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On vertex-stabilizers of bipartite dual polar graphs

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Abstract

Let X, Y denote vertices of a bipartite dual polar graph, and let G_X and G_Y denote the stabilizers of X and Y in the full automorphism group of this graph. In this paper, a description of the orbits of $G_X \cap G_Y$ in the cases when the distance between X and Y is 1 or 2, is given.

Keywords: Dual polar graphs, automorphism group, quadratic form, isotropic subspace.

Math. Subj. Class.: 05E18, 05E30

1 Preliminaries and introductory remarks

Let q denote a prime power, let $GF(q)$ denote a finite field with q elements, and let d denote a positive integer. Let $V = GF(q)^{2d}$ denote the vector space over $GF(q)$ of dimension $2d$, consisting of column vectors with entries in $GF(q)$. We define a map $Q : V \to GF(q)$ as follows. For $u = (u_1, u_2, ..., u_{2d})^t \in V$ we let

$$Q(u) = \sum_{i=1}^{d} u_{2i-1}u_{2i}. \tag{1.1}$$

The form Q is a *quadratic form* on V, that is, $Q(\lambda u) = \lambda^2 Q(u)$ $(\lambda \in GF(q), u \in V)$, and

$$f(u, v) = Q(u + v) - Q(u) - Q(v) \qquad (u, v \in V) \tag{1.2}$$

is a symmetric bilinear form on V. The form Q is usually called *hyperbolic quadric*. Note that for vectors $u = (u_1, u_2, ..., u_{2d})^t$ and $v = (v_1, v_2, ..., v_{2d})^t$ of V we have

$$f(u, v) = \sum_{i=1}^{d}(u_{2i-1}v_{2i} + u_{2i}v_{2i-1}). \tag{1.3}$$

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A vector $v \in V$ is called *isotropic*, if $Q(v) = 0$. A subspace U of V is called *isotropic*, if $Q(u) = 0$ for every $u \in U$, and it is called *maximal isotropic*, if it is maximal (with respect to inclusion) in the set of all isotropic subspaces of V. It turns out that the dimension of every maximal isotropic subspace is d (see, for example, [1, Theorem 3.10] or [10, Lemma 3]). Observe that if $u, v \in V$ belong to the same isotropic subspace of V, than $Q(\lambda u + \mu v) = 0$ for every $\lambda, \mu \in GF(q)$. Furthermore,

$$f(u, v) = Q(u + v) - Q(u) - Q(v) = 0. \tag{1.4}$$

Conversely, if u and v are isotropic with $f(u, v) = 0$, then $\langle u, v\rangle$ is an isotropic subspace of V. Indeed, for $\lambda, \mu \in GF(q)$ we have

$$Q(\lambda u + \mu v) = \lambda^2 Q(u) + \mu^2 Q(v) + \lambda\mu f(u, v) = 0. \tag{1.5}$$

We now define the dual polar graph $D_d(q)$ on V. The vertex-set $V(D_d(q))$ of $D_d(q)$ is the set of all maximal isotropic subspaces of V. Vertices $X, Y \in V(D_d(q))$ are adjacent in $D_d(q)$ if and only if the dimension of $X \cap Y$ is $d-1$. Let ∂ denote the path-length distance function on $D_d(q)$. It is easy to see that $\partial(X, Y) = i$ if and only if $\dim(X \cap Y) = d - i$ ($X, Y \in V(D_d(q))$). The following facts about $D_d(q)$ can be found, for example, in [2, Section 9.4]. The graph $D_d(q)$ is bipartite with diameter d and with $\prod_{i=0}^{d-1}(q^{d-i-1} + 1)$ vertices. For convenience let

$$b_i = q^i \frac{q^{d-i} - 1}{q - 1}, \qquad c_i = \frac{q^i - 1}{q - 1} \qquad \text{and} \qquad k_i = \frac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \tag{1.6}$$

for $0 \le i \le d$. The graph $D_d(q)$ is regular with valency $b_0 = k_1$. For $X \in V(D_d(q))$ and an integer $0 \le i \le d$ we set $S_i(X) = \{Z \in V(D_d(q)) \mid \partial(X, Z) = i\}$.

Let $GL(V)$ denote the general linear group of V. Then $\sigma \in GL(V)$ is called *isometry* of V, if $Q(\sigma(v)) = Q(v)$ for every $v \in V$. It follows from (1.2) that if σ is an isometry of V, then $f(u, v) = f(\sigma(u), \sigma(v))$ for $u, v \in V$. The group of all isometries of V is called the *orthogonal group* for Q, and is denoted by $O^+_{2d}(q)$. Note that every $\sigma \in O^+_{2d}(q)$ acts on $V(D_d(q))$ as an automorphism of $D_d(q)$. The full automorphism group G of $D_d(q)$ acts distance-transitively on $V(D_d(q))$, that is, for $X, Y, Z, W \in V(D_d(q))$ with $\partial(X, Y) = \partial(Z, W)$ there exists $\sigma \in G$ such that $\sigma(X) = Z$ and $\sigma(Y) = W$ (see, for example, [2, Table 6.1]). Recall that every distance-transitive graph is also distance-regular in the sense of [2, Section 4.1].

