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A study on supereulerian digraphs and spanning trails in digraphs

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Abstract

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A strong digraph D is eulerian if for any $v \in V(D)$, $d^+_D(v) = d^-_D(v)$. A digraph D is supereulerian if D contains a spanning eulerian subdigraph, or equivalently, a spanning closed directed trail. A digraph D is trailable if D has a spanning directed trail. This dissertation focuses on a study of trailable digraphs and supereulerian digraphs from the following aspects.

1. Strong Trail-Connected, Supereulerian and Trailable Digraphs.
For a digraph D, D is trailable digraph if D has a spanning trail. A digraph D is strongly trail-connected if for any two vertices u and v of D, D posses both a spanning (u, v)-trail and a spanning (v, u)-trail. As the case when $u = v$ is possible, every strongly trail-connected digraph is also supereulerian. Let D be a digraph. Let $S(D) = \{e \in A(D) : e \text{ is symmetric in } D\}$. A digraph D is symmetric if $A(D) = S(D)$. The symmetric core of D, denoted by $J(D)$, has vertex set $V(D)$ and arc set $S(D)$. We have found a well-characterized digraph family $\mathcal{D}$ each of whose members does not have a spanning trail with its underlying graph spanned by a $K_{2,n-2}$ such that for any strong digraph D with its matching number $\alpha'(D)$ and arc-strong-connectivity $\lambda(D)$, if $n = |V(D)| \geq 3$ and $\lambda(D) \geq \alpha'(D) - 1$, then each of the following holds.
(i) There exists a family $\mathcal{D}$ of well-characterized digraphs such that for any digraph D with $\alpha'(D) \leq 2$, D has a spanning trial if and only if D is not a member in $\mathcal{D}$.
(ii) If $\alpha'(D) \geq 3$, then D has a spanning trail.
(iii) If $\alpha'(D) \geq 3$ and $n \geq 2\alpha'(D) + 3$, then D is supereulerian.
(iv) If $\lambda(D) \geq \alpha'(D) \geq 4$ and $n \geq 2\alpha'(D) + 3$, then for any pair of vertices u and v of D, D contains a spanning (u, v)-trail.

2. Supereulerian Digraph Strong Products.
A cycle vertex cover of a digraph D is a collection of directed cycles in D such that every vertex in D lies in at least one dicycle in this collection, and such that the union of the arc sets of these directed cycles induce a connected subdigraph of D. A subdigraph F of a digraph D is a circulation if for every vertex v in F, the indegree of v equals its outdegree, and a spanning circulation if F is a cycle factor. Define $f(D)$ to be the smallest cardinality of a cycle vertex cover of the digraph D/F obtained from D by contracting all arcs in F, among all circulations F of D. In [International Journal of Engineering Science Invention, 8 (2019) 12-19], it is proved that if D_1 and D_2 are nontrivial strong digraphs such that D_1 is supereulerian and D_2 has a cycle vertex cover $\mathcal{C}'$ with $|\mathcal{C}'| \leq |V(D_1)|$, then the Cartesian product D_1 and D_2 is also supereulerian. We prove that for strong digraphs D_1 and D_2, if for some cycle factor F_1 of D_1, the digraph formed from D_1 by contracting arcs in F_1 is hamiltonian with $f(D_2)$ not bigger than $|V(D_1)|$, then the strong product D_1 and D_2 is supereulerian.

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Chapter 1

1 Preliminary

1.1 Notations and Terminology

In this chapter, we will provide the common terminology and notation used in this dissertation.
We consider finite and simple graphs and digraphs. Undefined terms and notations will follow [15] for graphs and [9] for digraphs. Usually, we use G to denote a graph and D a digraph. Undefined terms and notations will follow [15] for graphs and [9] for digraphs. A directed graph (or just digraph) D consists of a non-empty finite set $V(D)$ of elements called vertices and a finite set $A(D)$ of ordered pairs of distinct vertices called arcs. We call $V(D)$ the vertex set and $A(D)$ the arc set of a digraph D. Throughout our discussions, we use the notation (u, v) to denote an arc oriented from u to v in a digraph D; and use $[u, v]$ to denote either (u, v) or (v, u). When $[u, v] \in A(D)$, we say that u and v are adjacent. If two arcs of D have a common vertex, we say that these two arcs are adjacent in D. If (u, v) is an arc, we also say that u dominates v (v is dominated by u). We say that a vertex u is incident to an arc e if u is the head or tail of e. If X is a vertex subset or an arc subset of D, we use $D[X]$ to denote the subdigraph of D induced by X, $c(D)$ denotes the number of components of the underlying graph of D. If e is an edge in a graph G or an arc in a digraph D incident with vertices u and v, define $V(e) = \{u, v\}$. As in [9] , we define, for a vertex $v \in V(D)$, $N_D^+(v) = \{w \in V(D) : (v, w) \in A(D)\}$, $N_D^-(v) = \{u \in V(D) : (u, v) \in A(D)\}$. The sets $N_D^+(v)$, $N_D^-(v)$ and $N_D(v) = N_D^+(v) \cup N_D^-(v)$ are called the **out-neighbourhood**, **in-neighbourhood** and **neighbourhood** of v. We call the vertices in $N_D^+(v)$, $N_D^-(v)$ and $N_D(v)$ the **out-neighbours**, **in-neighbours** and **neighbours** of v. For a subset $X \subseteq V(D)$, define $N_D(X) = \cup_{x \in X} N_D(x)$.

For an arc subset $F \subseteq A(D)$, define $V(F) = \cup_{e \in F} V(e)$ to be the set of vertices incident with an edge of F in D. Following [9], for subsets $X, Y \subseteq V(D)$, define

$$(X, Y)_D = \{(x, y) \in A(D) : x \in X, y \in Y\}, \text{ and } (X, Y)_{G(D)} = (X, Y)_D \cup (Y, X)_D.$$

If $X = \{x\}$ or $Y = \{y\}$, we often use $(x, Y)_D$ for $(X, Y)_D$ or $(X, y)_D$ for $(X, Y)_D$, respectively. Hence $(x, y)_D = (\{x\}, \{y\})_D$. For a vertex $v \in V(D)$, let $\partial_D^+(v) = (v, V(D) - v)_D$ and $\partial_D^-(v) = (V(D) - v, v)_D$. Thus $d_D^+(v) = |\partial_D^+(v)|$ and $d_D^-(v) = |\partial_D^-(v)|$. We further define $c(D)$ denotes the number of components of the underlying graph of D. In addition, we define the **minimum out-degree** (**minimum in-degree**, respectively) of D to be

$$\delta^+(D) = \min\{d_D^+(v) : v \in V(D)\}(\delta^-(D) = \min\{d_D^-(v) : v \in V(D)\}, respectively).$$

Following [15], $\kappa(G)$, $\kappa'(G)$ and $\alpha(G)$ denote the **connectivity**, the **edge connectivity** and the **independence number** of a graph G; and $\kappa(D)$ and $\lambda(D)$ denotes the **vertex-strong connectivity** and the **arc-strong connectivity** of a digraph D, respectively. If D is a digraph, we often use $G(D)$ to denote the underlying undirected graph of D, the graph obtained from D by erasing all orientation on the arcs of D. The **stability number** $\alpha(D)$, and the **matching number** $\alpha'(D)$, of a digraph D are defined as

$$\alpha(D) = \alpha(G(D)) \text{ and } \alpha'(D) = \alpha'(G(D)),$$

By the definition of $\lambda(D)$ in [9], we note that for any integer $k \geq 0$ and a digraph D,

$$\lambda(D) \geq k \text{ if and only if for any nonempty proper subset } X \subset V(D), |\partial_D^+(X)| \geq k.$$

We use paths, cycles, and trails as defined in [15] when the discussion is on an undirected graph G, and to denote directed paths, directed cycles and directed trails when the discussion is on a digraph D. A directed trail (or path, respectively) from a vertex u to a vertex v in a digraph D is often refereed as to a (u, v)-trail (a (u, v)-path, respectively). For an integer n, we define $[n] = \{1, 2, \ldots, n\}$. A **walk** in D is an alternating sequence $W = x_1a_1x_2a_2x_3 \cdots x_{k-1}a_{k-1}x_k$ of vertices x_i and arcs a_j from D such that $a_j = (x_j, x_{j+1})$ for every $i \in [k]$ and $j \in [k-1]$. A walk W is **closed** if $x_1 = x_k$, and **open** otherwise. We use $V(W) = \{x_i : i \in [k]\}$ and $A(W) = \{a_j : j \in [k-1]\}$. We say that W is a walk from x_1 to x_k or an (x_1, x_k)-walk. If $x_1 \neq x_k$, then we say that the vertex x_1 is the **initail vertex** of W, the vertex x_k is the **terminal vertex** of W, and x_1 and x_k are end-vertices of W. The length of a walk is the number of its arcs. When the arcs of W are understood from the context, we will denote W by $x_1x_2 \cdots x_k$. A **ditrail** in D is a walk in which all arcs are distinct. A ditrail is often considered as a subdigraph induced by the arcs in the trail. If the vertices of W are distinct, then W is a **dipath**. If the vertices in the trail $x_1x_2 \cdots x_{k-1}$ are distinct, $k \geqslant 3$ and $x_1 = x_k$, then W is a **dicycle**. We say that an ordered pair of vertices (x, y) is **dominated** (**dominating**, respectively) if there exists $z \in V(D)$, with $(z, x), (z, y) \in A(D)((x, z), (y, z) \in A(D)$, respectively).

An **Eulerian trail** (or **Eulerian tour**) of G is a trail in G that visits every edge exactly once (allowing for revisiting vertices). For a graph G, denote $O(G) = \{v \in V(G) : d_G(v) \text{ is odd}\}$. A graph with $O(G) = \emptyset$ is called an **even graph**.

Theorem 1.1 *(Euler, 1736) The following are equivalent for a graph G.*
(i) G contains an Euler tour.
(ii) G is connected and $O(G) = \emptyset$.

A graph G is **eulerian** if G is a connected with $O(G) = \emptyset$. A graph G is **supereulerian** if G has a spanning eulerian subgraph. Thus a graph G is supereulerian if G has a spanning closed trail. The supereulerian graph problem, raised by Boesch, Suffel, and Tindell [16], seeks to characterize supereulerian graphs. Pulleyblank [43] showed that determining whether a graph is supereulerian, even when restricted to planar graphs, is $\mathcal{NP}$-complete. For more literature on supereulerian graphs, see Catlin's survey [17] and its supplement by Z.Chen et.al. [20] and the updating in [34]. The supereulerian graph problem is also motivated by the study of hamiltonian problems of graphs. A graph G is **hamiltonian** if G has a spanning cycle.

A walk (path, cycle) W is a **Hamilton (or hamiltonian)** walk (path, cycle) if $V(W) = V(D)$. A digraph D is **hamiltonian** if D contains a Hamilton cycle. A trail $W = x_1x_2 \ldots x_k$ is an **Euler (or eulerian)** trail if $A(W) = A(D)$, $V(W) = V(D)$ and $x_1 = x_k$. For digraphs, a strong digraph D is

eulerian if for any $v \in V(D)$, $d^+_D(v) = d^-_D(v)$. The following is well-known or immediately from the definition.

Theorem 1.2 *(Euler, see Theorem 1.7.2 of [9] and Veblen [46]) Let D be a digraph. The following are equivalent.*
(i) D is eulerian.
(ii) D is a spanning closed trail.
(iii) D is a disjoint union of dicycles and D is connected.

The supereulerian problem in digraphs was considered by Gutin [25]. A digraph D is supereulerian if D contains a spanning eulerian subdigraph, or equivalently, a spanning closed trail. Thus supereulerian digraphs must be strong, and every hamiltonian digraph is also a supereulerian digraph.

A digraph D is **trialable** if there exist $u, v \in (D)$, such that D has a spanning $(u, v)-$trail. A digraph D is a **strong** if, for every pair u, v of distinct vertices in D, there exists an $(u, v)-$walk; and D is a **weakly connected** if $G(D)$ is a connected.

A digraph D is **strongly trail-connected** if for any two vertices u and v of D, D posses both a spanning $(u, v)-$trail and a spanning $(v, u)-$trail. As the case when $u = v$ is possible, every strongly trail-connected digraph is also supereulerian.

Given a digraph D, we define the **path covering number** of D, $pc(D)$, as the minimum possible number of vertex-disjoint paths covering the vertices of D and the **trail covering number** of D, $\tau(D)$, as the minimum possible number of arc-disjoint trails covering the vertices of D . Note that some of these trails may consist of a single vertex.

A graph G is complete, if every pair of distinct vertices in G are adjacent. We will denote the complete graph on n vertices (which is unique up to isomorphism) by K_n. Its complement K_n^c has no edge. A digraph D is complete if, for every pair u, v of distinct vertices of D, both (u, v) and (v, u) are in D. The complete digraph on n vertices will be denoted by K_n^*.

Let $e = [v_1, v_2] \in A(D)$ be an arc of D. Define D/e to be the digraph obtained from $D - e$ by identifying v_1 and v_2 into a new vertex v_e, and deleting the possible resulting loop(s). If $W \subseteq A(D)$ is an arc subset, then define the **contraction** D/W to be the digraph obtained from D by contracting each arc $e \in W$, and deleting any resulting loops. Thus even D does not have parallel arcs, a contraction D/W is loopless but may have parallel arcs. If H is a subdigraph of D, then we often use D/H for $D/A(H)$. If L is a connected component of H and v_L is the vertex in D/H onto which L is contracted, then $D[V(L)]$ is the **contraction preimage** of v_L. We adopt the convention to define $D/\emptyset = D$, and define a vertex $v \in V(D/W)$ to be a **trivial vertex** if the preimage of v is a single vertex (also denoted by v) in D. Hence we often view trivial vertices in a contraction D/W as vertices in D.

For a graph G, a **matching** M of G is a subset of edges of G its elements are links and no two are adjacent in G. Let M be a matching in a graph G. A path P is an $M-$**augmenting path**, if the edges of P are alternately in M and in $E(G) - M$, and if both end vertices of P are not in $V(M)$. An $M-$augmenting path of a digraph D is an $M-$augmenting path of $G(D)$.

Definition 1.3 *[37] For a digraph D, an arc $[u, v] \in A(D)$ is a* **symmetric** *in D if both arcs (u, v)*

and (v,u) are in $A(D)$. In particular, a **symmetric dipath** *P is a dipath such that every arc of P is symmetric.*

Definition 1.4 *[4] Let D be a digraph such that either $D = K_1$ or $A(D) \neq \emptyset$. If for any $u, v \in V(D)$, D contains a symmetric dipath from u to v, then D is called a* **symmetrically connected** *digraph.*

Definition 1.5 *[4] Let $c \geq 2$ be an integer and let D be a weakly connected digraph and let $\{H_1, H_2, \ldots, H_c\}$ be the set of maximal symmetrically connected subdigraphs of D. If for any proper nonempty subset $\mathcal{J} \subset \{H_1, H_2, \ldots, H_c\}$, there exist an $H_i \in \mathcal{J}$ and a vertex $v \in V(H_i)$, and an $H_j \notin \mathcal{J}$ such that $N_D^+(v) \cap V(H_j) \neq \emptyset$ and $N_D^-(v) \cap V(H_j) \neq \emptyset$, then D is a* **partially symmetric**.

A digraph $D = (V, A)$ is a **semicomplete** if D is without nonadjacent vertices. Bang-Jenson and Gutin in [9] defined a locally semicomplete digraph as following, a digraph D is a **locally in-semicomplete (locally out-semicomplete)** if for every vertex x of D, the in-neighbours (out-neighbours) of x induce a **semicomplete** digraph. A digraph D is **locally semicomplete** if it is both locally in-semicomplete and locally out-semicomplete.

A digraph $D = (V, A)$ is a **semicomplete multipartite** if there is a partition $V_1, V_2, \ldots, V_c$ of V into independent sets so that every vertex in V_i shares an arc with every vertex in V_j for $1 \leq i < j \leq c$.

Definition 1.6 *[8] A* ***locally semicomplete multipartite*** *digraph D is obtained from a locally semicomplete digraph F with $V(F) = \{v_1, v_2, \ldots, v_c\}$ by blowing up each vertex $v_i \in V(F)$ into one independent set V_i in D, such that $N_D^\lambda(x) = V_{i_1} \cup \cdots \cup V_{i_p}$ for any $x \in V_i$ if and only if $N_F^\lambda(v_i) = \{v_{i_1}, \ldots, v_{i_p}\}$,where $\lambda \in \{+, -\}$ and $\{v_{i_1} \cup \cdots \cup v_{i_p}\} \subset V(F)$.*

Definition 1.7 *[9] A digraph D is* **transitive***, if for every pair (x, y) and (y, z) of arcs in D with $x \neq z$, the arc (x, z) is also in D. A digraph D is a* **quasi-transitive***, if for every triple x, y, z of distinct vertices of D such that (x, y) and (y, z) are arcs of D, there is at least one arc between x and z. Clearly, a semicomplete digraph is a quasi-transitive.*

The following theorem is an equivalent definition of a strong quasi-transitive digraph.

Theorem 1.8 *(***Canonical Decomposition***, Bang-Jenson and Huang, Theorem 3.5 of [13]) Let D be a strong quasi-transitive digraph, then there exist a strong semicomplete digraph S on s vertices and quasi-transitive digraphs $Q_1, \ldots, Q_s$ such that $D = S[Q_1, \ldots, Q_s]$.*

For an integer $k \geq 2$, let P_k denote the dipath on k vertices. A subdigraph H of a digraph D is a P_k-subdigraph if $H \cong P_k$. If D does not have an induced P_k, then for any P_k-subdigraph H of D, we must have $|A(D[V(H)])| \geq k$.

Definition 1.9 *[5] For integers $h \geq k \geq 2$, defined $\mathcal{F}(P_k, h)$ to be the family of all simple digraphs such that $D \in \mathcal{F}(P_k, h)$ if and only if D is strong and satisfies both of the following.*
(i) D contains at least one dipath P_k with $|A(D[V(P_k)])| = h$, and
(ii) for any dipath P_k in D, $|A(D[V(P_k)])| \geq h$.

A graph G to be locally connected, if for every vertex $v \in V(G)$, the vertices adjacent to v induce a connected subgraph in G. M. Algefari et al [3], defined the following.

Definition 1.10 *[3] For a vertex $v \in V(D)$ is k^+***-locally-arc-connected***, (or k^-***- locally-arc-connected***, or k***-locally-arc-connected***, respectively) if $\lambda(D[N^+(v)]) \geq k(\lambda(D[N^-(v)]) \geq k$, or $\lambda(D[N(v)]) \geq k$, respectively). A digraph D is k^+-locally-arc-connected, (or k^-- locally- arc-connected, or k- locally-arc-connected, respectively) if every vertex of D is k^+- locally-arc-connected, (or k^--locally-arc- connected, or k-locally-arc-connected, respectively).*

Definition 1.11 *[24] For any distinct four vertices c_1, c_2, c_3, c_4 of D, D is $\mathcal{H}_1$-quasi-transitive if $c_1 \rightarrow c_2 \leftarrow c_3 \leftarrow c_4$, c_1 and c_4 are adjacent; D is $\mathcal{H}_2$-quasi-transitive if $c_1 \leftarrow c_2 \rightarrow c_3 \rightarrow c_4$, c_1 and c_4 are adjacent; D is $\mathcal{H}_3$-quasi-transitive if $c_1 \rightarrow c_2 \rightarrow c_3 \rightarrow c_4$, c_1 and c_4 are adjacent; D is $\mathcal{H}_4$-quasi-transitive if $c_1 \rightarrow c_2 \leftarrow c_3 \rightarrow c_4$, c_1 and c_4 are adjacent There are four distinct possible orientations of a 3-path; therefore, $\mathcal{H}_i$-quasi-transitive digraphs as* **3-path-quasi-transitive** *digraphs for convenience, where $i \in \{1, 2, 3, 4\}$.*

Definition 1.12 *[6] Let D be a digraph, C_1 , C_2 , ..., C_k be cycle subdigraphs of D and set $\mathcal{F} = \{C_1, C_2, \ldots, C_k\}$, where $k > 0$ is an integer. $\mathcal{F}$ is called an* **cycle vertex cover** *of D, if both*
(i) $V(D) = \cup_{C_i \in \mathcal{F}} V(C_i)$; and
(ii) $\cup_{C_i \in \mathcal{F}} C_i$ is weakly connected.

Definition 1.13 *[36] Let D be a digraph. We define D to be a* **circulation** *if for any $v \in V(D)$, we have $d^+_D(v) = d^-_D(v) > 0$; and D is* **eulerian** *if D is a spanning connected circulation. A subdigraph F of D is a* ***cycle factor*** *if F is a spanning circulation, or equivalently, F is a collection of arc-disjoint cycles spanning $V(D)$.*

By definition, if D is a circulation, then every component of D is eulerian. By Theorem 1.2, we observe the following

$$\text{Every circulation is an arc-disjoint union of cycles.} \tag{1}$$

Thus, for a subdigraph F of D is a cycle factor, if F is a collection of arc-disjoint cycles spanning $V(D)$.

Definition 1.14 *[36] Let F be a circulation of a digraph D and D/F denote the digraph formed from D by contracting arcs in $A(F)$, for any circulation F of D, define*
(i) $f_D(F) = min\{|\mathcal{C}| : \mathcal{C}$ is a cycle vertex cover of $D/F\}$ and,
(ii) $f(D) = min\{f_D(F) : F$ is a circulation of $D\}$.

The following is well-known or immediately from the definition. Following [29], some digraph products are defined as follows.

Definition 1.15 *[29] Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two digraphs, such that*

$$V_1 = \{u_1, u_2, \ldots, u_{n_1}\} \text{ and } V_2 = \{v_1, v_2, \ldots, v_{n_2}\} \tag{2}$$

Then the Cartesian product, the Direct product and the Strong product of D_1 and D_2 are defined as following,
(i) ***The Cartesian product*** *denoted by $D_1 \Box D_2$ is the digraph with vertex set $V_1 \times V_2$ and*

$$A(D_1 \Box D_2) = \{(u_i, v_j)(u_s, v_t) : u_i = u_s \text{ and } v_j v_t \in A_2, \text{ or } u_i u_s \in A_1 \text{ and } v_j = v_t\}.$$

(ii) ***The Direct product*** *denoted by $D_1 \times D_2$ is the digraph with vertex set $V_1 \times V_2$ and*

$$A(D_1 \times D_2) = \{(u_i, v_j)(u_s, v_t) : u_i u_s \in A_1 \text{ and } v_j v_t \in A_2\}.$$

(iii) ***The Strong product*** *denoted by $D_1 \boxtimes D_2$ is the digraph with vertex set $V_1 \times V_2$ and*

$$A(D_1 \boxtimes D_2) = \{(u_i, v_j)(u_s, v_t) : u_i = u_s \text{ and } v_j v_t \in A_2, \text{ or } u_i u_s \in A_1 \text{ and } v_j = v_t \text{ or both } u_i u_s \in A_1 \text{ and } v_j v_t \in A_2\}.$$

v) ***The Lexicographic product*** *denoted by $D_1[D_2]$ is the digraph with vertex set $V_1 \times V_2$ and*

$$A(D_1[D_2]) = \{((u_i, v_j), (u_s, v_t)) : u_i = u_s and (v_j, v_t) \in A_2 \text{ or } (u_i, u_s) \in A_1\}.$$

The following figures illustrate the definition of the Cartesian product (Fig. 1.), the Direct product(Fig. 2.) and Strong product (Fig. 3.) of P_4 and C_3.

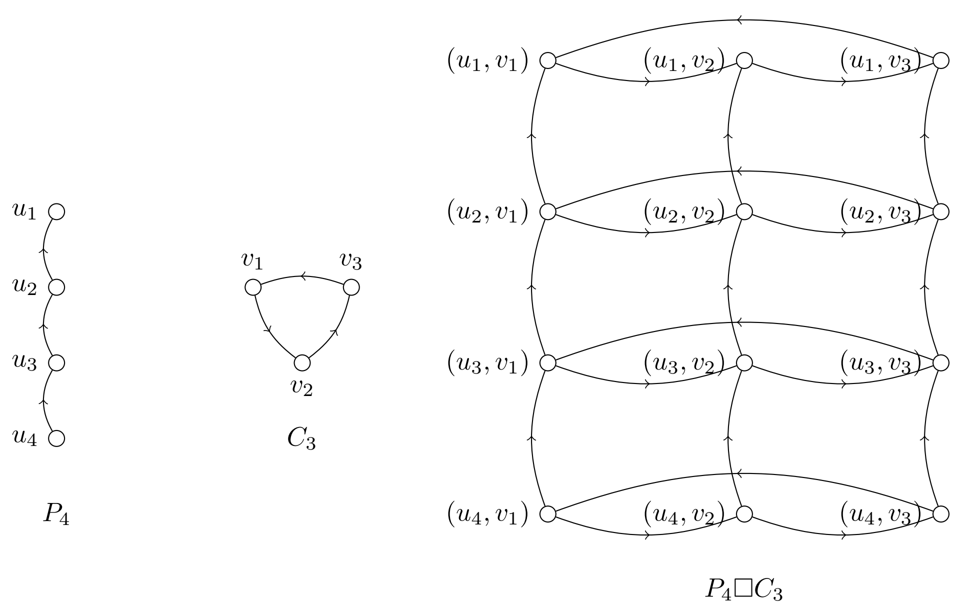


Figure 1. The digraphs P_4, C_3 and the Cartesion product $P_4 \Box C_3$

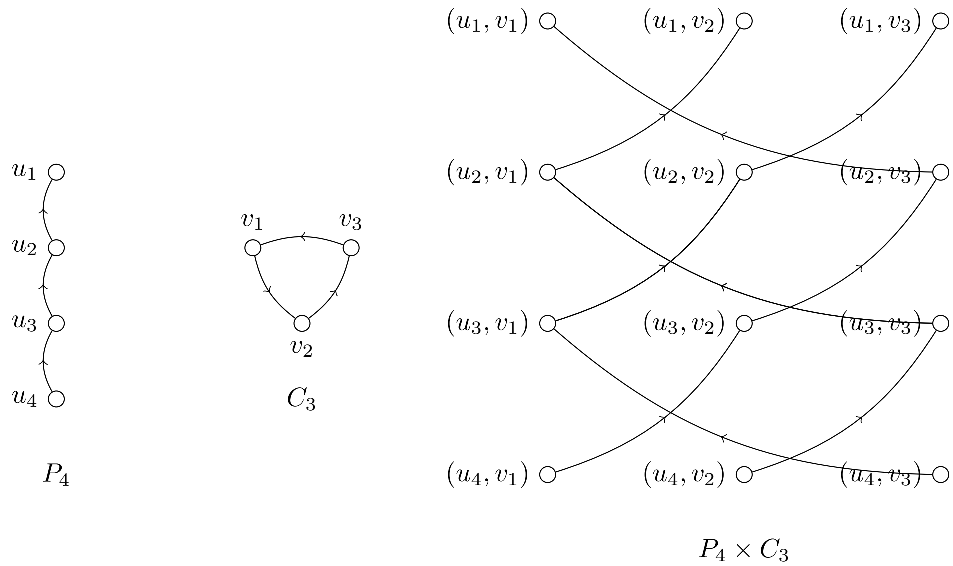


Figure 2. The digraphs P_4, C_3 and the Direct product $P_4 \times C_3$

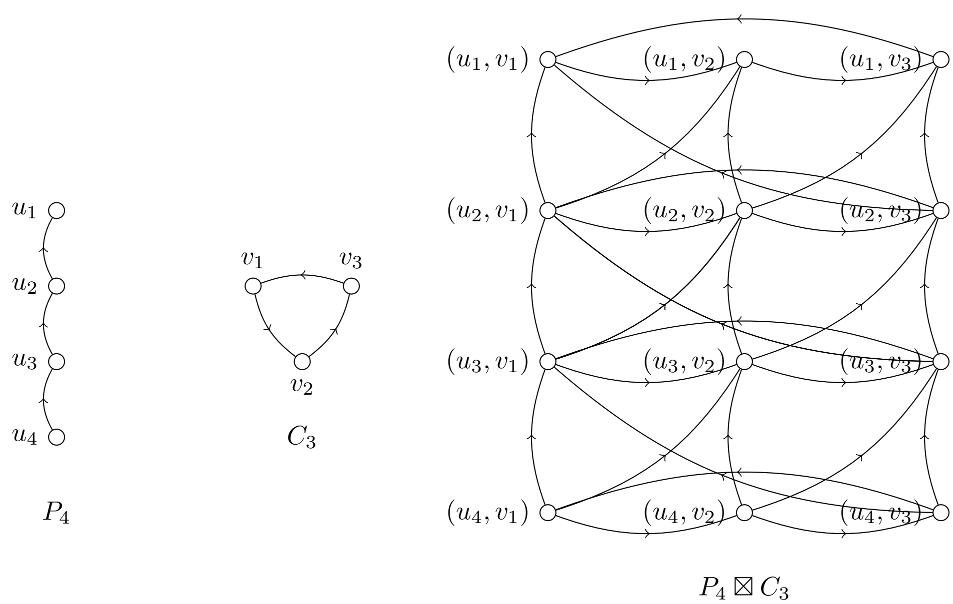


Figure 3. The digraphs P_4, C_3 and the Strong product $P_4 \boxtimes C_3$

1.2 Main Results

This dissertation focuses on a study of dicycle cover and supereulerian digraphs from the following aspects.

