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A NOTE ON GENERALIZED (m, n)-JORDAN CENTRALIZERS

Abstract. The aim of this paper is to define generalized (m, n)-Jordan centralizers and to prove that on a prime ring with nonzero center and $char(R) \neq 6mn(m+n)(m+2n)$ every generalized (m, n)-Jordan centralizer is a two-sided centralizer.

Throughout, R will represent an associative ring with a center $Z(R)$. Let $n \geq 2$ be an integer. A ring R is said to be n-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that R is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. A ring R is called semiprime if $aRa = \{0\}$ implies $a = 0$. An additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$ and is called a Jordan derivation if $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. One can easily prove that every derivation is a Jordan derivation, but converse is in general not true. A classical result due to Herstein [5, Theorem 3.3] asserts that a Jordan derivation on a prime ring of characteristic different from two is a derivation. A brief proof of Herstein's result can be found in [3]. This result was extended to 2-torsion free semiprime rings by Cusack [4] (see [2] for an alternative proof).

An additive mapping $T : R \to R$ is called a left (right) centralizer if $T(xy) = T(x)y$ ($T(xy) = xT(y)$) holds for all pairs $x, y \in R$. If R has the identity element, then $T : R \to R$ is a left centralizer if and only if T is of the form $T(x) = ax$ for all $x \in R$, where $a \in R$ is a fixed element. For a semiprime ring R, left centralizers are of the form $T(x) = qx$ for all $x \in R$, where q is a fixed element of a Martindale right ring of quotients Q_r (see, for example, Chapter 2 in [1]). An additive mapping $T : R \to R$ is called a left (right) Jordan centralizer if $T(x^2) = T(x)x$ ($T(x^2) = xT(x)$) holds for all $x \in R$. We call an additive mapping $T : R \to R$ a two-sided centralizer

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(a two-sided Jordan centralizer) if T is both a left and a right centralizer (a left and a right Jordan centralizer). If R is a semiprime ring with an extended centroid C and $T : R \to R$ is a two-sided centralizer, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$ for all $x \in R$ (see Theorem 2.3.2 in [1]). In [9], Zalar has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer.

In [8], Vukman defined (m, n)-Jordan centralizers in the following way.

Definition 1. Let $m \geq 0$, $n \geq 0$ with $m + n \neq 0$ be some fixed integers and R an arbitrary ring. An additive mapping $T : R \to R$ is called a (m, n)-Jordan centralizer if

$$(m + n)T(x^2) = mT(x)x + nxT(x) \tag{1}$$

holds for all $x \in R$.

Obviously, a $(1, 0)$-Jordan centralizer is a left Jordan centralizer. Similarly, a $(0, 1)$-Jordan centralizer is a right Jordan centralizer. In the case when $m = n = 1$ we have the relation

$$2T(x^2) = T(x)x + xT(x), \qquad x \in R.$$

Vukman [6] has proved that every additive mapping $T : R \to R$, where R is a 2-torsion free semiprime ring, satisfying the relation above, is a two-sided centralizer.

Motivated by these results, we introduce the following definition.

Definition 2. Let $m \geq 0$, $n \geq 0$ with $m + n \neq 0$ be some fixed integers and R an arbitrary ring. An additive mapping $T : R \to R$ is called a generalized (m, n)-Jordan centralizer if there exists an (m, n)-Jordan centralizer $T_0 : R \to R$ such that

$$(m + n)T(x^2) = mT(x)x + nxT_0(x) \tag{2}$$

holds for all $x \in R$.

Similar as above, a generalized $(1, 0)$-Jordan centralizer is a left Jordan centralizer.

In [8], Vukman proved that on a prime ring with a nonzero center $Z(R)$ and $char(R) \neq 6mn(m + n)$ every (m, n)-Jordan centralizer is a two-sided centralizer. The natural question here is whether an analogue holds true for generalized (m, n)-Jordan centralizers. Theorem 1 answers this question in the affirmative.

Theorem 1. *Let $m \geq 1$, $n \geq 1$ be some fixed integers, let R be a prime ring with $char(R) \neq 6mn(m + n)(m + 2n)$, and let $T : R \to R$ be a generalized (m, n)-Jordan centralizer. If $Z(R)$ is nonzero, then T is a two-sided centralizer.*

In the proof of Theorem 1 we will need the next lemma.

