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Some New Properties of g-Frame in Hilbert C^*-Modules

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Abstract

The theory of frames which appeared in the last half of the century, has been generalized rapidly and various generalizations of frames in Hilbert spaces and Hilbert C^*-modules. In this paper, we will give some new properties of modular Riesz basis and modular g-Riesz basis that present a generalization of the results established in a Hilbert space.

Keywords: Frame, modular Riesz basis, modular g-Riesz basis, C^*-algebra, Hilbert $\mathcal{A}$-modules.

1. INTRODUCTION

Frame theory has a great revolution for recent years, this theory has several properties applicable in many fields of mathematics and engineering and play a significant role in signal and image processing, which leads to many applications in informatics, medicine and probability. Frame theory has been extended from Hilbert spaces to Hilbert C^*-modules and began to be study widely and deeply. The basic idea was to consider module over C^*-algebra instead of linear spaces and to allow the inner product to take values in the C^*-algebra.

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [4] in 1952 to study some deep problems in nonharmonic Fourier series. After the fundamental paper [3] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [6]. Frames have been used in signal processing, image processing, data compression and sampling theory.

This theory has been extended by Frank and Larson [5] in 2000 for the elements of C^*-algebras and Hilbert C^*-modules. Eventually, frames with C^*-valued bounds in Hilbert C^*-modules have been considered in [1].

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The theory of frames has been generalized rapidly and there are various generalizations of frames in Hilbert spaces and Hilbert C^*-modules. The notions of modular Riesz basis and modular g-Riesz basis has been introced by Khosravi A and Khosravi B in [10].

The aim of this paper is to extend results of Khosravi A and Farmani M. R [8], given in the Hilbert space to Hilbert C^*-module.

Let us recall some definitions and basic properties of a Hilbert C^*-module that we need in the rest of the parer.

For a C^*-algebra $\mathcal{A}$ if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of all positive elements of $\mathcal{A}$.

Definition 1.1. [11] Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \to \mathcal{A}$, such that is sesquilinear, positive definite and respects the module action. In the other words,

(i) $\langle x, x \rangle_{\mathcal{A}} \geq 0$, for all $x \in \mathcal{H}$, and $\langle x, x \rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.

(ii) $\langle ax + y, z \rangle_{\mathcal{A}} = a\langle x, z \rangle_{\mathcal{A}} + \langle y, z \rangle_{\mathcal{A}}$, for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.

(iii) $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^*$, for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define the norm of x by $||x|| = ||\langle x, x \rangle_{\mathcal{A}}||_{\mathcal{A}}^{\frac{1}{2}}$. If $\mathcal{H}$ is complete with $||.||$, it is called a Hilbert $\mathcal{A}$-module or a Hilbert C^*-modules over $\mathcal{A}$.
For every a in C^*-algebra $\mathcal{A}$, we have $|a|_{\mathcal{A}} = (a^*a)^{\frac{1}{2}}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $|x| = \langle x, x \rangle_{\mathcal{A}}^{\frac{1}{2}}$, for all $x \in \mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules, a map $T : \mathcal{H} \to \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \to \mathcal{H}$ such that $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^*y \rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$.

Throughout this paper, We reserve the notation $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ for the set of all adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $End_{\mathcal{A}}^*(\mathcal{H}, \mathcal{H})$ is abbreviated to $End_{\mathcal{A}}^*(\mathcal{H})$.

For a unital C^*-algebra $\mathcal{A}$, let I and J be a finite or countable subset of $\mathbb{Z}$ and $\{\mathcal{H}_i\}_{i \in I}$ be a sequence of Hilbert $\mathcal{A}$-modules. Let $l^2(\{\mathcal{H}_i\}_{i \in I})$ be the Hilbert $\mathcal{A}$-module defined by

$$l^2(\{\mathcal{H}_i\}_{i \in I}) = \Big\{\{x_i\}_{i \in I} : x_i \in \mathcal{H}_i, \sum_{i \in I} \langle x_i, x_i \rangle_{\mathcal{A}} \; converge \; in \; \|.\| \Big\}.$$

Let $l^2(\mathcal{A})$ be the Hilbert $\mathcal{A}$-module defined by

$$l^2(\mathcal{A}) = \Big\{\{a_j\}_{j\in J} \subseteq \mathcal{A} : \sum_{j\in J} a_j a_j^* \; converge \; in \; \|.\|\Big\}$$

The following lemmas will be used to prove our mains results.