1. **Strong trail-connected, Supereulerian and Trailable Digraphs.** Digraphs.
   For a digraph D, D is trailable digraph if D has a spanning trail. A digraph D is strongly trail-connected if for any two vertices u and v of D, D posses both a spanning (u, v)-trail and a spanning (v, u)-trail. As the case when $u = v$ is possible, every strongly trail-connected digraph is also supereulerian. Let D be a digraph. Let $S(D) = \{e \in A(D) : e \text{ is symmetric in } D\}$. A digraph D is symmetric if $A(D) = S(D)$. The symmetric core of D, denoted by $J(D)$, has vertex set $V(D)$ and arc set $S(D)$. We have found a well-characterized digraph family $\mathcal{D}$ each of whose members does not have a spanning trail with its underlying graph spanned by a $K_{2,n-2}$ such that for any strong digraph D with its matching number $\alpha'(D)$ and arc-strong-connectivity $\lambda(D)$, if $n = |V(D)| \geq 3$and $\lambda(D) \geq \alpha'(D) - 1$, then each of the following holds.
   (i) There exists a family $\mathcal{D}$ of well-characterized digraphs such that for any digraph D with $\alpha'(D) \leq 2$, D has a spanning trial if and only if D is not a member in $\mathcal{D}$.
   (ii) If $\alpha'(D) \geq 3$, then D has a spanning trail.
   (iii) If $\alpha'(D) \geq 3$ and $n \geq 2\alpha'(D) + 3$, then D is supereulerian.
   (iv) If $\lambda(D) \geq \alpha'(D) \geq 4$ and $n \geq 2\alpha'(D) + 3$, then for any pair of vertices u and v of D, D contains a spanning (u, v)-trail.

2. **Supereulerian Digraph Strong Products.** A cycle vertex cover of a digraph D is a collection of directed cycles in D such that every vertex in D lies in at least one dicycle in this collection, and such that the union of the arc sets of these directed cycles induce a connected subdigraph of D. A subdigraph F of a digraph D is a circulation if for every vertex v in F, the indegree of v equals its outdegree, and a spanning circulation if F is a cycle factor. Define $f(D)$ to be the smallest cardinality of a cycle vertex cover of the digraph D/F obtained from D by contracting all arcs in F, among all circulations F of D. In [International Journal of Engineering Science Invention, 8 (2019) 12-19], it is proved that if D_1 and D_2 are nontrivial strong digraphs such that D_1 is supereulerian and D_2 has a cycle vertex cover $\mathcal{C}'$ with $|\mathcal{C}'| \leq |V(D_1)|$, then the Cartesian product D_1 and D_2 is also supereulerian. We prove that for strong digraphs D_1 and D_2, if for some cycle factor F_1 of D_1, the digraph formed from D_1 by contracting arcs in F_1 is hamiltonian with $f(D_2)$ not bigger than $|V(D_1)|$, then the strong product D_1 and D_2 is supereulerian.

Chapter 2

2 Literature Review

2.1 Related Results in undirected Graphs

In this section, we will give a brief review of supereulerian undirected graphs. In 1962, a Chinese mathematician called Kuan Mei-Ko was interested in a postman delivering mail to a number of streets such that the total distance walked by the postman was as short as possible. Motivated by the Chinese Postman Problem, Boesch et al. [16] proposed the supereulerian problem which determines of a graph has a spanning eulerian subgraph. They indicated that this might be a difficult problem. Pulleyblank [43] showed that such a decision problem, even when restricted to planar graphs, is $\mathcal{NP}$-complete. Since then, there have been lots of researches on this topic. Catlin [17] in 1992 presented the first survey on supereulerian graphs. Later Chen et al. [20] gave an update in 1995, specifically on the reduction method associated with the supereulerian problem. A latest survey on supereulerian graphs is given in [34].

The following corollary provides a sufficient condition for the existence of edge-disjoint spanning trees of cardinality k.

Corollary 2.1 *([42], [33], [28]) Every finite $2k$-edge-connected graph has k edge-disjoint spanning trees.*

Jaeger [32] and Catlin [18] independently showed the following theorem.

Theorem 2.2 *(Jeager [32], Catlin [18]) Every 4-edge-connected graph is supereulerian.*

Theorem 2.3 *(Catlin, Corollary 1 of [18]) There exist graph families $\mathcal{F}$ such that if every edge of a connected graph G lies in a subgraph of G isomorphic to a member in $\mathcal{F}$, then G is supereuplerian. In particular, if every edge of G lies in a 3-cycle of G, then G is supereulerian.*

For $X \subset E(G)$, the contraction G/X is obtained from G by contracting each edge of X and deleting the resulting loops. If $H \subset G$, we write G/H for $G/E(H)$. If H is connected, let v_H denote the vertex in G/H to which H is contracted, in this case, H is called the preimage of v_H. A graph G is a **collapsible** [18], if for every even subset $R \subset V(G)$, G has a spanning connected subgraph H_R of G with $O(H_R) = R$. In particular, K_1 is both supereulerian and collapsible and any collapsible graph G is supereulerian. In [18], Catlin showed that every graph G has a unique collection of pairwise disjoint maximal collapsible subgraphs $H_1, H_2, ..., H_c$. The graph obtained from G by contracting each H_i into a single vertex ($1 \leq i \leq c$), is called the **reduction** of G. A graph is reduced if it is the reduction of some other graph. For undirected graph G, Catlin [18] proved that if G has two edge-disjoint spanning tree, then G is collapsible which implies that G is supereulerian. Earlier, Jaeger in [32] proved that such graphs must be supereulerian .

Theorem 2.4 *[32] If a graph G has two edge-disjoint spanning trees, then G is supereulerian.*

Catlin, in [18], showed the following theorem.

Theorem 2.5 *(Catlin's Reduction Method)[18] Let G be a connected graph and G' be the reduction of G. Let H be a collapsible subgraph of G. Then each of the following holds.*
(i) G is collapsible if and only if G/H is collapsible. In particular, G is collapsible if and only if $G' = K_1$.
(ii) G is supereulerian if and only if G/H is supereulerian. In particular, G is supereulerian if and only if G' is supereulerian.

Let $F(G)$ denote the minimum number of edges that must be added to G in order to obtain a graph that has two edge-disjoint spanning trees. Thus, Theorem 2.4 says that if $F(G) = 0$, then G is supereulerian. Catlin [18] defined the reduction of a graph.

Theorem 2.6 *(Theorem 7 of Catlin [18]). If $F(G) \leq 1$; then either G is supereulerian or G can be contracted to K_2.*

Theorem 2.7 *(Theorem 1.5 of Catlin et al. [19]). Let G be a connected graph. If $F(G) \leq 2$, then exactly one of the following holds*
(i) G is supereulerian;
(ii) G has a cut-edge(bridge);
(iii)The reduction of G is $K_{2,s}$ for some odd integer $s \geq 3$.

Motivated by the above result, H-J. Lai and H. Yan [35] obtained the following result for 2-edge-connected simple graphs.

Theorem 2.8 *(Lai and Yan, Theorem 2 of [35]) If G is a 2-edge-connected simple graph and $\alpha'(G) \leq 2$, then G is supereulerian if and only if G is not $K_{2,t}$ for some odd number t.*

2.2 Necessary Condition for Supereulerian Digraphs

In this section, we introduce necessary conditions to a digraph to be supereulerian. The first necessary condition for a digraph to be supereulerian is presented by Y. Hong et al. [30]. In [30] they introduced the following definition.

Definition 2.9 *[30] Let D be a strong digraph and $U \subset V(D)$. Then $D[U]$, the digraph induced by U, has ditrails $P_1, ..., P_t$ such that*
(i) $\bigcup_{i=1}^{t} V(P_i) = U$; and
(ii) $A(P_i) \cap A(P_j) = \emptyset$ for any $i \neq j$.
Let $\tau(U)$ be the minimum value of such t. Then $c(G(D[U])) \leq \tau(U) \leq |U|$ where $c(G(D[U]))$ is the number of components of the underlying graph of $D[U]$.
For any $A \subset V(D) - U$, denote $B := V(D) - U - A$ and let
$h(U, A) := min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} + min\{|(U, B)_D|, |(B, U)_D|\} - \tau(U)$, and
$h(U) := min\{h(U, A) : A \cap U = \emptyset\}$.

The next proposition has been provided by Y. Hong et al. [30] as a necessary condition for a digraph D to be supereulerian. It has been used to show that there exists a families of strong digraphs each of which contains no spanning eulerian subdigraphs (non-supereulerin).

Proposition 2.10 *(Hong, Lai and Liu, Proposition 2.1 of [30]) If D has a spanning eulerian subdigraph, then for any $U \subset V(D)$, $h(U) \geq 0$.*

In the rest, we will display some of the results that have used Proposition 2.10 to construct the infinity families of non-supereulerian digraphs.

Example 2.11 *[30] Let $k_1, k_2, l \geq 2$ be integers, and D_1 and D_2 be two disjoint complete digraphs of order k_1+1 and k_2+1, respectively, and let U be an independent set of size ℓ such that $(V(D_1) \cup V(D_2)) \cap U = \emptyset$. Let $\mathcal{D}(k_1, k_2, \ell)$ denote the family of digraphs such that $D \in \mathcal{D}(k_1, k_2, \ell)$ if and only if D is the digraph obtained from $D_1 \cup D_2 \cup U$ by adding all arcs directed from every vertex in U and D_2 to every vertex in D_1, and all arcs directed from every vertex in D_2 to every vertex in U, and then by adding an set of $l-1$ arcs directed from some vertices in D_1 to some vertices in D_2. Assume $k_1, k_2 \geq \ell - 1$. For any $D \in \mathcal{D}(k_1, k_2, \ell)$, $V(D) = k_1 + k_2 + \ell + 2$, and D is a strong digraph with minimum degree $\delta^+(D) = k_1$ and $\delta^-(D) = k_2$. Let $A = V(D_1)$.Then*

$$h(U, A) = |\partial_D^+(A)| + |(U, V(D) - U - A)_D| - \tau(U) = (\ell - 1) - \ell < 0.$$

By Proposition 2.10, D does not have a spanning eulerian subdigraph.

Example 2.12 *[31] Let $k_1, k_2 \geq 2$ be integers and for any $i \in \{1, 2\}$. Let $\mathcal{D}(i, k_2, 3)$ and $\mathcal{D}(k_1, i, 3)$ be infinity families defined as Example 2.11. Let $\mathcal{D}_2 \subset \cup_{i=1}^{2}(\mathcal{D}(i, k_2, 3) \cup \mathcal{D}(k_1, i, 3))$ be the family of digraphs with $\delta^+(D) = \delta^-(D) = 2$ for each $D \in \mathcal{D}_2$. As each $D \in \cup_{i=1}^{2}(\mathcal{D}(i, k_2, 3) \cup \mathcal{D}(k_1, i, 3))$, $D \in \mathcal{D}(k_1, k_2, \ell)$. By Example 2.11, D contains no spanning closed ditrails. Thus, every one in $\mathcal{D}_2$ is non-supereulerian.*

Example 2.13 *[31] Let $k_1, k_2 \geq 2$ be integers, Let $\mathcal{D}(0, k_2, 2)$ and $\mathcal{D}(k_1, 0, 2)$ be infinity families defined as Example 2.11 where $U = \{u_1, u_2\}$. let $\mathcal{D}_3$ be the set of digraphs obtained from digraphs in $\mathcal{D}(0, k_2, 2) \cup \mathcal{D}(k_1, 0, 2)$ by replacing a vertex in U by a dicycle $u_1u_2u_1$ of length 2 and adding all the arcs from $\{u_1, u_2\}$ to $V(D_1)$ and all the arcs from $V(D_2)$ to $\{u_1, u_2\}$. Let $D \in \mathcal{D}_3$, let $A = V(D_1)$ and $V(D) - U - A = V(D_2)$. As $\tau(U) = 2$, then*

$$h(U, A) = min\{|\partial_D^+(A)|, |\partial_D^-(A)|\} + min\{|(U, V(D) - U - A)_D|, |(V(D) - U - A, U)_D|\} - \tau(U) = 1 + 0 - 2 < 0.$$

Thus, D is non-supereulerian by Proposition 2.10.

Example 2.14 *[4] Let $\alpha, \beta, k > 0$ be integers with $\alpha, \beta \geq k+1$, and let A and B be two disjoint set of vertices with $|A| = \alpha$ and $|B| = \beta$. Let $l \geq \alpha\beta + 1$ be an integer and let U be an independent set of size ℓ such that $(A \cup B) \cap U = \emptyset$. Let $D = D(\alpha, \beta, k, \ell)$ is a digraph obtained from $V(D) = A \cup B \cup U$ by adding all arcs directed from every vertex in U and in B to every vertex in A and all arcs directed from* *every vertex in B to every vertex in U, and then by adding all arcs directed from every vertex in A to every vertex in B. (See Fig.4.). Thus $D[A \cup B] \cong K^*_{\alpha+\beta}$ and for any $u \in U$, $N^+_D(u) = A$, $N^-_D(u) = B$. As $|\partial^+_D(A)| = \alpha\beta$, and $|(U, B)_D| = 0$ and so $\tau(U) = |U| > \alpha\beta$. Therefore we have*

$$h(U, A) = |\partial^+_D(A)| + |(U, B)_D| - \tau(U) = \alpha\beta - |U| < 0.$$

It follows from Proposition 2.10, D is non-supereulerian.

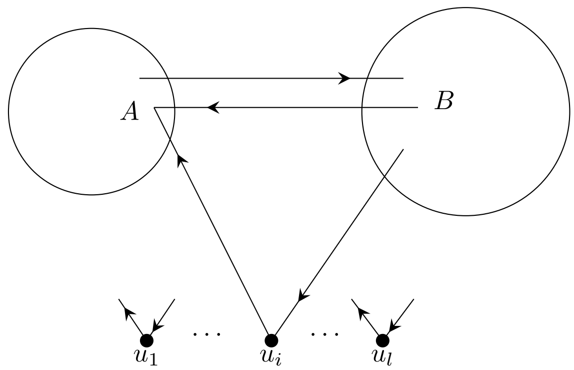


Figure 4. The digraph $D = D(\alpha, \beta, k, \ell)$.

From Example 2.14, M. Algefari et al. [4] showed that there exists an infinite family of non-supereulerian digraphs with arbitrarily high arc-strong connectivity such that every arc of each of these digraphs lies in a directed 3-cycle. Hence both Theorem 2.2 and Theorem 2.3 cannot be directly extended to digraphs. Moreover, it follows from Definition 1.9 that the previous example investigated forbidden induced subdigraph conditions to assure the existence of non-supereulerian digraphs where Algefari et al. in [5] proved that digraphs in $\mathcal{F}(P_3, h)$ with $3 \leq h \leq 4$ are not necessarily supereulerian, as can be seen in the Example 2.14 above. Since any $D \in D(\alpha, \beta, k, \ell)$ is non-supereulerian. By Definition 1.9, $D \in \mathcal{F}(P_3, 4)$.

The k-locally-arc-connected digraphs are defined at Definition 1.10, M. Algefari, H-J. Lai, J. Xu [3] showed that Proposition 2.10 can be applied to show that there exists a family of strong and locally k^+-arc-connected which is non-supereulerian digraphs and non-supereulerian locally k-arc-connected digraphs. The following have been proved by Algefari et al. [3] to show that the following digraph $D = D(n_1, n_2, \ell) \in \mathcal{D}(k, \ell)$ is a locally k^+- arc-connected digraph that is non-supereulern, also they proved that D is a locally k- arc-connected digraph.

Example 2.15 *[3] Let $k > 0$, $\ell > (k+1)^2$ and $n_1 \geq n_2 \geq k+2$ be integers, D_1 and D_2 be two vertex disjoint complete digraphs on n_1 and n_2 vertices, respectively, $X \subset V(D_1)$ and $Y \subset V(D_2)$ with $|X| = |Y| = k+1$ and let U be a set of independent vertices of size ℓ such that $(V(D_1) \cup V(D_2)) \cap U = \emptyset$. Let $\mathcal{D}(k, \ell)$ denote the family of digraphs such that $D = D(n_1, n_2, \ell) \in \mathcal{D}(k, \ell)$ if and only if D is the digraph obtained from the disjoint union $D_1 \cup D_2 \cup U$ by adding all arcs directed from every vertex in U and D_2 to every vertex in D_1, and all arcs directed from every vertex in D_2 to every vertex in U, and then by adding $(k+1)^2$ arcs from X to Y. (See Fig. 5.). In [3], they proved that D is a locally k^+-arc-connected digraph. By applying Proposition 2.10, Let $A = V(D_1)$. Then*

$$h(U,A) = |\partial_D^+(A)| + |(U, V(D) - U - A)_D| - \tau(U) = (k+1)^2 + 0 - \ell < 0.$$

Thus, D is non-supereulerian.

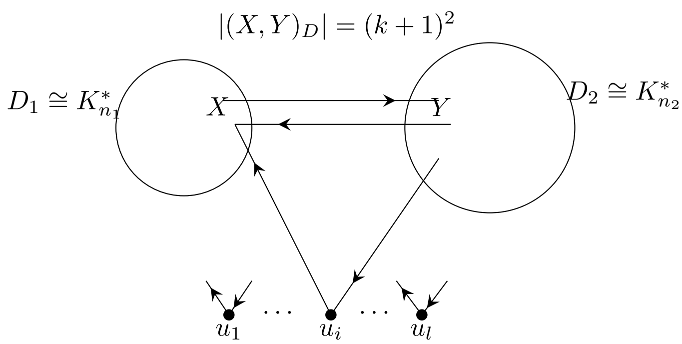


Figure 5. The digraph $D = D(n_1, n_2, \ell)$, with $n_1, n_2 \geq k+2$, and $\ell > (k+1)^2$.

The following example indicate that there exists a family of non-supereulerian bipartite digraphs.

Example 2.16 *[48] Let $k > 0$ and $\ell \geq \lfloor \frac{k}{2} \rfloor 2+1$ be integers, a, b be even integers with $a \leq b$ and $a+b = 2k$, and let A and B be two disjoint sets of vertices with $|A| = a$ and $|B| = b$. Let U be an independent set of size ℓ such that $(A \cup B) \cap U = \emptyset$. Define a digraph $D = D(a, b, k, \ell)$ such that $V(D) = A \cup B \cup U$ and $A(D)$ consists exactly the arcs satisfying the following (See Fig. 6).*

(D1) *$D[A]$ is a complete bipartite digraph $k(\frac{a}{2}, \frac{a}{2})$ with vertex bipartition (X_1, Y_1) such that $|X_1| = |Y_1| = \frac{a}{2}$; and $D[B]$ is a complete bipartite digraph $k(\frac{b}{2}, \frac{b}{2})$ with vertex bipartition (X_2, Y_2) such that $|X_2| = |Y_2| = \frac{b}{2}$.*

(D2) *$|(X_1, Y_2)_D \cup (Y_1, X_2)_D| = \lfloor \frac{k}{2} \rfloor$ and $|(X_2, Y_1)_D \cup (Y_2, X_1)_D| = \lfloor \frac{k}{2} \rfloor$.*

(D3) *for every vertex $u \in U$, and for every $x' \in X_1$ and $x'' \in X_2$, we have both $(u, x'), (x'', u) \in A(D)$.*

From (D1),(D3), D is a bipartite digraph with vertex bipartition (X, Y), where $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2 \cup U$. Moreover, D is non-supereulerian, since $|(X_1, Y_2)_D \cup (Y_1, X_2)_D| = \frac{k}{2}$, and $|\partial_D^+(A)| = \frac{k}{2}$. By (D3), $|(U, B)_D| = 0$ and $\tau(U) = \ell \geq \frac{k}{2} + 1$. By applying Proposition 2.10, it follows that

$$h(U,A) = |\partial_D^+(A)| + |(U, B)_D| - \tau(U) = \tfrac{k}{2} - |U| < 0.$$

Thus $h(U) < 0$, and so by Proposition 2.10, D is not supereulerian.

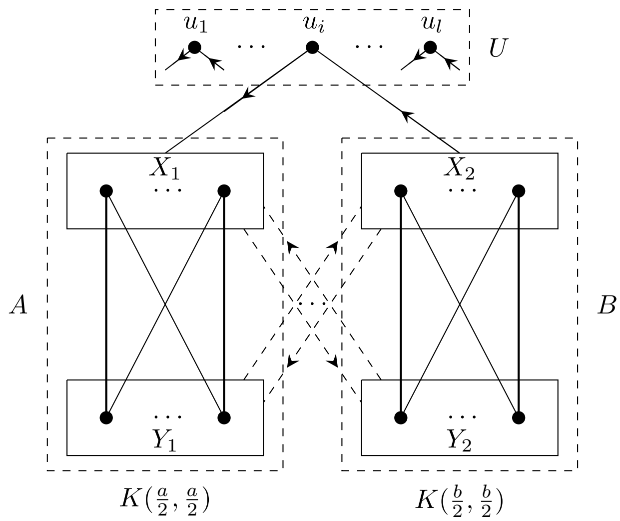


Figure 6. The digraph $D(a, b, k, \ell)$.

The following is another necessary condition for a digraph to be supereulerian has been investigated by Alsatami et al. [7].

Lemma 2.17 *(K.A. Alsatami et al., Lemma 2 of [7]) A digraph D is not supereulerian if for some integer $m > 0$, $V(D)$ has vertex disjoint subsets $\{B, B_1, \ldots, B_m\}$ satisfying both of the following:*
i) $N_D^-(B_i) \subset B$, for all $i \in \{1, 2, \ldots, m\}$.
ii) $|\partial_D^-(B)| \leq m - 1$.

Lemma 2.17 has been helped many researchers to investigate the non-supereulerianicity for some families of digraphs, the following examples showed that.

Example 2.18 *[7] Let $n_1, n_2 \geq 3$ be integers and $C_{n_1} = v_{11}v_{12}\ldots v_{1n_1}v_{11}$ and $C_{n_2} = v_{21}v_{22}\ldots v_{2n_2}v_{12}$ be to dicycles of length n_1 and n_2, respectively, such that $V(C_{n_1}) \cap V(C_{n_2}) = \emptyset$. Consider D' is a digraph obtained from C_{n_1} and C_{n_2} by identifying the arc (v_{11}, v_{12}) in C_{n_1} with the arc (v_{21}, v_{22}) in C_{n_2}. Let $V(B) = \{v_{12}\}$, $V(B_1) = \{v_{13}\}$ and $V(B_2) = \{v_{23}\}$ be a subdigraphs of D'. By applying Lemma 2.17, so D' is non-supereulerian.*

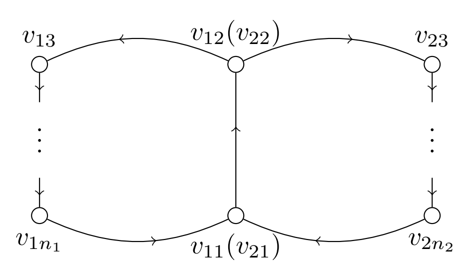


Figure 7. The digraph D'

The families $\mathcal{F}(P_4, 5)$, $\mathcal{F}(P_4, 6)$ and $\mathcal{F}(P_4, 7)$ are defined at Definition 1.9. The following examples have been showed the families $\mathcal{F}(P_4, 5)$, $\mathcal{F}(P_4, 6)$ and $\mathcal{F}(P_4, 7)$ are non-supereulerian digraphs.