Pick $X, Y \in V(D_d(q))$ and let G_X and G_Y denote the stabilizers of X and Y in G, respectively. Since G acts distance-transitively on $V(D_d(q))$, the orbits of G_X are precisely the sets $S_i(X)$ $(0 \le i \le d)$. In this paper we examine the orbits of $G_X \cap G_Y$. These orbits play an important role in the theory of Terwilliger algebras of $D_d(q)$. This role is especially important in the case when $\partial(X, Y) \in \{1, 2\}$, see [6]. For the definition and more background on Terwilliger algebras of distance-regular graphs see [3, 4, 7, 8, 9].

In this paper we give a description of the orbits of $G_X \cap G_Y$ when $\partial(X, Y) \in \{1, 2\}$. To do this, we consider the following situation for the rest of this paper.

Notation 1.1. Let q denote a prime power, let $GF(q)$ denote a finite field with q elements, and let d denote a positive integer. Let $V = GF(q)^{2d}$ denote the vector space over $GF(q)$ of dimension $2d$, consisting of column vectors with entries in $GF(q)$. Let Q and f be as defined in (1.1) and (1.2). Let $D_d(q)$ denotes the bipartite dual polar graph over V, and let b_i, c_i and k_i be as in (1.6). Fix $X, Y \in V(D_d(q))$. For $0 \leq i, j \leq d$ let $D^i_j = D^i_j(X,Y) = S_i(X) \cap S_j(Y)$. Let G_X and G_Y denote the stabilizers of X and Y in the full automorphism group G of $D_d(q)$.

Our paper is organised as follows. In Section 2 we state some results about maximal isotropic subspaces that we need later. In Section 3 (Section 4, respectively) we describe the orbits of $G_X \cap G_Y$ in the case when $\partial(X,Y) = 1$ ($\partial(X,Y) = 2$, respectively). In what follows we use the same symbols (capital letters) for the vertices of $D_d(q)$ and for the maximal isotropic subspaces of V; this should cause no confusion.

2 Maximal isotropic subspaces

In this section we state some results about maximal isotropic subspaces of V that we need later. The first one is known as *Witt's lemma* (see, for example, [1, Theorem 3.9]).

Lemma 2.1. *With reference to Notation 1.1, let U and W be subspaces of V, and let $\sigma_U : U \rightarrow W$ be a bijective linear map satisfying $Q(\sigma_U(u)) = Q(u)$ for every $u \in U$. Then there is an isometry of V which extends σ_U.*

Lemma 2.2. *With reference to Notation 1.1, let U and W be maximal isotropic subspaces of V with $\dim(U \cap W) = d - i$ for some $1 \leq i \leq d$. Pick linearly independent vectors $u_1, \dots, u_i \in U \setminus W$ and linearly independent vectors $w_1, \dots, w_i \in W \setminus U$. Let F be the $i \times i$ matrix with (j,ℓ)-entry equal to $f(u_j, w_\ell)$. Then the determinant of F is nonzero.*

Proof. First note that F is a nonzero matrix. Namely, if $f(u_j, w_\ell) = 0$ for every $1 \leq j, \ell \leq i$, then a subspace generated by U and W is isotropic subspace of dimension $d + i$, a contradiction. Suppose now that $\det(F) = 0$. Then the columns of F are linearly dependent vectors of $GF(q)^i$, that is, there exist scalars λ_j $(1 \leq j \leq i)$ which are not all equal to zero, such that for each $1 \leq \ell \leq i$ we have

$$0 = \lambda_1 f(u_\ell, w_1) + \lambda_2 f(u_\ell, w_2) + \dots + \lambda_i f(u_\ell, w_i) = f(u_\ell, \lambda_1 w_1 + \lambda_2 w_2 + \dots + \lambda_i w_i).$$

Note that $w = \lambda_1 w_1 + \lambda_2 w_2 + \dots + \lambda_i w_i$ is nonzero, since $w_1, w_2, \dots, w_i$ are linearly independent. Multiplying the above equation with an arbitrary scalar μ_ℓ gives us $\mu_\ell f(u_\ell, w) = 0$. Adding the obtained equations we get

$$\sum_{\ell=1}^{i} \mu_\ell f(u_\ell, w) = f(\mu_1 u_1 + \mu_2 u_2 + \dots + \mu_i u_i, w) = 0.$$

This implies that $f(u, w) = 0$ for every $u \in U$. By (1.5), the subspace generated by U and w is isotropic with dimension $d + 1$, a contradiction. Therefore, $\det(F) \neq 0$. □

