Conversions Between Trigonometric Representation Systems by Pre-service Secondary School Teachers

Enrique Martín-Fernández, Luis Rico, and Juan F. Ruiz-Hidalgo

Understanding trigonometry relational system is a school mathematics demanding topic. The angle, the unit circle and the trigonometric functions are its foundational notions. Trigonometric contents meaning and their understanding involve these three concepts and their relationships. This research aims to deepen in the pre-service teachers' understanding about the angle, the unit circle and the trigonometric function when converting notions between two trigonometric representation systems based on the unit circle and the trigonometric functions. The results indicate that pre-service mathematics teachers' present a lack of connections between the goniometric and the analytical representation systems.

Keywords: Conversions Between Representations; Relational System School Mathematical Content Meaning; Trigonometric Contents Understanding Modes

Conversiones entre sistemas de representación trigonométricos por profesores de secundaria en formación inicial

Comprender el sistema relacional de trigonometría es un tópico exigente en las matemáticas escolares. El ángulo, la circunferencia unidad y las funciones trigonométricas son sus nociones fundamentales. El significado de los contenidos trigonométricos y su comprensión involucran estos tres conceptos y sus relaciones. Proponemos como objetivo profundizar en la comprensión de los profesores en formación sobre el ángulo, la circunferencia goniométrica y la función trigonométrica al convertir nociones entre dos sistemas de representación trigonométrica basados en el círculo unitario y las funciones trigonométricas. Los resultados indican que los profesores en formación presentan una carencia de conexiones entre los sistemas de representación analítico y goniométrico.

Términos clave: Conversiones entre representaciones; Modos de comprensión de contenidos trigonométricos; Significado del contenido matemático escolar; Sistema relacional

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This research report describes the understanding modes that an empirically available group of high school pre-service teachers express about the angle concept and its cosine, when considering the links between two trigonometric representation systems.

Understanding a mathematical content in depth implies developing its concepts within a mathematical system, using its structure and performing its procedures in a coherent way; in other words, being able to manage it within an existing mathematics relational system with meaning (Martín-Fernández et al., 2019, p. 858; Skemp, 1987, p. 56).

Recent studies on mathematics education have shown that the wealth and diversity of its cognitive contents and the semantic complexity of their meanings help delve deeper into the design of teaching tasks, and into the pupils' understanding of school mathematical concepts, procedures, and their blends (Thompson, 2016; Rico, 2018). In fact, "Mathematical knowledge that matters most for teachers resides in the meaning of (its) mathematical content" (Thompson, 2016, p. 437). Moreover, if meaning is the key for the organization of content (Kumar, 2017, p. 559), consequently it should be the pillar for the mathematics relational systems students' learning (Castro-Rodriguez et al., 2016; Thompson, 2016, p. 461).

We have adapted an existing content semantic framework thought to design school tasks, to look into and to interpret mathematics concepts' meanings, which consists of three categories: structural including the concepts, properties, propositions, and relationships between concepts involved in a mathematical topic, representational –talking about several modes of expressing and symbolizing numerical structures by means of some specific signs, rules and statements (Rico, 2009)–, and contextual –referring to those phenomena, situations, terms that give sense to a mathematical concept– (Bunge, 2008, p. 24-25). This content semantic framework has also been conceived as dimension of a curricular system (Martín-Fernández et al., 2016, p. 55; Rico, 2018).

A way of broadening the understanding of concepts and procedures includes using and blending different systems of representation in solving problems to convert and process one representation into a different one (Camacho & Depool, 2003, p. 2; Even, 1990, p. 105; Skemp, 1987, pp. 55-56; Rico, 2009). In fact, as Kaput (1992) states, "all aspects of a complex idea cannot be adequately represented by a single notation system, and hence require multiple systems for their full expression, meaning that multiple" (p. 530). Similarly, Duval (1993) claims the need for various systems linked to the same mathematical content. This plurality leads to consider the relationships between different systems for the same mathematical content.

Given that converting representations plays a crucial role to improve understanding, we concentrate on the representational semantic category in this paper. To deepen this subject, we have selected as our prototype a teachers' training setting, exemplified by a secondary school relevant mathematics relational system –trigonometry –, and, more specifically, by the angle cosine concept.

Following the same line of focusing on the conversions among relational systems, the purpose of the present study is to deepen the understanding of pre-service teachers in a situation of conversions between two trigonometric representation systems based on the unit circle and the trigonometric functions, respectively.

In order to achieve this purpose, we describe notions by means of semantic categories, which at the same time are classified by themes, according to specific contents and components, whose meanings characterize them (Reinhardt & Soeder, 1984, p. 37; Rico et al., 2020). Concretely, we claim that the trigonometry relational system adopted by Freudenthal (1983) is the mathematical content system under our study (RAC, 1996). This system is basically composed of the three selected angle concepts, its corresponding meanings, and the established links among them (Table 1). As we will see, our work is focused on conversions between these concepts' representations.

For data collection we designed a semantic questionnaire, from which we selected and highlighted a task as a catalyst item. The proposed situation implies converting a graphically given angle, its measure and the value of its cosine, when moving between two partial trigonometry systems. The task also involves changes in notations and in their relations, in the utilized contents and in their sense, provoking changes of meaning. Concretely, as is shown in Task 3, we intend to study and interpret the conversions between two trigonometric representation systems.

The results show that these prospective teachers have found difficulties converting trigonometric notions and they have not developed enough ability to move between partial trigonometry systems. Consequently, this paper deepens on an unanswered question of the literature on the relation of the construction of the cosine function with is graphical representation (Martinez-Planell & Cruz Delgado, 2016).

THEORETICAL FRAMEWORK

Reviewing the mathematics education literature on conversions reveals the existence of only a small number of studies that focus on the ability of converting between trigonometric representation systems or that emphasize its importance (e.g. Brown, 2005; Steckroth 2007; Challenger, 2009; Chin, 2013; Çekmez, 2020; Demir, 2012; Marchi, 2012; Martinez-Planell & Cruz Delgado, 2016). Finally, all of them differ in their scope, in their methodological approach, and in the used reference frameworks compared with the one in the present paper.