Example 2.19 *[5] Let $M = xzy$ be a symmetric dipath, $Q = xuy$ be a dipath and $H_i = xv_iy$, $i \geq 1$ be dipaths. Let $D_1 = M \cup Q \cup H_1 \cup \{(u, z)\}$. For any P_4 in D_1, $|A(D[V(P_4)])| \geq 5$ and $|A(D[V(uzxv_1)])| = 5$ and by Lemma 2.17 $B = \{x\}$, $B_1 = \{u\}$ and $B_2 = \{v_1\}$. Thus, D_1 is not supereulerian. Let $D_\ell = D_1 \cup \{H_2, \ldots, H_\ell\}$. Then $D_\ell \in \mathcal{F}(P_4, 5)$ and by Lemma 2.17, D_ℓ is non-supereulerian.*

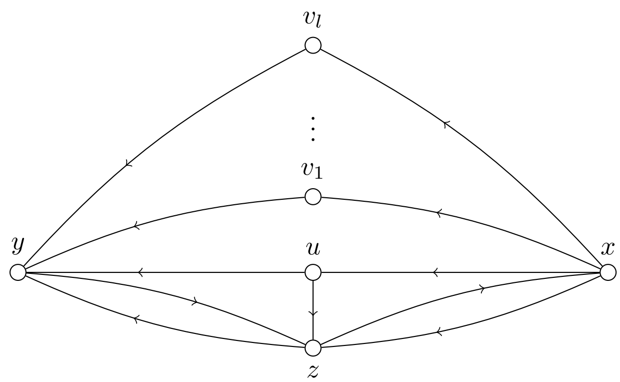


Figure 8. The digraph family D_ℓ

Example 2.20 *[5] Let $M = xzy$ be a symmetric dipath, $Q = xuy$ be a dipath and $H_i = xv_iy$, $i \geq 1$ be dipaths. Let $D_1 = M \cup Q \cup H_1$. For any P_4 in D_1, $|A(D[V(P_4)])| = 6$ and by Lemma 2.17 let $B = \{x\}$, $B_1 = \{u\}$ and $B_2 = \{v_1\}$. Thus, D_1 is not supereulerian. Let $D_\ell = D_1 \cup \{H_2, ..., H_\ell\}$. Then $D_\ell \in \mathcal{F}(P_4, 6)$ and by Lemma 2.17, D_ℓ, is non-supereulerian*

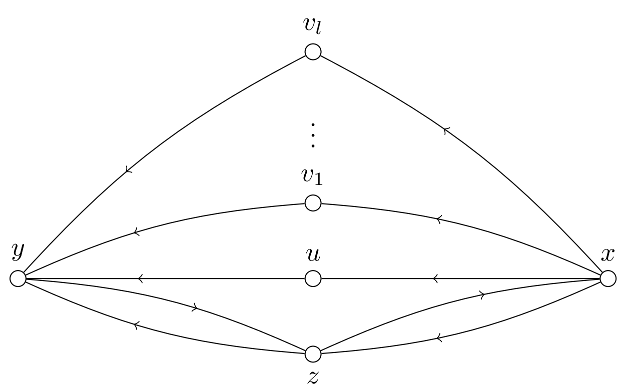
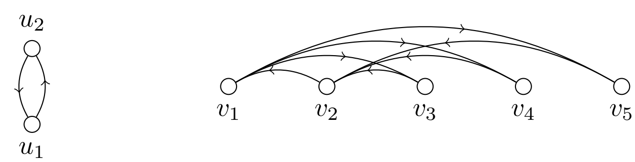


Figure 9. The digraph family D_ℓ

Example 2.21 *[5] Let $M = xzy$ be a symmetric dipath, $Q = xuy$ be a dipath and $H_i = xv_iy$, $l \geq i \geq 1$ be dipaths. Let $D_l = M \cup Q \cup \{\cup_{i=1}^{l} H_i\} \cup \{(x, y)\}$, $D_l \in \mathcal{F}(P_4, 7)$. By Lemma 2.17, let $B = D[x]$, $B_1 = D[u]$ and $B_2 = D[v_1]$, we have D_l is non-supereulerian.*

As we mentioned on previous chapter for Definition 1.15 of product digraphs and Definition 1.12 of a cycle vertex cover of a digraph D, Alsatami et al. [6] used Lemma 2.17 to show that the Cartesian product of supereulerian digraph D_1 and a strong digraph D_2, which has an eulerian vertex cover with m eulerian subdigraphs and $m > |V(D_1)|$, that the Cartesian product$D_1 \Box D_2$ is non-supereulerian.

Example 2.22 *[6] Let D_1 be a supereulerian digraph with $V(D_1) = \{u_1, u_2\}$ and $A(D_1) = \{(u_1, u_2), (u_2, u_1)\}$. Let D_2 be a strong digraph with $V(D_2) = \{v_1, v_2, v_3, v_4, v_5\}$ and $A(D_2) = \{(v_2, v_1), (v_1, v_3), (v_3, v_2), (v_1, v_4),$ $(v_4, v_2), (v_1, v_5), (v_5, v_2)\}$, which has an eulerian vertex cover with 3 eulerian subdigraphs. By Definition 1.15, we can obtain the Cartesian product $D_1 \Box D_2$ of D_1 and D_2 (See Fig. 10). Let B, B_1, B_2 and B_3 be vertex-disjoint subsets of $V(D_1 \times D_2)$ with $B = \{(u_1, v_1), (u_2, v_1)\}$, $B_1 = \{(u_1, v_3), (u_2, v_3)\}$, $B_2 = \{(u_1, v_4), (u_2, v_4)\}$ and $B_3 = \{(u_1, v_5), (u_2, v_5)\}$. We find that $N^-_D(B_i) \subset B$ for $i \in \{1, 2, 3\}$ and $|\partial^-_D(B)| = 2$. By Lemma 2.17, the Cartesian product $D_1 \Box D_2$ is non-supereulerian.*



D_1 $\qquad$ D_2

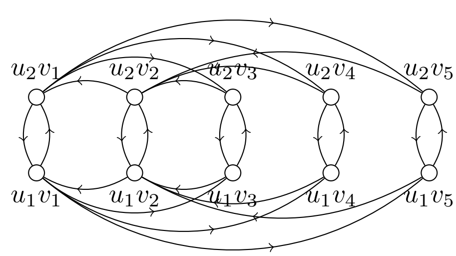
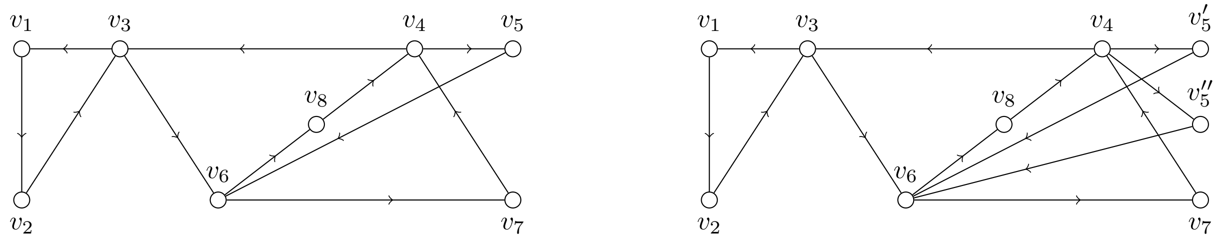


Figure 10. $D_1 \Box D_2$

The following two examples have been used Lemma 2.17 to show that the extended digraph of an eulerian digraph and the digraphs under some degree condition are non-supereuleian.

Example 2.23 *[23] (Extended digraphs) Let D be an eulerian digraph with $V(D) = \{v_1, v_2, ..., v_8\}$ and let D' be a digraph obtained from of D by splattering one vertex say v_5 to v'_5 and v''_5 such that $N^+_{D'}(v'_5) = N^+_{D'}(v''_5) = N^+_D(v_5)$ and $N^-_{D'}(v'_5) = N^-_{D'}(v''_5) = N^-_D(v)$, so $V(D') = \{v_1, v_2, v_3, v_4, v'_5, v''_5, v_6, v_7, v_8\}$ (see Fig. 13). Let B, B_1, B_2, B_3 be vertex disjoint subsets of $V(D')$ with $B = \{v_4\}$, $B_1 = \{v_1, v_2, v_3\}$, $B_2 = \{v'_5\}$ and $B_3 = \{v''_5\}$. We find that $N^-_D(B_i) \subset B$ for $i \in \{1, 2, 3\}$ and $|\partial^-_D(B)| = 2$. By Lemma 2.17, the digraph D' is non-supereulerian.*



An eulerian digraph D An extended eulerian digraph D'

Figure 11. The digraph D and D'

Example 2.24 *[1] Let G, H be two digraphs isomorphic to K^*_m , where $m \geq 2$. Let $u, x \in V(G)$ and $v, y \in V(H)$. Let $D_m = G \cup H \cup \{(z_1, u), (z_1, v), (x, z_2), (y, z_2), (z_2, z_1)\}$. Then $V(D_m) = n = 2m + 2$. (See Fig. 12. for $m = 3$). By Lemma 2.17 with $A = D[z_1]$, $B_1 = G$ and $B_2 = H$, we conclude that D_m is not supereulerian eventhough $d^+_D(x) + d^+_D(y) + d^-_D(u) + d^-_D(v) = 4m = 2n - 4$.*

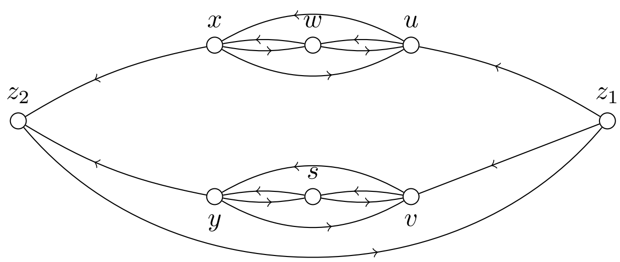


Figure 12. The digraph family D_m

Finally, there is another necessary condition of some specific digraphs to be supereulerian which is also sufficient condition. Follows from the definition of semicompete multipartite digraphs and Definition 1.13 of a cycle factor, Bang-Jensen and Maddaloni [10] proved the following theorem for a semicomplete multipartite digraph to be supereulerian.

Theorem 2.25 *[10] Let D be a semicomplete multipartite digraph. Then D is supereulerian if and only if it is strong and has a cycle factor.*

Next example showed the existences of a cycle factor is the necessary condition of a strong semicomplete multipartite digraphs to be supereulerian.

Example 2.26 *[10] Let D be the semicomplete multipartite digraph with five partitesets U, W, W', Z, Z', where U has size $k+1$ and the others have size k. W has all the possible arcs from all the other partite sets and so does W'. Z has all the possible arcs to all the other partite sets and so does Z'. Moreover there* *is a matching from W to Z. Since D has no cycle factor; then by Theorem 2.25, D is not supereulerian. (Fig. 13. shows an example with $k = 3$ where the thick arcs between sets represent complete adjacency in the direction of the arc, double arcs indicate arcs in both directions).*

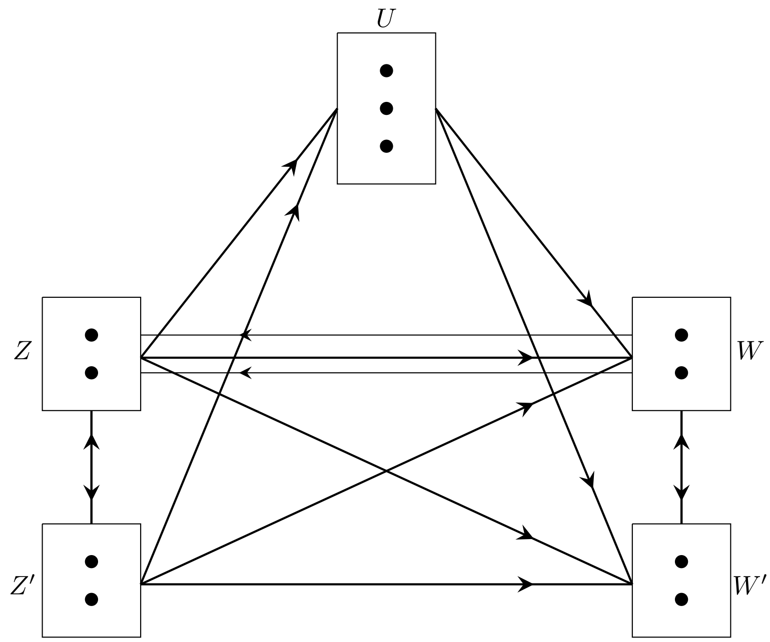


Figure 13. A non-supereulerian semicomplete multipartite digraph D with $\alpha(D) = 3$ and $\lambda(D) = 2$.

Follows Definition 1.6 of a locally semicomplete multipartite digraphs, F. Liu, Z-X. Tian, D. Li [38] generalized the result of Bang-Jensen and Maddaloni for a semicomplete mulltipartite digraph that they used the same approach that Bang-Jensen and Maddaloni used in [10] and they proved the following result.

Theorem 2.27 *(Liu, Tian and Li, Theorem 2.5 of [38]) Let D be a locally semicomplete multipartite digraph. Then D is supereulerian if and only if it is strong and has a cycle factor.*

Follows from Definition 1.7 of a quasi-transitive digraph and Definition 1.13 of a cycle factor, the following theorem has been proved by [10] of any quasi-transitive digraphs to be supereulerian. In [10] proved that the existences of a cycle factor if the necessary condition of a strong quasi-transitive digraphs to be supereulerian and it is a sufficient condition as well.

Theorem 2.28 *(Bang-Jenson and Maddaloni, Theorem 2.12 of [10]) Let D be a quasi-transitive digraph. D is supereulerian if and only if it is strong, with canonical decomposition $D = S[Q_1, ..., Q_s]$, and the semicomplete directed multigraph S_1 obtained from D by contracting each Q_i into a single vertex v_i has an cycle factor $\mathcal{E}'$ such that $d^+_{D[\mathcal{E}']}(v_i) \geq \tau(Q_i)$ for every $i = 1, \ldots, s$.*

Next example showed a existences of a cycle factor is a necessary condition of a strong quasi-transitive digraphs to be supereulerian.

Example 2.29 *[10] Let D be the quasi-transitive digraph with vertex set given by an independent set U on k vertices, together with two complete digraphs W, Z on $k-1$ vertices and all the arcs from U to W, all the arcs from Z to $W \cup U$ and a matching from W to Z. Since D does not even have a cycle factor; then by Theorem 2.28, D is not supereulerian. (Fig. 14. shows an example with $k = 3$ where the thick arcs between sets represent complete adjacency in the direction of the arc, double arcs indicate arcs in both directions).*

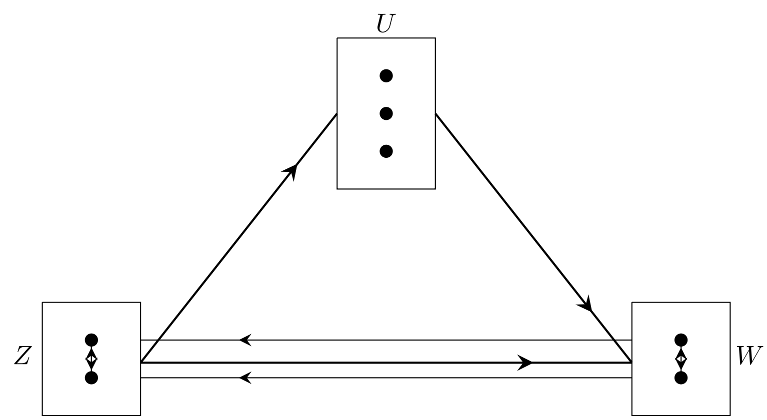


Figure 14. A non-supereulerian semicomplete multipartite digraph D with $\alpha(D) = 3$ and $\lambda(D) = 2$.

Theorem 2.30 *(Dong and Liu, Thorem1.3 of [23]) An extended cycle D' is supereulerian if and only if $D'$is strong and has a cycle factor.*

C. Dong et al. in [24] gave a necessary and a sufficient conditions involving 3-path-quasi-transitive digraphs to be supereulerian.

The 3-path-quasi-transitive digraphs are defined in Definition 1.11 where the following theorem is a necessary condition of the 3-path-quasi-transitive digraphs to be supereulerian and it is also a sufficient condition. In [24] proved the following theorem to for each $i \in \{1, 2, 3, 4\}$.

Theorem 2.31 *[24] Let D be a strong $\mathcal{H}_i$-quasi-transitive digraph, then D is supereulerian if and only if D contains a cycle factor.*

2.3 Degree Condition for Supereulerian Digraphs

In this section will give the brief discussion of sufficient degree conditions for supereulerian of digraphs. One of the motivation of the studies of supereulerian digraphs is the study of hamiltonian digraphs, as hamiltonian graphs are also supereulerian. We start with the main origin of the degree condition idea, Diract condition and Ore conditions, which are commonly used to study hamiltonian (di)graphs. For any graph G, a path that contains every vertex of G is called a Hamilton path of G; similarly, a Hamilton cycle of G is a cycle that contains every vertex of G. A graph is hamiltonian if it contains a Hamilton cycle. The result of Dirac in 1952 introduced in [15] as sufficient conditions for a graph G to be hamiltonian which is a useful result of hamiltonian graphs.

Theorem 2.32 *(Dirac's Theorem)[15] If G is a simple graph with $n \geq 3$ and $\delta(G) \geq \frac{n}{2}$, then G is hamiltonian.*

Ore [41] generalized the previous theorem to introduce the degree condition of graphs to be hamiltonian.

Theorem 2.33 *(Ore's Theorem)[41] A graph satisfying $d(x)+d(y) \geq n$ for every pair x, y of nonadjacent vertices is hamiltonian.*

As it is the case for undirected graphs, some sufficient degree conditions for hamiltonicity in digraphs can be (slightly) weakened to become sharp sufficient conditions for supereulerianity. The property of being supereulerian is at the same time relaxation of being hamiltonian: being supereulerian digraph means having a closed ditrail covering all the vertices of the digraph; being hamiltonian means having a closed ditrail covering all vertices of the digraph without using a vertex twice. In this section, we display some sufficient conditions for a digraph to be supereulerian. For a digraph part, there are many results of digraphs to be hamiltonian.

Theorem 2.34 *(Nash-Williams)[40] Let D be a digraph of order $n \geq 3$ such that for every vertex x, $d^+(x) \geq \frac{n}{2}$ and $d^-(x) \geq \frac{n}{2}$, then D is hamiltonian.*

Theorem 2.35 *(Ghouila-Houri)[27] Let D be a strongly connected digraph of order $n \geq 3$. If $d(x) \geq n$ for all vertices $x \in V(D)$, then D is hamiltonian.*

Theorem 2.36 *(Woodall)[45] Let D be a digraph of order $n \geq 3$. If $d^+(x) + d^-(y) \geq n$ for all pair of non-adjacent vertices, then D is hamiltonian.*

There are two generalzation of Woodall theorem. The first generalization by Meyniel.

Theorem 2.37 *(Meyniel)[39] Let D be a strongly connected digraph of order $n \geq 2$. If $d(x)+d(y) \geq 2n-1$ for all pairs of non-adjacent vertices in D, then D is hamiltonian.*

The second generalization by Bong-Jenson, Gutin and Li in [12] .

Theorem 2.38 *(Bang-Jensen, Gutin, Li, Theorem 4.1 of [12]) Let D be a strongly connected digraph of order $n \geq 2$. Suppose that $min\{d(x), d(y)\} \geq n-1$ and $d(x)+d(y) \geq 2n-1$ for every pair of non-adjacent vertices x, y with a common in-neighbor. Then D is hamiltonian.*

Bang-Jensen, Maddaloni[10] proved the analogue of Meyniel's theorem for supereulerian part which is the degree condition for digraphs to be supereulerian, where they gave some sufficient Ore-type conditions to be supereulerian. In the theorems below, we always assume D is a digraph on n vertices. A pair of vertices x and y are adjacent in D if (x, y) or (y, x) is in $A(D)$.

Theorem 2.39 *(Bang-Jensen, Maddaloni, Theorem 3.6 of [10]) A strong digraph such that $d(x)+d(y) \geq 2n-3$ for all of non-adjacent vertices x, y is supereulerian.*

In [30], Y. Hong, H. Lai, Q. Liu define the the family $\mathcal{D}_0(k_1, k_2, 2)$ is the set of spanning subdigraphs D' of the digraphs D in $\mathcal{D}(k_1, k_2, 2)$ defined in Example 2.11, which satisfy $\delta^+(D') + \delta^-(D') = |V(D')| - 4$. Y. Hong et al. [30] proved that no digraph in $D \in \mathcal{D}_0(k_1, k_2, 2)$ has a spanning eulerian subdigraph. Moreover, Y. Hong, H. Lai, Q. Liu [30] investigated the Ore-type sufficient condition of supereulerian digraphs and proved the following theorem.

Theorem 2.40 *(Hong, Lai, Liu, Theorem 3.4 of [30]) Let D be a strong digraph of order n and minimum out-degree $\delta^+(D) \geq 4$ and minimum in-degree $\delta^-(D) \geq 4$. If $\delta^+(D) + \delta^-(D) \geq n-4$, then the following are equivalent.*
(i) D has a spanning eulerian subdigraph.
(ii) Either $\delta^+(D)+\delta^-(D) > n-4$, or for some integer k_1, k_2,$\delta^+(D) = k_1$,$\delta^-(D) = k_2$ but $D \notin \mathcal{D}_0(k_1, k_2, 2)$.

Follows from the previous theorem, Hong et al in [30] showed that Example 2.11 shows that the bound in Theorem 2.40 is a best possible lower bound of the minimum degree.

There are other degree conditions for supereulerian digraphs. Another Ore-type condition has been investigated. Y. Hong, H. Lai, Q. Liu [31] characterized families of digraphs, let $\mathcal{D}_1$ be the family $\mathcal{D}(k_1, k_2, 2)$ as defined in Example 2.11 which proved that a simple digraph D satisfying $min\{\delta^+(D), \delta^-(D)\} \geq 4$ and $\delta^+(D) + \delta^-(D) \geq n-4$, then D is supereulerian if and only if D is not a member in $\mathcal{D}_1$.

Let $\mathcal{D}_2$ as defined in Example 2.12 which is non-supereulerian and let $\mathcal{D}_3 \subset \mathcal{D}(0, k_2, 2) \cup \mathcal{D}(k_1, 0, 2)$ as Example 2.13. Thus for $i = 1, 2, 3$ none of the spanning subdigraphs of digraphs in $\mathcal{D}_i$ has a spanning eulerian subdigraph. Y. Hong in [31] defined that for $i = 1, 2, 3$, let $\mathcal{F}_i$ be the family of digraphs such that $D \in \mathcal{F}_i$ if and only if for some member $D' \in \mathcal{D}_i$, D is a strong spanning subdigraph of D' satisfying $d^+_D(x) + d^-_D(y) \geq n-4$ for any pair of vertices x, y with $xy \notin A(D)$. Then, each $\mathcal{F}_i$ is also a family of non-supereulerian digraphs, it follows the following theorem.

Theorem 2.41 *(Hong, Liu, Lai, Theorem 3.4 of [31]) Let D be a strong digraph of order $n \geq 11$. If $d^+_D(u)+d^-_D(v) \geq n-4$ for any pair of vertices u, v with $(u, v) \notin A(D)$, then D is supereulerian if and only if it $D \notin \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$.*

Recall that an ordered pair of vertices x, y is dominated (dominating, respectively) if there exists $z \in V(D)$, with $(z, x), (z, y) \in A(D)((x, z), (y, z) \in A(D)$, respectively). Next theorem is due to Zhao and Meng.

Theorem 2.42 *[49] Let D be a strong digraph of order $n \geq 2$. If $d^+_D(x)+d^+_D(y)+d^-_D(u)+d^-_D(v) \geq 2n-1$ for every pair x, y of dominating vertices and every pair u, v of dominated vertices, then D is hamiltonian.*

Algefari [1] studied this kind of sufficient conditions in Theorem 2.42, for a digraph to be supereulerian, and proved the following theorem.

Theorem 2.43 *[1] Let D be a strong digraph of order $n \geq 2$. If $d_D^+(x) + d_D^+(y) + d_D^-(u) + d_D^-(v) \geq 2n - 3$ for every pair x, y of dominating non-adjacent vertices and every pair u, v of dominated non-adjacent vertices, then D is supereulerian.*

In addition, Algefari [1] define infinite family of nonsupereulerian digraphs as seen in Example 2.24 which makes Theorem 2.43 sharp.

2.4 Bang-Jensen and Thomassé Conjecture for Digraphs to be Supereulerian

In this section, we start with a well known theorem of Chvátal Erdös [21] states that every 2-connected graph G with $\kappa(G) \geq \alpha(G)$ is hamiltonian. Thomassen [44] gave an infinite family of non-hamiltonian (but supereulerian) digraphs such that $\kappa(D) = \alpha(D) = 2$, showing that the the Chvátal Erdös theorem does not extend to digraphs. This result motivates Bang-Jensen and Thomassè (2011, unpublished, see [11]) to make the following conjecture.

Conjecture 2.44 *Let D be a digraph. If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.*

Bang-Jensen and Maddaloni [10] indicated that the above condition is not necessary, and considered a directed cycle on four vertices C_4 as an example that where C_4 is eulerian digraph, and hence supereulerian, but $\lambda(C_4) = 1$ and $\alpha(C_4) = 2$. Moreover, they showed that Conjecture 2.44 is true for undirected graph.

Theorem 2.45 *(Bang-Jensen, Maddaloni, Theorem 2.3 of [10]) Let G be an undirected graph on at least three vertices. If $\lambda(D) \geq \alpha(D)$, then G is supereulerian.*

Conjecture 2.44 has motivated many researchers to verified it for many digraph families. Let start with Bang-Jensen and Maddaloni [10], who proved that Conjecture 2.44 is true for semicomplete multipartite digraphs and for quasi-transitive digraphs.

Theorem 2.46 *(Bang-Jensen, Maddaloni, Theorem 2.10 of [10]) Let D be a semicomplete multipartite digraph. If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.*

Theorem 2.47 *(Bang-Jensen, Maddaloni, Theorem 2.13 of [10]) Let D be a quasi-transitive digraph. If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.*

Bang-Jensen and Maddaloni [10] proved the following useful theorem where they used flow theory to show that the condition $\lambda(D) \geq \alpha(D)$ guarantees the existence of a cycle factor. The follow is used to prove Theorem 2.25 and Theorem 2.28.

Theorem 2.48 *(Bang-Jensen, Maddaloni, Theorem 2.4 of [10]) Let D be a digraph. If $\lambda(D) \geq \alpha(D)$, then D has a cycle factor.*

Bang-Jensen and Maddaloni [10] provided Example 2.26 and Example 2.29 to show that there exists infinite families of digraphs with $\lambda(D) \geq \alpha(D) - 1$ that are not supereulerian. Hence, Example 2.26, Example 2.29, respectively, showed that Conjecture 2.44 would be best possible for both semicomplete multipartite digraphs and quasi-transitive digraphs.

Following definition of locally semicomplete multipartite digraph, Definition 1.6, F. Liu, Z. Xian, D. Li [38] generalized the result of Bang-Jensen and Maddaloni for a semicomplete mulltipartite digraph and they proved the following result for a locally semicomplete multipartite digraphs, they used the same approach that Bang-Jensen and Maddaloni used in [10] where F. Liu, et al.[38] used Theorem 2.48 and Theorem 2.27 to drive the following theorem.

Theorem 2.49 *(Liu, Xian, Li, Theorem 2.6 of [38]) Let D be a locally semicomplete multipartite digraph. If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.*

Following the definition of 3-path-quasi-transitive digraphs provided in Definition 1.11, Dong, Liu, Meng,[24], showed that Conjecture 2.44 has been verified for 3-path-quasi-transitive in [24], where the following theorem to for each $i \in \{1, 2, 3, 4\}$.

Theorem 2.50 *(Dong, Liu, Meng, Theorem 1.2 of [24]) Let D be a strong $\mathcal{H}_i$-quasi-transitive digraph for $i \in \{1, 2, 3, 4\}$. If $\lambda(D) \geq \alpha(D)$, then D is supereulerian.*

C. Dong et al. [24] have used Theorem 2.48 and Theorem 2.31 to prove that Conjecture 2.44 is true for 3-path-quasi-transitive digraphs.

Many other researchers have investigated Conjecture 2.44. In particular, Algefari et al.[2] proved the following result.

