JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **30**, Issue **1**, June **2022**, pp. 71–87, DOI 10.15421/142203

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

NONLINEAR EVOLUTIONARY PROBLEM OF FILTRATION CONSOLIDATION WITH THE NON-CLASSICAL CONJUGATION CONDITION

Olha R. Michuta[*], Petro M. Martyniuk[†]

Abstract. Finite-element solutions of the initial-boundary value problem for a nonlinear parabolic equation in an inhomogeneous domain with the conjugation condition of a non-ideal contact were found. The initial boundary value problem is a mathematical model of an important technical problem of filtration consolidation of inhomogeneous soils. Inhomogeneity is considered in terms of the presence of thin inclusions, physico-chemical characteristics of which differ from those of the main soil. The problem of long-term consolidation is especially pronounced in soils with low filtration coefficient. Low permeability of the porous medium causes deviation from the linear relationship between the pressure gradient and the filtration rate. Weak formulation of the problem is suggested, and the accuracy of the approximate finite element solution, its existence and uniqueness are substantiated for the case of Darcy's nonlinear law. A test example and the effect of the nonlinear filtration law for thin inclusion on the dynamics of scattering of excess pressures in the entire area of the problem are considered.

Key words: nonlinear initial-boundary value problem, finite element method, consolidation, threshold gradient, nonlinear filtration law, conjugation condition.

2010 Mathematics Subject Classification: 49J20, 49K20, 58J37.

Communicated by Prof. V. O. Kapustyan

1. Introduction

A nonlinear initial-boundary value problem is investigated for the parabolic equation in the inhomogeneous domain $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$, where Ω_1, Ω_2 are some given domains. By inhomogeneity we mean the presence of a contact interface $\omega = \overline{\Omega_1} \cap \overline{\Omega_2}$ which from a physical view point means a thin inclusion of the third material. Differences in the physical characteristics of the materials of the inclusion ω and regions Ω_i, $i = 1, 2$, can lead to the discontinuity of the solutions of the initial boundary value problem at the inclusion. Regarding the study of problems in inhomogeneous environments of this type, we will focus on the methodology where the study of processes at the thin inclusion itself is taken outside the general initial-boundary value problem. The physical characteristics of the thin inclusion material and the inclusion thickness are taken into account. The presence of a thin inclusion is taken into account in the general initial-boundary

[*]Department of Computer Science and Applied Mathematics , National University of Water and Environmental Engineering, 11, Soborna st., Rivne, 33028, Ukraine, o.r.michuta@nuwm.edu.ua

[†]Department of Computer Science and Applied Mathematics , National University of Water and Environmental Engineering, 11, Soborna st., Rivne, 33028, Ukraine, p.m.martyniuk@nuwm.edu.ua

value problem by the so-called conjugation conditions for an unknown function. This approach, when using numerical methods, avoids solving the problem in the inclusion itself and thus simplifies the solution process.

The above approach to simulating inhomogeneities in soils began to develop in the work of I.I. Lyashko, I. V. Sergienko, V. V. Skopetskii, V. S. Deineka and is quite fully described in review monographs [6, 15, 16]. The works [2, 11, 14–17] are also worth mentioning in this direction which develop both the solution methods and the qualitative theory of initial-boundary value problems with possible discontinuous solutions.

As noted above, the initial-boundary value problem, more specifically the conjugation with non-ideal contact, include physical characteristics of the material of the thin inclusion (inclusions themselves may be both of natural origin and artificial). The parameters of the material of thin soil inclusions (filtration coefficient, porosity, thermal conductivity, etc.) are nonlinearly dependent on the effect of external factors. Considering the initial-boundary value problems as mathematical models of physico-chemical processes in porous soil media, the presence of such dependences requires modification of the conjugation conditions. The above-mentioned works [2, 6, 11, 14–17] assumed the parameters of inclusions to be constant, which is reflected in the conjugation conditions with non-ideal contact. Mathematical models for the distribution of inorganic chemicals in porous media and modification of conjugation conditions taking into account nonlinear dependences of material parameters of thin geobarriers on the effect of physicochemical factors, including chemical suffusion [23] were developed in [18, 19].