Lemma 2.3. *With reference to Notation 1.1, let U, U_1, W and W_1 be maximal isotropic subspaces of V with $\dim(U \cap W) = \dim(U_1 \cap W_1) = d - i$ for some $1 \leq i \leq d$. Let $u_1, u_2, \dots, u_d$ be a basis of U such that $u_{i+1}, \dots, u_d$ is a basis of $U \cap W$. Let $w_1, \dots, w_i \in W$ be such that $w_1, \dots, w_i, u_{i+1}, \dots, u_d$ is a basis of W. Let $v_1, v_2, \dots, v_d$ be a basis of U_1 such that $v_{i+1}, \dots, v_d$ is a basis of $U_1 \cap W_1$. Let $z_1, \dots, z_i \in W_1$ be such that $z_1, \dots, z_i, v_{i+1}, \dots, v_d$ is a basis of W_1. Then there exists an isometry σ of V, such that $\sigma(u_j) = v_j$ $(1 \leq j \leq d)$ and $\sigma(w_j) \in \langle z_1, \dots, z_i \rangle$ $(1 \leq j \leq i)$.*

Proof. We first define a bijective linear map $\overline{\sigma}$ from a subspace generated by U and W to a subspace generated by U_1 and W_1, such that $\overline{\sigma}(u_j) = v_j$ $(1 \leq j \leq d)$ and $\overline{\sigma}(w_j) \in \langle z_1, \ldots, z_i \rangle$ $(1 \leq j \leq i)$. We will then show that $\overline{\sigma}$ extends to an isometry of V. We now define $\overline{\sigma}(w_j)$ $(1 \leq j \leq i)$. Let F denote an $i \times i$ matrix with (j, ℓ)-entry equal to $f(v_j, z_\ell)$. For $1 \leq \ell \leq i$ consider the following system of linear equations in variables $\alpha_1^\ell, \alpha_2^\ell, \ldots, \alpha_i^\ell$:

$$F(\alpha_1^\ell, \alpha_2^\ell, \ldots, \alpha_i^\ell)^t = (f(u_1, w_\ell), f(u_2, w_\ell), \ldots, f(u_i, w_\ell))^t. \tag{2.1}$$

Note that this system has a unique solution since F is nonsingular by Lemma 2.2. For convenience, we denote the solutions of this system also by $\alpha_1^\ell, \alpha_2^\ell, \ldots, \alpha_i^\ell$. For $1 \leq \ell \leq i$ we let

$$\overline{\sigma}(w_\ell) = \alpha_1^\ell z_1 + \alpha_2^\ell z_2 + \cdots + \alpha_i^\ell z_i. \tag{2.2}$$

We extend $\overline{\sigma}$ to a linear map from $\langle U, W \rangle$ to $\langle U_1, W_1 \rangle$ in a natural way:

$$\overline{\sigma}(\lambda_1 u_1 + \cdots + \lambda_d u_d + \mu_1 w_1 + \cdots + \mu_i w_i) = \\ \lambda_1 \overline{\sigma}(u_1) + \cdots + \lambda_d \overline{\sigma}(u_d) + \mu_1 \overline{\sigma}(w_1) + \cdots + \mu_i \overline{\sigma}(w_i)$$

for $\lambda_1, \ldots, \lambda_d, \mu_1, \ldots, \mu_i \in GF(q)$.

We now show that $\overline{\sigma}$ is a bijection. To do this, it is enough to show that $\overline{\sigma}(w_\ell)$ $(1 \leq \ell \leq i)$ are linearly independent. Let A be an $i \times i$ matrix with (j, ℓ)-entry equal to α_j^ℓ. Observe that $\overline{\sigma}(w_\ell)$ $(1 \leq \ell \leq i)$ are linearly independent if and only if A is nonsingular. Let F_1 denote an $i \times i$ matrix with (j, ℓ)-entry equal to $f(u_j, w_\ell)$. The matrix F_1 is nonsingular by Lemma 2.2. Furthermore, it follows from (2.1) that $F \cdot A = F_1$, implying that A is nonsingular.

We now show that $\overline{\sigma}$ preserves Q. Pick arbitrary $v \in \langle U, W \rangle$:

$$v = \sum_{j=1}^{d} \alpha_j u_j + \sum_{j=1}^{i} \beta_j w_j.$$

By (1.2) and (1.4),

$$Q(v) = \sum_{r=1}^{i} \sum_{s=1}^{i} \alpha_r \beta_s f(u_r, w_s).$$

Let us now compute $Q(\overline{\sigma}(v))$. By (1.2) and (1.4) we first get

$$Q(\overline{\sigma}(v)) = \sum_{r=1}^{i} \sum_{s=1}^{i} \alpha_r \beta_s f(\overline{\sigma}(u_r), \overline{\sigma}(w_s)).$$

By (2.2) and since $\sigma(u_r) = v_r$ we further find

$$f(\overline{\sigma}(u_r), \overline{\sigma}(w_s)) = f(v_r, \alpha_1^s z_1 + \cdots + \alpha_i^s z_i) = \alpha_1^s f(v_r, z_1) + \cdots + \alpha_i^s f(v_r, z_i).$$