Theorem 2.51 *(Algefari, Lai, Xu, Theorem 1.5 of [2]) Let D be a strong digraph. If $\lambda(D) \geq \alpha'(D)$, then D is supereulerian.*

As $\alpha'(D) = \alpha'(G(D))$, Algefari et al. [2] used the following fundamental theorem of graph theory to prove Theorem 2.51.

Theorem 2.52 *(Berge, 1957)[14] A matching M in G is a maximum matching if and only if G does not have $M-$ augmenting paths.*

X. D. Zhang, J. Liu, L. Wang, H.-J. Lai [48] proved that Conjecture 2.44 holds for a bipartite digraph with the lower bound begin half of the conjecture bound by proving the following result.

Theorem 2.53 *(Zhang, Liu, Lai, Theorem 1.5 of [48]) Let D be a strong bipartite digraph. If $\lambda(D) \geq \left\lfloor \frac{\alpha(D)}{2} \right\rfloor + 1$, then D is supereulerian.*

X. Zhang et al. [48] provided the following theorem as a tool to prove Theorem 2.53.

Theorem 2.54 *(Zhang, Liu, Lai, Theorem 1.4 of [48]) Let D be a strong bipartite digraph with a vertex bipartition (X, Y) satisfying $|X| \le |Y|$. Each of the following holds.*
(i) If $\delta(D) \ge \left\lfloor \frac{\alpha'(D)}{2} \right\rfloor + 1$, then D is supereulerian.
(ii) Suppose that $\alpha'(D)$ is even and $\alpha'(D) < |X|$. If $\delta(D) \ge \frac{\alpha'(D)}{2}$, then D is supereulerian.

As $\alpha(D) \ge |Y| \ge |X| \ge \alpha'(D)$, X. Zhang et al. [48] concloude that Theorem 2.53 followes from Theorem 2.54 (i). Also, as $\delta(D) \ge \lambda(D) \ge \kappa(D)$, thus $\delta(D)$ can be repleased by either $\lambda(D)$ or $\kappa(D)$ in Theorem 2.54. Moreover, Example 2.16 showed that Theorem 2.53 is sharp in some sense of non-supereulerian strong bipartite digraphs.

In [23], Bang-Jensen and Thomasse's conjecture has also been verified for several extended digraph such as extended hamiltonian, an arc-locally semicomplete digraph, an extended arc-locally semicomplete digraph.

2.5 Supereulerian Digraphs with Global or Local Density Conditions

In this section, we will introduce some local structures of some digraphs to be supereulerian. The following theorem proved by [4].

Theorem 2.55 *(Algefari, Alsatami, Lai, Liu, Theorem 1.3 (i) of [4]) Every symmetrically connected digraph is supereulerian.*

Follows from Definition 1.9, Algefari et al. in [5] observed that if $D \in \mathcal{F}(P_2, 2) \cup \mathcal{F}(P_3, 5)$, then D is symmetrically connected, and so by Theorem 2.55, every digraph in $\mathcal{F}(P_2, 2) \cup \mathcal{F}(P_3, 5)$ is supereulerian.

Theorem 2.56 *(Algefari, Alsatami, Lai, Liu, Theorem 1.3 (ii) of [4]) Every partially symmetric digraph is supereulerian.*

Another result has been proved by Algefari, Lai, Liu and Zhang [5] who studied the supereulerianicity of digraphs in $\mathcal{F}(P_4, h)$, and determined the smallest value of h_4 such that every digraph in $\mathcal{F}(P_4, h_4)$ is supereulerian by proving the following theorem.

Theorem 2.57 *(Algefari et al, Theorem 3.1 (i) of [5]) Every digraph D in $\mathcal{F}(P_4, 8)$ is supereulerian.*

As in Example 2.21 showed that there exist at least one non-supereulerian digraph in $\mathcal{F}(P_4, 7)$ which showed that Theorem 2.57 is sharp in some sense.

As well known, for any digraph D, $0 \le diam(D) \le \infty$. If a digraph D with $diam(D) = 0$, that is, $D \cong k_1^*$, then D is supereulerian. If a digraph D on $n > 1$ vertices with $diam(D) = 1$, that is, $D \cong k_n^*$, then D is supereulerian. In 2018, C. Dong, J. Liu, X. Zhang [22] obtained sufficient condition on digraphs to be supereulerian for a given diameter.

Theorem 2.58 *(Dong, Liu and Zhang, Theorem 3.1 of [22]) A digraph D with $|V(D)| \geq 3$ and $diam(D) \leq 2$ is supereulerian.*

Moreover, Example 2.14 indicated that there are infinitely many non-supereulerian digraphs with $diam(D) = 3$, so Theorem 2.58 is sharp in some sense.

Another result provided in [22], they discussed the supereulerian bipartite digraph with diameter 3 and proved the following theorem of bipartite digraph.

Theorem 2.59 *(Dong, Liu and Zhang, Theorem 4.1 of [22]) A bipartite digraph D with $|V(D)| \geq 4$ and $diam(D) \leq 3$ is supereulerian.*

2.6 Supereulerian Sums and Products of Digraphs

In this section, we introduce the definition of 2-sum digraph and display results of sufficient conditions of 2-sum digraph and product of two digraphs D_1, D_2 to be supereulerian.

2.6.1 Digraph 2-Sum

K. Alsatami, X. Zhang, J.Liu and H-J. Lai in [7] displayed a 2-sum digraph as the following.

Definition 2.60 *Let D_1 and D_2 be two vertex disjoint digraphs, and let $a_1 = (v_{11}, v_{12}) \in A(D_1)$and $a_2 = (v_{21}, v_{22}) \in A(D_2)$ be two distinguished arcs. The 2-sum $D_1 \bigoplus a_1, a_2 D_2$ of D_1 and D_2 with base arcs a_1 and a_2 is obtained from the union of D_1 and $D_2 - a_2$ by identifying v_{11} with v_{21} and v_{12} with v_{22} , respectively. When the arcs a_1 and a_2 are not emphasized or is understood from the context, often used $D_1 \bigoplus_2 D_2$ for $D_1 \bigoplus a_1, a_2 D_2$.*

By Definition 2.60, D' in Example 2.18 is $C_{n_1} \bigoplus_2 C_{n_2} = C_{n_1} \bigoplus_{a_1,a_2} C_{n_2}$ such that $a_1 = (v_{11}, v_{12})$ and $a_2 = (v_{21}, v_{22})$ which is non-supereulerian. Alsatami et al. in [7] obtained several sufficient conditions on D_1 and D_2 for $D_1 \bigoplus a_1, a_2 D_2$ to be supereulerian. In particular, they showed that if D_1 and D_2 are symmetrically connected or partially symmetric, then $D_1 \bigoplus a_1, a_2 D_2$ is supereulerian. Their main result of this direction, is to show that the digraph 2-sums of symmetrically connected or partially symmetric digraphs are supereulerian. The following lemma has been proved in [7].

Lemma 2.61 *[7] Let D_1 and D_2 be two vertex disjoint digraphs with $a_1 = (v_{11}, v_{12}) \in A(D_1)$ and $a_2 = (v_{21}, v_{22}) \in A(D_2)$ and let $C_{n_1} \bigoplus_2 C_{n_2}$ denote $D_1 \bigoplus a_1, a_2 D_2$. Each of the following holds.*
i) If D_1 and D_2 are symmetrically connected, then $D_1 \bigoplus a_1, a_2 D_2$ is symmetrically connected.
ii) If D_1 and D_2 are partially symmetric, then $D_1 \bigoplus a_1, a_2 D_2$ is partially symmetric.
iii) If D_1 is symmetric and D_2 is partially symmetric, then $D_1 \bigoplus a_1, a_2 D_2$ is partially symmetric.

By using Theorem 2.55 and Theorem 2.56 with Lemma 2.61, then the following has been proved.

Theorem 2.62 *(K. A. alsatami et al., Theorem 4 of [7]) Let D_1 and D_2 be two digraphs. Each of the following holds.*
(i) If D_1 and D_2 are symmetrically connected, then $D_1 \bigoplus_2 D_2$ is supereulerian.
(ii) If D_1 and D_2 are partially symmetric, then $D_1 \bigoplus_2 D_2$ is supereulerian.
(iii) If D_1 is symmetric and D_2 is partially symmetric, then $D_1 \bigoplus_2 D_2$ is supereulerian.

2.6.2 Product Digraph

In [26], an open problem (Problem 6 of [26]) was raised to find natural conditions for the product of graphs to be hamiltonian. Motivated by this problem, K.A. Alsatami, J. Liu and X.D. Zhang [6], proposed to seek natural conditions on digraphs D_1 and D_2 such that the product of D_1 and D_2 is supereulerian. K.A. Alsatami et al. [6] investigated sufficient conditions on D_1 and D_2 for $D_1 \square D_2$ and $D_1[D_2]$ to be supereulerian or trailable investigated. The following useful theorem has been used as a tool to show the results of K. Alsatami et al.[6].

Theorem 2.63 *[47] Let D_1 and D_2 be eulerian digraphs. Then the Cartesian product $D_1 \square D_2$ is eulerian.*

K. Alsatami et al.[6] have been proved the following theorem, whose sharpness is showed in Example 2.22.

Theorem 2.64 *(Alsatami, Liu and Zhang, Theorem 2.3 of [6]) Let D_1 and D_2 be two strong digraphs with $min\{|V(D_1)|, |V(D_2)|\} \geq 2$ such that D_1 is supereulerian and D_2 has an eulerian vertex cover with m eulerian subdigraphs such that $m \leq |V(D_1)|$. Then the Cartesian product $D_1 \square D_2$ is supereulerian.*

Corollary 2.65 *[6] Let D_1 be a supereulerian digraph and D_2 be a digraph.*
(i) If D_2 is supereulerian, then the Cartesian product $D_1 \square D_2$ is supereulerian.
(ii) If D_2 is trailable, then the Cartesian product $D_1 \square D_2$ is trailable.

Follows from Definition 1.15(v) of the Lexicographic product $D_1[D_2]$ of two digraphs D_1 and D_2, the following two results have proved by [6].

Theorem 2.66 *(Alsatami, Liu and Zhang, Theorem 2.5 of [6]) Let D_1 and D_2 be two digraphs. If D_1 is supereulerian with $|V(D_1)| \geq 2$, then the Lexicographic product $D_1[D_2]$ is supereulerian.*

Theorem 2.67 *(Alsatami, Liu and Zhang, Theorem 2.6 of [6]) Let D_1 and D_2 be two strong digraphs with $min\{|V(D_1)|, |V(D_2)|\} \geq 2$ such that D_1 is trailable. Then the Lexicographic product $D_1[D_2]$ is supereulerian.*

Follows from Definition 1.15(iii) of the Strong product digraph $D_1 \boxtimes D_2$ of digraphs D_1 and D_2, the following results has been verified in this dissertation.

Theorem 2.68 *(H-J Lai et al., Theorem 1.6 of [36]) Let D_1 and D_2 be strong digraphs. If for some cycle factor F of D_1, D_1/F is hamiltonian with $f(D_2) \leq |V(D_1)|$, then the strong product $D_1 \boxtimes D_2$ is supereulerian.*

Chapter 3

3 Matching and Spanning Trail in Digraphs

In this chapter, we motivated the result of Bang-Jensen and Thomassé conjecture 2.44 ; if $\lambda(D) \geq \alpha(D)$, then D is supereulerian. Algefari et al in [2], motivated Bang-Jensen and Thomassé conjecture and proved Theorem 2.51 in the previous chapter, for a strong digraph D; if $\lambda(D) \geq \alpha'(D)$, then D is supereulerian. This motivates us to study for strong digraphs with $\lambda(D) \geq \alpha'(D) - 1$ and we show the following theorem which is the main result of this chapter.

Theorem 3.1 *Let D be a strong digraph on $n \geq 12$ vertices satisfying $\lambda(D) \geq \alpha'(D) - 1$. Each of the following holds.*
(i) There exists a family $\mathcal{D}$ of well-characterized digraphs such that for any digraph D with $\alpha'(D) \leq 2$, D has a spanning trial if and only if D is not a member in $\mathcal{D}$.
(ii) If $\alpha'(D) \geq 3$, then D has a spanning trail.
(iii) If $\alpha'(D) \geq 3$ and $n \geq 2\alpha'(D) + 3$, then D is supereulerian.
(iv) If $\lambda(D) \geq \alpha'(D) \geq 4$ and $n \geq 2\alpha'(D) + 3$, then for any pair of vertices u and v of D, D contains a spanning (u, v)-trail.

3.1 The symmetric core of digraphs

In this section, we intreduse the symmetric core of digraphs and some of its proprieties. We use $\mathbb{Z}_n$ to denote the (additive) group of integers modulo n.

Definition 3.2 *[37] For a digraph D, an arc $[u, v] \in A(D)$ is a* **symmetric** *in D if both arcs (u, v) and (v, u) are in $A(D)$. Let $S(D) = \{e \in A(D) : e$ is symmetric in $D\}$. A digraph D is a* **symmetric** *if $A(D) = S(D)$. The* **symmetric core** *of D, denoted by $J(D)$, has vertex set $V(D)$ and arc set $S(D)$.*

Lemma 3.3 *Let D be a digraph, $J = J(D)$ and J_0 be a symmetric subdigraph of J.*
(i) For any $v \in V(J_0)$, $d^+_{J_0}(v) = d^-_{J_0}(v)$.
(ii) If J_0 is connected, then J_0 is an eulerian subdigraph of D and so J_0 is strongly connected.
(iii) Suppose that J_0 is connected. Then for any vertices $u, v \in V(J_0)$, J_0 contains a spanning (u, v)-trail.
(iv) If D is strong and for some vertices $u, v \in V(D)$, D has a (u, v)-trail P such that $D - A(P)$ contains a connected symmetric subdigraph J' of J such that $V(P) \cup V(J') = V(D)$, $u, v \notin V(J')$ and there exist two vertices $v^+, v^- \in V(J')$ with $(v, v^+), (v^-, u) \in A(D)$, then D is supereulerian.
(v) If D/J_0 has a hamiltonian cycle, then D is supereulerian. In particular, if D is strong and J_0 is a spanning subdigraph of D with at most two connected components, then D is supereulerian.
(vi) If D is strong and $D[A(D) - A(J_0)]$ has a trail T' that intersects every component of J_0 with $V(D) - V(J_0) \subseteq V(T')$, then $T = D[A(T') \cup A(J_0)]$ is a spanning trail in D.
(vii) Suppose $\lambda(D) \geq 2$. If $G(D - V(J_0))$ is spanned by a 3-cycle, then D is supereulerian.

Proof. As (i) and (ii) are immediate consequences of the definitions, it suffices to justify the other conclusions. Let $u, v \in V(J_0)$. By (ii), we assume that J_0 is strong and $u \neq v$. Let P be a shortest (v, u)-path in J_0. As P is shortest, if an arc $e = (x, y) \in A(P)$, then $(y, x) \notin A(P)$. By (i), $T = J_0 - A(P)$ is a connected digraph such that $d^+_T(u) = d^-_T(u) + 1$, $d^+_T(v) = d^-_T(v) - 1$ and for any vertex $w \in V(T) - \{u, v\}$, $d^+_T(w) = d^-_T(w)$. Thus T is a spanning (u, v)-trail of J_0. This proves (iii).

By assumption, J' is a connected symmetric subdigraph, and so J' is the symmetric core of itself. By (iii) with $J_0 = J'$, J' contains a spanning (v^+, v^-)-trail T. As $A(T) \cap A(P) \subseteq A(J') \cap A(P) = \emptyset$, the arc set $A(T) \cup A(P) \cup \{(v, v^+), (v^-, u)\}$ induces a spanning closed trail of D, and so D is supereulerian. Hence (iv) is justified.

To prove (v), let $D' = D/J_0$ and denote $n = |V(D')|$. Suppose that D' has a Hamilton cycle C with $V(C) = \{v_1, v_2, ..., v_n\}$ and $A(C) = \{e_i = (v_i, v_{i+1}) : i \in \mathbb{Z}_n\}$. Let $J_1, J_2, ..., J_n$ be the preimage of $v_1, v_2, ..., v_n$, respectively. By definition, each J_i is a connected component of J_0, and so a connected symmetric subdigraph of J. By the definition of contraction, $A(D') \subseteq A(D)$, and so for each $i \in \mathbb{Z}_n$, the arc $e_i \in A(D)$. Therefore, there exist vertices $v'_i \in V(J_i)$ and $v''_{i+1} \in V(J_{i+1})$ with $e_i = (v'_i, v''_{i+1}) \in A(D)$. Since each J_i is a connected symmetric subdigraph of J, it follows by (iii) that J_i has a spanning (v''_i, v'_i)-trail T_i. Let $A_1 = \{(v'_i, v''_{i+1}) : i \in \mathbb{Z}_n\}$. Then $H = D[A_1 \cup (\bigcup_{i\in\mathbb{Z}_n} A(T_i))]$ is a spanning closed trail of D, and so D is supereulerian. Now we assume that D is strong and J_0 is a spanning subdigraph of D with at most two connected components. Then D/J_0 is strong with $|V(D/J_0)| \leq 2$. It follows that D/J_0 is hamiltonian, and so D is supereulerian. Thus (v) follows.

Let T' be a trail of $D[A(D) - A(J_0)]$ that intersects every component of J_0 with $V(D) - V(J_0) \subseteq V(T')$, and let $J_1, J_2, ..., J_c$ be the connected components of J_0. Since for each i with $1 \leq i \leq c$, $V(T') \cap V(J_i) \neq \emptyset$ and so $T = D[A(T') \cup A(J_0)]$ is connected. As $V(D) - V(J_0) \subseteq V(T')$, $T = D[A(T') \cup A(J_0)]$ is spanning in D. Let $v \in V(T)$. If $v \in V(D) - V(T')$, we define $d^+_{T'}(v) = d^-_{T'}(v) = 0$. By (i), $d^+_T(v) = d^+_{T'}(v) + d^+_{J_0}(v) = d^-_{T'}(v) + d^-_{J_0}(v) = d^-_T(v)$, and so T is a spanning trail of D. This justifies (vi).

To prove (vii), we assume that $\lambda(D) \geq 2$ and $V(D - V(J_0)) = \{v_1, v_2, v_3\}$ such that $G(D - V(J_0))$ has a Hamilton cycle. Suppose first that $D[\{v_1, v_2, v_3\}]$ is spanned by a 3-cycle. Then as D is strong, there must be arcs $(v', v^-), (v^+, v'') \in A(D)$ for some $v', v'' \in \{v_1, v_2, v_3\}$ and $v^-, v^+ \in V(J_0)$. It follows by Lemma 3.3(iv) or (vi) that D is supereulerian. Hence we assume that $D[\{v_1, v_2, v_3\}]$ does not contain a 3-cycle. Since D is a digraph, we may assume, by symmetry, that $(v_1, v_2), (v_2, v_3), (v_1, v_3) \in A(D)$ and $(v_3, v_1) \notin A(D)$. Since $d^-_D(v_1) \geq \lambda(D) \geq 2$, we must have $(v^+, v_1) \in A(D)$ for some $v^+ \in V(J_0)$. Likewise, as $d^+_D(v_3) \geq \lambda(D) \geq 2$, we must have $(v_3, v^-) \in A(D)$ for some $v^- \in V(J_0)$. It follows by Lemma 3.3(iv) that D is supereulerian. This justifies (vii) and completes the proof of the lemma. ∎

3.2 Structural properties

The rest of this section is devoted to the structural analysis for strong graphs whose arc-strong connectivity is at least as big as the matching number minus one. We start with a definition.

Definition 3.4 *Let M be a matching of D. For each $w \in V(D) - V(M)$, define*

$$\begin{aligned} M_w^{2,2} &= \{e = [u_w(e), v_w(e)] \in M : \ |(w, \{u_w(e), v_w(e)\})_{G(D)}| = 4\}, \\ M_w^{2,1} &= \{e = [u_w(e), v_w(e)] \in M : \ |(w, \{u_w(e), v_w(e)\})_{G(D)}| = 3\}, \\ M_w^{2,0} &= \{e = [u_w(e), v_w(e)] \in M : \\ & \quad \textit{for some } v \in \{u_w(e), v_w(e)\}, |(w, v)_{G(D)}| = |(w, \{u_w(e), v_w(e)\})_{G(D)}| = 2\}, \\ M_w^{1,1} &= \{e = [u_w(e), v_w(e)] \in M : \ |(w, u_w(e))_{G(D)}| = |(w, v_w(e))_{G(D)}| = 1\}, \\ M_w^{1,0} &= \{e = [u_w(e), v_w(e)] \in M : \\ & \quad \textit{for some } v \in \{u_w(e), v_w(e)\}, |(w, v)_{G(D)}| = |(w, \{u_w(e), v_w(e)\})_{G(D)}| = 1\}, \\ M_w^{0,0} &= \{e = [u_w(e), v_w(e)] \in M : \ |(w, u_w(e))_{G(D)}| = |(w, v_w(e))_{G(D)}| = 0\}. \end{aligned} \tag{3}$$

The following observation follows from Definition 3.4 and Theorem 2.52 (Brege Theorem).

Observation 3.5 *Let $n = |V(D)|$ and $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$ be a maximum matching of D.*
(i) As M is a maximum matching, $V(D) - V(M)$ is a stable set. This implies that for any $w \in V(D) - V(M)$, $N_D(w) \subseteq V(M)$, and so by Definition 3.4, $d_D(w) = 4|M_w^{2,2}| + 3|M_w^{2,1}| + 2(|M_w^{2,0}| + |M_w^{1,1}|) + |M_w^{1,0}|$, and $|M_w^{2,2}| + |M_w^{2,1}| + |M_w^{2,0}| + |M_w^{1,1}| + |M_w^{1,0}| + |M_w^{0,0}| = k$.
(ii) Let $x, y \in V(D) - V(M)$ are distinct vertices, and $[u, v] \in M$. By Theorem 2.52, D does not have an M-augmenting path, and so if $x \in N_D(u)$, then $y \notin N_D(v)$.
(iii) As a consequence of (ii), if $x, y \in V(D) - V(M)$ are distinct vertices, then

$$(M_x^{2,2} \cup M_x^{2,1} \cup M_x^{1,1}) \cap (M_y^{2,2} \cup M_y^{2,1} \cup M_y^{2,0} \cup M_y^{1,1} \cup M_y^{1,0}) = \emptyset.$$

Throughout the rest of this section, we always assume that D is a digraph with $k = \alpha'(D) \geq 3$, $n = |V(D)| \geq 2k+3$, $J = J(D)$ is the symmetric core of D, and let $X = V(D) - V(M)$. For each $x \in X$, define

$$k_1(x) = |M_x^{2,2}| + |M_x^{2,1}| + |M_x^{1,1}| \textit{ and } k_2(x) = |M_x^{2,0}| + |M_x^{1,0}|. \tag{4}$$

Lemma 3.6 *Let D be a digraph with $k = \alpha'(D) \geq 3$ and $\delta(D) \geq 2k-2$, and M be a maximum matching of D. If for some vertex $x_1 \in X$, both $d_D(x_1) \geq 2k-1$ and $k_1(x_1) > 0$, then each of the following holds.*
(i) $k_1(x_1) = 1$, $k_2(x_1) \in \{k-2, k-1\}$, and for any vertex $x \in X - \{x_1\}$, $k_1(x) = 0$.
(ii) D has a stable set $\{v_1, v_2, ..., v_k\}$ such that $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$ with $M_{x_1}^{2,2} \cup M_{x_1}^{2,1} \cup M_{x_1}^{1,1} = \{[u_1, v_1]\}$ and $\{u_1, u_2, ..., u_{k-1}, v_1\} \subseteq N_D(x_1) \subseteq \{u_1, u_2, ..., u_k, v_1\}$, and such that J has a connected component J' with $(X - \{x_1\}) \cup \{u_2, u_3, ..., u_k\} \subseteq V(J')$.
(iii) $\{v_2, ..., v_k\} \subseteq V(J')$. Moreover, if $k \geq 4$, then v_1 lies in a nontrivial connected component of J.
(iv) If $\lambda(D) \geq 2$, then D is supereulerian.
(v) If, in addition, $d_D(x_1) \geq 2k$, then either $(x_1, v_1), (v_1, x_1) \in A(D)$, or there exist at least $k-1$ vertices $u \in \{u_1, u_2, ..., u_k\}$ with $(x_1, u), (u, x_1) \in A(D)$.

Proof. Throughout the proof of this lemma, we let $k_1 = k_1(x_1)$ and $k_2 = k_2(x_1)$. Denote $M_{x_1}^{2,2} \cup M_{x_1}^{2,1} \cup M_{x_1}^{1,1} = \{[u_1, v_1], ..., [u_{k_1}, v_{k_1}]\}$ and $M_{x_1}^{2,0} \cup M_{x_1}^{1,0} = \{[u_{k_1+1}, v_{k_1+1}], ..., [u_{k_1+k_2}, v_{k_1+k_2}]\}$ with $\{u_{k_1+1}, ..., u_{k_1+k_2}\} \subseteq N_D(x_1)$.

Choose $x_2 \in X - \{x_1\}$ such that

$$k_1(x_2) = \max\{k_1(x) : x \in X - \{x_1\}\}, \text{ and let } k_2'' = \left|\bigcup_{j=1}^{2}(M_{x_j}^{2,0} \cup M_{x_j}^{1,0})\right|.$$

By Observation 3.5(i) and (iii),

$$\begin{aligned}
2k-1 &\leq d_D(x_1) = 4|M_{x_1}^{2,2}| + 3|M_{x_1}^{2,1}| + 2(|M_{x_1}^{2,0}| + |M_{x_1}^{1,1}|) + |M_{x_1}^{1,0}| \leq 4k_1 + 2k_2, \\
2k-2 &\leq d_D(x_2) = 4|M_{x_2}^{2,2}| + 3|M_{x_2}^{2,1}| + 2(|M_{x_2}^{2,0}| + |M_{x_2}^{1,1}|) + |M_{x_2}^{1,0}| \leq 4k_1(x_2) + 2k_2''.
\end{aligned}$$

By adding the inequalities above side by side, and by Observation 3.5(iii), we have

$$4k - 3 \leq 4(k_1 + k_1(x_2) + k_2'') \leq 4k - 4(|M_{x_1}^{0,0}| + |M_{x_2}^{0,0}|).$$

It follows that $|M_{x_1}^{0,0}| + |M_{x_2}^{0,0}| = 0$. By Observation 3.5(iii),

$$\bigcup_{j=1}^{2}(M_{x_j}^{2,0} \cup M_{x_j}^{1,0}) \subseteq M - \left(\bigcup_{j=1}^{2}(M_{x_j}^{2,2} \cup M_{x_j}^{2,1} \cup M_{x_j}^{1,1})\right),$$

and so by Observation 3.5(i) and by $k_1 > 0$, we have

$$N_D(x) \subseteq \bigcup_{j=1}^{2}\left(V(M_{x_j}^{2,0} \cup M_{x_j}^{1,0}) \cap N_D(x_j)\right), \text{ for any } x \in X - \{x_1, x_2\}, \tag{5}$$

$$k - 1 - k_1(x_2) \geq k - (k_1 + k_1(x_2)) \geq \left|\bigcup_{j=1}^{2}(M_{x_j}^{2,0} \cup M_{x_j}^{1,0})\right|. \tag{6}$$

If $k_1 = 1$ and $k_1(x_2) = 0$, then as $d_D(x_1) \geq 2k-1$, it would follow that $k_2 \in \{k-2, k-1\}$. Hence to prove Lemma 3.6(i), it suffices to show that $k_1 = 1$ and $k_1(x_2) = 0$. By contradiction, we assume that either $k_1 \geq 2$ or $k_1(x_2) > 0$. Then by (6), $k - 2 \geq |\bigcup_{j=1}^{2} V(M_{x_j}^{2,0} \cup M_{x_j}^{1,0})|$. Since $n = |V(D)| \geq 2k+3$, there exists a vertex $x_3 \in X - \{x_1, x_2\}$. By $\delta(D) \geq 2k-2$, (5) and by Observation 3.5(iii), $2(k-1) \leq |N_D(x_3)| \leq 2|\bigcup_{j=1}^{2} V(M_{x_j}^{2,0} \cup M_{x_j}^{1,0})| \leq 2(k-2)$, a contradiction. This proves that Lemma 3.6(i).

