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A note on a conjecture on consistent cycles

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Abstract

Let Γ denote a finite digraph and let G be a subgroup of its automorphism group. A directed cycle $\vec{C}$ of Γ is called G*-consistent* whenever there is an element of G whose restriction to $\vec{C}$ is the 1-step rotation of $\vec{C}$. In this short note we prove a conjecture on G-consistent directed cycles stated by Steve Wilson.

Keywords: Digraphs, consistent directed cycles.

Math. Subj. Class.: 05C20, 05C38, 05E18

1 Introduction

Let Γ denote a finite digraph (without loops and multiple arcs). By a *directed cycle* in Γ we mean a cyclically ordered set $\vec{C} = \{v_0, v_1, v_2, \ldots, v_{r-1}\}$, $r \geq 3$, of pairwise distinct vertices of Γ such that (v_i, v_{i+1}) is an arc of Γ for every $i \in \mathbb{Z}_r$ (the addition being mod r). Let G be a subgroup of the automorphism group of Γ. Directed cycle $\vec{C}$ is called G-*consistent*, if there exists $g \in G$ such that $v_i^g = v_{i+1}$ for each $i \in \mathbb{Z}_r$. In this case g is called a *shunt* for $\vec{C}$. Clearly, G acts on the set of G-consistent directed cycles: for $h \in G$, $\vec{C}^h = \{v_0^h, v_1^h, v_2^h, \ldots, v_{r-1}^h\}$ is G-consistent with a shunt $h^{-1}gh$.

Consistent cycles in finite arc-transitive graphs were introduced by J. H. Conway in one of his public lectures [3]. Since then a number of papers on consistent cycles and their applications appeared, see [1, 2, 4, 5, 6, 7, 8, 9, 10, 11].

Observe that if (u, v) is an arc of Γ and $g \in G$ is such that $u^g = v$, then the orbit of u under g induces a G-consistent directed cycle $\{u, v = u^g, u^{g^2}, \ldots\}$ (provided that $u^{g^2} \neq u$). Steve Wilson [12] stated the following conjecture on consistent cycles.

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Conjecture 1.1. *Let Γ denote a finite digraph (without loops and multiple arcs) and let G be an arc-transitive subgroup of its automorphism group. Pick vertices u, v of Γ, such that (u, v) is an arc of Γ. For a G-orbit $\mathcal{A}$ of G-consistent directed cycles, let $B_{\mathcal{A}}$ denote the set of all automorphisms $g \in G$, such that $u^g = v$, and the orbit of u under g is in $\mathcal{A}$. Let $G_{(u,v)}$ denote the G-stabilizer of the arc (u, v). Then the number of elements in $B_{\mathcal{A}}$ is independent of $\mathcal{A}$, and is equal to the order of $G_{(u,v)}$.*

In this short note we prove the above conjecture.

2 Proof of the conjecture

In this section we prove Conjecture 1.1. We prove Conjecture 1.1 in two steps. In Proposition 2.1 we prove that $|G_{(u,v)}| \leq |B_{\mathcal{A}}|$, and in Proposition 2.2 we prove that $|B_{\mathcal{A}}| \leq |G_{(u,v)}|$.

Proposition 2.1. *With the notation of Conjecture 1.1, we have $|G_{(u,v)}| \leq |B_{\mathcal{A}}|$.*

Proof. Since G is arc-transitive, there exists a G-consistent directed cycle $\vec{C}$ in $\mathcal{A}$, which contains the arc (u, v). Let g denote a shunt for $\vec{C}$. Let $G_{\vec{C}}$ denote the pointwise stabiliser of $\vec{C}$ and let k be the index of $G_{\vec{C}}$ in $G_{(u,v)}$. Let $g_1, \dots, g_k$ be representatives of cosets of $G_{\vec{C}}$ in $G_{(u,v)}$.

Observe that for each $1 \leq i \leq k$ and each $h \in G_{\vec{C}}$, the automorphism $g_i^{-1} g h g_i$ sends u to v. Furthermore, the orbit of u under $g_i^{-1} g h g_i$ is the directed cycle $\vec{C}^{g_i}$. Namely, since g is a shunt for $\vec{C}$ and $h \in G_{\vec{C}}$, the image of $v^{g^j g_i}$ under $g_i^{-1} g h g_i$ is $v^{g^{j+1} g_i}$. Moreover, $\vec{C}^{g_i}$ is clearly in $\mathcal{A}$. Therefore, $g_i^{-1} g h g_i \in B_{\mathcal{A}}$.

We claim that if either $i \neq j$ or $h_1 \neq h_2$ $(h_1, h_2 \in G_{\vec{C}})$, then $\alpha = g_i^{-1} g h_1 g_i$ and $\beta = g_j^{-1} g h_2 g_j$ are distinct. Indeed, assume first that $i \neq j$. Note that $\vec{C}^{g_i} \neq \vec{C}^{g_j}$ since g_i and g_j are from different cosets of $G_{\vec{C}}$ in $G_{(u,v)}$. Moreover, α is a shunt for $\vec{C}^{g_i}$ and β is a shunt for $\vec{C}^{g_j}$. Since $\vec{C}^{g_i} \neq \vec{C}^{g_j}$ (and since $\vec{C}^{g_i}$ and $\vec{C}^{g_j}$ have at least the arc (u, v) in common), it follows that also $\alpha \neq \beta$. On the other hand, if $i = j$ and $\alpha = \beta$, then $h_1 = h_2$. Therefore, if $h_1 \neq h_2$ and $i = j$, then $\alpha \neq \beta$. This proves the claim.