Finally, by (2.1), the above expression is equal to $f(u_r, w_s)$. Therefore, $Q(v) = Q(\overline{\sigma}(v))$. By Lemma 2.1 there exists an isometry σ of V which extends $\overline{\sigma}$. This completes the proof. ∎

Lemma 2.4. *With reference to Notation 1.1, let U be a $(d-1)$-dimensional isotropic subspace of V. Then U is contained in exactly two maximal isotropic subspaces of V.*

Proof. By [2, p. 274], the number of isotropic k-dimensional subspaces of V containing a given isotropic $(k-1)$-dimensional subspace of V is $(q^{d-k+1}-1)(q^{d-k}+1)/(q-1)$. The result follows. □

3 The case $\partial(X, Y) = 1$

With reference to Notation 1.1, in this section we describe the orbits of $G_X \cap G_Y$ when $\partial(X, Y) = 1$. We first determine the size of the D^i_j $(0 \le i, j \le d)$.

Lemma 3.1. *With reference to Notation 1.1 assume that $\partial(X, Y) = 1$. Then the following* (i), (ii) *hold.*

(i) $|D^i_{i-1}| = |D^{i-1}_i| = c_i k_i / b_0$ $(1 \le i \le d)$.

(ii) $D^i_j = \emptyset$ *if* $|i-j| \neq 1$ $(0 \le i, j \le d)$.

Proof. (i) This follows from [5, Lemma 4.1(i)].
(ii) By the triangle inequality we find $D^i_j = \emptyset$ if $|i-j| \ge 2$. Since $D_d(q)$ is bipartite, we also have $D^i_i = \emptyset$. □

Lemma 3.2. *With reference to Notation 1.1 assume that $\partial(X, Y) = 1$. Pick $u \in X \setminus Y$ and $v \in Y \setminus X$. Then $f(u, v) \neq 0$. In particular, u and v are not contained in a common isotropic subspace.*

Proof. Suppose on the contrary that $f(u, v) = 0$. Pick $\lambda, \mu \in GF(q)$ and $w \in X \cap Y$. Consider $\lambda u + w + \mu v \in \langle X, Y \rangle$. By (1.2) and (1.4) we have

$$Q(\lambda u + w + \mu v) = Q(\lambda u + w) + Q(\mu v) + f(\lambda u + w, \mu v) = \lambda\mu f(u, v) + \mu f(w, v) = 0.$$

This shows that $\langle X, Y \rangle$ is an isotropic subspace of dimension $d+1$, a contradiction. □

Theorem 3.3. *With reference to Notation 1.1 assume that $\partial(X, Y) = 1$. Then the following* (i), (ii) *hold for $1 \le i \le d$.*

(i) *For every $Z, Z' \in D^{i-1}_i$ there exists $\sigma \in G_X \cap G_Y$ which maps Z to Z'.*

(ii) *For every $Z, Z' \in D^i_{i-1}$ there exists $\sigma \in G_X \cap G_Y$ which maps Z to Z'.*

Proof. (i) If $i = 1$ then the result is clear. Assume now that $i \ge 2$. Since $\dim(X \cap Z) = d - i + 1$ and $\dim(Y \cap Z) = d - i$, it follows from Lemma 3.2 that $X \cap Y \cap Z = Y \cap Z$ with $\dim(X \cap Y \cap Z) = d - i$, and $X \cap Z = \langle X \cap Y \cap Z, u \rangle$ for some $u \in X \setminus Y$. Pick $w \in Y \setminus X$. Let $v_1, \ldots, v_{d-1}$ be a basis of $X \cap Y$, such that $v_i, \ldots, v_{d-1}$ is a basis of $X \cap Y \cap Z$. Let $z_1, \ldots, z_{i-1} \in Z$ be such that $u, v_i, \ldots, v_{d-1}, z_1, \ldots, z_{i-1}$ is a basis of Z. Note that $u, v_1, \ldots, v_{d-1}$ is a basis of X and that $w, v_1, \ldots, v_{d-1}$ is a basis of Y.

Similarly as above, let $u' \in X \setminus Y$ be such that $X \cap Z' = \langle X \cap Y \cap Z', u' \rangle$. Let $v'_1, \ldots, v'_{d-1}$ be a basis of $X \cap Y$, such that $v'_i, \ldots, v'_{d-1}$ is a basis for $X \cap Y \cap Z'$. Let $z'_1, \ldots, z'_{i-1} \in Z'$ be such that $u', v'_i, \ldots, v'_{d-1}, z'_1, \ldots, z'_{i-1}$ is a basis for Z'. Observe that $u', v'_1, \ldots, v'_{d-1}$ is a basis for X and that $w, v'_1, \ldots, v'_{d-1}$ is a basis for Y.