By (i), $k_1 = 1$. Let $[u_1, v_1]$ denote the only arc in $M_{x_1}^{2,2} \cup M_{x_1}^{2,1} \cup M_{x_1}^{1,1}$. As $k_2 \in \{k-2, k-1\}$, we can label the vertices and denote $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$ such that $\{u_1, u_2, ..., u_{k-1}\} \subseteq N_D(x_1)$, and such that if $(X, \{u_k, v_k\})_{G(D)} \neq \emptyset$, then $(X, \{u_k\})_{G(D)} \neq \emptyset$. Hence $\{u_1, u_2, ..., u_{k-1}, v_1\} \subseteq N_D(x_1) \subseteq \{u_1, u_2, ..., u_k, v_1\}$. Fix a vertex $x \in X - \{x_1\}$. By $k_1 = 1$ and by Observation 3.5(i) and (ii), $(x, \{u_1, v_1, v_2, ..., v_k\})_D = \emptyset$, and so by $\delta(D) \geq 2k-2$, $N_D(x) = \{u_2, ..., u_k\}$. It follows by $\delta(D) \geq 2k-2$ that $\{(u_j, x), (x, u_j) \in A(D)\}$ for any $2 \leq j \leq k$, and so J has a connected component J' containing the vertices $(X - \{x_1\}) \cup \{u_2, u_3, ..., u_k\}$. As $N_D(x) = \{u_2, u_3, ..., u_k\}$, $k \geq 3$ and $u_1, v_1 \in N_D(x_1)$, We conclude by Theorem 2.52 that $\{v_1, v_2, ..., v_k\}$ is a stable set of D as any arc in D incident with two distinct vertices in $\{v_1, v_2, ..., v_k\}$ would give rise to an M-augmenting path in D. This proves Lemma 3.6(ii).

For any v_i with $2 \leq i \leq k$, as $\{v_1, v_2, ..., v_k\}$ is a stable set, $N_D(v_i) \subseteq V(D) - \{v_1, ..., v_k\}$. By Observation 3.5(iii) and by Lemma 3.6(ii), we further conclude that $N_D(v_i) \subseteq \{u_2, u_3, ..., u_k\}$. This, together with $\delta(D) \geq 2k-2$, forces that $\{(u_j, v_i), (v_i, u_j)\} \subseteq A(D)$, for any j with $2 \leq j \leq k$. Hence $\{v_2, ..., v_k\} \subseteq V(J')$. By Observation 3.5, $(\{X - \{x_1\}\}, \{v_1\})_{G(D)} = \emptyset$, and so $N_D(v_1) \subseteq \{u_1, u_2, u_3, ..., u_k, x_1\}$. It follow that

$|(\{u_1, u_2, u_3, ..., u_k, x_1\}, \{v_1\})_{G(D)}| \geq |N_D(v_1)| \geq 2k-2$, and so there exist at least $(2k-2)-(k+1) \geq k-3$ vertices $z \in \{u_1, u_2, u_3, ..., u_k, x_1\}$ satisfying $(z, v_1), (v_1, z) \in A(D)$. Hence if $k \geq 4$, then v_1 lies in a non-trivial connected component of J. This proves Lemma 3.6(iii).

Let $J_0 = J[V(D) - \{u_1, v_1, x_1\}]$. By (ii) an (iii), J_0 is a connected symmetric subdigraph of J. As $[u_1, v_1], [v_1, x_1], [x_1, u_1] \in A(D)$, it follows by $\lambda(D) \geq 2$ and Lemma 3.3(vii) that D is supereulerian. This proves (iv).

Finally, we assume that $d_D(x_1) \geq 2k$ but $|(\{x_1\}, \{v_1\})_{G(D)}| = 1$. Then $|(\{x_1\}, \{u_1, ..., u_k\})_{G(D)}| \geq 2k-1$, implying that there exist at least $k-1$ vertices $u \in \{u_1, u_2, ..., u_k\}$ with $(x_1, u), (u, x_1) \in A(D)$. Hence (v) holds. This completes the proof of Lemma 3.6. ■

For a digraph D with vertex set $V = V(D)$, recall D is a complete digraph if for any pair of distinct vertices $u, v \in V$, $(u, v), (v, u) \in A(D)$. A complete digraph on n vertices will be denoted by K_n^*. Define D_0 to be the vertex disjoint union of three complete digraphs of order 3.

Lemma 3.7 *Let D be a digraph with $k = \alpha'(D) \geq 3$, $\delta(D) \geq 2k-2$ and M be a maximum matching of D. Suppose that $\delta(D) \geq 2k-2$ holds.*
(i) If, for some vertex $x_1 \in X$, $d_D(x_1) \geq 2k-1$ and $k_1(x_1) = 0$, then for any $x \in X$, $k_1(x) = 0$.
(ii) If for some vertex $x_1 \in X$, $k_1(x_1) > 0$, then either $D \cong D_0$, or $k_1(x_1) = 1$ and $k_1(x) = 0$ for any $x \in X - \{x_1\}$.

Proof. Arguing by contradiction to prove (i), we may assume that $x_2 \in X - \{x_1\}$ and $k_1(x_2) > 0$. Let $[u_2, v_2] \in M_{x_2}^{2,2} \cup M_{x_2}^{2,1} \cup M_{x_2}^{1,1}$. Then by Observation 3.5(i), $N_D(x_1) \subseteq V(M - \{[u_2, v_2]\})$. As $d_D(x_1) \geq 2k-1$, and as $|M - \{[u_2, v_2]\}| = k-1$, there exists an arc $[u_1, v_1] \in M - \{[u_2, v_2]\}$ such that $|(x_1, \{u_1, v_1\})_D| \geq 3$. Hence we must have $k_1(x_1) > 0$, contrary to the assumption that $k_1(x_1) = 0$. This proves Lemma 3.7(i).

Now assume that for some vertex $x_1 \in X$, $k_1(x_1) > 0$. Then there exists an arc $[u_1, v_1] \in M$ such that $u_1, v_1 \in N_D(x_1)$. By Observation 3.5(ii), for any $x \in X - \{x_1\}$, $u_1, v_1 \notin N_D(x)$. Suppose that we have another vertex $x_2 \in X - \{x_1\}$ with $k_1(x_2) > 0$, or we have $k_1(x_1) \geq 2$. Then there must be an arc $[u_2, v_2] \in M - \{[u_1, v_1]\}$ such that $u_2, v_2 \in N_D(x_2)$ (if $k_1(x_2) > 0$), or $u_2, v_2 \in N_D(x_1)$ (if $k_1(x_1) \geq 2$). If there exists a vertex $x \in X$ with $k_1(x) = 0$, then by $d_D(x) \geq 2k-2$, either $(x, \{u_1, v_1\})_{G(D)} \neq \emptyset$ or $(x, \{u_2, v_2\})_{G(D)} \neq \emptyset$. In either case, a contradiction to Observation 3.5(ii) is obtained. Thus, either $k_1(x) > 0$ for any $x \in X$, or $k_1(x_1) = 1$ and $k_1(x) = 0$ for any $x \in X - \{x_1\}$.

To complete the proof of (ii), in the following we, assume that $k_1(x) > 0$ for any $x \in X$. If $D \cong D_0$, then done. Hence we by contradiction assume that $D \ncong D_0$. Define $S = \cup_{x \in X}(M_x^{2,0} \cup M_x^{1,0})$, $m' = \min\{k_1(x) : x \in X\}$ and $m'' = \sum_{x \in X, k_1(x) > 0}(k_1(x) - 1)$. Since $k_1(x) > 0$ for any $x \in X$, $m' > 0$. By Observation 3.5(iii), $(\bigcup_{x \in X}(M_x^{2,2} \cup M_x^{2,1} \bigcup M_x^{1,1})) \cup S$ is a disjoint union and is a subset of M. This, together with $|X| = n - 2k$, implies that

$$k = |M| \geq \sum_{x \in X} k_1(x) + |S| = m'' + (n - 2k) + |S|. \tag{7}$$

Claim 1 *We have $m'' = 0$, $n = 2k+3$, $|X| = 3$.*

By (7), $k \geq m'(n-2k)+|S|$. Let $x' \in X$ satisfying $k_1(x') = m'$. Then $4m'+2|S| \geq d_D(x') \geq 2k-2$, and so $|S| \geq k-1-2m'$. Hence we have

$$k \geq m'(n-2k)+|S| \geq m'(n-2k)+k-1-2m' = m'(n-2k-2)+k-1. \tag{8}$$

With $n \geq 2k+3$, (8) leads to the conclusion that $1 \geq m'(n-2k-2) \geq m' \geq 1$, forcing $m'=1$ and $n=2k+3$. Thus $|X| = n-2k = 3$. By (7) and by $|S| \geq k-1-2m' = k-3$, we have $k \geq m''+3+(k-3) = m''+k$. This implies $m''=0$ and proves Claim 1.

By Claim 1, we may assume that $X = \{x_1, x_2, x_3\}$. As $m''=0$, for any $x \in X$, $k_1(x) = 1$. Fix an $x_i \in X$ for $1 \leq i \leq 3$. As $k_1(x_i) = 1$, we may assume that $u_i, v_i \in N_D(x_i)$, and $(\{x_i\}, \{v_j\})_{G(D)} = \emptyset$ for any j with $j \neq i$. By Observation 3.5(ii), we observe that $(\{x_i\}, \{u_h, v_h\})_{G(D)} = \emptyset$ for any $1 \leq i \leq 3$ and $h \neq i$. This implies that $4+2(k-3) \geq |(\{x_i\}, \{u_i, v_i\})_{G(D)}| + \sum_{j=4}^{k} |(x_i, u_j)_{G(D)}| = d_D(x_i) \geq 2k-2$, and so we must have $d_D(x_i) = 2k-2$, $|(\{x_i\}, \{u_i, v_i\})_{G(D)}| = 4$, and for j with $4 \leq j \leq k$, $|(x_i, u_j)_{G(D)}| = 2$.

We further claim that $\{v_1, ..., v_k\}$ is a stable set in D. By contradiction, we assume that there exists an arc $[v_i, v_j] \in A(D)$ for some $1 \leq i < j \leq k$. If $j \leq 3$, then $\{[x_i, u_i], [u_i, v_i], [v_i, v_j], [v_j, u_j], [u_j, x_j]\}$ induces an M-augmenting path in D. If $i \leq 3 < j$, then choosing an index $i' \neq i$ and $1 \leq i' \leq 3$, then $\{[x_i, u_i], [u_i, v_i], [v_i, v_j], [v_j, u_j], [u_j, x_{i'}]\}$ induces an M-augmenting path in D. If $i \geq 4$, then $\{[x_1, u_i], [u_i, v_i], [v_i, v_j], [v_j, u_j], [u_j, x_2]\}$ induces an M-augmenting path in D. In any case, Theorem 2.52 is violated. Hence $\{v_1, ..., v_k\}$ must be a stable set.

If $k \geq 4$, then $N_D(v_4) \subseteq \{u_1, u_2, ..., u_k\}$. Since $d_D(v_4) \geq 2k-2$, there must be an i with $1 \leq i \leq 3$ such that $[u_i, v_4] \in A(D)$. Pick $i' \neq i$ and $1 \leq i' \leq 3$. Then $\{[x_i, v_i], [u_i, v_i], [u_i, v_4], [v_4, u_4], [u_4, x_{i'}]\}$ induces an M-augmenting path in D, violating Theorem 2.52. Hence we must have $k=3$. Recall that for each $i \in \{1,2,3\}$, $|(\{x_i\}, \{u_i, v_i\})_{G(D)}| = 4$. Since $D \not\cong D_0$ and $d_D(u_i) \geq 2k-2 = 4$, we may assume that, either $[u_i, v_j] \in A(D)$ or $[u_i, u_j] \in A(D)$, for $1 \leq i, j \leq 3$ with $i \neq j$. Once again, $\{[x_i, v_i], [v_i, u_i], [u_i, v_j], [v_j, u_j], [u_j, x_j]\}$ or $\{[x_i, v_i], [v_i, u_i], [u_i, u_j], [u_j, v_j], [v_j, x_j]\}$ induces an M-augmenting path in D. These contradictions indicate that if $k_1(x) > 0$ for any $x \in X$, then we must have $D \not\cong D_0$. This proves Lemma 3.7(ii). ∎

Corollary 3.8 *Let $k \geq 4$ be an integer, D be a digraph with $\lambda(D) \geq \alpha'(D) = k$ and $n = |V(D)| \geq 2k+3$. Then $J = J(D)$ is connected.*

Lemma 3.9 *Let D be a digraph with $k = \alpha'(D) \geq 3$ and M be a maximum matching of D. Suppose that for some vertex $x_1 \in X$, $d_D(x_1) \geq 2k-1$ with $k_1(x_1) = 0$. If $\delta(D) \geq 2k-2$, then there exists a labeling of the vertices of $V(M)$ such that $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$ and each of the following holds.*
(i) $N_D(x_1) = \{u_1, u_2, u_3, ..., u_k\}$, $(X, \{v_1, v_2, ..., v_k\})_{G(D)} = \emptyset$, and there exist at least $k-1$ vertices $u \in \{u_1, u_2, ..., u_k\}$ with $(x_1, u), (u, x_1) \in A(D)$. Moreover, if $d_D(x_1) \geq 2k$, then for any $u \in \{u_1, u_2, ..., u_k\}$, we have $(x_1, u), (u, x_1) \in A(D)$.
(ii) For any $x \in X - \{x_1\}$, $N_D(x) \subseteq \{u_1, u_2, ..., u_k\}$; and there exist at least $k-2$ vertices $u \in \{u_1, u_2, ..., u_k\}$ satisfying $(x, u), (u, x) \in A(D)$.
(iii) The vertex subset $\{v_1, v_2, ..., v_k\}$ is a stable set in D. Furthermore, for each v_j with $1 \leq j \leq k$, $N_D(v_j) \subseteq \{u_1, u_2, ..., u_k\}$ and there exist at least $k-2$ vertices $u \in \{u_1, u_2, ..., u_k\}$ satisfying $(v_j, u), (u, v_j) \in A(D)$.
(iv) J has at most two components; and if $\lambda(D) \geq 1$, then D is supereulerian.

Proof. By Lemma 3.7(i), for any $x \in X$, $k_1(x) = 0$. By Observation 3.5(i), $N_D(x_1) \subseteq V(M)$. Hence by $d_D(x_1) \geq 2k-1$ and $k(x_1) = 0$, we can label $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$ so that $N_D(x_1) = \{u_1, u_2, u_3, ..., u_k\}$. Again by $d_D(x_1) \geq 2k-1$, there must be at least $k-1$ vertices $u \in \{u_1, u_2, ..., u_k\}$ satisfying $(x_1, u), (u, x_1) \in A(D)$. Similarly, if $d_D(x_1) \geq 2k$, then for any $u \in \{u_1, u_2, ..., u_k\}$, we have $(x_1, u), (u, x_1) \in A(D)$. It follows by $N_D(x_1) = \{u_1, u_2, u_3, ..., u_k\}$ and by Observation 3.5 that $(X, \{v_1, v_2, ..., v_k\})_{G(D)} = \emptyset$. This verifies Lemma 3.9(i).

By (i), $N_D(x_1) = \{u_1, u_2, u_3, ..., u_k\}$. For any $x \in X - \{x_1\}$, by Observation 3.5(i) and (ii), $N_D(x) \subseteq \{u_1, u_2, ..., u_k\}$. By $\delta(D) \geq 2k-2$, $d_D(x) \geq 2k-2$, and so there must be at least $k-2$ vertices $u \in \{u_1, u_2, ..., u_k\}$ with $(x, u), (u, x) \in A(D)$. This proves Lemma 3.9(ii).

To prove (iii), we argue by contradiction and assume that for some $1 \leq i < j \leq k$, an arc $[v_i, v_j]$ is in $A(D)$. Since $n \geq 2k+3$, there exists a vertex $x_2 \in X - \{x_1\}$. By Lemma 3.9(ii), $N_D(x_2) \subseteq \{u_1, u_2, ..., u_k\}$. As $d_D(x_2) \geq 2k-2$, we may assume that $u_i \in N_D(x_2)$, and so $\{[x_2, u_i], [u_i, v_i], [v_i, v_j], [v_j, u_j], [u_j, x_1]\}$ induced an M-augmenting path in D, contrary to Theorem 2.52. Hence $\{v_1, v_2, ..., v_k\}$ must be a stable set in D. Likewise, by Lemma 3.9(i) and (ii), and arc in $(X, \{v_1, v_2, ..., v_k\})_{G(D)}$ will give rise to an M-augmenting path, contrary to Theorem 2.52. Thus $(X, \{v_1, v_2, ..., v_k\})_{G(D)} = \emptyset$. Consequently, for each v_j with $1 \leq j \leq k$, $N_D(v_j) \subseteq \{u_1, u_2, ..., u_k\}$. By $d_D(v_j) \geq 2k-2$, there exist at least $k-2$ vertices $u \in \{u_1.u_2, ..., u_k\}$ satisfying $(v_j, u), (u, v_j) \in A(D)$.

To show (iv), we first assume by (i) and by symmetry that for any i with $1 \leq i \leq k-1$, (x_1, u_i) is a symmetric arc in D and $[x_1, u_k] \in A(D)$. Thus J has a connected component of J' with $\{x_1, u_1, ..., u_{k-1}\} \subseteq V(J')$. Let J'' denote the connected component of J with $u_k \in V(J'')$. As $k \geq 3$, it follows by (ii) that, for every $x \in X - \{x_1\}$, either $x \in V(J')$ or $x \in V(J'')$. Similarly, by (ii), for every $v \in \{v_1, v_2, ..., v_k\}$, either $v \in V(J')$ or $v \in V(J'')$. Hence J has at most two connected components J' and J''. It now follows by Lemma 3.3(v) that if D is strong, then D must be supereulerian. This completes the proof of the lemma.
■

Lemma 3.10 *Let D be a digraph with $k = \alpha'(D) \geq 3$, $\delta(D) \geq 2k-2$ and let M be a maximum matching of D and $J = J(D)$ be the symmetric core of D. If for any $x \in X$, $k_1(x) = 0$, and if there exists an arc $e \in M$ with $(X, V(e))_{G(D)} = \emptyset$, then there exists a labeling of the vertices of $V(M)$ with $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$ and $e = [u_k, v_k]$ such that each of the following holds.*
(i) $(X, \{v_1, v_2, ..., v_k\})_{G(D)} = \emptyset$, $\{v_1, v_2, ..., v_{k-1}\}$ is a stable set in D and J has a connected component J' with $X \cup \{u_1, u_2, ..., u_{k-1}\} \subseteq V(J')$.
(ii) If $\{v_1, v_2, ..., v_k\}$ is a stable set in D, then for any $j \in \{1, 2, ..., k\}$, there exist $k-2$ vertices $u \in \{u_1, u_2, ..., , u_k\}$ with $(v_j, u), (u, v_j) \in A(D)$, and J has at most two connected components.
(iii) Suppose that $\{v_1, v_2, ..., v_k\}$ is not a stable set in D and $[v_{k-1}, v_k] \in A(D)$. Then $(u_k, \{v_1, ..., v_{k-2}\})_{G(D)}$ $= \emptyset$. Moreover, if $k \geq 4$, then $\{v_1, ..., v_{k-2}\} \subseteq V(J')$; and if $\lambda(D) \geq 2$, then D is supereulerian.

Proof. By Observation 3.5(i), for any $x \in X$, $N_D(x) \subseteq V(M)$. As for some $e \in M$, we have $(X, V(e))_{G(D)} = \emptyset$. By $k_1(x) = 0$ and $d_D(x) \geq 2k-2$, we can label $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$ with $e = [u_k, v_k]$ such that for any $x \in X$, $N_D(x) = \{u_1, u_2, ..., u_{k-1}\}$, and for any i with $1 \leq i \leq k-1$, $(x, u_i), (u_i, x) \in A(D)$. As $k \geq 3$ and $|X| = n-2k \geq 3$, it follows that J has a connected component J' with $X \cup \{u_1, u_2, ..., u_{k-1}\} \subseteq V(J')$. As $k_1(x) = 0$ for any $x \in X$, we conclude that $(X, \{v_1, v_2, ..., v_k\})_{G(D)} = \emptyset$.

We argue by contradiction to show that $\{v_1, v_2, ..., v_{k-1}\}$ is a stable set in D. Suppose that for some $1 \leq i < j \leq k-1$, $[v_i, v_j] \in A(D)$. As $n-2k \geq 3$, $D[\{[x_1, u_i], [u_i, v_i], [v_i, v_j], [v_j, u_j], [u_j, x_2]\}]$ is an

M-augmenting path, contrary to Theorem 2.52. This proves (i).

In the proof of (ii) and (iii), we let J^2, J^3 and J^4 be connected components of J such that $u_k \in V(J^2)$, $v_k \in V(J^3)$ and $v_{k-1} \in V(J^4)$.

Assume that $\{v_1, v_2, ..., v_k\}$ is a stable set in D. Fix an arbitrary vertex v_j with $1 \le j \le k$. By (i), we have $N_D(v_j) \subseteq \{u_1, u_2, ..., u_{k-1}, u_k\}$, and so by $\delta(D) \ge 2k-2$, there must be at least $k-2$ vertices $u \in \{u_1, u_2, ..., u_k\}$ with $(v_j, u), (u, v_j) \in A(D)$. It follows by $k \ge 3$ and by (i) that either $v_j \in V(J')$ (if $u \ne u_k$) or $v_j \in V(J^2)$ (if $u = u_k$). Hence every vertex in D is either in J' or in J^2, and so J has at most two connected components. This proves (ii).

To prove (iii), we assume by symmetry that $[v_{k-1}, v_k] \in A(D)$. Fix a vertex v_j with $1 \le j \le k-2$. If $[u_k, v_j] \in A(D)$, then by (i) and by $n \ge 2k+3$, $D[\{[x_1, u_j], [u_j, v_j], [v_j, u_k], [u_k, v_k], [v_k, v_{k-1}], [v_{k-1}, u_{k-1}], [u_{k-1}, x_2]\}]$ is an M-augmenting path, contrary to Theorem 2.52. Hence $(u_k, v_j)_{G(D)} = \emptyset$. This proves that $(u_k, \{v_1, ..., v_{k-2}\})_{G(D)} = \emptyset$, and so $N_D(v_j) \subseteq \{u_1, ..., u_{k-1}, v_k\}$. By $d_D(v_j) \ge 2k-2$, there exist at least $k-2$ vertices $u' \in \{u_1, ..., u_{k-1}, v_k\}$ such that $(u', v_j), (v_j, u') \in A(D)$. If $k \ge 4$ then $u' \in \{u_1, ..., u_{k-1}\} \subseteq V(J')$, and so $v_j \in V(J')$. Thus $\{v_1, ..., v_{k-2}\} \subseteq V(J')$.

In the following, we assume that $\lambda(D) \ge 2$ to prove the following claim, which completes the proof of the lemma.

Claim 2 *Under the assumption of Lemma 3.10(iii), if $\lambda(D) \ge 2$, then each of the following holds.*
(a) If $k \ge 5$, then J has at most two components, and so by Lemma 3.3(v), D is supereulerian.
(b) If $[u_k, v_{k-1}] \in A(D)$, then $(\{v_k\}, \{v_1, ..., v_{k-2}\})_{G(D)} = \emptyset$.
(c) If $k = 4$, then J has at most two components, and so by Lemma 3.3(v), D is supereulerian.
(d) If $k = 3$, then J has a symmetric subdigraph J_0 such that $G(D - V(J_0))$ is spanned by a 3-cycle, and so by Lemma 3.3(vii), D is supereulerian.

Assume that $k \ge 5$. If $J^2 = J^3 = J^4$, then J has at most two components. Hence we assume that either $J^2 \ne J^3$, whence $|(\{u_k\}, \{v_k\})_{G(D)}| \le 1$; or $J^2 \ne J^4$, whence $|(\{u_k\}, \{v_{k-1}\})_{G(D)}| \le 1$. Since $(u_k, \{v_1, ..., v_{k-2}\})_{G(D)} = \emptyset$ and $(X, \{u_k, v_k\})_{G(D)} = \emptyset$, we have $N_D(u_k) \subseteq \{u_1, ..., u_{k-1}, v_{k-1}, v_k\}$. This, together with $d_D(u_k) \ge 2k-2$, implies that $|(u_k, \{u_1, ..., u_{k-1}\})_{G(D)}| \ge 2k-5$, and so there exists at least $k-4$ vertices $u'' \in \{u_1, ..., u_{k-1}\}$ such that $(u_k, u''), (u'', u_k) \in A(D)$. As $k \ge 5$, $u_k \in V(J')$. Similarly, by (i), $N_D(v_{k-1}) \subseteq \{u_1, ..., u_{k-1}, u_k, v_k\}$ and so $|(v_{k-1}, \{u_1, ..., u_{k-1}, u_k\})_{G(D)}| \ge 2k-4$. Again by $k \ge 5$, there exists at least $k-4$ vertices $u^3 \in \{u_1, ..., u_{k-1}, u_k\}$ such that $(v_{k-1}, u^3), (u^3, v_{k-1}) \in A(D)$, and so $v_{k-1} \in V(J')$. This indicates that $V(D) - V(J') \subseteq \{v_k\}$, and so Claim 2(a) follows.

By contradiction, we assume that $[u_k, v_{k-1}], [v_j, v_k] \in A(D)$ for some $j \in \{1, 2, ..., k-2\}$. Then $\{[x_1, u_j], [u_j, v_j], [v_j, v_k], [v_k, u_k], [u_k, v_{k-1}], [v_{k-1}, u_{k-1}], [u_{k-1}, x_2]\}$ induces an M-augmenting path in D, contrary to Theorem 2.52. Hence (b) holds.