Lemma 1.2. [2]. *Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules and $T \in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{K})$. The following statements are equivalent:*

(i) *T is surjective.*

(ii) *T^* is bounded below with respect to norm, i.e. there is $m > 0$ such that*

$$m\|x\| \leq \|T^*x\| \qquad x \in \mathcal{K}.$$

(iii) *T^* is bounded below with respect to the inner product, i.e. there is $m' > 0$ such that,*

$$m'\langle x, x\rangle_{\mathcal{A}} \leq \langle T^*x, T^*x\rangle_{\mathcal{A}} \qquad x \in \mathcal{K}.$$

Lemma 1.3. [11]. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and $T \in End^*_{\mathcal{A}}(\mathcal{H})$, then we have for all $x \in \mathcal{H}$,*

$$\langle Tx, Tx\rangle_{\mathcal{A}} \leq \|T\|^2\langle x, x\rangle_{\mathcal{A}}.$$

Lemma 1.4. [7] *Let $\mathcal{A}$ be a C^*-algebra. Suppose that $\{a_j\}_{j\in J}$ and $\{b_j\}_{j\in J}$ are two sequences of $\mathcal{A}$ such that both $\sum_{j\in J} a_j a_j^*$ and $\sum_{j\in J} b_j b_j^*$ converge in $\mathcal{A}$, then,*

$$\sum_{j\in J}(a_j + b_j)(a_j + b_j)^* \leq 2\sum_{j\in J}(a_j a_j^* + b_j b_j^*)$$

2. G-frame in Hilbert C*-Module

Definition 2.1. [5]. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module. A family $\{x_i\}_{i\in I}$ of elements of $\mathcal{H}$ is called a frame for $\mathcal{H}$, if there exist two positive constants A, B, such that for all $x \in \mathcal{H}$,

$$A\langle x, x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle x, x_i\rangle_{\mathcal{A}}\langle x_i, x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}}. \tag{2.1}$$

The numbers A and B are called lower and upper bound of the frame, respectively.

If $A = B = \lambda$, the frame is called λ-tight. If $A = B = 1$, it is called a normalized tight frame or a Parseval frame. If only upper inequality of (2.1) hold, then $\{x_i\}_{i\in I}$ is called a Bessel sequence for $\mathcal{H}$.

Let $\{x_i\}_{i\in I}$ be a Bessel sequence in a Hilbert C^*-module $\mathcal{H}$, we define the analysis operator by

$$T:\mathcal{H}\longrightarrow \quad l^2(\mathcal{A})$$
$$x\longrightarrow \{\langle x,x_i\rangle_{\mathcal{A}}\}_{i\in I}$$

T is a bounded linear operator, the adjoin operator called the synthesis operator is defined by

$$T^*:l^2(\mathcal{A})\longrightarrow \mathcal{H}$$
$$\{a_i\}_{i\in I}\longrightarrow \sum_{i\in I} a_i x_i$$

By composing T and T^*, the frame operator S is given by

$$S:\mathcal{H}\longrightarrow \mathcal{H}$$
$$x\longrightarrow Sx=T^*Tx=\sum_{i\in I}\langle x,x_i\rangle_{\mathcal{A}}x_i$$

Proposition 2.2. *Let $\{x_i\}_{i\in I}$ be a frame in a Hilbert C^*-module $\mathcal{H}$. Then the frame operator S thus defined is bounded, selfadjoint, positive and invertible.*

Definition 2.3. [9]. Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module and $(\mathcal{H}_i)_{i\in I}$ be a sub-modules of $\mathcal{H}$. We call a sequence $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{H}_i), i\in I\}$ a g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if there exist two positive constants A and B, such that for all $x\in\mathcal{H}$,

$$A\langle x,x\rangle_{\mathcal{A}}\leq \sum_{i\in I}\langle \Lambda_i x,\Lambda_i x\rangle_{\mathcal{A}}\leq B\langle x,x\rangle_{\mathcal{A}}. \tag{2.2}$$

The numbers A and B are called lower and upper bound of the g-frame, respectively.