The influence of organic substances on the development of microorganisms and the effect of bioclogging processes on the value of pressure jumps at a thin geobarrier were studied in [20, 21]. Modified conjugation conditions and mathematical models of moisture transfer in inhomogeneous porous media are presented in [5, 12]. The method of modification of conjugation conditions for the partial case of Darcy's nonlinear law is shown in [13].

Here, we investigate the initial-boundary value problem for a quasilinear parabolic equation as a mathematical model of soil consolidation. The problem of consolidation (compaction) of soils is especially relevant for water-saturated clays. This is caused by the low filtration coefficient of clay soils and, as a result, the long time for the transition of clay bases of civil and industrial buildings to a stable state. Weak permeability of clay soils raises questions about the limits of Darcy's filtration law in its classical form [9]. It will be recalled that Darcy's classical law mathematically records the linear relationship between the filtration rate and the pressure gradient. The linearity of Darcy's basic filtration law has its physically determined limits. The linearity is violated both for highly permeable porous media and for weakly permeable ones. In particular, for weakly permeable porous media, this is manifested in the presence of the so-called "threshold gradient", below which the relationship between the filtration rate and the pressure gradient becomes nonlinear.

The nonlinearity of Darcy's law for consolidation problems is taken into account in e.g. [10, 24, 25]. However, only homogeneous media without thin inclusions are considered there. Additionally, power laws were considered nonlinear (dependence of the filtration rate on the pressure gradient raised to a certain degree other than unity). Quasilinear filtration processes were studied in [3, 4] where the linear dependence of the filtration rate on the pressure gradient is preserved, but the filtration coefficient nonlinearly depends on the physico-chemical parameters of the porous medium.

Thus, the objectives of this work are: 1) modification of the conjugation condition on a thin inclusion under Darcy's nonlinear law; 2) formation of a mathematical model of filtration consolidation of inhomogeneous soil in the presence of the threshold gradient; 3) investigation of finite element solutions of the corresponding boundary value problem, numerical experiments and analysis of the significance of the nonlinearity of Darcy's law on the value of excess pressures and their jumps.

2. The problem of nonlinear filtration through a thin inclusion

It is suggested in [9] to generalize nonlinear filtration laws in the form (in one-dimensional case)

$$u = -k\left(\frac{\partial h}{\partial x} - \frac{I}{\gamma\left(\frac{1}{\alpha}\right)}\gamma\left(\frac{1}{\alpha}, \left(\frac{i}{I^*}\right)^{\alpha}\right)\operatorname{sgn}\left(\frac{\partial h}{\partial x}\right)\right), \tag{2.1}$$

where i is the absolute value of the pressure gradient, i.e. in the one-dimensional case $i = \left|\frac{\partial h}{\partial x}\right|$; k is the filtration coefficient of the porous medium; u is the filtration rate; I is the absolute value of the pressure gradient below which the linearity of Darcy's law is violated (the so-called threshold gradient); α is an empirical parameter;

$$I^* = \frac{\alpha}{\gamma\left(\frac{1}{\alpha}\right)}I,$$

$$\gamma\left(a, x\right) = \int_0^x s^{a-1}e^{-s}ds,$$

$$\gamma\left(a\right) = \int_0^\infty s^{a-1}e^{-s}ds.$$

Here $\gamma\left(a\right)$, $\gamma\left(a, x\right)$ are the so-called gamma function and the lower incomplete gamma function.

As noted in [9], Eq. (2.1) includes previously proposed nonlinear filtration laws for permeable soils, i.e. Hansbo's law (1960), Swartzendruber's law (1961), Zou's law (1996).