Assume that $k = 4$. Then $v_1, v_2 \in V(J')$ and $(u_k, \{v_1, v_2\})_{G(D)} = \emptyset$. Hence $N_D(u_4) \subseteq \{u_1, u_2, u_3, v_3, v_4\}$. Since $d_D(u_4) \ge 6$, for some $w \in \{u_1, u_2, u_3, v_3, v_4\}$, both $(w, u_4), (u_4, w) \in A(D)$. Hence either $J^2 = J'$ (if $w \in \{u_1, u_2, u_3\}$), or $J^2 = J^3$ (if $w = v_4$), or $J^2 = J^4$ (if $w = v_3$), and so J has at most three connected components J', J^3 and J^4. Similarly, $N_D(v_3) \subseteq \{u_1, u_2, u_3, u_4, v_4\}$. As $d_D(v_3) \ge 6$, for some $w' \in \{u_1, u_2, u_3, u_4, v_4\}$, both $(w', v_3), (v_3, w') \in A(D)$. Hence either $J^2 = J^4 = J'$, or $J^2 = J^4 = J^3$, or $J^2 = J^4$ with $V(J^4) \cap (V(J') \cup V(J^3)) = \emptyset$. It follows that either J has at most two connected components

J' and J^3, or $J^2 = J^4$ and J has at most three connected components J', J^3 and J^4. When $J^2 = J^4$, we have $[u_4, v_3] \in A(D)$, and so by (b), $N_D(v_4) \subseteq \{u_1, u_2, u_3, u_4, v_3\}$. By $d_D(v_4) \geq 6$, we must have $J^3 = J'$ or $J^3 = J^4$ and so J has at most two connected components J' and J^4. This proves (c).

We now assume that $k = 3$. Assume first that $(u_3, v_2)_{G(D)} = \emptyset$. Then for each $z \in \{v_1, v_2, u_3\}$, as $N_D(z) \subseteq \{u_1, u_2, v_3\}$, $z \in V(J')$ or $z \in V(J^3)$. Hence J has at most two connected components J' and J^3. and so by Lemma 3.3(v), D is supereulerian. Therefore, we assume that $[u_3, v_2] \in A(D)$. By (b), $|(\{v_1\}, \{v_3\})_{G(D)}| = 0$. By (i), $|(\{v_1\}, \{v_2\})_{G(D)}| = 0$. Hence $N_D(v_1) \subseteq \{u_1, u_2\}$. By $d_D(v_1) \geq 4$, $(v_1, u_1), (u_1, v_1) \in A(D)$, and so $v_1 \in V(J')$. Let $J_0 = J'[V(D) - \{u_3, v_2, v_3\}]$. As $[u_3, v_2], [v_2, v_3], [u_3, v_3] \in A(D)$, it follows from $\lambda(D) \geq 2$ and Lemma 3.3(vii) that D is supereulerian. This completes the justification of Claim 2. ∎

Lemma 3.11 *Let D be a digraph with $k = \alpha'(D) \geq 3$ and $\delta(D) \geq 2k-2$, and M be a maximum matching of D. If for any $x \in X$, $k_1(x) = 0$ and for any arc $e \in M$, $(X, V(e))_{G(D)} \neq \emptyset$, then there exists a labeling of the vertices of $V(M)$ such that $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$, $N_D(X) = \{u_1, u_2, ..., u_k\}$, and each of the following holds.*
(i) $(X, \{v_1, v_2, ..., v_k\})_{G(D)} = \emptyset$, and for any $x \in X$, there exists $k-2$ vertices $u \in \{u_1, u_2, ..., u_k\}$ with $(x, u), (u, x) \in A(D)$.
(ii) $\{v_1, v_2, ..., v_k\}$ is a stable set in D, and for any v_j with $1 \leq j \leq k$, there exist at least $k-2$ vertices $u \in \{u_1, u_2, ..., u_k\}$ with $(u, v_j), (v_j, u) \in A(D)$.
(iii) If $\lambda(D) \geq 2$, then D is supereulerian.

Proof. For any vertex $x \in X$, by Observation 3.5(i), $N_D(x) \subseteq V(M)$; by assumption, $k_1(x) = 0$ and

$$\text{for any arc } e \in M,\ (X, V(e))_{G(D)} \neq \emptyset. \tag{9}$$

This, together with Observation 3.5(ii), implies that every arc in M has exactly one vertex in $N_D(X)$. Thus we can denote $V(M) \cap N_D(X) = \{u_1, u_2, ..., u_k\}$ and $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$. This labeling of vertices in $V(M)$ implies that $N_D(X) \subseteq \{u_1, u_2, ..., u_k\}$, and so $(X, \{v_1, v_2, ..., v_k\})_{G(D)} = \emptyset$. Fix an $x \in X$. Since $d_D(x) \geq 2k-2$, for at least $k-2$ vertices $u \in \{u_1, u_2, ..., u_k\}$, both (u, x) and (x, u) are in $A(D)$. Thus (i) holds.

By contradiction, assume that $\{v_1, v_2, ..., v_k\}$ is not a stable set in D. By symmetry, we may assume that $[v_1, v_2] \in A(D)$. For i with $1 \leq i \leq k$, let $X_i = X \cap N_D(u_i)$. By (9), $X_i \neq \emptyset$, and so there exists a vertex $x_1 \in X_1$. If there exists a vertex $x_2 \in X_2 - \{x_1\}$, then $D[\{[x_1, u_1], [u_1, v_1], [v_1, v_2], [v_2, u_2], [u_2, x_2]\}]$ is an M-augmenting path, contrary to Theorem 2.52. Hence $X_2 = \{x_1\}$. By the same argument, we conclude that $X_1 = X_2 = \{x_1\}$. Since $n \geq 2k+3$, we have $|X| \geq 3$, and so $X - \{x_1\} \neq \emptyset$. For any vertex $x \in X - \{x_1\}$, as $N_D(X) \subseteq \{u_1, u_2, ..., u_k\}$ and $X_1 = X_2 = \{x_1\}$, we conclude that $N_D(x) \subseteq \{u_3, u_4, ..., u_k\}$, which implies that $2k-2 = 2\lambda(D) \leq d_D(x) \leq 2(k-2)$, a contradiction. Thus $\{v_1, v_2, ..., v_k\}$ must be a stable set in D.

Fix a vertex v_j with $1 \leq j \leq k$. By (i), $(X, \{v_1, v_2, ..., v_k\})_{G(D)} = \emptyset$. As $\{v_1, v_2, ..., v_k\}$ is a stable set, we must have $N_D(v_j) \subseteq \{u_1, u_2, ..., u_k\}$. Since $\delta(D) \geq 2k-2$, there exist at least $k-2$ vertices $u \in \{u_1, u_2, ..., u_k\}$ with $(u, v_j), (v_j, u) \in A(D)$. This proves (ii).

We now assume that $\lambda(D) \geq 2$. By contradiction, we assume that D is not supereulerian. Pick a vertex $x_1 \in X$ and let J_1 be the connected component of J with $x_1 \in V(J_1)$. By (i), we may assume that

$u_1, ..., u_{k-2} \in V(J_1)$. Let J_2 and J_3 be connected components of J with $u_{k-1} \in V(J_2)$ and $u_k \in V(J_3)$. By (i) and (ii), and by $k \geq 3$, for every vertex $v \in X \cup \{v_1, v_2, ..., v_k\}$, there exists an $i \in \{1, 2, 3\}$ such that either $v \in V(J_i)$. It follows that J has at most three connected components J_1, J_2 and J_3. By Lemma 3.3(v), if J has at most two connected components, then D is supereulerian. Hence J must have exactly three components J_1, J_2 and J_3.

Case 1 $k \geq 4$.

If there exists a vertex $v \in X \cup \{v_1, v_2, ..., v_k\}$ such that for distinct $i, j \in \{1, 2, 3\}$, $v \in V(J_i) \cup V(J_j)$, then as $k - 2 \geq 2$, we have either $J_1 = J_2$, or $J_1 = J_3$, or $J_2 = J_3$, contrary to the assumption that J has exactly three components. Therefore, for any $k \geq 4$, we have

$$V(J_1) = V(D) - \{u_{k-1}, u_k\}, V(J_2) = \{u_{k-1}\} \text{ and } V(J_3) = \{u_k\}. \tag{10}$$

Thus for any $x \in X$, and $u \in \{u_1, ..., u_{k-2}\}$ and any $v \in \{v_1, v_2, ..., v_k\}$, the arcs $(x, u), (u, v)$ are symmetric in D. As $\delta(D) \geq 2k - 2$, we conclude that for any $v \in X \cup \{v_1, v_2, ..., v_k\}$, $d_D(v) = 2k - 2$ and $|(v, u_{k-1})_{G(D)}| = |(v, u_k)_{G(D)}| = 1$. If $[u_{k-1}, u_k] \in A(D)$, then by $\lambda(D) > 0$ and by Lemma 3.3(iv), D is supereulerian. Thus $(u_{k-1}, u_k)_{G(D)} = \emptyset$. If $D - A(J_1)$ has a cycle C containing both u_{k-1} and u_k, then $D[A(J_1) \cup D(C)]$ is a spanning closed trail of D, and so D is supereulerian. Hence we assume $D - A(J_1)$ does not have a cycle or disjoint cycles containing both u_{k-1} and u_k.

Since $\lambda(D) \geq 2$, there exist vertices $v^-, v^+, w^-, w^+ \in V(J_1)$ such that

$$(v^-, u_{k-1}), (w^-, u_k), (u_{k-1}, v^+), (u_k, w^+) \in A(D). \tag{11}$$

Since J_1, J_2 and J_3 are distinct components of J, thus, we assume that $w^- \neq w^+$ and $v^- \neq v^+$.

If $v^-, w^+ \in X \cup \{v_1, ..., v_k\}$, then $(w^+, u_1), (u_1, w^+), (u_1, v^-), (v^-, u_1) \in A(J_1)$. Let $J_1' = J_1 - \{(w^+, u_1), (u_1, w^+), (u_1, v^-), (v^-, u_1)\}$. As $|X| \geq 3$ and $k \geq 4$, J_1' is a connected symmetric subdigraph of D, and by (11), $D - A(J_1')$ has a trail $w^- u_k w^+ u_1 v^- u_{k-1} v^+$. By Lemma 3.3(iv) with $J' = J_1'$, D is supereulerian.

Suppose that $|\{u_1, ..., u_{k-2}\} \cap \{v^-, w^+\}| = 1$ and $|(X \cup \{v_1, ..., v_k\}) \cap \{v^-, w^+\}| = 1$ By symmetry, we assume that $v^- = u_1$ and $w^+ \in X \cup \{v_1, ..., v_k\}$. As $(w^+, u_1) \in A(J_1)$ is symmetric arcs of D. Let $J_2' = J_1 - \{(w^+, u_1), (u_1, w^+)\}$. As $|X| \geq 3$ and $k \geq 4$, J_2' is a connected symmetric subdigraph of D, and by (11), $D - A(J_2')$ has a trail $w^- u_k w^+ u_1 u_{k-1} v^+$. It follows from Lemma 3.3(iv) with $J' = J_2'$ that D is supereulerian. Hence we may assume that $v^-, w^+ \in \{u_1, ..., u_{k-2}\}$. By (10), $(w^+, x_1), (x_1, v^-) \in A(J_1)$ are symmetric arcs of D. As $|X| \geq 3$ and $k \geq 4$, $J_1 - x_1$ is a connected symmetric subdigraph of D, and by 11), $D - A(J_1 - x_1)$ has a trail $w^- u_k w^+ x_1 v^- u_{k-1} v^+$. By Lemma 3.3(iv) with $J' = J_1 - x_1$, D is supereulerian.

Case 2 $k = 3$.

By definition, for each $i \in \{1, 2, 3\}$, $u_i \in V(J_i)$. By relabeling the vertices u_1, u_2 and u_3, we assume that $u_i \in V(J_i)$. By (ii) and by $\delta(D) \geq 4$, every v_i is adjacent to a u_j by a pair of symmetric arcs. Therefore, we may relabel v_1, v_2, v_3 and assume that $(u_i, v_i) \in A(J_i)$ is a symmetric arc of D.

Let $D' = D/J$, and denote $V(D') = \{z_1, z_2, z_3\}$, where $z_i \in V(D')$ be the vertex onto which J_i is contracted. If D' has a Hamilton cycle, then by Lemma 3.3(v), D is supereulerian. Hence we may assume that D is not Hamiltonian. By (i), (ii), $\lambda(D) \geq 2$, and the fact that for $i \in \{1, 2, 3\}$, $d_D(v_i) = 4$, we observe that

$$\text{if } \{i', i'', i'''\} = \{1, 2, 3\}, \text{ then } |(v_{i'}, \{u_{i''}, u_{i'''}\})_D| = 1 \text{ and } |(\{u_{i''}, u_{i'''}\}, v_{i'})_D| = 1. \tag{12}$$

By (12) and by symmetry, we assume that $(v_1, u_2), (u_3, v_1) \in A(D)$. Thus $(z_1, z_2), (z_3, z_1) \in A(D')$. As D' is not hamiltonian, we assume that $(z_2, z_3) \notin A(D')$. By (12) and since $(z_2, z_3) \notin A(D')$, we conclude that $(u_3, v_2), (v_3, u_2) \in A(D)$. These force, by (12), that $(v_2, u_1), (u_1, v_3) \in A(D)$. As $(u_1, v_3), (v_3, u_2), (v_2, u_1) \in A(D)$, it follows that D' must be hamiltonian, a contradiction. This proves that in Case 2, D is also supereulerian. This completes the proof of the lemma. ∎

Lemma 3.12 *Let $k \geq 3$ be an integer, D be a digraph with $k = \alpha'(D) \geq 3$, $\delta(D) \geq 2k - 2$, and M be a maximum matching of D. Suppose that for some $x_1 \in X$, $k_1(x_1) > 0$. Then each of the following holds.*
(i) Either $D \cong D_0$, or J has a connected component J' such that the subdigraph $D_1 = D - V(J')$ satisfies $|V(D_1)| \leq 3$ and that $G(D_1)$ is spanned by a 3-cycle or a K_2.
(ii) If, in addition, $\lambda(D) \geq 2$, then D is supereulerian.

Proof. As $k_1(x_1) > 0$, there exists an arc $e = [u_1, v_1] \in M$ with $u_1, v_1 \in N_D(x_1)$. By Lemma 3.7(ii), $D \cong D_0$, or $k_1(x_1) = 1$ and $k_1(x) = 0$ for any $x \in X - \{x_1\}$. Thus to prove (i), it suffices to assume that $k_1(x_1) = 1$ and $k_1(x) = 0$ for any $x \in X - \{x_1\}$ to show that the desired J' and D_1 exist.

Fix a vertex $x \in X - \{x_1\}$. By Observation 3.5(ii), $N_D(x) \subseteq V(M) - \{u_1, v_1\}$; and by $k_1(x) = 0$, for any $e \in M$, $|N_D(x) \cap V(e)| \leq 1$. Hence we can label $M = \{[u_1, v_1], [u_2, v_2], ..., [u_k, v_k]\}$ such that $N_D(x) \subseteq \{u_2, ..., u_k\}$. By $\delta(D) \geq 2k-2$, we conclude that for any u_i with $2 \leq i \leq k$, $(x, u_i), (u_i, x) \in A(D)$. It follows that J has a connected component J' such that $(X - \{x_1\}) \cup \{u_2, ..., u_k\} \subseteq V(J')$.

We claim that $\{v_1, v_2, ...v_k\}$ is a stale set. Assume by contradiction that for some $1 \leq i < j \leq k$, $[v_i, v_j] \in A(D)$. If $i = 1$, then $D[\{[x_1, u_1], [u_1, v_1], [v_1, v_j], [v_j, u_j], [u_j, x_2]\}]$ is an M-augmenting path; If $i > 1$, then $D[\{[x_2, u_i], [u_i, v_i], [v_i, v_j], [v_j, u_j], [u_j, x_3]\}]$ is an M-augmenting path. In either case, a contradiction to Theorem 2.52 is obtained. Hence $\{v_1, v_2, ...v_k\}$ is a stable set.

Fix a vertex v_j with $2 \leq j \leq k$. If $[u_1, v_j] \in A(D)$, then $\{[x_1, v_1], [v_1, u_1], [u_1, v_j], [v_j, u_j], [u_j, x_2]\}$ induces an M-augmenting path in D, contrary to Theorem 2.52. Hence $(u_1, \{v_2, ..., v_k\})_{G(D)} = \emptyset$ and so $N_D(v_j) \subseteq \{u_2, .., u_k\}$. As $d_D(v_j) \geq 2k - 2$, we conclude that for any $u \in \{u_2, ..., u_k\}$ with $(u, v_j), (v_j, u) \in A(D)$, and so $(X - \{x_1\}) \cup \{u_2, ..., u_k\} \cup \{v_2, ..., v_k\} \subseteq V(J')$. As $[x_1, u_1], [x_1, v_1], [u_1, v_1] \in A(D)$, Lemma 3.12(i) is justified.

By Lemma 3.12(i) and since $\lambda(D) \geq 2$, we observe that $D \ncong D_0$ and so $J(D)$ has a connected component J' such that the subdigraph $D_1 = D - V(J')$ satisfies $|V(D_1)| \leq 3$ and that $G(D_1)$ is spanned by a 3-cycle or a K_2. If $G(D_1)$ is spanned by a 3-cycle, then by Lemma 3.3(vii), D is supereulerian. If $G(D_1)$ is spanned by a K_2, then then by Lemma 3.3(iv), D is supereulerian. Hence Lemma 3.12(ii) holds. ∎

3.3 Spanning trails in digraphs with small matching numbers

In this subsection, we will identify a family $\mathcal{D}$ of digraphs, and use it to prove Theorem 3.1(i). Let D be a digraph and let X denote a set of arcs not in $A(D)$ satisfying $\cup_{e \in X} V(e) \subset V(D)$. Define $D + X$ to be the digraph with vertex set $V(D)$ and arc set $A(D) \cup X$. If $X \subset A(D)$ (or $X \subset V(D)$, respectively), then define $D - X = D[A(D) - X]$ (or $D - X = D[V(D) - X]$, respectively). We often use $D + e$ for $D + \{e\}$, $D - e$ for $D - \{e\}$ and $D - v$ for $D - \{v\}$. We start with some examples.

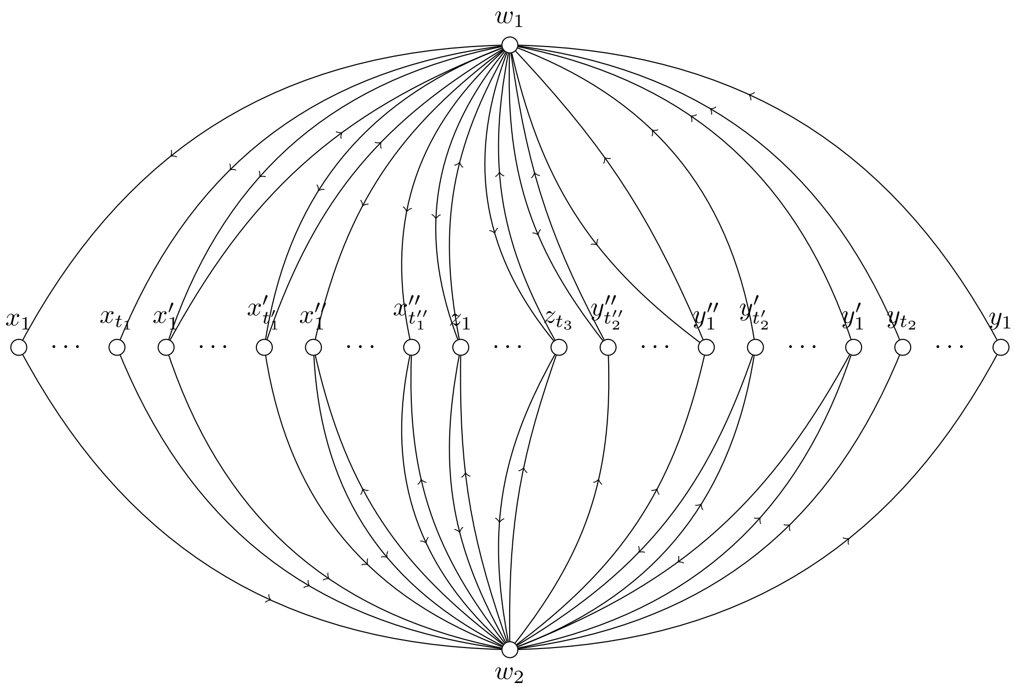


Figure 15. Digraph family $D(t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3)$

Example 3.13 *Let $n, t_1, t'_1, t''_1, t_2, t'_2, t''_2, t_3$ be nonnegative integers with $n = 2+t_1+t'_1+t''_1+t_2+t'_2+t''_2+t_3$. Define mutually disjoint vertex sets X, Y and Z as follows,*

$$\begin{aligned} X &= \{x_1, x_2, ..., x_{t_1}, x'_1, x'_2, ..., x'_{t'_1}, x''_1, x''_2, ..., x''_{t''_1}\} \\ Y &= \{y_1, y_2, ..., y_{t_2}, y'_1, y'_2, ..., y'_{t'_2}, y''_1, y''_2, ..., y''_{t''_2}\} \\ Z &= \{z_1, z_2, ..., z_{t_3}\} \end{aligned}$$

and w_1, w_2 be two vertices not in $X \cup Y \cup Z$; and define mutually disjoint arc sets A_X, A_Y and A_Z as *follows,*

$$
\begin{aligned}
A_X &= \left(\bigcup_{i=1}^{t_1}\{(w_1,x_i),(x_i,w_2)\}\right) \cup \left(\bigcup_{i=1}^{t_1'}\{(w_1,x_i'),(x_i',w_1),(x_i',w_2)\}\right) \qquad (13)\\
&\quad \cup \left(\bigcup_{i=1}^{t_1''}\{(w_1,x_i''),(w_2,x_i''),(x_i'',w_2)\}\right)\\
A_Y &= \left(\bigcup_{i=1}^{t_2}\{(w_2,y_i),(y_i,w_1)\}\right) \cup \left(\bigcup_{i=1}^{t_2'}\{(w_2,y_i'),(y_i',w_2),(y_i',w_1)\}\right)\\
&\quad \cup \left(\bigcup_{i=1}^{t_2''}\{(w_2,y_i''),(w_1,y_i''),(y_i'',w_1)\}\right)\\
A_Z &= \bigcup_{i=1}^{t_3}\{(w_1,z_i),(z_i,w_1),(w_2,z_i),(z_i,w_2)\}.
\end{aligned}
$$

Define a digraph $D = D(t_1,t_1',t_1'',t_2,t_2',t_2'',t_3)$ *with* $V(D) = \{w_1,w_2\} \cup X \cup Y \cup Z$ *and arc set* $A(D) = A_X \cup A_Y \cup A_Z$*.(See Fig. 15.)*

Observation 3.14 *Define a digraph* $D = D(t_1,t_1',t_1'',t_2,t_2',t_2'',t_3)$ *with that* $n \geq 4$ *and* $\lambda(D) > 0$*. Then each of the following holds.*
(i) D *is supereulerian if and only if both* $t_1 \leq t_2 + t_2' + t_2'' + t_3$ *and* $t_2 \leq t_1 + t_1' + t_1'' + t_3$*.*
(ii) D *has a spanning trail if and only if one of the following holds.*

$$\text{both } t_1 \leq t_2 + t_2' + t_2'' + t_3 + 1 \text{ and } t_2 \leq t_1 + t_1' + t_1'' + t_3; \qquad (14)$$

$$\text{both } t_1 \leq t_2 + t_2' + t_2'' + t_3 \text{ and } t_2 \leq t_1 + t_1' + t_1'' + t_3 + 1. \qquad (15)$$

Proof. We are to justify the conclusions of Example 3.13. By inspection, the conclusions (i) and (ii) holds if $n = 4$. Thus we assume that $n \geq 5$. Let $J = J(D)$ be the symmetric core of D.

We assume that both $t_1 \leq t_2+t_2'+t_2''+t_3$ and $t_2 \leq t_1+t_1'+t_1''+t_3$ to show by induction on t_1+t_2 that D is supereulerian. If $t_1+t_2 = 0$, then J has at most two connected components, and so by Lemma 3.3(v), D is supereulerian. Assume that $t_1+t_2 > 0$ and that for smaller values of t_1+t_2, D is supereulerian. By symmetry, we may assume that $t_1 \geq t_2$, and so $t_1 > 0$. If $t_2 > 0$, then let $D_1 = D - \{x_1, y_1\}$. Then as $D_1 = D(t_1-1, t_1', t_1'', t_2-1, t_2', t_2'', t_3)$, by induction, D_1 has a spanning eulerian subdigraph H_1, and so $D[A(H_1) \cup \{(w_1,x_1),(x_1,w_2),(w_2,y_1),(y_1,w_1)\}]$ is a spanning eulerian subdigraph of D. Hence we assume that $t_2 = 0$. Since $t_1 \leq t_2 + t_2' + t_2'' + t_3 = t_2' + t_2'' + t_3$, there exists a $v \in \{y_1', y_2', ..., y_{t_2'}', y_1'', y_2'', ..., y_{t_2''}'', z_1, z_2, ..., z_{t_3}\}$ such that $(w_2, v), (v, w_1) \in A(D)$. Let $D_2 = D - \{x_1, v\}$. By induction, D_2 has a spanning eulerian subdigraph H_2, and so $D[A(H_2) \cup \{(w_1,x_1),(x_1,w_2),(w_2,v),(v,w_1)\}]$ is a spanning eulerian subdigraph of D.

Conversely, we assume that D has a spanning eulerian subdigraph H. We again argue by induction on $t_1 + t_2$ to show that both $t_1 \leq t_2 + t_2' + t_2'' + t_3$ and $t_2 \leq t_1 + t_1' + t_1'' + t_3$. As these inequalities holds when $t_1 = t_2 = 0$, we assume by symmetry, that $t_1 \geq t_2$ and $t_1 > 0$. If $t_2 > 0$, then $(w_1,x_1),(x_1,w_2),(w_2,y_1),(y_1,w_1) \in A(H)$, and so $H - \{x_1,y_1\}$ is a spanning eulerian subdigraph of

$D - \{x_1, y_1\}$, and so by induction. $t_1 - 1 \le (t_2 - 1) + t_2' + t_2'' + t_3$ and $t_2 - 1 \le (t_1 - 1) + t_1' + t_1'' + t_3$. Hence we assume that $t_2 = 0$. As H is a spanning eulerian subdigraph, there must be a $v \in \{y_1', y_2', ..., y_{t_2'}',\ y_1'', y_2'', ..., y_{t_2''}'',\ z_1, z_2, ..., z_{t_3}\}$ such that $(w_2, v), (v, w_1) \in A(H)$. Let H' denote the nontrivial component of $H - \{(w_1, x_1), (x_1, w_2), (w_2, v), (v, w_1)\}$ and D' the nontrivial component of $D - \{(w_1, x_1), (x_1, w_2), (w_2, v), (v, w_1)\}$. Then H' is a spanning eulerian subdigraph of D', and so by induction, we have $t_2 = 0$ and $t_1 - 1 \le t_2' + t_2'' + t_3 - 1$. Hence (i) holds by induction.