Applying Lemma 2.3 (with $U = X = \langle u, v_1, \ldots, v_{d-1} \rangle$, $W = Z = \langle u, v_i, \ldots, v_{d-1}, z_1, \ldots, z_{i-1} \rangle$, $U_1 = X = \langle u', v'_1, \ldots, v'_{d-1} \rangle$ and $W_1 = Z' = \langle u', v'_i, \ldots, v'_{d-1},$

$z'_1, \dots, z'_{i-1}\rangle$) we find that there exists an isometry σ such that $\sigma(u) = u'$, $\sigma(v_j) = v'_j$ $(1 \le j \le d-1)$, and $\sigma(z_j) \in \langle z'_1, \dots, z'_{i-1}\rangle$ $(1 \le j \le i-1)$. Clearly, σ preserves X (and thus also $X \cap Y$), and maps Z to Z'. To finish the proof we have to show that σ preserves Y. Observe that $X \cap Y$ is a $(d-1)$-dimensional isotropic subspace of V. By Lemma 2.4, the only two maximal isotropic subspaces containing $X \cap Y$ are X and Y. Since X and $X \cap Y$ are both preserved by σ, also Y is preserved by σ.

(ii) Similar as (i) above. □

Proposition 3.4. *With reference to Notation 1.1 assume that* $\partial(X,Y) = 1$. *Then the following* (i), (ii) *hold.*

(i) *The set* D_i^{i-1} $(1 \le i \le d)$ *is an orbit of* $G_X \cap G_Y$.

(ii) *The set* D_{i-1}^{i} $(1 \le i \le d)$ *is an orbit of* $G_X \cap G_Y$.

Proof. It is clear that two vertices from different sets from (i) and (ii) above could not be in the same orbit of $G_X \cap G_Y$. The result now follows from Theorem 3.3. □

4 The case $\partial(X,Y) = 2$

With reference to Notation 1.1, in this section we describe the orbits of $G_X \cap G_Y$ when $\partial(X,Y) = 2$. We first determine the size of the sets D_j^i $(0 \le i, j \le d)$. The proposition below follows from [5, Lemma 4.1(ii)–(iv)].

Proposition 4.1. *With reference to Notation 1.1 assume that* $\partial(X,Y) = 2$. *Then the following* (i)–(iv) *hold.*

(i) $|D_{i-2}^i| = |D_i^{i-2}| = k_i c_{i-1} c_i/(b_0 b_1)$ $\quad (2 \le i \le d)$;

(ii) $|D_0^0| = 0$ *and* $|D_i^i| = k_i(c_i(b_{i-1}-1) + b_i(c_{i+1}-1))/(b_0 b_1)$ $\quad (1 \le i \le d-1)$;

(iii) $|D_d^d| = k_d(b_{d-1}-1)/b_1$;

(iv) $|D_j^i| = 0$ *if* $|i-j| \notin \{0,2\}$ $\quad (0 \le i, j \le d)$.

Lemma 4.2. *With reference to Notation 1.1 assume that* $\partial(X,Y) = 2$. *Then the following* (i), (ii) *hold.*

(i) *Let* $u_1, u_2 \in X \setminus Y$ *be linearly independent, and let* $w \in Y \setminus X$. *Then* u_1, u_2 *and* w *are not contained in a common isotropic subspace of* V.

(ii) *Let* $w_1, w_2 \in Y \setminus X$ *be linearly independent, and let* $u \in X \setminus Y$. *Then* w_1, w_2 *and* u *are not contained in a common isotropic subspace of* V.

Proof. (i) Suppose on contrary that u_1, u_2 and w are contained in a common isotropic subspace. Pick $\lambda_1, \lambda_2, \mu \in GF(q)$ and $v \in X \cap Y$. Consider $\lambda_1 u_1 + \lambda_2 u_2 + v + \mu w \in \langle X, w\rangle$. By (1.2) and (1.4) we have

$$Q(\lambda_1 u_1 + \lambda_2 v_2 + v + \mu w) = Q(\lambda_1 u_1 + \lambda_2 u_2 + v) + Q(\mu w) + f(\lambda_1 u_1 + \lambda_2 u_2 + v, \mu w) =$$

$$\lambda_1 \mu f(u_1, w) + \lambda_2 \mu f(u_2, w) + \mu f(v, w) = 0.$$

Therefore, $\langle X, w\rangle$ is an isotropic subspace of dimension $d+1$, a contradiction.
(ii) Similar as (i) above. □

Theorem 4.3. *With reference to Notation 1.1 assume that* $\partial(X,Y)=2$. *Then the following* (i), (ii) *hold for* $2 \le i \le d$.

(i) *For every* $Z, Z' \in D_i^{i-2}$ *there exists* $\sigma \in G_X \cap G_Y$ *which maps* Z *to* Z'.

(ii) *For every* $Z, Z' \in D_{i-2}^{i}$ *there exists* $\sigma \in G_X \cap G_Y$ *which maps* Z *to* Z'.