As it has been said, our efforts to work on conversions between representation systems and to examine the understanding of pre-services teachers several framework references have been worked. We underline the following three basis: firstly, Rico's (2018, p. 306) semantic characterization for the meaning of school mathematics contents; secondly, the structured categories system with two fields useful to analyze and display cognitive contents developed by Bell et al. (1983, pp. 77-79) and by Hiebert and Lefevre (1986, pp. 1-8). Finally, we stress that the trigonometry relational system that we work on comes from the blend of the unit circle and analytical function (RAC, 1996).

About Meaning's Notion for School Mathematical Contents

Following the semantic tradition about the meaning's interpretation developed by Frege, Klein, Bunge and others, Rico (2012) postulates and sustains a didactic notion of meaning for a school mathematical content.

According to Rico (2018), the meaning of a school mathematical content implies knowing and giving its definitions, representing its relationships, establishing and processing its operations as well as giving them sense in different contexts and situations.

On the one hand, the content structures include facts, concepts, properties, propositions, and relationships involved in the mathematical topic (Martín-Fernández et al., 2016; 2019). On the other hand, the representation systems are articulated through sets of notations, signs and graphs ruled by syntactic procedures and grammatical laws needed to express and present concepts through valid outlines, useful to argue and to establish logical relationships between them. The wealth of concepts within a representation system may be revealed through its symbols equivalence, the manipulations and processing of its signs, the operations through its rules and conventions, and the coherent changes and transformations within the system. Links and properties appear among the systems of representation, and thus representations may be converted and utilized for proving and solving. Furthermore, "depending of the representation used for a concept, different meanings may be identified" (Martín-Fernández et al., 2019).). In fact, Morgan and Kynigos (2014, pp. 359-360) consider that representations and the way they are manipulated are ways of expressing meaning. In our framework, representations are also conceived as an instrument to produce meaning (Castro-Rodriguez et al., 2016, p. 131).

Finally, each representational form may embody different senses, but it also may be included within different representation systems, among which links appear as well (Skemp, 1983, pp. 47-51).

Conceptual and Procedural Contents

Following Bell et al. (1983, pp. 77-79), Hiebert and Lefevre (1986, pp. 1-8), and Rico (1997) from a cognitive point of view we consider two fields to classify mathematical contents: conceptual and procedural. Conceptual content is organized in three levels of increasing complexity: basic, middle and higher, that correspond to three different kinds of components: facts, concepts and structures (Skemp, 1987, pp. 53-55).

Procedural content corresponds to operations, properties and mathematical methods, the way of handling them as well as rules, the logical reasoning, and strategies (Hiebert & Lefevre, 1986). The procedural field is also organized in three corresponding components and increasing levels of complexity: skills, reasonings and strategies. Skills are procedures to manipulate facts; reasonings consists of logical procedures to infer between concepts or among concepts and facts, and strategies are procedures to work within and between structures.

School Relational Trigonometry System

In the second half of the 20th century several authors reviewed the definition of the trigonometry relational system to give answer to a variety of theoretical and practical problems derived from angles measure (Choquet, 1964, pp. 117-116; Dieudonnè, 1971, pp. 181-190; Freudenthal, 1973, pp. 476-494). Concretely, Freudenthal (1973) describes the instrumental way in which angles have been measured and organized, highlighting their main definitions and changes throughout the history of mathematics. As can be seen in table 1, he establishes the following subsystems within the global trigonometry relational system: elementary geometry, goniometry, and analytic geometry.

Table 1
Partial Trigonometry Systems or Subsystems According to the Angle Concept

	Elementary geometry	Goniometry	Analytic geometry
Angles sides	A non-ordered pair of half-lines	An ordered pair of half-lines	An ordered pair of lines
Type of plane	Non-oriented	Oriented	Oriented
Representation model	Right triangle	Oriented unit circle	Analytic function
Module	Determined between 0º and 180º	Mod 2π	Mod π

The trigonometry relational system consists of several subsystems, each with its own notation and rules of representation. Trigonometry relational system combines these rules and notations and share them. Such combined notations are considered and used as a representation system for the trigonometry relational system as a whole. Given a concept, some of the afore mentioned studies allow to interpret its conversion between trigonometric representation systems as change in its meaning. The moving between partial trigonometry systems is structured by the organization of the contents, and it can be used as methodological tool to analyse meanings and their related understanding.

The complexity of trigonometry relational system, with its wealth of contents, which may convey several senses such as instrument, ratio, distance, coordinate, value, … (Brown, 2003; Martín-Fernández et al., 2019), and the variety of notations and rules, make this global system an ideal choice to approach this research about conversions between representation systems together with their relationships (Reihardt & Soeder, 1984, p. 37). However, there are several possible movements between the partial systems aforementioned that involves a variety of concepts and relationships. In order not to enlarge our research report excessively, we chose the cosine and focus on the conversions from the goniometric representation system to the analytical representation system.

Research Questions and Goals

Under the global school trigonometry relational system, theoretical framework based on the blending of goniometric contents, angle meanings, and their relationships, we have worked with a seventy-two Pre-service Secondary Teachers –the participants– on movings between two partial trigonometry systems, proposed to them as part of the global trigonometry relational system.

This approach is expressed by the following research questions.

- How do the participants represent an angle, its cosine in the goniometric representation system?
- How do they give meaning to the angle and its cosine in the goniometric representation system?
- What contents do participants utilize to represent a point P as an angle?
- How do they convert the angle from the unit circle representation system to the analytical representation system?
- How do participants represent the cosine of that angle in the analytical representation system?

In order to respond these questions, this study has established the following general aims: to identifying meanings of the concept angle and its cosine and describe the conceptual and procedural content and how they are managed by Pre-service Mathematics Teachers when converting notions between two trigonometric representation systems.