If $A=B=\lambda$, the g-frame is called a λ-tight. If $A=B=1$, it is called a g-Parseval frame.

Let $\mathcal{H}$ and $\mathcal{K}$ be a Hilbert $\mathcal{A}$-modules. We recall that $\mathcal{H}\oplus\mathcal{K}=\{(x,y):x\in\mathcal{H},y\in\mathcal{K}\}$ is a Hilbert $\mathcal{A}$-module with pointwise operations and inner product

$$\langle (x_1,y_1),(x_2,y_2)\rangle=\langle x_1,x_2\rangle_{\mathcal{A}}+\langle y_1,y_2\rangle_{\mathcal{A}}\quad x_1,x_2\in\mathcal{H};\ y_1,y_2\in\mathcal{K}.$$

Let $\mathcal{U}$ and $\mathcal{V}$ be two Hilbert $\mathcal{A}$-modules. For $T\in End^*_{\mathcal{A}}(\mathcal{H},\mathcal{U})$ and $L\in End^*_{\mathcal{A}}(\mathcal{K},\mathcal{V})$ we define

$$T\oplus L\in End^*_{\mathcal{A}}(\mathcal{H}\oplus\mathcal{K},\mathcal{U}\oplus\mathcal{V})\quad by\quad (T\oplus L)(x,y):=(Tx,Ly)\quad x\in\mathcal{H},\ y\in\mathcal{K}.$$

Theorem 2.4. *Let* $\{x_i\}_{i\in I}$ *be a frame for* $\mathcal{H}$ *with bounds* A, B *and frame operator* S_x. *Let* $\{y_j\}_{j\in I}$ *be a frame for* $\mathcal{K}$ *with bounds* C, D *and frame operator* S_y. *Then* $\{(x_i \oplus y_j)\}_{(i\in I,\, j\in J)}$ *is a frame for* $\mathcal{H}\oplus\mathcal{K}$ *with frame operator* $S_{(x\oplus y)} = S_x \oplus S_y$.

Proof. Let $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ be a frames as they were defined in the last Theorem. Then for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$ we have

$$A\langle x,x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle x,x_i\rangle_{\mathcal{A}}\langle x_i,x\rangle_{\mathcal{A}} \leq B\langle x,x\rangle_{\mathcal{A}}. \tag{2.3}$$

$$C\langle y,y\rangle_{\mathcal{A}} \leq \sum_{j\in J}\langle y,y_i\rangle_{\mathcal{A}}\langle y_i,y\rangle_{\mathcal{A}} \leq D\langle y,y\rangle_{\mathcal{A}}. \tag{2.4}$$

From (2.3), (2.4) and Lemma (1.4), we have

$$\begin{aligned} min\{A,C\}(\langle(x,y),(x,y)\rangle) &= min\{A,C\}(\langle x,x\rangle_{\mathcal{A}} + \langle y,y\rangle_{\mathcal{A}}) \\ &\leq \sum_{(i,j)\in IxJ} |\langle(x,y),(x_i,y_j)\rangle|^2 \\ &= \sum_{(i,j)\in IxJ} |\langle x,x_i\rangle_{\mathcal{A}} + \langle y,y_j\rangle_{\mathcal{A}}|^2 \\ &\leq 2max\{B,D\}(\langle x,x\rangle_{\mathcal{A}} + \langle y,y\rangle_{\mathcal{A}}) \\ &= 2max\{B,D\}(\langle(x,y),(x,y)\rangle), \end{aligned}$$

which shows that $\{(x_i \oplus y_j)\}_{(i\in I,\, j\in J)}$ is a frame for $\mathcal{H}\oplus\mathcal{K}$.
Moreover, we have for all $(x\oplus y) \in \mathcal{H}\oplus\mathcal{K}$,

$$\begin{aligned} S_{(x\oplus y)}(x,y) &= \sum_{i\in I,\, j\in J} \langle(x,y),(x_i,y_j)\rangle(x_i,y_j) \\ &= \sum_{i\in I,\, j\in J} (\langle x,x_i\rangle_{\mathcal{A}} + \langle y,y_i\rangle_{\mathcal{A}})(x_i,y_j) \\ &= S_x(x)\oplus S_y(y) = (S_x\oplus S_y)(x\oplus y). \end{aligned}$$

