

# Algorithmic Graph Theory

A very brief and informal overview of some basic notations and definitions

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## I. Graphs and graph parameters

- **graph**:  $G = (V, E)$  where  $V$  is a finite set of **vertices** and  $E$  is a subset of pairs of vertices (elements of  $E$  are referred to as **edges**)  
 $N(v) = \{u \in V : uv \in E\}$  is the set of **neighbors** of a vertex  $v$ ,  $d(v) = d_G(v) = |N(v)|$  is its **degree**  
 $\Delta(G)$ : maximum degree of a vertex in  $G$   
 $\delta(G)$ : minimum degree of a vertex in  $G$
- $K_n$ : a complete graph on  $n$  vertices;  $C_n$ : a cycle on  $n$  vertices;  $P_n$ : a path on  $n$  vertices;  $K_{m,n}$ : a complete bipartite graph with  $m + n$  vertices.
- **independent set (stable set)**: a subset of pairwise non-adjacent vertices in a graph  
 $\alpha(G) =$  **independence number** of  $G$ : maximum size of an independent set in  $G$
- **clique**: a subset of pairwise adjacent vertices in a graph  
 $\omega(G) =$  **clique number** of  $G$ : maximum size of a clique in  $G$
- **dominating set**: a subset of vertices such that every vertex not in the set has a neighbor in the set  
 $\gamma(G) =$  **domination number** of  $G$ : minimum size of a dominating set in  $G$
- **vertex cover**: a subset of vertices such that every edge of the graph has at least one of its endpoints in the set  
 $\tau(G) =$  **vertex cover number** of  $G$ : minimum size of a vertex cover in  $G$
- **matching**: a subset of pairwise disjoint edges  
**perfect matching**: a matching covering all vertices of the graph  
 $\nu(G) =$  **matching number** of  $G$ : maximum size of a matching in  $G$
- **$k$ -(vertex) coloring**: a mapping  $c : V \rightarrow \{1, \dots, k\}$  such that for all  $u, v \in V$  such that  $\{u, v\} \in E$ , it holds  $c(u) \neq c(v)$   
 **$k$ -colorable graph**: a graph that admits a  $k$ -coloring  
 $\chi(G) =$  **chromatic number** of  $G$ : minimum  $k$  such that  $G$  is  $k$ -colorable
- **$k$ -edge coloring**: a mapping  $c : E \rightarrow \{1, \dots, k\}$  such that for all  $e, f \in E$  such that  $e \neq f$  and  $e \cap f \neq \emptyset$ , it holds that  $c(e) \neq c(f)$   
 **$k$ -edge colorable graph**: a graph that admits a  $k$ -edge coloring  
 $\chi'(G) =$  **chromatic index** of  $G$ : minimum  $k$  such that  $G$  is  $k$ -edge colorable
- **list-chromatic number** of  $G$ : minimum  $k$  such that for all choices of sets  $S_v, v \in V$ , with  $|S_v| \geq k$ , there exists an  $S$ -coloring (a mapping  $c : V \rightarrow \cup_{v \in V} S_v$  such that  $c(v) \in S_v$  for all  $v \in V$  and  $c(u) \neq c(v)$  for all  $uv \in E$ )
- **simplicial vertex**: a vertex whose neighborhood is a clique

## II. Graph operations

- $H = (V', E')$  is a **subgraph** of  $G = (V, E)$  if  $V' \subseteq V$  and  $E' \subseteq E$
- $H = (V', E')$  is an **induced subgraph** of  $G = (V, E)$  if  $V' \subseteq V$  and  $E' = \{e \in E \mid e \subseteq V'\}$ ; notation:  $H < G$
- **disjoint union** of two graphs  $G$  and  $H$ : the graph obtained by adding to  $G$  a disjoint copy of  $H$  and no additional edges
- **join** of two graphs  $G$  and  $H$ : the graph obtained by adding to  $G$  a disjoint copy of  $H$  and all possible edges between  $G$  and  $H$

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- **complement** of a graph  $G = (V, E)$ : a graph  $\overline{G}$  with vertex set  $V$  in which two vertices are adjacent if and only if they are non-adjacent in  $G$
  - the **line graph** of a graph  $G = (V, E)$  is the graph with vertex set  $E$  in which two distinct edges  $e$  and  $f$  are adjacent if and only if they have a common endpoint in  $G$

### III. Graph classes

- **hereditary graph class**: a set of graphs closed under vertex deletions (equivalently, closed under induced subgraphs)
- **forest**: a graph without cycles; **tree**: a connected forest
- **bipartite graph**: a graph such that there exists two disjoint sets  $A$  and  $B$  with  $V = A \cup B$  such that  $|e \cap A| = |e \cap B| = 1$  for every edge  $e \in E$ ; equivalently, a 2-colorable graph
- **perfect graph**: a graph such that  $\chi(H) = \omega(H)$  for all its induced subgraphs  $H$ ; equivalently: a  $\{C_5, C_7, \overline{C_7}, C_9, \overline{C_9}, \dots\}$ -free graph
- **threshold graph**: a graph such that there exists non-negative weights  $w : V \rightarrow \mathbb{R}_+$  and a threshold  $t$  such that for every  $I \subseteq V$ ,  $\sum_{v \in I} w(v) \leq t$  if and only if  $I$  is an independent set; equivalently: a  $\{2K_2, C_4, C_5\}$ -free graph
- **split graph**: a graph that admits a partition of its vertex set into a clique and an independent set; equivalently: a  $\{2K_2, C_4, C_5\}$ -free graph
- **cograph**: a graph that can be recursively built from copies of the one-vertex graph by iteratively applying the operations of disjoint union and join; equivalently: a  $P_4$ -free graph
- **chordal graph**: a graph in which every cycle of length at least 4 has a **chord** (an edge connecting two non-consecutive vertices of the cycle); equivalently: a  $\{C_4, C_5, \dots\}$ -free graph
- **interval graph**: intersection graph of closed intervals on the real line
- **planar graph**: a graph that can be drawn in the plane without edge crossings

### IV. Graph problems

- **INDEPENDENT SET**: Given a graph  $G$  and an integer  $k$ , is  $\alpha(G) \geq k$ ?
- **CLIQUE**: Given a graph  $G$  and an integer  $k$ , is  $\omega(G) \geq k$ ?
- **VERTEX COVER**: Given a graph  $G$  and an integer  $k$ , is  $\tau(G) \leq k$ ?
- **MATCHING**: Given a graph  $G$  and an integer  $k$ , is  $\nu(G) \geq k$ ?
- **DOMINATING SET**: Given a graph  $G$  and an integer  $k$ , is  $\gamma(G) \leq k$ ?
- **INDEPENDENT DOMINATING SET**: Given a graph  $G$  and an integer  $k$ , is there an independent dominating set of size at most  $k$ ?
- **COLORABILITY**: Given a graph  $G$  and an integer  $k$ , is  $\chi(G) \leq k$ ?
- **$k$ -COLORABILITY**: Given a graph  $G$ , is  $\chi(G) \leq k$ ?
- **CHROMATIC INDEX**: Given a graph  $G$  and an integer  $k$ , is  $\chi'(G) \leq k$ ?
- **RECOGNITION OF GRAPHS IN  $X$** : Given a graph  $G$ , is  $G \in X$ ?