METHODOLOGY

Participants and Settings

The participants that were selected for this study were seventy-two graduate students following a Pre-service Secondary Mathematics Teachers training program at a large Spanish public university. The program has four modules: a generic educational module, a specific module (in which students take courses from the Department of Mathematics and from the Department of Mathematics Education), an elective module, and another module, which comprises a practicum, together with a final project (MEC, 2007). Prior to this empirical work, each participant had completed three general and compulsory courses from the educational module. The research was conducted during the specific module. None of the participants had any previous teaching experience in a school. While 53% of the sample possessed a bachelor's degree in mathematics, the remaining possessed others bachelor's degrees such as: civil engineering, architecture, physic, electrical engineering, chemical engineering, and statistics. We highlight the variety of bachelor's degrees within our sample. The participants had developed notions about mathematics as students in high school, and in college mathematics courses. Thus, the instructional experiences of the participants were multiple and varied previously to the study.

The reason why we work with pre-service teachers is partially strategic, given that previously we had studied the meaning of the sine and cosine of an angle in a group of secondary school students (Martín-Fernández et al., 2016; Martín-Fernández et al., 2019), and we utilise their results in order to shed light on pre-service teachers' findings.

Data Collection Tools

We designed a semantic questionnaire associated with the sine and cosine of an angle, consisting of 10 items, which sought to gather evidence of issues such as constructions of angles associated with a value of the sine or cosine, conversions and reasoning between some of the frequent trigonometric representation systems, reasoning and identification of students' mistakes, or how pre-service teachers make sense to the sine and cosine.

This research was designed following studies of Fi (2003) and Brown (2005). We also consulted tasks used in other studies in order to design the questionnaire, and some items were taken or modified from them.

In addition, two specialists in the field of mathematics education were asked about the adequacy of the tasks proposed, their order, their presentation, and their aims, with regards to our theoretical perspective. It is highlighted that components of the three semantic categories stated by Rico (2012, pp. 51-53) can be recognized in the responses to the questionnaire.

The questionnaire was delivered and completed during an ordinary college class period of 60 minutes. The questionnaire was implemented during the winter term of the academic year 2016-2017. The tasks were presented in a booklet, where ten open-ended tasks were included, some of which comprised more than one question. Participants were told to respond the questions to the best of their ability.

In this study, we focused on analysing answers received to item 3, as shown in figure 1. It has been chosen as a catalyst given that it has the potential to elicit relationships between the configurations of two partial trigonometry systems, as required by the core of this paper. Furthermore, this item was also chosen because of its complexity and its similarity to a task closely related to participants' abilities to answer related questions trigonometry relational system, as stated by Brown (2005, p. 139).

Item 3. Point out in the trigonometric function the point that corresponds to the point P. Define the cosine of the angle corresponding to the point P.

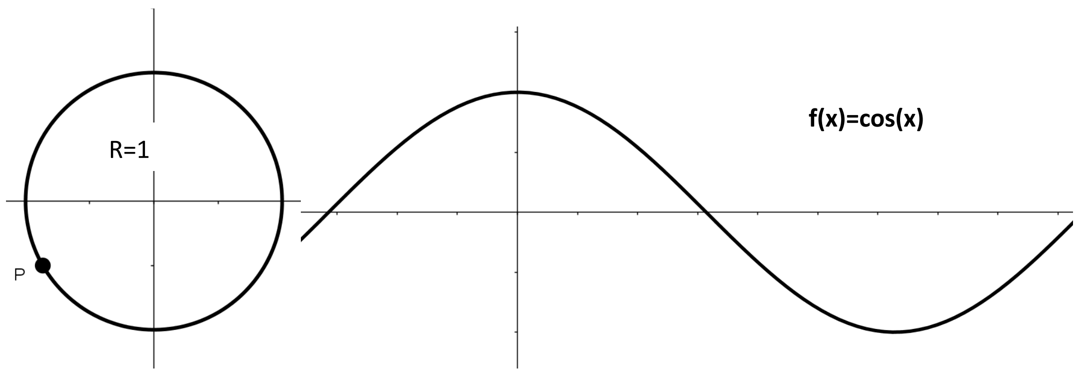


Figure 1. Analysed Question of the Questionnaire

Task Design

The analysed task of the questionnaire asks students not only to convert a point P from the unit circle to the representation of its corresponding one on the trigonometric analytic function, but also to define the cosine of the related angle to P in one or both of the involved partial trigonometry systems depending on their ability. It is obvious that the two systems are mainly linked by the concept of angle, which is the central concept in trigonometry relational system, whose representation may be made by a point in both systems. The definition and construction of the cosine help as a support to identify the representation of the angle corresponding to P, since the cosine of the angle is a coordinate of the point P in both graphical representations. Implicitly, this task also asks students to process the point P in the first partial trigonometry system. Concretely, students should process the point P into the measure of an angle, into a number associated with it, into an ordered pair of coordinates, into its cosine, etc. Then, by means of converting the angle, its measure, its cosine, etc., between representation systems, students will point out the point in the trigonometric function. Normally, the mentioned conversions consist of going from angle as measure of an arc in the goniometric circumference to angle measured as numerical value, and of going from cosine as an abscissa to cosine as y-coordinate.

The chosen situation of the task is also based on the reconceptualization of the cosine of an angle, when the domain of the angle is expanded from the interval $[0,2\pi)$ to the real numbers field and its meaning changes. A field extension occurs in mathematics when a partial system is generalized and included into a broader system, as is the case of the nesting of the finite decimal numbers in the rational numbers (Feferman, 1989). A generalization is a new combination of concepts and procedures with different levels of extension, which focuses on certain essential notions that maintain their sense by their application to extended situations. Some of the notations and rules of the previous worked representation system are maintained, some are generalized to fit the enlarged concepts, but others are not. In other words, the trigonometric representation systems are not bijective, emphasizing different properties and highlighting uniqueness by means of specific signs, which increases the difficulty when converting notions between them (Lakoff & Nunes, 2000). Consequently, learners cope with the changes of meaning caused by conversions (Chin, 2013, p. 44; Skemp, 1987, pp. 40-41).