To prove (ii), it suffices to investigate spanning trails in a nonsupereulerian D. By (i), any strong digraph $D(0, t_1', t_1'', 0, t_2', t_2'', t_3)$ is supereulerian, and so we assume that $\max\{t_1, t_2\} > 0$. We make the following claim.

Claim 3 *Let $D = D(t_1, t_1', t_1'', t_2, t_2', t_2'', t_3)$ with $\lambda(D) > 0$ be a non supereulerian digraph. If D has a spanning trail, then D has a spanning (u, v)-trail T satisfying*

$$\text{both } u \in \{x_1, x_2, ..., x_{t_1}\} \text{ and } v = w_2, \text{ or both } u \in \{y_1, y_2, ..., y_{t_2}\} \text{ and } v = w_1. \tag{16}$$

Proof. Since D is not supereulerian, by Observation 3.14(i), $\max\{t_1, t_2\} > 0$. We assume that $t_1 > 0$. Let T' be a spanning (u', v')-trail of D. We construct a spanning trail satisfying(16) from the following cases.
We note that as T' is a (u', v')-trail, we have

$$d^+_{T'}(u') - d^-_{T'}(u') = 1 \text{ and } d^-_{T'}(v') - d^+_{T'}(v') = 1. \tag{17}$$

Case 1 $\{u', v'\} = \{w_1, w_2\}$.
if $u' = v'$, then D is supereulerian, contrary to the assumption of Claim 3. If T' ia a (w_1, w_2)-trail and $d^+_{T'}(w_1) \ge 2$,then $T' - (w_1, x_1)$ is a spanning (x_1, w_2)-trail of D satisfying(16). If T' is $(w_1 w_2)$-trail and $d^+_{T'}(w_1) = 1$, the there exists a vertex $y \in X \cup Y \cup Z$ such that $(y, w_2) \in A(T')$ and $(y, w_1) \in A(D) - A(T')$, so $T' - (y, w_2) + (y, w_1)$ ia an eulerian digraph of D, contrary the assumption of Claim 3. The proof for the case when both T' is a (w_2, w_1)-trail and $t_2 > 0$ is similar so it is omitted. Hence we assume that T' a (w_2, w_1)-trail and $t_2 = 0$. As $t_1 > 0$, $(w_1, x_1), x_1, w_2) \in A(T')$. Since $n \ge 4$and T' is a spanning in D, there must be a vertex $y \in V(D)$ such that $(w_2, y), (y, w_1) \in A(T')$. It follows that $y \in Y \cup Z$ and $T' - y$ is an eulerian subdigraph of D. Since $t_2 = 0$, we have $y \in \{y_1', y_2', \ldots, y_{t_2'}', y_1'', y_2'', \ldots, y_{t_2''}'\} \cup Z$, and so y is incident with a pair of symmetric arcs $(y, w), (w, y)$ for some $w \in \{w_1, w_2\}$. It follows that $(T' - y) + \{(y, w), (w, y)\}$ is a spanning closed trail of D, contrary the assumption of Claim 3.

Case 2 Both $u' \in \{w_1, w_2\}$ and $v' \in X \cup Y \cup Z$, or both $u' \in X \cup Y \cup Z$ and $v' \in \{w_1, w_2\}$.
Suppose first that $u' \in \{w_1, w_2\}$ and $v' \in X \cup Y \cup Z$. If $d^-_{T'}(v') = 1$, then by (17)and by (10), for some for some $i \in \{1, 2\}$, $(v', w_i) \notin A(D) - A(T')$. It follows that $T' + (v', w_i)$ is a spanning (u', w_i)-trail. By Case 3.3, we are done. Hence we assume that $d^-_{T'}(v') = 2$. Then by (17) and by (13), for some $i' \in \{1, 2\}$, $(w_1, v'), (w_2, v'), (v', w_{i'}) \in A(T')$. It follows that $T' - (w_{3-i'}, v')$ is a spanning $(u', w_{3-i'})$-trail. By Case 3.3, we are done. Thie proof for the case when both $u' \in X \cup Y \cup Z$ and $v' \in \{w_1, w_2\}$ is similar and so it is omitted.

Case 3 $u', v' \in X \cup Y \cup Z$.
By (17), either $d^+_{T'}(u') = 1$ and for some $j_1 \in \{1, 2\}$, $(w_{j_1}, u') \in A(D) - A(T')$, or $d^+_{T'}(u') = 2$ and for some $j_2 \in \{1, 2\}$, $(u', w_1), (u', w_2), (w_{j_2}, u') \in A(T')$. Likewise, either $d^-_{T'}(v') = 1$ and for some $j_3 \in$

$\{1,2\}$, $(v', w_{j_3}) \in A(D) - A(T')$, or $d^-_{T'}(v') = 2$ and for some $j_4 \in \{1,2\}$, $(w_1, v'), (w_2, v'), (v', w_{j_4}) \in A(T')$. It follows that

$$T'' = \begin{cases} T' + \{(w_{j_1}, u'), (v', w_{j_3})\} & \text{if } d^+_{T'}(u') = 1 \text{ and } d^-_{T'}(v') = 1, \\ (T' - \{(u', w_{3-j_2})\}) + \{(v', w_{j_3})\} & \text{if } d^+_{T'}(u') = 2 \text{ and } d^-_{T'}(v') = 1, \\ (T' - \{(w_{3-j_4}, v')\}) + \{(w_{j_1}, u')\} & \text{if } d^+_{T'}(u') = 1 \text{ and } d^-_{T'}(v') = 2, \\ T' - \{(u', w_{3-j_2}), (w_{3-j_4}, v')\} & \text{if } d^+_{T'}(u') = 2 \text{ and } d^-_{T'}(v') = 2, \end{cases}$$

is a spanning (w', w'')-trail of D, for some $w', w'' \in \{w_1, w_2\}$. By Case 3.3, we are done.

Assume that (14) holds. Then $t_1 \geq 1$ and so $D - \{x_1\}$ satisfies the inequalities in Observation 3.14(i). By the definition of D in Observation 3.14, $\lambda(D - \{x_1\}) > 0$ if and only if either $t_3 > 0$, or both $(t_1 - 1) + t_1' + t_1'' > 0$ and $t_2 + t_2' + t_2'' > 0$. As $\lambda(D) > 0$, if $t_3 = 0$, then $t_2 + t_2' + t_2'' > 0$. Therefore, if $\lambda(D - \{x_1\}) = 0$, then $t_3 = 0$ and $t_2 + t_2' + t_2'' > 0$, and so by (14), we must have $t_1 = 1$ and $t_1' + t_1'' = 0$. These, together with (14), imply that D itself satisfies the inequalities in Observation 3.14(i), and so D is supereulerian, a contradiction. Hence we must have $\lambda(D - \{x_1\}) > 0$. By Observation 3.14(i), $D - \{x_1\}$ has a spanning closed trail Q. It follows that $Q + \{(x_1, w_2)\}$ is a spanning (x_1, w_2)-trail of D. With a similar argument, if (15) holds, then D also has a spanning trail.

Conversely, assume that D has a spanning trail. If D has a spanning closed trail, then by Observation 3.14(i), each of (14) and (15) is satisfied. Hence we assume that D is not supereulerian. By Claim 3, we assume by symmetry that D has a spanning (x_1, w_2)-trail. Then $D - x_1$ has a spanning closed trail, and so (14) follows from Observation 3.14(i). ∎

Definition 3.15 *Using the notation used in Observation 3.14, we introduce a digraph family $\mathcal{D}(n)$ for each $n \geq 4$. Define a digraph $D \in \mathcal{D}(n)$ if and only if each of the following holds.*
(F1) D has a subdigraph D', (called the **corresponding digraph of** *D), such that there exist nonnegative integers $t_1, t_1', t_1'', t_2, t_2', t_2'', t_3$ satisfying $|V(D')| = 2 + t_1 + t_1' + t_1'' + t_2 + t_2' + t_2'' + t_3 \geq 4$ and $D' = D(t_1, t_1', t_1'', t_2, t_2', t_2'', t_3)$ (as defined in Observation 3.14) such that both (14) and (15) are violated.*
(F2) For each $i \in \{1,2\}$, let s_i be a nonnegative integer and D_i be digraph with $V(D_i) = \{w_i, w_1^i, ..., w_{s_i}^i\}$ and $A(D_i) = \{(w_i, w_j^i), (w_j^i, w_i) : 1 \leq j \leq s_i\}$, such that $V(D_1) \cap V(D_2) = \emptyset$ and $V(D_i) \cap V(D') = \{w_i\}$. When $s_i = 0$, then D_i consists of a single vertex w_i.
(F3) Define D to be the digraph with $V(D) = V(D') \cup V(D_1) \cup V(D_2)$ and $A(D) = A(D') \cup A(D_1) \cup A(D_2)$, and let $n = |V(D)|$.

By Lemma 3.3(vii) and using the notation in Definition 3.15, a digraph $D \in \mathcal{D}(n)$ has a spanning trail if and only if the corresponding D' of D has a spanning trail. The following follows from Observation 3.14.

$$\text{For any digraph } D \in \mathcal{D}(n), D \text{ does not have a spanning trail.} \tag{18}$$

Corollary 3.16 *Let D be a digraph obtained from a digraph $D' = D(t_1, t_1', t_1'', t_2, t_2', t_2'', t_3)$ (as defined in Observation 3.14) with $4 = |V(D')| = 2 + t_1 + t_1' + t_1'' + t_2 + t_2' + t_2'' + t_3$ by attaching a number of 2-cycles to each vertex of $V(D')$. Then D is supereulerian if and only if D is strong.*

Proof. By Lemma 3.3 (vii), it suffices to examine these properties for D'. Since D is strong, by the way we form D from D', D' is also strong. By Example 3.13, D' is strong if and only if both $t_1 + t_1' + t_1'' + t_3 > 0$

and $t_2 + t_2' + t_2'' + t_3 > 0$. As $2 = t_1 + t_1' + t_1'' + t_2 + t_2' + t_2'' + t_3$, we have both $t_1 \le t_2 + t_2' + t_2'' + t_3$ and $t_2 \le t_1 + t_1' + t_1'' + t_3$. Thus Corollary 3.16 follows from Observation 3.14(i). ∎

Lemma 3.17 *Let D be a digraph with $|V(D)| = 5$ such that $G(D)$ has a Hamilton cycle. If D is strongly connected, then D has a spanning trail.*

Proof. If D is supereulerian, then D has a spanning trail. Hence we assume that D is not supereulerian to show that D has a spanning trail. Let c be the length of a longest cycle in D. As D is not supereulerian, we have $3 \le c \le 4$. Suppose first that $c = 3$. Let C be a 3-cycle with arcs $A(C) = \{(z_1, z_2), (z_2, z_3), (z_3, z_1)\}$. Fix a vertex $x \in V(D) - V(C)$. Since D is strong, there exist vertices $z_x', z_x'' \in \{z_1, z_2, z_3\}$ such that D contains a (x, z_x')-path P_x' and a (z_x'', x)-path P_x''. If for any $x \in V(D) - V(C)$, we always have $z_x' = z_x''$, then D would be supereulerian, a contradiction. Hence there exists a vertex x_1 such that $z_{x_1}' \ne z_{x_1}''$. By symmetry, we assume that $z_2 = z_{x_1}'$ and $z_3 = z_{x_1}''$. Since $c = 3$, D does not have a 4-cycle and so we must have $(x_1, z_2), (z_3, x_1) \in A(D)$. Let x_2 denote the only vertex in $V(D) - \{z_1, z_2, z_3, x_1\}$. If $z_{x_2}' = z_{x_2}''$, then we must have $(x_2, z_{x_2}'), (z_{x_2}', x_2) \in A(D)$, and so D has a spanning trail induced by the arcs $\{(z_1, z_2), (z_2, z_3), (z_3, x_1), (x_2, z_{x_2}'), (z_{x_2}', x_2)\}$. Therefore, we assume that $z_{x_2}' \ne z_{x_2}''$. If $z_1 \in \{z_{x_2}', z_{x_2}''\}$, then we may assume by symmetry that $\{z_1, z_3\} = \{z_{x_2}', z_{x_2}''\}$. It follows by $c = 3$ that $(z_1, x_2), (x_2, z_3) \in A(D)$, and so D has a spanning closed trail induced by the arcs $\{(x_1, z_2), (z_2, z_3), (z_3, x_1), (x_2, z_1), (z_3, x_2), (z_3, z_1)\}$. If $z_1 \notin \{z_{x_2}', z_{x_2}''\}$, then by $c = 3$ and as D is not supereulerian, we must have that $(x_2, z_2), (z_3, x_2) \in A(D)$. Since $G(D)$ has a 5-cycle, there must be an arc $e \in A(D)$ incident with two vertices in $\{z_1, x_1, x_2\}$. By symmetry, assume that $(x_1, x_2) \in A(D)$, then D has a spanning trail induced by the arcs $\{(x_1, x_2), (x_2, z_2), (z_2, z_3), (z_3, z_1)\}$. This completes the proof of the lemma. ∎

A **block** of a graph G is a maximal subgraph H of G such that H contains no cut vertices of itself. By definition, if B is a block of a graph G with at least 3 vertices, then B must be 2-connected. Also by definition, if D is strong, then every block of $G(D)$ must either be 2-connected, or spanned by a 2-cycle. The main purpose of this subsection is to prove Theorem 3.18 below, which implies Theorem 3.1(i).

Theorem 3.18 *Let $n > 1$ be an integer, D be a strong digraph on n vertices with $n = |V(D)|$, $\alpha'(D) \le 2$ and $\kappa(G(D)) \ge 2$, and $G = G(D)$. Then one of the following holds.*
(i) $\alpha'(D) = 1$ and D is strongly trail-connected.
(ii) $\alpha'(D) = 2$ and the following are equivalent.
  (ii-1) D has a spanning trail.
  (ii-2) $D \notin \mathcal{D}(n)$.

Proof. Suppose first that $\alpha'(D) = 1$. Then G is spanned by a $K_{1,n-1}$. As (i) holds trivially if $n = 2$, we assume that $n \ge 3$. Let v_0 be the vertex of degree $n - 1$ in this $K_{1,n-1}$. If G does not have a cycle of length longer than 2, then v_0 is incident with every arc in $A(D)$. As D is strong, every arc of D is symmetric, and so D is the symmetric core of itself. It follows from Lemma 3.3(iii) that D is strongly trail-connected. Hence we assume that G contains a cycle of length at least 3. Then D has an arc that is not incident with v_0. By $\alpha'(D) = 1$, we must have $n = 3$ and so D is spanned by a directed 3-cycle. Once again we have that D is strongly trail-connected. This proves (i).

To prove (ii), we assume that $\alpha'(D) = 2$. By (18), every member $D \in \mathcal{D}(n)$ does not have a spanning trail, and so (ii-1) implies (ii-2). Hence we assume that $D \notin \mathcal{D}(n)$ to show that D has a spanning trail.

As it is routine to verify that every strong digraph with at most 3 vertices is supereulerian, we assume that $n \geq 4$.

Let $c = c(G)$ denote the length of a longest cycle of G. Since D is strong and $\alpha'(G) = \alpha'(D) = 2$, $2 \leq c \leq 5$. If $c = 2$, then $\tilde{G}$, the simplification of G, must be a tree and so every pair of adjacent vertices $u, v \in V(D)$ are vertices of a 2-cycle in D. It follows by Lemma 3.3(i) that $D = J(D)$ is supereulerian. Thus we may assume that $3 \leq c \leq 5$. Let B be a block of G that contains a longest cycle of G.

Claim 4 *Each of the following holds.*
(i) If $c = 5$, then $G = B$ with $|V(G)| = 5$.
(ii) If $c = 4$, then either $G = B$, or B is spanned by a $K \cong K_{2,t}$ for some $t \geq 2$ with w_1, w_2 being two nonadjacent vertices of degree t in K, such that every block B' of G other than B is a 2-cycle in D and contains exactly one vertex $v_{B'} \in V(K)$. Furthermore, if $t \geq 3$, then $v_{B'} \in \{w_1, w_2\}$.

Suppose that $c = 5$ and let C be a cycle of length 5. If $|V(B)| > 5$, then as B is connected, an edge $e \in E(B) - E(C)$ together with a matching of size 2 not adjacent with e forms a matching of sizes 3 in B, leading to a contradiction that $2 = \alpha'(G) \geq \alpha'(B) \geq 3$. Hence we must have $|V(B)| = 5$. Assume that G has a block B_1 other than B. Then there must be an edge $e' \in E(B_1)$. By definition of blocks, $|V(B) \cap V(B_1)| \leq 1$. Since C contains a matching M' of size 2. It follows that $2 = \alpha'(G) \geq |M' \cup \{e'\}| = 3$, a contradiction. Hence we must have $G = B$.

Now we assume that $c = 4$, and so B contains a $K_{2,2}$ as a subgraph. Choose a maximum value t such that B contains a subgraph K isomorphic to a $K_{2,t}$. Let w_1, w_2 denote two nonadjacent vertices of degree t in K and let $V(K) - \{w_1, w_2\} = \{v_1, v_2, ..., v_t\}$. If there exists a vertex $z \in V(B) - V(K)$, then since $\kappa(B) \geq 2$, there will be two internally disjoint shortest paths from z to two distinct vertices z', z'' in $V(K)$, implying that either B has a cycle of length at least 5, or G has a subgraph isomorphic to a $K_{2,t+1}$. As either case leads to a contradiction, we conclude that B is spanned by K.

Assume that $G \neq B$. Let B' be an arbitrary block of G other than B. If $V(B') \cap V(B) = \emptyset$, then an edge in B' together with a 2-matching in B would lead to the contradiction $2 = \alpha'(D) \geq 3$. Hence every block B' other than B in G must contain a vertex $v_{B'}$ such that $V(B') \cap V(K) = V(B') \cap V(B) = \{v_{B'}\}$, and every edge in B' is incident with the vertex $v_{B'} \in V(K)$. Again by $\alpha'(D) = 2$, if $t \geq 3$, then we must have $v_{B'} \in \{w_1, w_2\}$ for any block B' other than B in G. As D is strong, G is 2-edge-connected and so $\kappa'(B') \geq 2$. This implies that B' is a 2-cycle containing $v_{B'}$. Since D is strong, this 2-cycle in B' is a 2-cycle in D. This justifies Claim 4.

By Claim 4 and Lemma 3.17, if $c = 5$, then D has a spanning trail. Hence it suffices to assume that $3 \leq c \leq 4$ to prove Theorem 3.18(ii).

Claim 5 *Suppose that $c = 3$. Each of the following holds.*
(i) Every block of G has 2 or 3 vertices.
(ii) There are at most two blocks of order 3, and if G has two blocks B', B'' of order 3, then $|V(B') \cap V(B'')| = 1$.
(iii) D has a spanning closed trail.

Assume that $c = 3$. Let $B_1, B_2, ..., B_b$ be all the blocks of G such that for some b' with $1 \leq b' \leq b$, $|V(B_1)| \geq ... \geq |V(B_{b'})| \geq 3$ and $|V(B_{b'+1})| = ... = |V(B_b)| = 2$. For each $B \in \{B_1, ..., B_{b'}\}$, as $c = 3$,

B contains a 3-cycle C. If there exists a vertex $v \in V(B) - V(C)$, then as $\kappa(B) \geq 2$, there will be two internally disjoint shortest paths from v to two distinct vertices in $V(C)$, implying the B has a cycle of length at least 4. Hence we must have $V(B) = V(C)$, and so Claim 5(i) follows.

Since two distinct blocks B', B'' of G must satisfy $|V(B') \cap V(B'')| \leq 1$, it follows that $b' \leq \alpha'(D) = 2$. Furthermore, assume that $|V(B') \cap V(B'')| = 0$, then as G is connected, there must be an additional block B''' of G. It follows by $|V(B')| = |V(B'')| = 3$ and $|V(B''')| = 2$ that G has a matching of size 3, contrary to $\alpha'(D) = 2$. This justifies Claim 5(ii).

Since D is strong, every block B of G induces a strong subdigraph $D[V(B)]$ of D. It follows by $|V(B)| \leq 3$ that every $D[V(B)]$ is supereulerian. Thus D has a spanning closed trail. This completes the proof of Claim 5.

By Claims 4 and 5 and by Lemma 3.17, we may assume that $c = 4$. By Claim 4(ii), for some integer $t \geq 2$, $G(D)$ has a unique block B spanned by a $K_{2,t}$. If $t = 2$, then B is a 4-cycle. By Claim 4(ii) and Corollary 3.16, D is supereulerian, and so D has a spanning trail.

Hence we assume that $t \geq 3$. Let w_1, w_2 denote the two vertices of degree t in this $K_{2,t}$ such that every block of $G(D)$ other than B is a 2-cycle of D containing w_1 or w_2. By Example 3.13 (and using the notation in Example 3.13), $B = D(t_1, t_1', t_1'', t_2, t_2', t_2'', t_3)$ for some non negative integers $t_1, t_1', t_1'', t_2, t_2', t_2'', t_3$ satisfying $|V(B)| = 2 + t_1 + t_1' + t_1'' + t_2 + t_2' + t_2'' + t_3$. As $D \notin \mathcal{D}(n)$, we conclude that either (14) or (15) must hold. By Example 3.13(ii), D has a spanning trail. This completes the proof for Theorem 3.18(ii) ■

3.4 Supereulerian digraphs and strongly trail-connected digraphs

The main result of this subsection is to prove Theorem 3.1(iii) and (iv), restated in Theorem 3.19 below. Recall that D_0 denotes the vertex disjoint union of three complete digraphs of order 3.

Theorem 3.19 *Let D be a strong digraph on n vertices with $\alpha'(D) \geq 3$, and $n \geq 2\alpha'(D) + 3$, and let $J = J(D)$ be a symmetric core of D. Each of the following holds.*
(i) If $\lambda(D) \geq \alpha'(D) - 1$, then D is supereulerian.
(ii) If $\lambda(D) \geq \alpha'(D) \geq 4$, then J is a spanning subdigraph of D.

Proof. Let $k = \alpha'(D) \geq 3$ and $n = |V(D)| \geq 2k+3$. By Corollary 3.8, Theorem 3.19(ii) holds. It suffices to prove Theorem 3.19(i). As $\lambda(D) \geq k - 1 \geq 2$, $D \ncong D_0$ and for any vertex $v \in V(D)$, $d_D(v) \geq 2k - 2$. Suppose first that there exists a vertex $x_1 \in X$ such that $d_D(x_1) \geq 2k - 1$. If $k_1(x_1) > 0$, then by Lemma 3.6(iv), D is supereulerian; if $k_1(x_1) = 0$, then by Lemma 3.9(iv) and as $\lambda(D) \geq 2$, D is supereulerian. Therefore, we assume that for any vertex $x \in X$, $d_D(x) = 2k - 2$. If there exists a vertex $x_1 \in X$ with $k_1(x_1) > 0$, then by Lemma 3.12(ii), D is supereulerian. Now assume that for any vertex $x \in X$, $k_1(x) = 0$. By Lemmas 3.10(iii) and 3.11(iii), D must also be supereulerian. This completes the proof of Theorem 3.19. ■

3.5 Spanning trails in digraphs

The purpose of this subsection is to prove Theorem 3.1(ii). Throughout this subsection, D denotes a strong digraph on n vertices with $n = |V(D)| \geq 6$ and $\alpha'(D) = k \geq 3$.

In chapter 2, we presented Example 2.11 which showed that there exists a family of digraphs $\mathcal{D}(k_1, k_2, \ell)$ such that for every digraph in $D \in \mathcal{D}(k_1, k_2, \ell)$ is a not supereulerian, also Hong et al. [30] showed that every digraph in $\mathcal{D}_0(k_1, k_2, 2)$ is a not supereulerian.

Let $k \geq 3$ be an integer. It is routine to verify the following.

Observation 3.20 *Every digraph $D \in \mathcal{D}_0(k-1, k-1, 2)$ with $\lambda(D) \geq k-1$ has a spanning trail.*

By using Example 2.11 for the structure of D, we let $D_1 \cong D_2 \cong K_k^*$ and $U = \{u_1, u_2\}$ with an arc $(v', v'') \in (V(D_1), V(D_2))_D$, one can start with a vertex $w'' \in V(D_2) - \{v''\}$, traverses every vertices in D_2 and then passes u_2; then from u_2 to a vertex $w' \in V(D_1) - \{v'\}$ and traverses every vertex in $V(D_1)$ with the last vertex in v'; and finally completes the trail with the arcs $(v', v''), (v'', u_1)$. Thus D has a spanning trail.

To prove Theorem 3.1(ii), we used Example 2.11, Theorem 2.40 and Observation 3.20.

Proof of Theorem 3.1(ii). Assume that $n = |V(D)| \geq 12$, $\alpha'(D) = k \geq 3$ and $\lambda(D) \geq k - 1 \geq 2$. By Theorem 3.1(iii), if $n = |V(D)| \geq 2k + 3$, then D is supereulerian and so has a spanning trail. Hence we assume that $2k \leq n \leq 2k + 2$. If $n \in \{2k, 2k + 1\}$, then by Theorem 2.40, D is supereulerian. Therefore we assume that $n = 2k + 2$, and so by $n \geq 12$, $\min\{\delta^+(D), \delta^-(D)\} \geq \lambda(D) \geq k - 1 \geq \frac{n-4}{2} \geq 4$ and $\delta^+(D) + \delta^-(D) \geq n - 4$. By Theorem 2.40, either D is supereulerian or $D \in \mathcal{D}_0(k-1, k-1, 2)$. By Observation 3.20, D has a spanning trail. This completes the proof of Theorem 3.1(ii).

Chapter 4

4 Supereulerin Digraph Strong Product

In this chapter, we motivate an open problem Problem 6 of [26], which was raised to find natural conditions for the product of graphs to be hamiltonian. Alsatami et al. [6] showed sufficient conditions on digraphs D_1 and D_2 and proved Theorem 2.64, in chapter 2, of Cartesian product of D_1 and D_2 is supereulerian. This motivates us to present sufficient conditions on digraphs D_1 and D_2 and prove the Strong product of D_1 and D_2 is supereulerian, which is following main result of this chapter.

Theorem 4.1 *Let D_1 and D_2 be strong digraphs. If $f(D_2) \leq |V(D_1)|$ and if for some cycle factor F of D_1, D_1/F is hamiltonian, then the strong product $D_1 \boxtimes D_2$ is supereulerian.*

4.1 Lemmas

In this section, we develop some lemmas which will be used in our arguments. The proof of Theorem 4.1 will be given in the last section.

Let $k \geq 0$ be an integer. We use $\mathbb{Z}_k = \{1, 2, \ldots, k\}$ to denote the cyclic group of order k and with the additive binary operation $+_k$ and with k being the additive identity in $\mathbb{Z}_k$. Let H and H' denote two digraphs. As we are to discuss product for digraphs D_1 and D_2 with $u \in V(D_1)$ and $v \in V(D_2)$, we save the notation (u, v) for a vertex in the product of D_1 and D_2. Define $H \cup H'$ to be the digraph with $V(H \cup H') = V(H) \cup V(H')$ and $A(H \cup H') = A(H) \cup A(H')$.