Then,

$$S_{(x\oplus y)} = S_x \oplus S_y$$

□

3. Modular G-frame in Hilbert C*-Module

Let $\{\Lambda_i\}_{i\in I}$ be a g-Bessel sequence for $\{End_{\mathcal{A}}^*(\mathcal{H},\mathcal{H}_i), i\in I\}$, we recall that the analysis operator for $\{\Lambda_i\}_{i\in I}$ is defined by

$$T_\Lambda : \mathcal{H} \longrightarrow l^2(\{\mathcal{H}_i\}_{i\in I})$$
$$x \longrightarrow T_\Lambda^* x = \{\Lambda_i x\}_{i\in I}$$

The adjoin of this operator is called the synthesis operator and defined by

$$T_\Lambda^* : l^2(\{\mathcal{H}_i\}_{i\in I}) \longrightarrow \mathcal{H}$$
$$\{x_i\}_{i\in I} \longrightarrow T_\Lambda^*\{x_i\}_{i\in I} = \sum_{i\in I}\Lambda_i^* x_i.$$

The g-frame operator S_Λ is defined by

$$S_\Lambda : \mathcal{H} \longrightarrow \mathcal{H}$$
$$x \longrightarrow S_\Lambda x = T_\Lambda^* T_\Lambda x = \sum_{i\in I}\Lambda_i^*\Lambda_i x.$$

The g-frame operator S_Λ is a positive and self-adjoin operator. Moreover, if $\{\Lambda_i\}_{i\in I}$ is a g-frame, then S_Λ is invertible, for more details see ([9])

Definition 3.1. [12] Let $K \in End_{\mathcal{A}}^*(\mathcal{H})$ and $\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H},\mathcal{H}_i)$ for all $i \in I$, then $\{\Lambda_i\}_{i\in I}$ is said to be a K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if there exist two constants $A, B > 0$ such that

$$A\langle K^*x, K^*x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leq B\langle x, x\rangle_{\mathcal{A}} \quad x \in \mathcal{H}.$$

Theorem 3.2. *Let $K \in End_{\mathcal{A}}^*(\mathcal{H})$ and let $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H},\mathcal{H}_i), i \in I\}$ be a g-frame in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ with bounds $A, B > 0$. Let $L_i \in End_{\mathcal{A}}^*(\mathcal{H},\mathcal{U}_i)$ where $\mathcal{U}_i$ is a Hilbert C^*-module for each $i \in I$. Suppose that there exist $C, D > 0$, such that*

$$C\langle x_i, x_i\rangle_{\mathcal{A}} \leq \langle L_i x_i, L_i x_i\rangle_{\mathcal{A}} \leq D\langle x_i, x_i\rangle_{\mathcal{A}} \quad for\ all \quad i \in I\ ,\ x_i \in \mathcal{H}_i$$

Then,

(a) *The sequence $\{L_i\Lambda_i K \in End_{\mathcal{A}}^*(\mathcal{H},\mathcal{U}_i) : i \in I\}$ is a K^*-g-frame.*

(b) *If K is invertible, then the sequence $\{L_i\Lambda_i K \in End_{\mathcal{A}}^*(\mathcal{H},\mathcal{U}_i) : i \in I\}$ is a g-frame.*

Proof. (a) For all $x \in \mathcal{H}$, we have

$$A\langle x, x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} \leq B\langle .x, x\rangle_{\mathcal{A}}. \tag{3.1}$$