It follows that $|B_{\mathcal{A}}| \geq k|G_{\vec{C}}| = |G_{(u,v)}|$. □

Proposition 2.2. *With the notation of Conjecture 1.1, we have $|B_{\mathcal{A}}| \leq |G_{(u,v)}|$.*

Proof. Let X denote the set of all G-consistent directed cycles in $\mathcal{A}$, containing the arc (u, v). Clearly, $B_{\mathcal{A}}$ is exactly the set of all shunts of directed cycles from X. Since all directed cycles from X have the arc (u, v) in common, every element of $B_{\mathcal{A}}$ is a shunt for exactly one directed cycle from X. Note also that X is nonempty as G is arc-transitive. We now define a mapping Ψ from $B_{\mathcal{A}}$ to $G_{(u,v)}$ as follows.

Fix $\vec{C} \in X$ and a shunt $g_{\vec{C}}$ of $\vec{C}$. For each $\vec{D} \in X$ there exists an element of G which sends $\vec{D}$ to $\vec{C}$. Pick such an element and denote it by $h(\vec{D})$. Composing $h(\vec{D})$ with an appropriate power of $g_{\vec{C}}$, we could assume that $h(\vec{D}) \in G_{(u,v)}$. For each $g \in B_{\mathcal{A}}$, let $\vec{D}(g)$ denote the unique directed cycle in X, for which g is a shunt (see Figure 1). For $g \in B_{\mathcal{A}}$ define $\Psi(g) = g h(\vec{D}(g)) g_{\vec{C}}^{-1}$ and note that $\Psi(g) \in G_{(u,v)}$.

We now show that Ψ is an injection. Pick $g_1, g_2 \in B_{\mathcal{A}}$ and assume that $\Psi(g_1) = \Psi(g_2)$. Let $\vec{D}(g_1) = \{u, v, v_1, v_2, \dots, v_{n-1}\}$ and $\vec{D}(g_2) = \{u, v, w_1, w_2, \dots, w_{n-1}\}$. We first

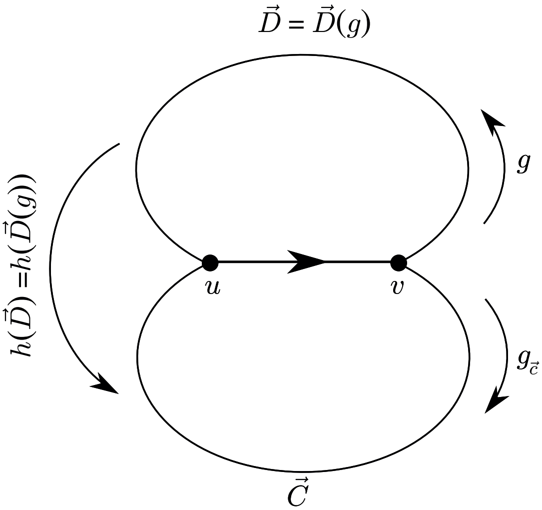

Figure 1: Directed consistent cycles $\vec{C}$ and $\vec{D}$.

show that $\vec{D}(g_1) = \vec{D}(g_2)$. Since $\Psi(g_1) = g_1 h(\vec{D}(g_1)) g_{\vec{C}}^{-1} = g_2 h(\vec{D}(g_2)) g_{\vec{C}}^{-1} = \Psi(g_2)$, we have $g_2^{-1} g_1 = h(\vec{D}(g_2)) h(\vec{D}(g_1))^{-1}$. This implies that $g_2^{-1} g_1$ is in $G_{(u,v)}$. We claim that $v_{n-i} = w_{n-i}$ for $i = 0, 1, \ldots n-1$, where $v_n = w_n = u$. We prove our claim using induction on i. Note that our claim is true for $i = 0$. Assume that our claim is true for $i = 0, 1, \ldots, t$, where $0 \leq t \leq n-2$. Note that $h(\vec{D}(g_2)) h(\vec{D}(g_1))^{-1}$ fixes the arc $(v_{n-t}, v_{n-t+1}, \ldots v_{n-1}, u, v)$, and therefore also $g_2^{-1} g_1$ fixes this arc. But since

$$v_{n-t-1}^{g_1} = v_{n-t} = v_{n-t}^{g_2^{-1} g_1} = w_{n-t-1}^{g_1},$$

we have $v_{n-t-1} = w_{n-t-1}$, verifying the claim. It follows that $\vec{D}(g_1) = \vec{D}(g_2)$. But since $\vec{D}(g_1) = \vec{D}(g_2)$, also $h(\vec{D}(g_1)) = h(\vec{D}(g_2))$. As $g_1 h(\vec{D}(g_1)) g_{\vec{C}}^{-1} = g_2 h(\vec{D}(g_2)) g_{\vec{C}}^{-1}$, it follows that $g_1 = g_2$. Therefore Ψ is an injection and so $|B_{\mathcal{A}}| \leq |G_{(u,v)}|$. $\square$

Corollary 2.3. *With the notation of Conjecture 1.1, we have* $|B_{\mathcal{A}}| = |G_{(u,v)}|$.

Proof. Immediately from Propositions 2.1 and 2.2. $\square$

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