Let $T = v_1v_2 \cdots v_k$ denote a trail. We use $T[v_1, v_k]$ to emphasize that T is oriented from v_1 to v_k. For any $1 \leq i \leq j \leq k$, we use $T[v_i, v_j] = v_iv_{i+1} \cdots v_{j-1}v_j$ to denote the sub-trail of T. Likewise, if $Q = u_1u_2 \cdots u_ku_1$ is a closed trail, then for any i, j with $1 \leq i < j \leq k$, $Q[u_i, u_j]$ denotes the subtrail $u_iu_{i+1} \cdots u_{j-1}u_j$. If $T' = w_1w_2 \cdots w_{k'}$ is a trail with $v_k = w_1$ and $V(T) \cap V(T') = \{v_k\}$, then we use TT' or $T[v_1, v_k]T'[v_k, w_{k'}]$ to denote the trail $v_1v_2 \cdots v_kw_2 \cdots w_{k'}$. If $V(T) \cap V(T') = \emptyset$ and there is a path $z_1z_2 \cdots z_t$ with $z_2, \ldots, z_{t-1} \notin V(T) \cup V(T')$ and with $z_1 = v_k$ and $z_t = w_1$, then we use $Tz_1 \cdots z_tT'$ to denote the trail $v_1v_2 \cdots v_kz_2 \cdots z_tw_2 \cdots w_{k'}$. In particular, if T is a (v, w)-trail of a digraph D and $uv, wz \in A(D) - A(T)$, then we use $uvTwz$ to denote the (u, z)-trail $D[A(T) \cup \{uv, wz\}]$. The subdigraphs uvT and Twz are similarly defined.

Lemma 4.2 *Let $J_1, J_2, \ldots, J_k$ be vertex disjoint strong subdigraphs of a digraph D, and $J = \bigcup_{i=1}^{k} J_i$ is the disjoint union of these subdigraphs. Let $v_1, v_2, \ldots, v_k$ be vertices in $V(D/J)$ such that for each $i \in [k]$, J_i is the preimage of v_i. Suppose that $C' = v_{i_1}v_{i_2} \cdots v_{i_s}$ be a cycle of D/J. Each of the following holds.*
(i) D has a cycle C with $A(C') \subseteq A(C)$ such that for each $i \in [k]$, $V(C) \cap V(J_i) \neq \emptyset$. (Such a cycle C is called a **lift** *of the cycle C'.*
(ii) If for each $i \in \mathbb{Z}_s$, $e_i = v_i''v_{i+1}' \in A(C')$ is an arc in D with $v_i'' \in V(J_i)$ and $v_{i+1}' \in V(J_{i+1})$, then $C[v_i', v_i'']$ is a path in J_i.

Proof. As (i) implies (ii), it suffices to prove (i). Let $C' = v_1v_2 \cdots v_sv_1$ be a cycle of D/J, and for each

$i \in \mathbb{Z}_s$. By definition, the arc $e_i := v_i v_{i+1} \in A(C')$ is an arc in D, and so we may assume that there exist vertices $v_i', v_i'' \in A(J_i)$ such that $e_i = v_i'' v_{i+1}' \in A(D)$. If J_i is trivial, then we have $v_i' = v_i''$. Since J_i is strong, J_i contains a (v_i', v_i'')-path P_i. Thus

$$C := P_1 v_1'' v_2' P_2 v_2'' v_3' \cdots v_{i-1}'' v_i' P_i v_i'' v_{i+1}' P_{i+1} \cdots v_{s-1}'' v_s' P_s v_s'' v_1'$$

is a cycle of D with $C[v_i', v_i'']$ being a path in J_i, for each $i \in \mathbb{Z}_s$. ∎

Following [9], we define a digraph to be **cyclically connected** if for every pair x, y of distinct vertices of D there is a sequence of cycles $C_1, C_2, \ldots, C_k$ such that x is in C_1, y is in C_k, and C_i and C_{i+1} have at least one common vertex for every $i \in [k-1]$. The following results are useful.

Lemma 4.3 *Let D be a digraph.*
(i) (Exercise 1.17 of [9]) A digraph D is strong if and only if it is cyclically connected.
(ii) If H_1 and H_2 are strong subdigraphs of D with $V(H_1) \cap V(H_2) \neq \emptyset$, then $H_1 \cup H_2$ is also strong.

Lemma 4.3 (ii) follows immediately from definition of strong digraphs.

Proposition 4.4 *(Alsatami, Liu and Zhang, Proposition 2.1 of [6]) Let D be a weakly connected digraph. Then the following are equivalent.*
(i) D has a cycle vertex cover.
(ii) D is strong.
(iii) D is cyclically connected.
(iv) For any vertices $u, v \in V(D)$, there exists an eulerian chain joining u and v.

Lemma 4.5 *Let D_1 and D_2 be digraphs. Each of the following holds.*
(i) If D_1 and D_2 are cycles, then $D_1 \times D_2$ is a circulation.
(ii) If H_1 and H_2 are arc-disjoint subdigraphs of D_1, then $H_1 \times D_2$ and $H_2 \times D_2$ are arc-disjoint subdigraphs of $D_1 \times D_2$.
(iii) If each of D_1 and D_2 has a cycle factor, then $D_1 \times D_2$ has a cycle factor.

Proof. For (i), let V_1 and V_2 be the vertex sets of D_1 and D_2, respectively. It suffices to prove that for each $(u_i, v_j) \in V_1 \times V_2$, $d^+_{D_1 \times D_2}((u_i, v_j)) = d^-_{D_1 \times D_2}((u_i, v_j))$. Let $(u_i, v_j) \in V_1 \times V_2$. Since D_1 and D_2 are cycles, we have $|N^+_{D_1}(u_i)| = |N^-_{D_1}(u_i)|$ and $|N^+_{D_2}(v_j)| = |N^-_{D_2}(v_j)|$. By Definition 1.15 (ii) (Direct Product

$D_1 \times D_2$), we have the following, which implies (i).

$$\begin{aligned}
d^+_{D_1\times D_2}((u_i,v_j)) &= |N^+_{D_1\times D_2}((u_i,v_j))| = |\{(u_s,v_t)\in V_1\times V_2 : (u_i,v_j)(u_s,v_t)\in A(D_1\times D_2)\}| \\
&= |\{(u_s,v_t)\in V_1\times V_2 : u_iu_s\in A(D_1) \text{ and } v_jv_t\in A(D_2)\}| \\
&= \sum_{u_s\in N^+_{D_1}(u_i)} \sum_{v_t\in N^+_{D_2}(v_j)} |\{(u_s,v_t)\in V_1\times V_2\}| \\
&= |N^+_{D_1}(u_i)|\cdot|N^+_{D_2}(v_j)| = |N^-_{D_1}(u_i)|\cdot|N^-_{D_2}(v_j)| \\
&= \sum_{u_s\in N^-_{D_1}(u_i)} \sum_{v_t\in N^-_{D_2}(v_j)} |\{(u_s,v_t)\in V_1\times V_2\}| \\
&= |\{(u_s,v_t)\in V_1\times V_2 : u_su_i\in A(D_1) \text{ and } v_tv_j\in A(D_2)\}| \\
&= |N^-_{D_1\times D_2}((u_i,v_j))| = |\{(u_s,v_t)\in V_1\times V_2 : (u_s,v_t)(u_i,v_j)\in A(D_1\times D_2)\}| \\
&= d^-_{D_1\times D_2}((u_i,v_j)).
\end{aligned}$$

To prove (ii), let H_1 and H_2 be an arc-disjoint subdigraphs of D_1. If there exists an arc

$$(u_i,v_j)(u_s,v_t)\in(H_1\times D_2)\cap(H_2\times D_2),$$

then by Definition 1.15, we must have $(u_i,u_s)\in H_1\cap H_2$. Hence if H_1 and H_2 are arc-disjoint subdigraphs of D_1, then $H_1\times D_2$ and $H_2\times D_2$ are arc disjoint subdigraphs of $D_1\times D_2$.

To prove (iii), let F_1 and F_2 be the spanning circulations of D_1 and D_2, respectively. By Definition 1.15 (ii) (Direct product $D_1\times D_2$), $F_1\times F_2$ is spanning subdigraph of $D_1\times D_2$. By (i), $F_1\times F_2$ is a circulation, and so $F_1\times F_2$ is the spanning circulation of $D_1\times D_2$. Thus $F_1\times F_2$ is a cycle factor of $D_1\times D_2$. ∎

Lemma 4.6 *Let D_1, D_2 be digraphs and F be a subdigraph of D_1. Then $A(F\Box D_2)\cap A(F\times D_2)=\emptyset$.*

Proof. Suppose that there exists an arc $(u_i,v_j)(u_s,v_t)\in A(F\Box D_2)\cap A(F\times D_2)$. By Definition 1.15 (Cartesian Product $D_1\Box D_2$) (i), as $(u_i,v_j)(u_s,v_t)\in A(F\Box D_2)$, we have either $u_i=u_s$ and $v_jv_t\in A(D_2)$, or $u_iu_s\in A(F)$ and $v_j=v_t$. By Definition 1.15 (ii), if $u_i=u_s$, or if $v_j=v_t$, then $(u_i,v_j)(u_s,v_t)\notin A(F\times D_2)$. It follows that $A(F\Box D_2)\cap A(F\times D_2)=\emptyset$. ∎

Theorem 4.7 *(Hammack, Theorem 10.3.2 of [29]) Let m and n be integers with $m\ge n\ge 2$ and let C_m and C_n denote the cycles of order m and n, respectively. Let $gcd(m,n)$ and $lcm(m,n)$ be the greatest common divisor and the least common multiplier of m and n, respectively. Then the direct product $C_m\times C_n$ is a vertex disjoint union of $gcd(m,n)$ cycles, each of which has length $lcm(m,n)$.*

We can show a bit more structural properties in the direct product revealed by Theorem 4.7, which are stated in Lemma 4.8.

Lemma 4.8 *Let D_1 and D_2 be digraphs with vertex set $V_1=\{u_1,u_2,\dots,u_{n_1}\}$ and $V_2=\{v_1,v_2,\dots,v_{n_2}\}$ (notation in (2)).*
(i) Suppose that D_1 and D_2 are cycles and $v\in V(D_2)$ is an arbitrarily given vertex. Then for any cycle

C in $D_1 \times D_2$, there exists a vertex $u \in V(D_1)$ such that the vertex $(u, v) \in V(C)$.
(ii) Suppose that D_1 and D_2 are circulations and $v \in V(D_2)$ is an arbitrarily given vertex. Then $D_1 \times D_2$ is also a circulation. Moreover, for any eulerian subdigraph F in $D_1 \times D_2$, there exists a vertex $u \in V(D_1)$ such that the vertex $(u, v) \in V(F)$.

Proof. Suppose $D_1 = u_1 u_2 \cdots u_{n_1} u_1$ and $D_2 = v_1 v_2 \cdots v_{n_2} v_1$ are cycles, and by symmetry, assume that $v = v_1$. Let C be a cycle in $D_1 \times D_2$. Thus C contains a vertex (u_i, v_j). It follows by Definition 1.15 (ii) that

$$C = \cdots (u_i, v_j)(u_{i+1}, v_{j+1}) \cdots (u_{i+n_2-j}, v_{n_2})(u_{i+n_2-j+1}, v_1) \cdots$$

where the subscripts of vertices in D_1 are taken in $\mathbb{Z}_{n_1}$ and those of vertices in D_2 are taken in $\mathbb{Z}_{n_2}$. It follows that $u = u_{i+n_2-j+1}$. This proves (i). Suppose that D_1 and D_2 are circulations. As every circulationis an arc-disjoint union of cycles (nothation (1)), each of D_1 and D_2 is an arc-disjoint union of cycles. By Lemma 4.5, $D_1 \times D_2$ is also a circulation. Let F be an eulerian subdigraph in $D_1 \times D_2$. By (1), F is also an arc-disjoint union of cycles $C_1, C_2, \cdots$. Applying Lemma 4.8 (i) to each cycle C_i, we conclude that (ii) holds as well. ∎

4.2 Proofs of Theorem 4.1

Assume that D_1 and D_2 are two strong digraphs, and for some cycle factor F of D_1, D_1/F is hamiltonian with $f(D_2) \le |V(D_1)|$. We start with some notation for the copies of factors in the Cartesian product.

Definition 4.9 *Let $D_1 = (V_1, A_1)$ and $D_2 = (V_2, A_2)$ be two strong digraphs with $V_1 = \{u_1, u_2, \ldots, u_{n_1}\}$ and $V_2 = \{v_1, v_2, \ldots, v_{n_2}\}$. For $i \in \{1, 2\}$, let H_i be a subdigraph of D_i.*
(i) For each $u \in V_1$, let D_2^u be the subdigraph of $D_1 \Box D_2$ induced by $V(D_2^u) = \{(u, v_i) : 1 \le i \le n_2\}$. The subdigraph D_2^u is called the u-copy of D_2 in $D_1 \Box D_2$.
(ii) For each $v \in V_2$, let D_1^v be the subdigraph of $D_1 \Box D_2$ induced by $V(D_1^v) = \{(u_i, v) : 1 \le i \le n_1\}$. The subdigraph D_1^v is called the v-copy of D_1 in $D_1 \Box D_2$.
(iii) More generally, for each $u \in V_1$ (or $v \in V_2$, respectively), let H_2^u (or H_1^v, respectively) be the subdigraph of D_2^u (or D_1^v, respectively) induced by $A(H_2^u) = \{(u, v_i)(u, v_i') : v_i v_i' \in A(H_2)\}$ (or $A(H_1^v) = \{(u_i, v)(u_i', v) : u_i u_i' \in A(H_1)\}$, respectively). The subdigraph H_1^v is called the v-copy of H_1 in $D_1 \Box D_2$ and the subdigraph H_2^u is called the u-copy of H_2 in $D_1 \Box D_2$.

If two digraphs D and H are isomorphic, then we write $D \cong H$. The following is an immediate observation from Definition 4.9 for the Cartesian product $D_1 \Box D_2$ of two digraphs D_1 and D_2.

$$\text{for any } v \in V(D_2),\ D_1 \cong D_1^v, \text{ and for any } u \in V(D_1),\ D_2 \cong D_2^u. \tag{19}$$

Let F be a cycle factor of D_1 such that D_1/F has a Hamilton cycle. Since F is a cycle factor of D_1, each component of F is an eulerian subdigraph of D_1. Let

$$F_1, F_2, \ldots, F_k \text{ be the components of } F, \text{ and } J = D_1/F. \tag{20}$$

Then $V(J) = \{w_1, w_2, \ldots, w_k\}$, where for each $i \in [k]$, w_i is the contraction image in J of the eulerian subdigraph F_i in D_1. Since J is hamiltonian, we may by symmetry assume that $C' = w_1 w_2 \cdots w_k w_1$ is a hamilton cycle of J. It follows by Lemma 4.2 that

$$D_1 \text{ has a cycle } C \text{ with } A(C') \subseteq A(C). \tag{21}$$

Now we consider D_2. Let $f(D_2) = m \le |V(D_1)|$ and F' be a circulation of D_2 such that D_2/F' has a cycle vertex cover $\mathcal{C}' = \{C'_1, C'_2, \ldots, C'_m\}$. Let $F'_1, F'_2, \ldots, F'_{k'}$ be the components of F', $w'_{k'+1}, \ldots, w'_t$ be the vertices in $V(D_2) - V(F')$. We define, for each i with $k'+1 \le i \le t$, F'_i to be the digraph with $V(F'_i) = \{w'_i\}$ and $A(F'_i) = \emptyset$. With these definitions, we have

$$V(D_2/F') = \{w'_1, w'_2, \ldots, w'_{k'}, w'_{k'+1}, \ldots w'_t\}. \tag{22}$$

By Lemma 4.2, for each $j \in [m]$, C'_j in $\mathcal{C}'$ can be lifted to a cycle C_j in D_2. To construct a spanning eulerian subdigraph of $D_1 \boxtimes D_2$, we start by justifying the following claims.

Claim 6 *Each of the following holds.*
(i) For any $i \in [k]$, and $j \in [t]$, $F_i \times F'_j$ is a circulation.
(ii) For any $i \in [k]$, and $j \in [t]$, $F_i \Box F'_j$ is an eulerian digraph.
(iii) For each $i \in [k]$, and each $j \in [t]$, if $v \in V(F'_j)$, then $F_i^v \cup (F_i \times F'_j)$ is a spanning eulerian subdigraph of $F_i \boxtimes F'_j$.

Proof. For each $i \in [k]$, F_i is an eulerian subdigrah of D_1, so F_i is a disjoint union of cycles. Similarly, for each $j \in [k']$, F'_j is an eulerian sudigraph of D_2, so F'_j is a disjoint union of cycles. By Lemma 4.8, $F_i \times F'_j$ is a circulation.

By assumption, for each $i \in [k]$, F_i is an eulerian subdigrah of D_1. If $j \in [k']$, then as F'_j is an eulerian subdigraph of D_2, it follows by Theorem 2.63 that $F_i \Box F'_j$ is an eulerian digraph. Now assume that $k'+1 \le j \le t$. Then $V(F'_j) = \{w'_j\}$, and so by (19), $F_i \Box F'_j = F_i^{w'_j} \cong F_i$ is eulerian. This proves (ii).

For each $i \in [k]$, each $j \in [t]$ and a fixed vertex $v \in V(F'_j)$, let $J' = F_i^v \cup (F_i \times F'_j)$. By (i), $F_i \times F'_j$ is a circulation. By (19), $F_i^v \cong F_i$ is an eulerian digraph. By Lemma 4.6, $A(F_i^v) \cap A(F_i \times F'_j) = \emptyset$. It follows that for any vertex $z \in V(J')$,

$$d^+_{J'}(z) = d^+_{F_i^v}(z) + d^+_{F_i \times F'_j}(z) = d^-_{F_i^v}(z) + d^-_{F_i \times F'_j}(z) = d^-_{J'}(z)$$

and so J' is a circulation. Without loss of generality, we denote $V(F_i) = \{u_{i_1}, u_{i_2}, \ldots, u_{i_{t_i}}\}$ and $V(F'_j) = \{v_{j_1}, v_{j_2}, \ldots, v_{j_{s_j}}\}$ with $v = v_{j_1}$. To prove that J' is connected, let $(u_{i_1}, v_{j_1}) \in V(J')$ and let J_1 be the connected component of J' that contains (u_{i_1}, v_{j_1}). If J' is not connected, then by symmetry, we may assume that there exists a vertex $(u_{j_2}, v_{j_2}) \in V(J') - V(J_1)$. As $F_i \times F'_j$ is a circulation, there must be an eulerian subdigraph F of $F_i \times F'_j$ with $(u_{i_2}, v_{j_2}) \in V(F)$. By Lemma 4.8$(ii)$, there exist a vertex $u' \in V(D_1)$ such that $(u', v_{j_1}) \in V(F)$. Thus by Definition 4.9(ii), $V(F) \cap V(F_i^v) \ne \emptyset$. By (19) and (20), $F_i^v \cong F_i$ is connected, and so both (u_{i_1}, v_{j_1}) and (u', v_{j_1}) must be in the same component of J'. This implies that $(u', v_{j_1}) \in V(J_1)$. Since (u_{i_2}, v_{j_2}) and (u', v_{j_1}) are in the same component of J', it follows that $(u_{i_2}, v_{j_2}) \in V(J_1)$ also, contrary to the assumption that $(u_{i_2}, v_{j_2}) \in V(J') - V(J_1)$. Hence J' must be connected, and so $F_i^v \cup (F_i \times F'_j)$ is a spanning eulerian subdigraph $F_i \boxtimes F'_j$. ∎

Claim 7 *Let C' be a Hamilton cycle of J and C be a lift of C' in D_1 as warranted by (21). For each*

$v \in V(D_2)$, let C^v denote the v-copy of C in $D_1 \Box D_2$. For each $j \in [t]$, if $v, v' \in V(F'_j)$ are two distinct vertices, then

$$H_{v,v';j} := \bigcup_{i=1}^{k} (F_i^{v'} \cup (F_i \times F'_j)) \cup C^v$$

is a spanning eulerian subdigraph $D_1 \boxtimes F'_j$.

Proof. By Lemma 4.2. for any $v \in V(D_2)$, C^v has the property that for any $i \in [k]$, $V(C^v) \cap V(F_i^v) \neq \emptyset$. By Claim 6 (*iii*), for any $i \in [k]$ and for any $j \in [t]$, $F_i^{v'} \cup (F_i \times F'_j)$ is a spanning eulerian subdigraph $F_i \boxtimes F'_j$, and so $F_i^{v'} \cup (F_i \times F'_j)$ is a strong subdigraph of $D_1 \boxtimes F'_j$. Since for any $i \in [k]$, $V(C^v) \cap V(F_i^v) \neq \emptyset$, we may assume that for some vertex $u \in V(F_i)$, $(u, v) \in V(C^v) \cap V(F_i^v)$. As $v \in V(F'_j)$, we have $(u, v) \in V(C^v) \cap V(F_i^{v'} \cup (F_i \times F'_j))$, and so $F_i^{v'} \cup (F_i \times F'_j) \cup C^v$ is connected. Since $v \neq v'$, $A(C^v) \cap A(F_i^{v'} \cup (F_i \times F'_j)) = \emptyset$, we conclude from the facts that C^v and $F_i \times F'_j$ are circulations (see Claim 6(i)) that $F_i^{v'} \cup (F_i \times F'_j) \cup C^v$ is eulerian. As $i \in [k]$ is arbitrarily, we conclude that $H_{v,v';j} = \bigcup_{i=1}^{k} (F_i^{v'} \cup (F_i \times F'_j)) \cup C^v$ is an eulerian subdigraph with vertex set $V(H_{v,v';j}) = \bigcup_{i=1}^{k} (F_i \times F'_j) = V(D_1 \boxtimes F'_j)$. This proves Claim 7.

Claim 8 *Let $u \in V(D_1)$ be an arbitrary vertex, F' be a circulation of D_2 such that D_2/F' has a cycle vertex cover $\mathcal{C}' = \{C'_1, C'_2, \ldots, C'_m\}$ with $m = f(D_2) \leq |V(D_1)|$. Each of the following holds.*
(i) F'^u is a circulation of D_2^u.
(ii) For any $j \in [m]$, $C_j'^u$ is a cycle of D_2^u/F'^u, and $\{C_1'^u, C_2'^u, \ldots, C_m'^u\}$ is a cycle vertex cover of D_2^u/F'^u.
(iii) Let $u \in V(D_1)$ be a vertex, $h \in [m]$ be arbitrarily given. For any vertex $w'_j \in V(C'_h)$, let $v(j), v'(j)$ be two distinct vertices in $V(F'_j)$, and C_h be a lift of C'_h in D_2. Then

$$H_h^u = \left[\bigcup_{w'_j \in V(C'_h)} H_{v(j),v'(j);j} \right] \cup C_h^u$$

is an eulerian digraph with $V(H_h^u) = \displaystyle\bigcup_{v_j \in V(C_h)} V(D_1^{v_j})$.

Proof. Each of (*i*) and (*ii*) follows from (19) and the definition of $\mathcal{C}'$. It remains to prove (*iii*). By Lemma 4.2, C'_h can be lifted to a cycle C_h in D_2. For any $w'_j \in V(C'_h)$, pick two distinct vertices $v, v' \in V(F'_j)$. By Claim 7, $H_{v,v';j}$ defined in Claim 7 is a spanning eulerian subdigraph $D_1 \boxtimes F'_j$. By Lemma 4.6, $C_h^u = D_1[\{u\}] \Box C_h$ is arc-disjoint from each $H_{v,v':j}$, and so by the facts that C_h^u is a directed cycle and $H_{v,v':j}$ is eulerian, it follows that H_h^u is a circulation. By Definition 4.9 (*iii*) and by Lemma 4.6, a vertex $w'_j \in V(C'_h)$ if and only if $V(C_h^u) \cap V(F_j'^u) \neq \emptyset$. This is equivalent to saying that a vertex $w'_j \in V(C'_h)$ if and only if for some vertex $v'' \in V(F'_j)$, $(u, v'') \in V(C_h^u)$. Since C_h^u is a cycle, and since, for each $w'_j \in V(C'_h)$, there exists some vertex $v'' \in V(F'_j)$ with $(u, v'') \in V(C_h^u)$, we observe that $V(H_{v,v';j}) \cap V(C_h^u)$ contains a vertex (u, v''), it follows that H_h^u must be connected. Hence H_h^u is a connected circulation, and so it must be eulerian. To complete the justification of Claim8 (*iii*), we note that by definition,

$$V(C_h^u) \subseteq \bigcup_{w'_j \in V(C'_h)} V(D_1 \boxtimes F'_j).$$

This, together with Claim 7, implies

$$V(H_h^u) = \bigcup_{w'_j \in V(C'_h)} V(H_{v(j),v'(j);j}) \cup V(C_h^u) = \bigcup_{w'_j \in V(C'_h)} V(D_1 \boxtimes F'_j) = \bigcup_{v_j \in V(C_h)} V(D_1^{v_j}).$$

This completes the proof of Claim 8. ∎

Recall that $V(D_1) = \{u_1, u_2, \ldots, u_{n_1}\}$ with $n_1 \geq m = f(D_2)$. We will complete the proof of Theorem 3.1 by proving that

$$H = \bigcup_{h=1}^{m} H_h^{u_h}$$

is a spanning eulerian subdigraph of $D_1 \boxtimes D_2$. By Claim 8 (*iii*), we conclude that $V(H) = \bigcup_{j=1}^{t} V(D_1 \boxtimes F'_j) = V(D_1 \boxtimes D_2)$. As $u_1, \ldots, u_m$ are mutually distinct, and as $F'_1, F'_2, \ldots, F'_t$ are mutually vertex disjoint, we conclude that the $H_h^{u_h}$'s are mutually arc-disjoint. By Claim 8 (*iii*), each $H_h^{u_h}$ is eulerian, and so H is a circulation. It remains to show that H is connected. By Claim 8 (*iii*), H has a component H' that contains $H_1^{u_1}$. If $H = H'$, then done. Assume that $V(H) - V(H') \neq \emptyset$. Since H' is a component, if some $H_h^{u_h}$ contains a vertex in H', then H' contains $H_h^{u_h}$ as a subdigraph. Thus every $H_h^{u_h}$ is either contained in H' or totally disjoint from H'. Let $W = \{w'_j \in V(D_2/F') : H_j^{u_j}$ is contained in $H'\}$. Then as $H \neq H'$, $V(D_2/F') - W \neq \emptyset$. Since $\mathcal{C}'$ is a cycle vertex cover of D_2/F', it follows by Definition 1.12 (ii) that there must be a cycle $C'_j \in \mathcal{C}'$ such that C'_j contains a vertex $w' \in W$ and a vertex $w'' \in V(D_2/F') - W$. Since $w' \in W$, $H_j^{u_j}$ is contained in H'. Since $w', w'' \in V(C'_j)$, it follows that $w'' \in W$, contrary to the fact that $w'' \in V(D_2/F') - W$. This contradiction indicates that we must have $H = H'$, and so H is a spanning eulerian subdigraph of $D_1 \boxtimes D_2$. ∎

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