The chosen partial trigonometry systems have been selected because the unit circle and the trigonometric function are essential contents for representing angles, and the values of their trigonometric lines, whose meanings are required as a prior basis for their understanding, and for solving trigonometry relational system-related tasks (Koyunkaya, 2016, p. 1471). Furthermore, they are usually introduced in classroom following that order (Demir, 2012, p. 1).

Data Analysis

The respondents' answers to the items were analysed qualitatively. For this study, the content analysis approach was utilized in order to establish patterns, procedures associated to the tasks and explanations of participants' answers through analysis of data (Cohen et al., 2011, p. 563; Krippendorff, 1990).

First, we coded each data source by participant and by the bachelor's degree held by each of them. After that, we examined the task proposed, and identified units of content in the responses. Then, we defined criteria to organize the variety of knowledge associated with each response and coded the data. After comparing our system of coding, it became clear that some of the criteria needed to be reviewed and refined. After making these revisions, we arranged our system of organization by categorizing the criteria into themes by means of their contents and components, which are identified in annex 1.

RESULTS: CRITERIA AND CLASSIFICATIONS

We present the results in two sections. The sections correspond to the contents used as criteria to classify the types of participants' answers in the goniometric and analytical representation systems respectively.

We distinguish six different content patterns chosen as criteria to classify the responses, three in each one of the partial trigonometry systems employed. These are: (1) identification of the angle corresponding to the point *P* in the unit circle, (2) strategies to build the cosine of the angle corresponding to *P*, and (3) attributed sense to the cosine corresponding to *P* in the unit circle; (4) identification of the angle corresponding to *P* in the analytical function, (5) strategies to represent *P* in the graph of the analytical function, and (6) attributed sense to the cosine corresponding to *P* in the analytical function.

We describe each of these six criteria in detail, categorizing into themes. In order to achieve this, we analysed the involved conceptual and procedural knowledge in the responses taking into account the facts, skills, concepts, reasoning, and strategies used by participants in their answers (see Annex 1). We choose and display some examples of individual pre-service teachers works that are very revealing of how they think.

First Section. Criteria for the Goniometric Representation System

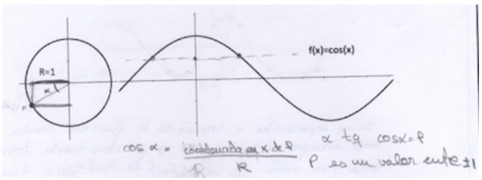
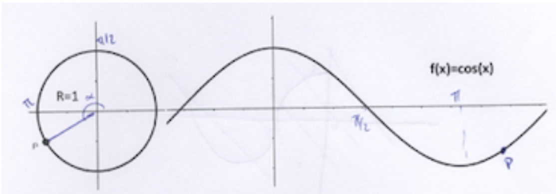
Related to the goniometric circumference, we recognize the criteria (1) to (3).

Criterion One: Identification of the Angle Corresponding to the Point P in the Unit Circle

The themes emerging from the responses can be described in terms of two different angle concepts: absolute geometric angle, and oriented goniometric angle, both didactically important according to Freudenthal (1973, p. 488-489). On the one hand, the absolute angle type corresponds to "elementary static angle" in the non-oriented plane, determined between 0° and 180°. On the other hand, the oriented angle type is considered as a dynamic version of angle, determined by an ordered pair of half-lines in the oriented plane or by means of a rotation higher than π (Clements & Burns, 2000, p. 31; Freudenthal, 1973, pp. 488-489; Hilbert, 1991; Russell, 1912, pp. 723-725). Therefore, the classification of the angle corresponding to P is based on the angle measure, and on the orientation of its pair of sides. Concretely, the theme "oriented goniometric angle" appears when the participants draw the angle in the third or first quadrant using sides, with origin in the positive x-axis or y-axis (only one participant), orientating it either clockwise (for positive values) or counter clockwise (for negative values, -a minority); when participants only label the unit circle so that the point P is included in the third quadrant, and finally if its measure is estimated and expressed by sexagesimals values higher than 180º. The theme "absolute geometric angle" is mainly related to angles whose measure is lower than 180º. Moreover, 15.27% of the participants do not identify the angle corresponding to the point *P*. Finally, one participant interprets *P* as an oriented angle and an absolute angle, and a small percentage of the participants represent two or more angles in the unit circle. Table 2 shows different types of responses under these themes, selected among those obtained.

Table 2

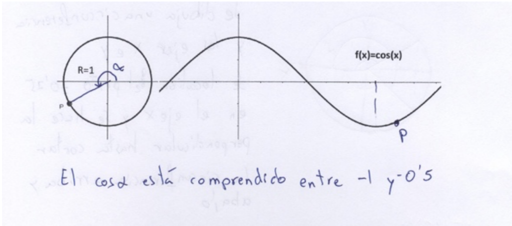
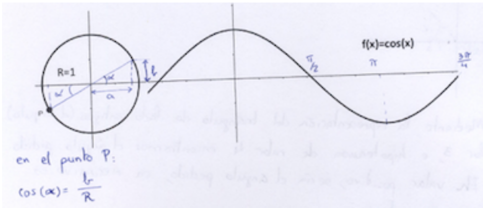
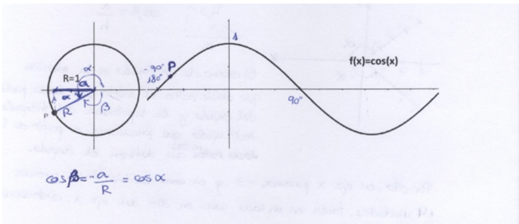
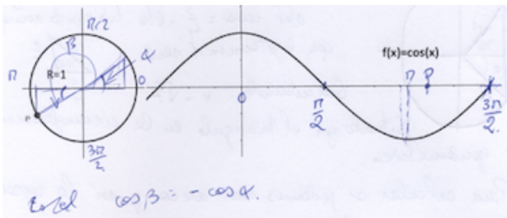
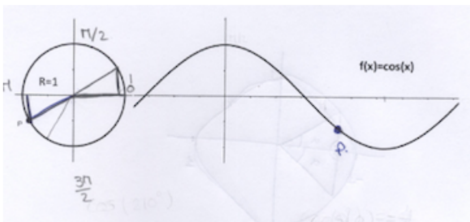
Percentage of Types of Answers for Criterion One