The use of the law in the form of (2.1) is quite inconvenient in terms of applying the finite element method. Given that $\frac{\partial h}{\partial x} = i\operatorname{sgn}\left(\frac{\partial h}{\partial x}\right)$, it follows from (2.1) that

$$u = -k(n)\left(1 - \frac{I}{\left(i+\varepsilon^2\right)\gamma\left(\frac{1}{\alpha}\right)}\gamma\left(\frac{1}{\alpha}, \left(\frac{i}{I^*}\right)^{\alpha}\right)\right)\frac{\partial h}{\partial x},$$

or

$$u = -k^*(n, I)\frac{\partial h}{\partial x}, \tag{2.2}$$

where

$$k^*(n, I) = k(n)\left(1 - \frac{I}{(i+\varepsilon^2)\,\gamma\left(\frac{1}{\alpha}\right)}\gamma\left(\frac{1}{\alpha}, \left(\frac{i}{I^*}\right)^{\alpha}\right)\right).$$

Here n is the soil porosity; $\varepsilon > 0$ is a small constant. Since in this work we consider the problem of soil compaction, the dependence of the filtration coefficient on porosity should be taken into account.

We assume (due to thinness of the inclusion ω, Fig. 1) that the filtration processes in the cross-section of this inclusion are stationary (or at least quasi-stationary). Thus, similarly to [5, 12, 13, 18–21], for the inclusion of thickness d, we consider the following filtration problem:

$$\frac{d}{d}\left(-k^*_\omega(n_\omega, I_\omega)\frac{dh}{d\xi}\right) = 0, 0 < \xi < d, \tag{2.3}$$

$$h(0) = h^-, h(d) = h^+. \tag{2.4}$$

Here, h^-, h^+ are the known values of pressures, and the sub-script ω means the corresponding characteristic for the inclusion ω (Fig. 4.1). Repeating the reasoning of [5, 12, 13, 18–21], we have

$$h(\xi) = \frac{\displaystyle\int_0^\xi \frac{dx}{k^*_\omega(n_\omega, I_\omega)}}{\displaystyle\int_0^d \frac{dx}{k^*_\omega(n_\omega, I_\omega)}}\left(h^+ - h^-\right) + h^-.$$

However, hereafter we are more interested not in the pressure itself, but in its gradient. As a result

$$\frac{dh}{d\xi} = \frac{1}{k^*_\omega(n_\omega, I_\omega)\displaystyle\int_0^d \frac{dx}{k^*_\omega(n_\omega, I_\omega)}}\left(h^+ - h^-\right). \tag{2.5}$$

3. Conjugation condition for nonlinear filtration law

According to [16], similarly to [5, 13] the conjugation condition is derived on the basis of the law of conservation of fluid flow through the cross-section area of the inclusion surface along the normal in time Δt. As the flow

$$q = -k^*_\omega(n_\omega, I_\omega)\frac{dh}{d}\Delta t = u\Delta t$$

and

$$q = q^+ = q^-,$$

then

$$u^{\pm}\big|_{x=\xi} = -k^*_\omega(n_\omega, I_\omega)\frac{dh}{d\xi}. \tag{3.1}$$

From (2.5) and (3.1) we have the final formula of the conjugation condition with non-ideal contact for pressures at the inclusion, with nonlinear filtration law in the form (2.2)

$$u^{\pm}\big|_{x=\xi} = -\frac{[h]}{\displaystyle\int_0^d \frac{dx}{k^*_\omega(n_\omega, I_\omega)}}. \tag{3.2}$$

Here $[h] = h^+ - h^-$ is the pressure jump at the inclusion.

Before formulating the mathematical model of the problem, we note that the porosity n (as well as n_ω) of the soil in the regions Ω_i, $i = 1, 2$, is related to the void ratio e as $n = \frac{e}{1+e}$. In its turn, and it is shown in the section of numerical experiment results, e depends on pressures $h(x,t)$. Thus $n = n(h)$, and in subsequent calculations $k^* = k^*(h, I)$, $k^*_\omega = k^*_\omega(h, I_\omega)$.