Lemma 1. *Let $m \geq 0$, $n \geq 0$ with $m+n \neq 0$ be some fixed integers, let R be a ring, and let $T : R \to R$ be a generalized (m,n)-Jordan centralizer. Then*

$$(3)\qquad \begin{aligned}2(m+n)^2T(xyx) &= mnT(x)xy + m(2m+n)T(x)yx - mnT(y)x^2\\ &\quad + 2mnxT_0(y)x - mnx^2T_0(y) + n(m+2n)xyT_0(x) + mnyxT_0(x)\end{aligned}$$

for all $x, y \in R$.

Proof. If we linearize the relation (2), we get

$$(4)\qquad (m+n)T(xy+yx) = mT(x)y + mT(y)x + nxT_0(y) + nyT_0(x)$$

for all $x, y \in R$. Similarly, if we linearize the relation (1), we get

$$(5)\qquad (m+n)T_0(xy+yx) = mT_0(x)y + mT_0(y)x + nxT_0(y) + nyT_0(x)$$

for all $x, y \in R$.

Now, if we put $(m+n)(xy+yx)$ instead of y in the relation (4), we get

$$\begin{aligned}(m+n)^2T(x^2y+yx^2+2xyx) &= m(m+n)T(x)(xy+yx) + m(m+n)T(xy+yx)x\\ &\quad + n(m+n)xT_0(xy+yx) + n(m+n)(xy+yx)T_0(x)\end{aligned}$$

for all $x, y \in R$. Applying the relation (4) and the relation (5) we obtain

$$\begin{aligned}2(m+n)^2T(xyx) &+ m(m+n)T(x^2)y + m(m+n)T(y)x^2\\ &+ n(m+n)x^2T_0(y) + n(m+n)yT_0(x^2)\\ &= m(m+n)T(x)(xy+yx)\\ &\quad + m(mT(x)y + mT(y)x + nxT_0(y) + nyT_0(x))x\\ &\quad + nx(mT_0(x)y + mT_0(y)x + nxT_0(y) + nyT_0(x))\\ &\quad + n(m+n)(xy+yx)T_0(x)\end{aligned}$$

for all $x, y \in R$. Using the relations (2) and (1) we get

$$\begin{aligned}2(m+n)^2T(xyx) &+ m(mT(x)x + nxT_0(x))y + m(m+n)T(y)x^2\\ &+ n(m+n)x^2T_0(y) + ny(mT_0(x)x + nxT_0(x))\\ = m(m+n)T(x)(xy+yx) &+ m(mT(x)y + mT(y)x + nxT_0(y) + nyT_0(x))x\\ + nx(mT_0(x)y + mT_0(y)x &+ nxT_0(y) + nyT_0(x)) + n(m+n)(xy+yx)T_0(x)\end{aligned}$$

for all $x, y \in R$. Collecting the terms we arrive at

$$\begin{aligned}2(m+n)^2T(xyx) &= mnT(x)xy + m(2m+n)T(x)yx - mnT(y)x^2\\ &\quad + 2mnxT_0(y)x - mnx^2T_0(y) + n(m+2n)xyT_0(x) + mnyxT_0(x)\end{aligned}$$

for all $x, y \in R$. This completes the proof. ∎

Proof of Theorem 1. If we put $(m+n)^2x^2$ for x in (2), we get

$$\begin{aligned}(m+n)^3T(x^4) &= m(m+n)^2T(x^2)x^2 + n(m+n)^2x^2T_0(x^2)\\ &= m(m+n)(mT(x)x + nxT_0(x))x^2 + n(m+n)x^2(mT_0(x)x + nxT_0(x))\\ &= m^2(m+n)T(x)x^3 + mn(m+n)xT_0(x)x^2 + mn(m+n)x^2T_0(x)x\\ &\quad + n^2(m+n)x^3T_0(x).\end{aligned}$$