Themes	Examples	Percentage N=72
Absolute geometric angle		37.50%
Oriented goniometric angle		45.83%

Criterion Two: Strategies to Build the Cosine of the Angle Corresponding to P

The emerged strategy themes are estimation, goniometric, metric-goniometric, metric, no apparent, and not built. Estimation strategy is detected when the contents utilized by participants are either giving a numerical value or limiting the value of the cosine within the confines of an interval without explanation. Metric strategy is developed when subjects calculate a ratio in a right-angle triangle included in the unit circle, but without expressing the negative value of the cosine, or when they only project the point *P* and define the cosine as a distance. Metric-goniometric strategy is considered when participants project, draw on some features of the unit circle by which they consider the negative value of the cosine, and either use it when calculating metric relationships or when estimating the value of the cosine of the angle. Goniometric strategy is used when subjects utilize relations between oriented angles or when they compare values of the cosine for certain angles in the unit circle. Moreover, there are some answers, which do not show exactly what the participants want to express; such responses are classified as "no apparent strategy". Finally, there are answers that do not build the cosine that we labelled "not built".

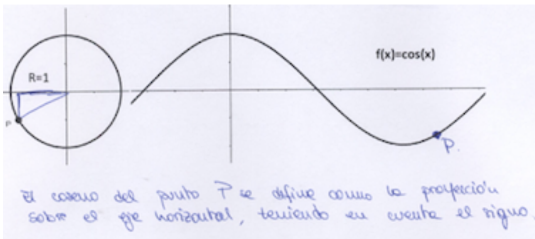
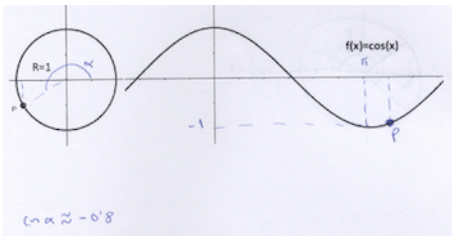
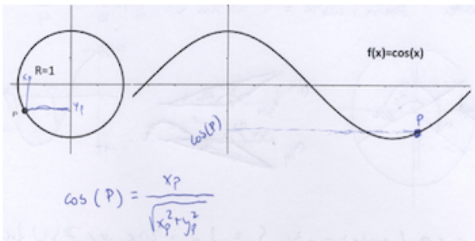
Table 3
Percentage of Types of Responses for Criterion Two

Themes	Examples	Percentage N=72
Estimation strategy		4.16%
Metric strategy		25%
Metric-goniometric strategy		25%
Goniometric strategy		9.72%
No apparent strategy		8.33%

Criterion Three: Attributed Sense to the Cosine Corresponding to P in the Unit Circle

The responses to the task show that the interpretation of the cosine of the angle in the unit circle was related to some of those from Brown (2005), and from Martín-Fernández et al. (2016; 2019). Concretely, the emerged themes from this criterion can be expressed as follows: length, value, ratio, and do not build (Table 4). The first theme was based on participants' reference to a length, as a cathetus, a height, a base, or a projection. Three subthemes were found: sides of the triangle with hypotenuse 1, segment on a unit circle, and cosine as a coordinate. The second theme is showed when students write a numerical value; when they limit the value of the cosine in an interval, and when participants use a property related to the unit circle to give a value. The third theme of the responses is connected to different ways of expressing a ratio such as a formula, or a relation. Finally, 36.11% of the answers do not interpret the cosine of the angle corresponding to *P* in the unit circle. It is highlighted that only a few answers reveal several senses.

Table 4
Percentage of the Types of Responses for Criterion Three

Themes	Examples	Percentages N=72
Length		23.60%
Value		20.82%
Ratio		6.94%

Second Section. Criteria for the Analytical Representation System

The trigonometry representation system associated with the analytical function is characterized by the criteria (4) to (6).

Criterion Four. Identification of the Angle Corresponding to P in the Analytical Function

After analyzing the responses, from this criterion emerged four themes: as *P*, as an angle and as *P*, as an angle, and not identified (Table 5). It is considered that when students mark the point *P* in the *x*-axis of the analytical function, they point out as *P* the value of the angle corresponding to *P*. There are some responses in which students label the *x*-axis as well (mostly in radians). Then, the value of the angle corresponding to P is indicated as an angle and as *P*. If participants label the *x*-axis and draw an auxiliary vertical line; if they mark the *x*-axis using a typical sign of an angle; and if they limit or bound somehow the value of the angle corresponding to *P*, we can state that they express the angle corresponding to *P* as an angle. Finally, it is suggested that participants do not identify the angle corresponding to *P* in the analytical function when none of the above conditions are found in the responses (68.05%).