Proof. (i) Note that the result is clear if $i=2$. Namely, for $i=2$ we have $Z=Z'=X$. Assume now $i \ge 3$. By Lemma 4.2, there exists a basis $v_1,\dots v_{d-2}$ of $X \cap Y$, vectors $u_1,u_2 \in X$, vectors $w_1,w_2 \in Y$, and vectors $z_1,\dots z_{i-2} \in Z$, such that $v_{i-1},\dots,v_{d-2}$ is a basis of $X \cap Y \cap Z$, $u_1,u_2,v_1,\dots v_{d-2}$ is a basis of X, $w_1,w_2,v_1,\dots v_{d-2}$ is a basis of Y, and $u_1,u_2,v_{i-1},\dots v_{d-2},z_1,\dots,z_{i-2}$ is a basis of Z. Without loss of generality we can assume that $f(u_1,w_1)=0$ (otherwise we replace w_1 by $w_1+\lambda w_2$ for an appropriate $\lambda \in GF(q)$). This implies that $\langle X \cap Y, u_1, w_1\rangle$ is maximal isotropic subspace.

Similarly, there exists a basis $v_1',\dots v_{d-2}'$ of $X \cap Y$, vectors $u_1',u_2' \in X$ and vectors $z_1',\dots z_{i-2}' \in Z'$, such that $v_{i-1}',\dots,v_{d-2}'$ is a basis of $X \cap Y \cap Z'$, $u_1',u_2',v_1',\dots v_{d-2}'$ is a basis of X, $w_1,w_2,v_1',\dots v_{d-2}'$ is a basis of Y, and $u_1',u_2',v_{i-1}',\dots v_{d-2}',z_1',\dots,z_{i-2}'$ is a basis of Z'. Without loss of generality we can assume that $f(u_1',w_1)=0$ (otherwise we replace u_1' by $u_1'+\lambda u_2'$ for an appropriate $\lambda \in GF(q)$). This implies that $\langle X \cap Y, u_1', w_1\rangle$ is maximal isotropic subspace.

Applying Lemma 2.3 (with $U=\langle u_1,w_1,v_1,\dots,v_{d-2}\rangle$, $U_1=\langle u_1',w_1,v_1',\dots,v_{d-2}'\rangle$, $W=Z$ and $W_1=Z'$) we find that there exists an isometry σ of V, such that $\sigma(u_1)=u_1'$, $\sigma(w_1)=w_1$, $\sigma(v_j)=v_j'$ for $1 \le j \le d-2$, and $\sigma(u_2),\sigma(z_j) \in \langle u_2',z_1',\dots,z_{i-2}'\rangle$ ($1 \le j \le i-2$). Clearly, σ maps Z to Z'. It remains to show that σ preserves X and Y. Consider the subspace $W=\langle X\cap Y, u_1\rangle$. Note that W is a $(d-1)$-dimensional isotropic subspace of V. By Lemma 2.4, the only two maximal isotropic subspaces containing W are X and $\langle W,w_1\rangle$. Isometry σ maps W to $W'=\langle X\cap Y,u_1'\rangle$. By Lemma 2.4, the only two maximal isotropic subspaces containing W' are X and $\langle W',w_1\rangle$. Since σ maps $\langle W,w_1\rangle$ to $\langle W',w_1\rangle$, it must map X to X. Similarly we show that σ maps Y to Y. It follows that $\sigma \in G_X \cap G_Y$, completing the proof of (i).

(ii) Similarly as (i) above. □

Let us now consider the sets D_i^i ($1 \le i \le d$). Pick $Z \in D_i^i$. By Lemma 4.2, two essentially different situations can occur: either $\dim(X\cap Y\cap Z)=d-i$ (and therefore $X\cap Z = Y\cap Z = X\cap Y\cap Z$), or $\dim(X\cap Y\cap Z)=d-i-1$ (and therefore $X\cap Z \ne Y\cap Z$).

Definition 4.4. With reference to Notation 1.1 assume that $\partial(X,Y)=2$. Let $Z \in D_i^i$ ($1 \le i \le d$). We say Z is of *positive* (*negative*, respectively) type, whenever $\dim(X\cap Y\cap Z)=d-i$ ($\dim(X\cap Y\cap Z)=d-i-1$, respectively).

Observe that all vertices of D_1^1 are of negative type, and that all vertices of D_d^d are of positive type. Moreover, every D_i^i ($2 \le i \le d-1$) is a disjoint union of the set of vertices of D_i^i of positive type, and the set of vertices of D_i^i of negative type.

Remark 4.5. In [6], the definition of the vertices of positive (negative, respectively) type is different from Definition 4.4 above. Namely, $Z \in D_i^i$ is defined to be of positive type, whenever all vertices in D_1^1 are at distance $i-1$ from Z. On the other hand, Z is defined to be of negative type, if there exists a vertex in D_1^1 which is at distance $i-1$ from Z, and all other vertices in D_1^1 are at distance $i+1$ from Z. However, these definitions are equivalent.

If $\dim(X \cap Y \cap Z) = d - i$, then Z is at distance at most i from every vertex in D_1^1. By the triangle inequality and since $D_d(q)$ is bipartite, Z is at distance $i - 1$ from every vertex of D_1^1. On the other hand, if $\dim(X \cap Y \cap Z) = d - i - 1$, then pick $u \in (X \cap Z) \setminus Y$ and $v \in (Y \cap Z) \setminus X$. Then $W = \langle X \cap Y, u, v\rangle$ is a vertex of $D_d(q)$, which belongs to D_1^1 and is at distance $i - 1$ from Z. Furthermore, all other vertices in D_1^1 are at distance $i + 1$ from Z.

