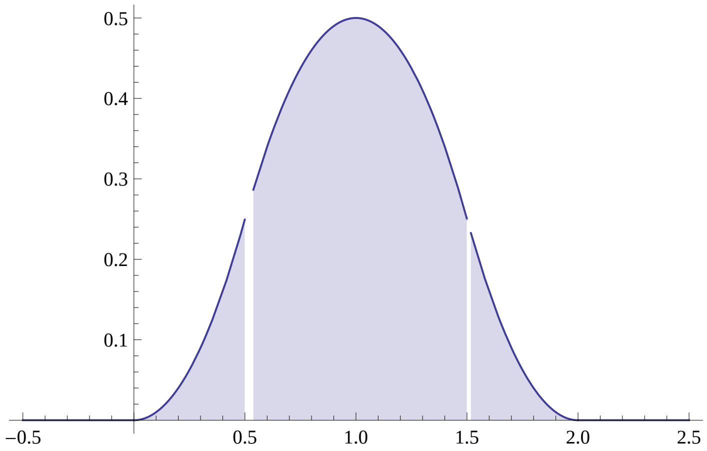


```
y = 0
```

Skica izhodnega signala y(t)

$$y[t\_] := \begin{cases} t^2 & 0 \le t \le 1/2 \\ -\frac{1}{2} + 2t - t^2 & 1/2 \le t \le 1 \\ -\frac{1}{2} + 2t - t^2 & 1 \le t \le 3/2 \\ 4 - 4t + t^2 & 3/2 \le t \le 2 \\ 0 & \text{True} \end{cases}$$

```
Plot[y[t], {t, -0.5, 2.5}, PlotRange → All, Filling → Axis]
```



Diskretna Fourierjeva transformacija

Definicija:

Imamo vzorčeni signal $f(t)$:
$\{f(nT)\} = \{f(0),\ f(T),\ f(2T),\ f(3T),\ \ldots,\ f((N-1)T)\}$,
$1/T$ je frekvenca vzorčenja

Diskretna Fourierjeva transformacija signala (DFT) *je* :

$$F_D(k\Omega) = \sum_{n=0}^{N-1} f(nT)\, e^{-jnTk\Omega} \qquad k = 0,\ 1,\ 2,\ \ldots,\ N-1$$

N	število vzorcev
$\Omega = (2\pi)/NT$	razmik med vzorci v frek. prostoru

Zveza s Fourierjevo transformacijo

$$F(\omega)\big|_{\omega = k\Omega} \doteq T \cdot F_D(k\Omega)$$

Inverzna DFT (IDFT) :

$$f(nT) = 1/N \sum_{k=0}^{N-1} F_D(k\Omega)\, E^{jnTk\Omega} \qquad n = 0,\ 1,\ 2,\ \ldots,\ N-1$$

- Prevajanje signalov skozi LSS:

Kako iz disretnega vhodnega signala $u(nT)$ in izhodnega signala $y(nT)$ dobimo prevajalno funkcijo sistema $h(nT)$?

Ker *velja* :

$$Y(\omega) = H(\omega) \cdot U(\omega) \Longrightarrow H(\omega) = \frac{Y(\omega)}{U(\omega)}$$

V diskretnem primeru to *pomeni* :

$$H(\omega)\,|_{\omega = k\Omega} = \frac{Y(\omega)}{U(\omega)}\,|_{\omega = k\Omega} \doteq \frac{Y_D(k\Omega)}{U_D(k\Omega)}$$

in velja :

$$H_D(k\Omega) \doteq \frac{1}{T} H(\omega)\,|_{\omega = k\Omega} \doteq \frac{1}{T} \cdot \frac{Y_D(k\Omega)}{U_D(k\Omega)}$$

Da dobimo h (nT) pa izračunamo inverzno DFT :

$$h(nT) = \frac{1}{N} \sum_{k=0}^{N-1} H_D(k\Omega)\, e^{jnTk\Omega}$$

Naloga 1:

Naloga:
DFT vhodnega signala u(t) v linearni stacionarni sistem je: $\{U_D(k\Omega)\} = \{2, 1+j, 1, 1-j\}$.
DFT odziva sistema *y(t)* na ta vhodni signal pa je: $\{Y_D(k\Omega)\} = \{2, -2j, -1, 2j\}$.
Pri določitvi obeh DFT smo signala vzorčili s časovnim presledkom $T = 1$.

a) Določi približno vrednost aplitudnega in faznega spektra prevajalne funkcije pri $\omega = \frac{\pi}{2}$.
b) Določi približno vrednost odziva sistema na enotin impulz pri $t = 1$.

Rešitev a):

Najprej *izračunamo* :

$$\Omega = \frac{2\pi}{NT} = \frac{2\pi}{4*1} = \frac{\pi}{2}.$$

Ker je $\omega = \frac{\pi}{2}$, *sledi* :

$$\omega = 1*\Omega \quad \Rightarrow \quad k = 1$$

$$H_D(k\Omega) = \frac{1}{T} \frac{Y_D(k\Omega)}{U_D(k\Omega)}$$

$$\{H_D(k\Omega)\} = \left\{1, \frac{-2j}{1+j}, \frac{-1}{1}, \frac{2j}{1-j}\right\} = \{1, -1-j, -1, -1+j\}$$

Ker velja :

$F(\omega)\,|_{\omega=k\Omega} \doteq T\,F_D(k\Omega)$

To v našem primeru pomeni :

$$H\left(\frac{\pi}{2}\right) \doteq 1 * H_D(1) = -1 - j$$

Amplitudni in fazni spekter sta :

$$\left|H\left(\frac{\pi}{2}\right)\right| = \sqrt{1+1} = \sqrt{2}$$

$$\phi_H\left(\frac{\pi}{2}\right) = \pm\pi + \mathrm{arctg}\left(\frac{-1}{-1}\right) = \pm\pi + \frac{\pi}{4} = -\frac{3\pi}{4}$$

- Rešitev b):

Iz a) smo že *izračunali* :

$$\{H_D(k\Omega)\} = \{1,\ -1-j,\ -1,\ -1+j\}$$

Potrebno je izračunati inverzno DFT za H_D pri $t = 1$.

Ker je $t = 1$ in velja $t = nT$, sledi $n * 1 = 1$, ali $n = 1$.

$$h(1) \doteq \frac{1}{N}\sum_{k=0}^{N-1} H_D(k\Omega) * e^{jk\Omega nT}$$

$$= \frac{1}{4}\sum_{k=0}^{3} H_D(k\Omega) * e^{jk\Omega*1}$$

$$= \frac{1}{4} * \left[1 + (-1-j)\,e^{j\frac{\pi}{2}} - 1 * e^{j\pi} + (-1+j)\,e^{j\frac{3\pi}{2}}\right]$$

$$= \frac{1}{4} * [1 + (-1-j) * j - 1 * (-1) + (-1+j) * (-j)]$$

$$= \frac{1}{4} * [1 - j + 1 + 1 + j + 1]$$

$$= \frac{1}{4} * [4]$$

$$= 1$$