Table 5

Percentage of Responses for Criterion Four

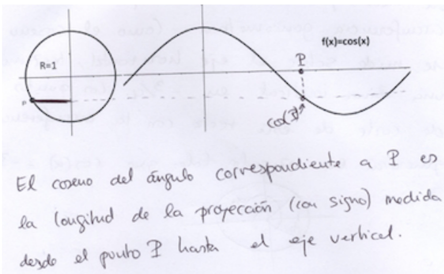
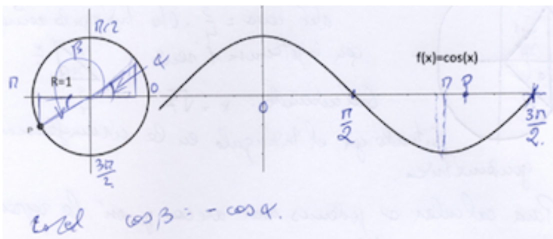
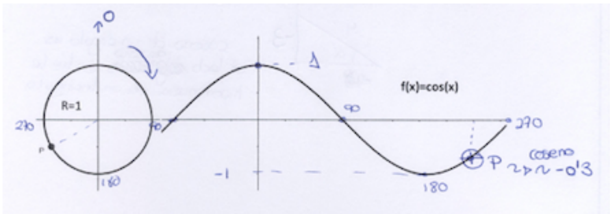
Themes	Examples	Percentage N=72
As *P*		5.55%
As *P* and as an angle		5.55%

Table 5
Percentage of Responses for Criterion Four

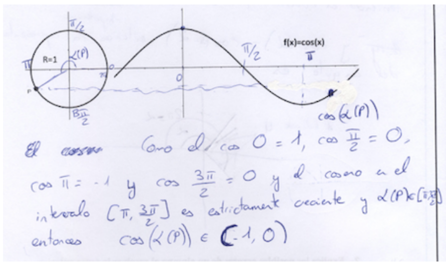
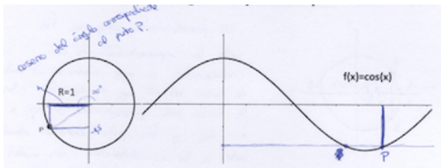
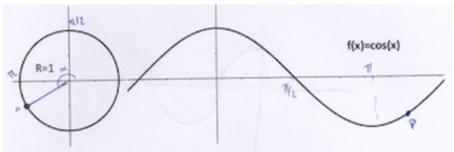
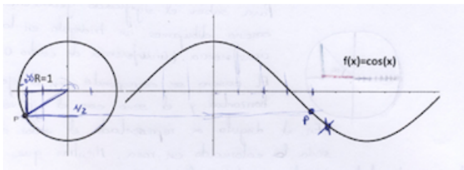
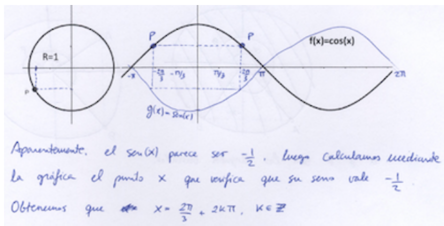
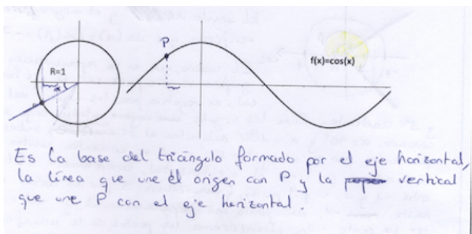
As an angle		20.82%

Criterion five. Strategies to represent P in the graph of the analytical function

In this criterion, we identify the chosen strategy to answer the task. Seven themes were identified in the responses: using the angle and the value, using the angle and the ordinate, using the angle, using the ordinate, using the value, not built, and no apparent strategy.

Using the angle and the value is a strategy based on drawing the angle related to P in the unit circle and/or projecting the point P towards the Cartesian axes. After that, respondents usually identify the value of the cosine in the unit circle. Eventually, all of them convert it to the graph of the trigonometric function taking into consideration its associated angle, determining the point P. If participants point out or express the cosine of the angle as a length and convert it to the second partial system as ordinate taking into account the associated angle, we consider that this strategy is based on the angle and on the ordinate. Using the angle is another strategy, which involves identifying the angle associated with P in the unit circle, and its subsequent use in the second partial system or conversion to the Cartesian axes of the graph of the trigonometric function. Then, based on this angle, the point P is determined with regards to the trigonometric function. Subjects utilize the strategy of using the ordinate when they perform a parallel line to the x-axis to represent a point or mark in the analytical function -they confuse in the second partial system the cosine with the sine-, and when they identify the cosine in the unit circle as a length, converting it to the graph of the trigonometric function determining the point P without expressing any information about the angle. Basing on the value means that participants identify points in the second partial system considering only the value of the sine or cosine associated to P in the first one. Furthermore, 12.5% of the responses are categorized as “not built” given that participants do not represent a point in the analytical function. Finally, the impossibility to infer how some subjects have solved the task in other productions makes us codify their responses into the theme “no apparent strategy”.

Table 6
Classification of Criterion Five

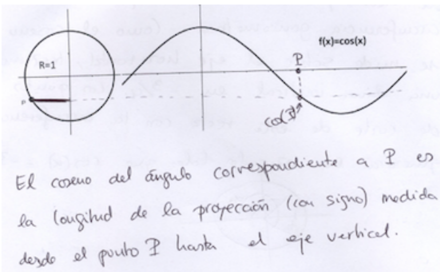
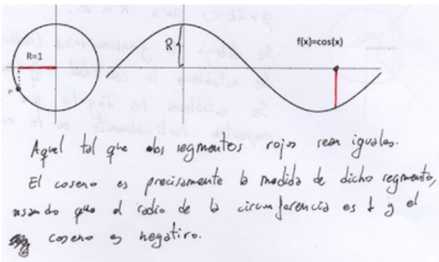
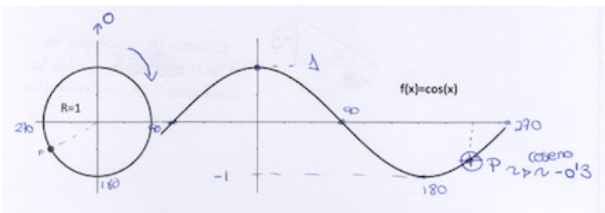
<table>
<tr><th>Themes</th><th>Examples</th><th>Percentage
N=72</th></tr>
<tr><td>Using the angle and the value</td><td>
</td><td>18.05%</td></tr>
<tr><td>Using the angle and the ordinate</td><td>
</td><td>4.16%</td></tr>
<tr><td>Using the angle</td><td>
</td><td>27.77%</td></tr>
<tr><td>Using the ordinate</td><td>
</td><td>23.61%</td></tr>
<tr><td>Based on the value</td><td>
</td><td>1.39%</td></tr>
<tr><td>No apparent strategy</td><td>
</td><td>13.88%</td></tr>
</table>

Criterion six. Attributed sense to the cosine corresponding to P in the analytical function.