Lemma 4.6. ([6, Theorem 5.3(iv),(v) and Proposition 6.3]) *With reference to Notation 1.1 assume that $\partial(X, Y) = 2$. Then the following* (i), (ii) *hold for $2 \le i \le d - 1$.*

(i) $|\{z \in D_i^i \mid z \text{ is of positive type}\}| = k_i(q - 1)c_i c_{i-1}/(b_0 b_1)$;

(ii) $|\{z \in D_i^i \mid z \text{ is of negative type}\}| = k_i b_i c_i c_2/(b_0 b_1)$.

Theorem 4.7. *With reference to Notation 1.1 assume that $\partial(X, Y) = 2$. Let $Z, Z' \in D_i^i$ $(1 \le i \le d-1)$ and assume Z, Z' are of negative type. Then there exists $\sigma \in G_X \cap G_Y$ which maps Z to Z'.*

Proof. Let $v_1, \dots, v_{d-2}$ be a basis of $X \cap Y$ such that $v_i, \dots, v_{d-2}$ is a basis of $X \cap Y \cap Z$. Let $u_1 \in X$ and $w_1 \in Y$ be such that $u_1, v_i, \dots, v_{d-2}$ is a basis of $X \cap Z$ and such that $w_1, v_i, \dots, v_{d-2}$ is a basis of $Y \cap Z$. Let $u_2 \in X$ and $w_2 \in Y$ be such that $u_1, u_2, v_1, \dots, v_{d-2}$ is a basis of X and such that $w_1, w_2, v_1, \dots, v_{d-2}$ is a basis of Y. Finally, let $z_1, \dots, z_{i-1} \in Z$ be such that $u_1, w_1, z_1, \dots, z_{i-1}, v_i, \dots, v_{d-2}$ is a basis of Z.

Similarly, let $v'_1, \dots, v'_{d-2}$ be a basis of $X \cap Y$ such that $v'_i, \dots, v'_{d-2}$ is a basis of $X \cap Y \cap Z'$. Let $u'_1 \in X$ and $w'_1 \in Y$ be such that $u'_1, v'_i, \dots, v'_{d-2}$ is a basis of $X \cap Z'$ and such that $w'_1, v'_i, \dots, v'_{d-2}$ is a basis of $Y \cap Z'$. Let $u'_2 \in X$ and $w'_2 \in Y$ be such that $u'_1, u'_2, v'_1, \dots, v'_{d-2}$ is a basis of X and such that $w'_1, w'_2, v'_1, \dots, v'_{d-2}$ is a basis of Y. Finally, let $z'_1, \dots, z'_{i-1} \in Z'$ be such that $u'_1, w'_1, z'_1, \dots, z'_{i-1}, v'_i, \dots, v'_{d-2}$ is a basis of Z'.

Applying Lemma 2.3 (with $U = \langle u_1, w_1, v_1, \dots, v_{d-2}\rangle$, $U_1 = \langle u'_1, w'_1, v'_1, \dots, v'_{d-2}\rangle$, $W = Z$ and $W_1 = Z'$) we find that there exists an isometry σ such that $\sigma(u_1) = u'_1$, $\sigma(w_1) = w'_1$, $\sigma(v_j) = v'_j$ $(1 \le j \le d - 2)$, and $\sigma(z_j) \in \langle z'_1, \dots, z'_{i-1}\rangle$ for $1 \le j \le i - 1$. Clearly, σ maps Z to Z'. It remains to show that σ preserves X and Y. Note that $W = \langle X \cap Y, u_1\rangle$ is a $(d-1)$-dimensional isotropic subspace of V. By Lemma 2.4, the only two maximal isotropic subspaces containing W are X and $\langle W, w_1\rangle$. Note that σ maps W to $W' = \langle X \cap Y, u'_1\rangle$, which is a $(d - 1)$-dimensional isotropic subspace of V. The only two maximal isotropic subspaces containing W' are X and $\langle W', w'_1\rangle$. Since σ maps $\langle W, w_1\rangle$ to $\langle W', w'_1\rangle$, it must map X to X. Similarly we show that σ maps Y to Y. Therefore $\sigma \in G_X \cap G_Y$ and the proof is completed. □

Theorem 4.8. *With reference to Notation 1.1 assume that $\partial(X, Y) = 2$. Let $Z, Z' \in D_i^i$ $(2 \le i \le d)$ and assume Z, Z' are of positive type. Then there exist $\sigma \in G_X \cap G_Y$ which maps Z to Z'.*