We have therefore

$$(6)\qquad (m+n)^3T(x^4) = m^2(m+n)T(x)x^3 + mn(m+n)xT_0(x)x^2 + mn(m+n)x^2T_0(x)x + n^2(m+n)x^3T_0(x)$$

for every $x \in R$. On the other hand, if we put $y = (m+n)x^2$ in the relation (3), we get

$$\begin{aligned}2(m+n)^3T(x^4) &= mn(m+n)T(x)x^3 + m(2m+n)(m+n)T(x)x^3\\ &\quad - mn(m+n)T(x^2)x^2 + 2mn(m+n)xT_0(x^2)x - mn(m+n)x^2T_0(x^2)\\ &\quad + n(m+2n)(m+n)x^3T_0(x) + mn(m+n)x^3T_0(x)\\ &= 2m(m+n)^2T(x)x^3 - mn(mT(x)x + nxT_0(x))x^2\\ &\quad + 2mnx(mT_0(x)x + nxT_0(x))x - mnx^2(mT_0(x)x + nxT_0(x))\\ &\quad + 2n(m+n)^2x^3T_0(x)\\ &= (2m(m+n)^2 - m^2n)T(x)x^3 + mn(2m-n)xT_0(x)x^2\\ &\quad + mn(2n-m)x^2T_0(x)x + (2n(m+n)^2 - mn^2)x^3T_0(x).\end{aligned}$$

We have therefore

$$(7)\qquad 2(m+n)^3T(x^4) = (2m(m+n)^2 - m^2n)T(x)x^3 + mn(2m-n)xT_0(x)x^2 + mn(2n-m)x^2T_0(x)x + (2n(m+n)^2 - mn^2)x^3T_0(x)$$

for every $x \in R$. By comparing (6) and (7) we get

$$mn(m+2n)T(x)x^3 - 3mn^2xT_0(x)x^2 - 3m^2nx^2T_0(x)x + mn(2m+n)x^3T_0(x) = 0$$

for every $x \in R$. The above equality reduces according to the requirements of the theorem to

$$(m+2n)T(x)x^3 - 3nxT_0(x)x^2 - 3mx^2T_0(x)x + (2m+n)x^3T_0(x) = 0.$$

Since T_0 is commuting on R (see the proof of Theorem 2 in [8]), i.e.,

$$[T_0(x), x] = T_0(x)x - xT_0(x) = 0$$

for all $x \in R$, we have

$$(m+2n)T(x)x^3 - (m+2n)T_0(x)x^3 = 0.$$

This yields that

$$(T(x) - T_0(x))x^3 = 0 \tag{8}$$

for all $x \in R$.

Let $F : R \to R$ be an additive mapping defined by $F(x) = T(x) - T_0(x)$, $x \in R$. We would like to show that $F(x) = 0$ for all $x \in R$. Namely, if $F(x) = T(x) - T_0(x) = 0$, then $T(x) = T_0(x)$ for all $x \in R$, which yields that T is a two-sided centralizer, since T_0 is a two-sided centralizer by [8, Theorem 2].

We already know that $F(x)x^3 = 0$ for all $x \in R$. Using full linearization of this relation one obtains

$$\sum_{\pi \in S_4} F(x_{\pi(1)})x_{\pi(2)}x_{\pi(3)}x_{\pi(4)} = 0 \tag{9}$$

for all $x_1, x_2, x_3, x_4 \in R$. Let c be a nonzero central element. Pick any $x \in R$ and set $x_1 = x_2 = x_3 = c$ and $x_4 = x$ in (9). We arrive at

$$(\alpha F(c)x + \beta F(x)c)c^2 = 0, \tag{10}$$

where $\alpha = 18$ and $\beta = 6$. Since R is prime, it follows that $\alpha F(c)x + \beta F(x)c = 0$ for all $x \in R$. In particular, $\alpha F(c)c + \beta F(c)c = 0$, which yields that $F(c) = 0$. Therefore from (10), we get $F(x) = 0$ for all $x \in R$. ∎

The above observations lead to the following conjecture.

Conjecture 1. Let $m \geq 1$, $n \geq 1$ be some fixed integers, let R be a semiprime ring with suitable torsion restrictions, and let $T : R \to R$ be a generalized (m,n)-Jordan centralizer. Then T is a two-sided centralizer.

At the end, let us also point out, that we do not know yet whether this conjecture is true even for (m,n)-Jordan centralizers.

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