On one hand we have

$$\begin{aligned}\sum_{i\in I}\langle L_i\Lambda_i Kx, L_i\Lambda_i Kx\rangle_{\mathcal{A}} &\leq D\sum_{i\in I}\langle \Lambda_i Kx, \Lambda_i Kx\rangle_{\mathcal{A}}\\ &\leq DB\langle Kx, Kx\rangle_{\mathcal{A}}\\ &\leq DB\|K\|^2\langle x, x\rangle_{\mathcal{A}}\end{aligned}$$

One the other hand, we have,

$$\begin{aligned}\sum_{i\in I}\langle L_i\Lambda_i Kx, L_i\Lambda_i Kx\rangle_{\mathcal{A}} &\geq C\sum_{i\in I}\langle \Lambda_i Kx, \Lambda_i Kx\rangle_{\mathcal{A}}\\ &\geq CA\langle Kx, Kx\rangle_{\mathcal{A}}\\ &= CA\langle (K^*)^*x, (K^*)^*x\rangle_{\mathcal{A}},\end{aligned}$$

which ends the proof.

(b) Let K be an invertible operator, we have for all $x \in \mathcal{H}$

$$\begin{aligned}\langle x, x\rangle_{\mathcal{A}} &= \langle K^{-1}Kx, K^{-1}Kx\rangle_{\mathcal{A}}\\ &\leq \|K^{-1}\|^2\langle Kx, Kx\rangle_{\mathcal{A}}\\ &\leq \frac{1}{A}\|K^{-1}\|^2\sum_{i\in I}\langle \Lambda_i Kx, \Lambda_i Kx\rangle_{\mathcal{A}}\\ &\leq \frac{1}{AC}\|K^{-1}\|^2\sum_{i\in I}\langle L_i\Lambda_i Kx, L_i\Lambda_i Kx\rangle_{\mathcal{A}}.\end{aligned}$$

So,

$$AC\|K^{-1}\|^{-2}\langle x, x\rangle_{\mathcal{A}} \leq \sum_{i\in I}\langle L_i\Lambda_i Kx, L_i\Lambda_i Kx\rangle_{\mathcal{A}},$$

which shows that $\{L_i\Lambda_i K \in End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{U}_i) : i \in I\}$ is a g-frame with bounds $AC\|K^{-1}\|^{-2}$ and $DB\|K\|^2$. □

Definition 3.3. [10] Let $\Lambda = \{\Lambda_i\}_{i\in I}$ be a sequence in $End^*_{\mathcal{A}}(\mathcal{H}, \mathcal{H}_i)$ for all $i \in I$

(1) If the $\mathcal{A}$-linear hull of $\bigcup_{i\in i}\Lambda^*(\mathcal{H}_i)$ is dense in $\mathcal{H}$, then $\{\Lambda_i\}_{i\in I}$ is g-complete.

(2) If $\{\Lambda_i\}_{i\in I}$ is g-complete and there exist $A, B > 0$ such that for any subset $J \subseteq I$ and $y_i \in \mathcal{H}_i$ we have

$$A\|\sum_{j\in J}|y_j|^2\| \leq \|\sum_{j\in J}\Lambda_j^* y_j\|^2 \leq B\|\sum_{j\in J}|y_j|^2\|,$$

then $\{\Lambda_i\}_{i\in I}$ is a modular g-Riesz basis for $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$. A and B are called bounds of $\{\Lambda_i\}_{i\in I}$.

Theorem 3.4. *Let $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H},\mathcal{H}_i), i\in I\}$ and let $\{x_{i,j}\}_{j\in J_i}$ be a Parseval frame for H_i for each $i \in I$. Then the following assertions hold*

(1) *The sequence $\{\Lambda_i \in End_{\mathcal{A}}^*(\mathcal{H},\mathcal{H}_i), i\in I\}$ is a g-frame (g-Bessel sequence) in Hilbert $\mathcal{A}$-module $\mathcal{H}$ with respect to $\{\mathcal{H}_i\}_{i\in I}$ if and only if the sequence $\{(\Lambda_i)^* x_{i,j} : i \in I, j \in J_i\}$ is a frame in $\mathcal{H}$ (Bessel sequence).*