Proof. Let $v_1, \dots, v_{d-2}$ be a basis of $X \cap Y$ such that $v_{i-1}, \dots, v_{d-2}$ is a basis of $X \cap Y \cap Z$. Let $u_1, u_2 \in X$ and $w_1, w_2 \in Y$ be such that $u_1, u_2, v_1, \dots, v_{d-2}$ is a basis of X and $w_1, w_2, v_1, \dots, v_{d-2}$ is a basis of Y. Without loss of generality we can assume that $f(u_1, w_1) = 0$ (otherwise we replace w_1 by $w_1 + \lambda w_2$ for an appropriate $\lambda \in GF(q)$). Note that $\langle X \cap Y, u_1, w_1\rangle \in D_1^1$. Since Z is of positive type we have $\dim(\langle X \cap Y, u_1, w_1\rangle \cap Z) = d - i + 1$. Therefore, there exist $\alpha, \beta \in GF(q)$ and $v \in X \cap Y$ such that $\langle X \cap Y, u_1, w_1\rangle \cap$

$Z = \langle \alpha u_1 + \beta w_1 + v, v_{i-1}, \ldots, v_{d-2} \rangle$. Since $\dim(X \cap Z) = \dim(Y \cap Z) = d - i$, we have $\alpha \neq 0$ and $\beta \neq 0$. Without loss of generality we can therefore assume that $\langle X \cap Y, u_1, w_1 \rangle \cap Z = \langle u_1 + w_1, v_{i-1}, \ldots, v_{d-2} \rangle$ (otherwise we replace u_1 by $\alpha u_1 + v$ and w_1 by βw_1). Finally, let $z_1, \ldots, z_{i-1} \in Z$ be such that $z_1, \ldots, z_{i-1}, u_1 + w_1, v_{i-1}, \ldots, v_{d-2}$ is a basis of Z.

Similarly, Let $v'_1, \ldots, v'_{d-2}$ be a basis of $X \cap Y$ such that $v'_{i-1}, \ldots, v'_{d-2}$ is a basis of $X \cap Y \cap Z'$. Let $u'_1, u'_2 \in X$ and $w'_1, w'_2 \in Y$ be such that $u'_1, u'_2, v'_1, \ldots, v'_{d-2}$ is a basis of X and $w'_1, w'_2, v'_1, \ldots, v'_{d-2}$ is a basis of Y. Without loss of generality we can assume that $f(u'_1, w'_1) = 0$ and that $\langle X \cap Y, u'_1, w'_1 \rangle \cap Z' = \langle u'_1 + w'_1, v'_{i-1}, \ldots, v'_{d-2} \rangle$. Let $z'_1, \ldots, z'_{i-1} \in Z'$ be such that $z'_1, \ldots, z'_{i-1}, u'_1 + w'_1, v'_{i-1}, \ldots, v'_{d-2}$ is a basis of Z'.

Applying Lemma 2.3 (with $U = \langle u_1, u_1 + w_1, v_1, \ldots, v_{d-2} \rangle, W = Z, U_1 = \langle u'_1, u'_1 + w'_1, v'_1, \ldots, v'_{d-2} \rangle$ and $W_1 = Z'$) we find that there exists an isometry σ of V such that $\sigma(u_1) = u'_1$, $\sigma(u_1 + w_1) = u'_1 + w'_1$ (and therefore also $\sigma(w_1) = w'_1$), $\sigma(v_j) = v'_j$ $(1 \leq j \leq d-2)$, and $\sigma(z_j) \in \langle z'_1, \ldots, z'_{i-1} \rangle$ for $1 \leq j \leq i-1$. Clearly, σ maps Z to Z'. It remains to show σ preserves X and Y.

Note that $W = \langle X \cap Y, u_1 \rangle$ is a $(d-1)$-dimensional isotropic subspace of V. By Lemma 2.4, the only two maximal isotropic subspaces containing W are X and $\langle W, w_1 \rangle$. Note that σ maps W to $W' = \langle X \cap Y, u'_1 \rangle$, which is a $(d-1)$-dimensional isotropic subspace of V. The only two maximal isotropic subspaces containing W' are X and $\langle W', w'_1 \rangle$. Since σ maps $\langle W, w_1 \rangle$ to $\langle W', w'_1 \rangle$, it must map X to X. Similarly we show that σ maps Y to Y. Therefore $\sigma \in G_X \cap G_Y$ and the proof is complete. □

Proposition 4.9. *With reference to Notation 1.1 assume that $\partial(X, Y) = 2$. Then the following* (i)–(iii) *hold.*

(i) *Each of D^1_1, D^d_d is an orbit of $G_X \cap G_Y$.*

(ii) *For $2 \leq i \leq d$ the sets D^i_{i-2} and D^{i-2}_i are orbits of $G_X \cap G_Y$.*

(iii) *For $2 \leq i \leq d-1$ the set of vertices in D^i_i that are of positive type (resp. negative type) is an orbit of $G_X \cap G_Y$.*

Proof. Observe that two vertices of $D_d(q)$, which are contained in distinct sets listed in (i), (ii) and (iii) above, cannot belong to the same orbit of $G_X \cap G_Y$. The result now follows from Theorems 4.3, 4.7 and 4.8. □

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