Analogous to criterion three, the determination of the themes is grounded on the same principles. However, there is another theme that arises: point. It is considered that this theme appears when participants identify the cosine of the angle as a point in the analytical function. Therefore, the themes identified are the following: point, length, value, and not built (Table 7). It is remarked that 54.17% of the responses belong to the last theme.

Table 7

Classification of Criterion Six

Themes	Examples	Percentages N=72
Point		5.56%
Length		16.67%
Value		23.61%

DISCUSSION

This research report describes the understanding modes that an available group of high school pre-service teachers express about the angle concept and that of its cosine, when they are required to convert them between the goniometric and analytical representation systems. For data collection we designed a semantic questionnaire, from which we selected and highlighted a task as a catalyst item (Figure 1). The proposed task involves changes of meaning.

On the one hand, regarding understanding of oriented angle concept, the results were similar to the responses of participants in Chaar's study (2015, p. 62). Although our results come from advanced students, this concept can be considered problematic given that drawing angles in the unit circle is a skill scarcely known to pre-service teachers. In fact, only 45.83% of the participants show the ability to draw oriented angles in the unit circle (Table 2). Even though the use of absolute angle does not allow students to approximate the cosine of a given angle, determine the quadrant in which the angle is included, and graph the trigonometric function on the Cartesian plane, nearly forty percent of participants draw this type of angle. Thus, the findings of this study also aligned with Fi (2003, p. 198), Chaar (2015, p. 125), and Martínez-Planell and Cruz Delgado, (2016, p. 130), which suggest the lack of understanding of the properties of the unit circle and of the advantages of its use by adult learners. In other words, as secondary school students, the unit circle remains as a scarcely used iconic form to draw angles (Martín-Fernández et al., 2019).

On the other hand, regarding the understanding of an angle in the analytic function, as it can be seen in the Table 5, only 26.37% of the subjects consider an angle in the x-axis associated with the marked point in the analytic function. Contrary to Marchi (2012, p. 217), a great part of the participants do not identify the x-coordinate value as the measure of the angle corresponding to P in the analytic function. Furthermore, given that only 22.21% of the subjects base their responses on the angle and the ordinate, or on the angle and the value of its cosine according to Table 6, consistent with Marchi, (2012, p. 216), it is argued that there is no solid evidence that participants understand what the point P in the analytic function means $(x, \cos(x))$. In other words, it is not evident that they understand that the y-value on the graph of the analytical function is the output for the formula $y = \cos(x)$. Besides, it is not clear that participants do connect the point corresponding to P in the analytical function as a point with two coordinates.

It is our conviction that it is the consequence of the poor perception of the point P as an oriented angle in the unit circle (Table 2), of the great percentage of participants that do not build the cosine of the angle (Table 3), and of the types and scarcity of combinations of senses of the cosine of the angle in the unit circle. Finally, participants generally consider radians when they work in the second partial trigonometry system; in short, apparently the preference for using degree measure over radian measure depends not only on the appearance of π (Akkoc, 2008, p. 860; Chaar, 2015, p. 277), but also on the partial system with which learners work.

The results show that a high percentage of the participants are only making superficial connections between the two partial trigonometry systems. Firstly, aligned with Marchi (2012, p. 212), we remark that participants incorrectly recalled information and made false connections when trying to connect the graph for $y = \cos(x)$ with the unit circle. Indeed, 22.22% of the subjects confuse the graph for $y = \sin(x)$ with the graph for $y = \cos(x)$. Secondly, more than half of the participants use a strategy invalid to solve the task (Table 6), and a few students argue properly their answers. Furthermore, only a low percentage of participants correctly convert the angle between studied trigonometric representation systems (Table 5). Thus, consistent with Chaar (2007, p. 267-268), it seems that the unit circle has not been taught to convert trigonometric notions. Besides, in contrast to Steckroth (2007, p. 173), there is no evidence that subjects who have connected partial systems consider the value of reference angles as vital to their link. In fact, using the angle (Table 6) is the most common strategy to solve the task. In brief, most of the students neither convert notions from one trigonometric representation system to another, nor are able to draw on foundational notions such as the analytical function, the unit circle, and the oriented angle concept with a coherent meaning.

The findings of this study shed light on the fragmented understanding of the cosine of the angle of the participants by the difficulty in building and linking meanings of the cosine of the angle when participants convert notions between trigonometric representation systems (Brown, 2005; Even, 1998, p. 109; Martín-Fernández, 2019). Firstly, as seen in Table 3, 50% of pre-service teachers utilize metric strategy somehow in order to build the cosine. Consequently, interpretations that emphasize metric senses (length and ratio) predominate over those that emphasize analytical ones (value) in the unit circle (Table 4). Secondly, there are few combinations of senses of the cosine of the angle in the first partial system. Thirdly, more than one third of participants do not build the cosine of the angle properly (Table 3). Additionally, while 36.11% of the participants do not interpret the cosine of the angle in the first partial system (Table 4), this percentage increases to 54.17% in the second one (Table 7). Thus, the emphasis on metric aspects, the scarcity of combinations and links of the meanings of the cosine in the responses in the first partial system may involve a lack of connections with the second one.