(2) *If $\{(\Lambda_i)^* x_{i,j} : i\in I, j\in J_i\}$ is a modular Riesz basis, then $\{\Lambda_i\}_{i\in I}$ is a modular g-Riesz basis. Conversely if $\{\Lambda_i\}_{i\in I}$ is a modular g-Riesz basis and there exist $m > 0$ such that for each $i \in I_i$ and $(c_{i,j})_{j\in I_1}$ for each finite $I_1 \subseteq J_i$,*

$$m\|\sum_{i\in S}\sum_{j\in J_i} c_{i,j}c_{i,j}^*\|^{\frac{1}{2}} \leq \|\sum_{i\in S}\sum_{j\in I_1} c_{i,j}\Lambda_i^* x_{i,j}\|$$

then $\{(\Lambda_i)^* x_{i,j} : i\in I, j\in J_i\}$ is a modular Riesz basis.

Proof. (1) Let $x \in \mathcal{H}$, for each $i \in I$ we have

$$\langle \Lambda_i x, \Lambda_i x\rangle_{\mathcal{A}} = \sum_{j\in J_i}\langle \Lambda_i x, x_{i,j}\rangle_{\mathcal{A}}\langle x_{i,j}, \Lambda_i x\rangle_{\mathcal{A}} = \sum_{j\in J_i}\langle x, \Lambda_i^* x_{i,j}\rangle_{\mathcal{A}}\langle \Lambda_i^* x_{i,j}, x\rangle_{\mathcal{A}}.$$

This last equality allows us to conclude that $\{\Lambda_i\}_{i\in I}$ is a g-frame if and only if $\{\Lambda_i^* x_{i,j}\}_{i\in I, j\in J_i}$ is a frame.

(2) Let $\{(\Lambda_i)^* x_{i,j} : i \in I, j \in J_i\}$ be a modular Riesz basis with bounds A and B. For each $y_i \in H_i$ we have

$$y_i = \sum_{j\in J_i}\langle y_i, x_{i,j}\rangle_{\mathcal{A}} x_{i,j} \quad \textit{and} \quad \langle y_i, y_i\rangle_{\mathcal{A}} = \sum_{j\in J_i}\langle y_i, x_{i,j}\rangle_{\mathcal{A}}\langle x_{i,j}, y_i\rangle_{\mathcal{A}} = \sum_{j\in J_i}|\langle y_i, x_{i,j}\rangle_{\mathcal{A}}|^2.$$

Furthermore, we have

$$\Lambda_i^* y_i = \sum_{j\in J_i}\langle y_i, x_{i,j}\rangle_{\mathcal{A}}\Lambda_i^* x_{i,j},$$

So, let $S \subseteq I$ a finite subset, we have

$$\begin{aligned} A\|\sum_{i\in S}\langle y_i, y_i\rangle_{\mathcal{A}}\| &= A\|\sum_{i\in S}\sum_{j\in J_i}|\langle y_i, x_{i,j}\rangle_{\mathcal{A}}|^2\| \\ &\leq \|\sum_{i\in S}\sum_{j\in J_i}\langle y_i, x_{i,j}\rangle_{\mathcal{A}}\Lambda_i^* y_i\|^2 \\ &= \|\sum_{i\in S}\Lambda_i^* y_i\|^2 \\ &\leq B\|\sum_{i\in S}\sum_{j\in J_i}|\langle y_i, x_{i,j}\rangle_{\mathcal{A}}|^2\| \\ &= B\|\sum_{i\in S}\langle y_i, y_i\rangle_{\mathcal{A}}\| \end{aligned}$$

Conversely, we assume that $\{\Lambda_i\}_{i\in I}$ be a modular g-Riesz basis for $\mathcal{H}$ with bounds A and B, it follows that for any finite subset $S \subseteq I$,

$$A\|\sum_{i\in S}\langle y_i, y_i\rangle_{\mathcal{A}}\| \leq \|\sum_{i\in S}\Lambda_i^* y_i\|^2 \leq B\|\sum_{i\in S}\langle y_i, y_i\rangle_{\mathcal{A}}\| \tag{3.2}$$

Since $y_i = \sum_{j\in J_i} c_{i,j}x_{i,j}$ and

$$\begin{aligned} \|\sum_{i\in S}\langle y_i, y_i\rangle_{\mathcal{A}}\|^2 &= \|\sum_{i\in S}\langle y_i, \sum_{j\in J_i} c_{i,j}x_{i,j}\rangle_{\mathcal{A}}\|^2 \\ &= \|\sum_{i\in S}\sum_{j\in J_i}\langle y_i, x_{i,j}\rangle_{\mathcal{A}}c_{i,j}^*\|^2 \\ &\leq \|\sum_{i\in S}\sum_{j\in J_i}|\langle y_i, x_{i,j}\rangle_{\mathcal{A}}|^2\|\|\sum_{i\in S}\sum_{j\in J_i}c_{i,j}c_{i,j}^*\| \\ &= \|\sum_{i\in S}\langle y_i, y_i\rangle_{\mathcal{A}}\|\|\sum_{i\in S}\sum_{j\in J_i}c_{i,j}c_{i,j}^*\|, \end{aligned}$$

then for each $i \in I$, on one hand, we have

$$\|\sum_{i\in S}\langle y_i, y_i\rangle_{\mathcal{A}}\| \leq \|\sum_{i\in S}\sum_{j\in J_i}c_{i,j}c_{i,j}^*\|.$$

On the other hand, from (3.2) we have

$$\|\sum_{i\in S}\Lambda_i^* y_i\|^2 = \|\sum_{i\in S}\sum_{j\in J_i}c_{i,j}\Lambda_i^* x_{i,j}\|^2 \leq B\|\sum_{i\in S}\langle y_i, y_i\rangle_{\mathcal{A}}\| \leq \|\sum_{i\in S}\sum_{j\in J_i}c_{i,j}c_{i,j}^*\|.$$

Which ends the proof. $\square$

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REFERENCES

1. Alijani A, Dehghan M, 2011. *$*$-Frames in Hilbert $\mathcal{C}^*$-modules*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **73**, no. 4, 89–106.
2. Arambašić L, 2007. *On frames for countably generated Hilbert $\mathcal{C}^*$-modules*, Proc. Am. Math. Soc. **135**, 469–478.
3. Daubechies I, Grossmann A, Meyer Y, 1986. *Painless nonorthogonal expansions*, J. Math. Phys. **27**, 1271–1283.
4. Duffin R. J, Schaeffer A. C, 1952. *A class of nonharmonic fourier series*, Trans. Am. Math. Soc. **72**, 341–366.
5. Frank M, Larson D. R, 2002. *Frames in Hilbert $\mathcal{C}^*$-modules and $\mathcal{C}^*$-algebras*", J. Oper. Theory, vol. 48, pp. 273-314.
6. Gabor D, 1946. *Theory of communications*, J. Elect. Eng. **93**, 429–457.
7. Han D, Jing W, Larson D, Mohapatra R. N, 2008. Riesz Bases and their dual mudular frames in Hilbert C^*-modules, Mathematical Analysis and Applications, Vol 343 , pp. 246-256.
8. Khosravi A, Farmani M. R, 2022. Frames and g-frame in Hilbert Spaces, Mathematics and computational sciences, Vol 3(1), pp.10-16.
9. Khorsavi A, Khorsavi B, 2008. *Fusion frames and g-frames in Hilbert $\mathcal{C}^*$-modules*, Int. J.Wavelet, Multiresolution and Information Processing 6, pp. 433-446.
10. Khosravi A, Khosravi B, 2012. g-frames and modular Riesz basis in Hilbert C^*-modules, International Journal of Wavelets, Multiresolution and Information Processing, Vol 10, No. 2, 12 pages.
11. Paschke W, 1973. *Inner product modules over B^*-algebras*, Trans. Am. Math. Soc. **182**, 443–468.
12. Xiang Z. Q, Li Y. M, 2016 *G-frames for operators in Hilbert C^*-modules*, Turkish Journal of Mathematics, vol 40, pp. 453-469.