CONCLUSION

This research expands and deepens previous studies describing the types of understanding on conversions between trigonometric representation systems. The discussion of results of the empirical study also shows deficiencies of prospective teachers about meanings within relational trigonometry system. In fact, several challenges that are documented in prior literature appear in the participants' responses, such as the graphing of trigonometric functions. Additionally, the analysis of responses using our framework reveals and helps interpret several insights regarding how subjects reason on, process, and convert trigonometric notions. Although there is extensive work by researchers on the role of representations in the learning and teaching of mathematics, a contribution of this study is to analyze the meaning of trigonometric contents within a relational trigonometric system.

A second specific result was achieved when the representation of the angle, its cosine and how surveyed participants give meaning to them were revealed. Considering the interpretative richness and relations of their contents, pre-service teachers' reason differently and show a determined progress. The modes were differentiated by the selected themes. Therefore, this research helps improve the theoretical knowledge about mathematical concepts understanding by prospective teachers and contributes to their characterization. Indeed, the different criteria provide us with information about what each one has understood and what should be enhanced. The difficulties of the participants partially stem from impoverished connections between similar notions involved in different partial systems that constitute the core of the trigonometry relational system (Akkoc, 2008; Moore, 2016). The findings of this study also indicate that meanings and contents were not paid enough attention in the training of the pre-service teachers, and they were therefore unable to convert notions and to move between partial systems. We stated that an underlying difficulty to master and comprehend trigonometry relational system is that surveyed subjects are required to reason on the absolute angle in the partial elementary geometry system, on the oriented angle in the partial goniometry system, and on the analytic angle in the partial analytic geometry system. In other words, participants must link different meanings of the angle concept, using contents of the trigonometry relational system. These meanings cause troubles to participants. Indeed, participants use a type of angle and contents associated with a representation model even though the task is situated in another one. We believe that the root of this problem is the treatment given for the teaching of the concept of angle, which imposes an unnecessary division in the trigonometry relational system. The manner in which the angle is introduced, in which situations and contexts, together with its order of appearance in the teaching are the keys for how participants understand the trigonometry relational system. We hold the view that oriented angles and absolute angles must be taught simultaneously emphasizing their differences. We sustain that the wrong use of the angle concept, the scarcity of combining meanings of the angle concept and of its cosine illustrate why this division is unnecessary. Finally, as Moore (2016) highlights, an underdeveloped angle measure understanding contributes to pre-service teachers' difficulties with relational trigonometry system, but we have also proved that so does an underdeveloped angle's concept.

Another aim of this paper was achieved when we gathered evidences of what contents must be mastered in order to move between studied partial trigonometry systems and to convert notions giving it meaning. Thus, contrary to Çekmez (2020), making a connection between the coordinates of a point and the values of the trigonometric functions is not enough to link the partial goniometry system to the partial analytic system. What is more, although the angle can be measured in the same units (degrees or real numbers in both partial systems), its sense, its sign and its content differ; in other words, its meaning varies. One of the strengths of our framework is that provides students with the opportunity to break down connections by means of contents, and by linking several meanings between partial systems. Although it is possible to establish alternative operational definitions of the trigonometric functions, which provide information about the rules of steps to be carried out (Çekmez, 2020), and which could be learnt by heart, our interest lies mainly in linking meanings. We hold that meanings become essential to develop useful mathematical knowledge (Thompson, 2013; Rico, 2019).

This paper therefore suggests practices that may help brush up teaching and learning of the moving between the partial goniometry system and the partial analytic system. In annex 1, the procedural and conceptual content to construct a point of the cosine function from the unit circle is shown. Teachers and students could use these contents to minimize students' mistakes. Activities should be implemented so that students can master these contents, linking meanings in order to avoid memorizing only facts and skills. The moving among these partial systems, relating a variety of facts, concepts, skills and reasoning, is a complex task which holds many difficulties for students.

Summarizing, these prospective teachers have found difficulties converting trigonometric notions and they have not developed enough ability to move between partial trigonometry systems. Although most of the participants only manage to develop essential skills to cope with the changing of meaning, it is much more important to understand how participants think when they work with different partial trigonometry systems. Knowing how trigonometry relational system is structured around central concepts and how to help learners organize and construct a solid core of contents is one of the principal parts of this research. However, our work reveals deficiencies that make evident a wide need for research on partial trigonometry systems along with the relationships and dependencies between the meanings of their notions. We believe that making links in one direction is insufficient. Therefore, one could extend the study using tasks in another direction, tasks in both directions, tasks which include other partial trigonometry systems, or tasks which imply moving among partial systems to fully grasp the conversions and the influence of their involved central notions.

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ANNEX 1

Table 1
Analysis of the School Mathematical Content for Question 3

Conceptual knowledge		Procedural knowledge
First level: Facts		First level: Skills
Terms	Unit circle	Drawing angles
	Quadrants	Identifying angles
	Coordinates	Estimating angles
	Trigonometric function	Relating angles
Notations	Degree sexagesimal (º), Radians	Comparing the values of the cosine of some angles
	sin, cos	Projecting points towards the axes
Conventions	Positive angle are represented anticlockwise	Calculating metric relationships
		Identifying the cosine of an angle in the unit circle
		Limiting the value of the cosine within the confines of an interval
		Labeling Cartesian axes, unit circle
		Bound the angle in an interval
		Identifying the cosine of an angle in the trigonometric function
		Estimating the cosine in the trigonometric function
		Identifying the negative value of the cosine in the unit circle and in the trigonometric function.
Results	The value of the cosine is between -1 and +1	

Table 1
Analysis of the School Mathematical Content for Question 3

Conceptual knowledge		Procedural knowledge
Second level: Abstractions and generalization		Second level: Reasoning
Concept	The cosine in the unit circle The cosine in the trigonometric function	Converting the cosine from the unit circle to the graph of the trigonometric function. Converting the angle from the unit circle to the graph of the trigonometric function.
Third level: Structuring		Third level: Strategies
Strategies	The technique to solve the task	Solving geometric task utilizing different results