4. Nonlinear mathematical model of filtration consolidation of porous medium with thin inclusion

Here we consider the process of filtration consolidation of the soil layer of total thickness l with a thin inclusion ω of thickness d which is located at the depth $x = \xi$ (Fig. 4.1). The material of the thin inclusion differs in its physico-chemical characteristics from those of the main soil.

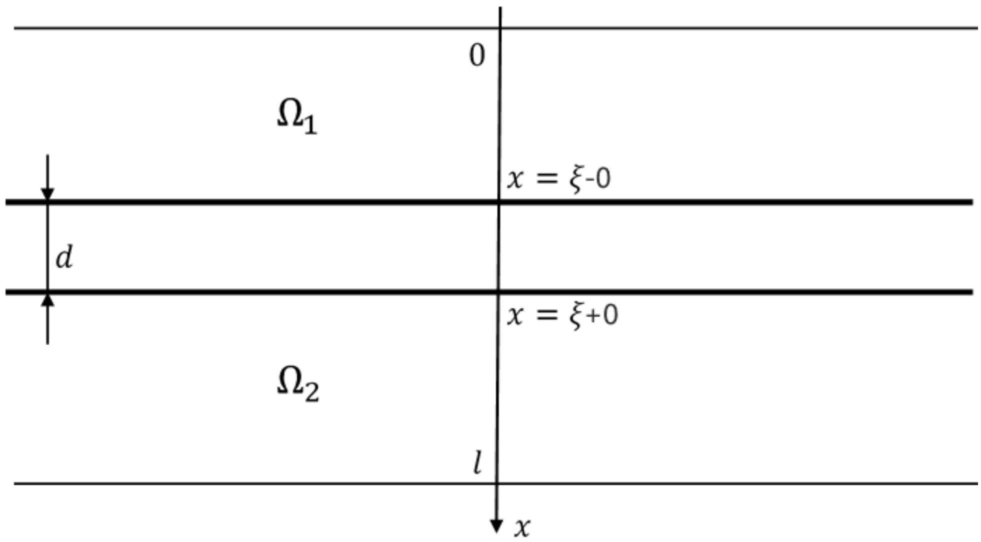

Fig. 4.1. A layer of soil of thickness l with a thin inclusion ω of thickness d ($d \ll l$).

The formulation of the mathematical model of the described problem will partially use the work of scientists reviewed in Introduction. The mathematical model will include the equations of filtration consolidation [8,22,23]. As a result, we have the following boundary value problem:

$$\frac{\partial h}{\partial t} = \frac{1+e}{\gamma a}\frac{\partial}{\partial x}\left(k^*(h,I)\frac{\partial h}{\partial x}\right),\ x\in\Omega_1\cup\Omega_2,\ t\in(0,T], \tag{4.1}$$

$$h\left(x,t\right)\big|_{x=0} = 0, t\in[0,T], \tag{4.2}$$

$$u\left(x,t\right)\big|_{x=l} = \left(-k^*(h,I)\frac{\partial h}{\partial x}\right)\bigg|_{x=l} = 0,\quad t\in[0,T], \tag{4.3}$$

$$h\left(x,0\right) = h_0\left(x\right),\quad x\in\overline{\Omega}_1\cup\overline{\Omega}_2, \tag{4.4}$$

$$u^{\pm}\big|_{x=\xi} = \left(-k^*(h,I)\frac{\partial h}{\partial x}\right)^{\pm}\bigg|_{x=\xi} = -\frac{[h]}{\displaystyle\int_0^d \frac{dx}{k_\omega^*(h,I_\omega)}}. \tag{4.5}$$

Here,

$$k^*(h,I) = k\left(h\right)\left(1-\frac{I}{\left(i+\varepsilon^2\right)\gamma\left(\frac{1}{\alpha}\right)}\gamma\left(\frac{1}{\alpha},\left(\frac{i}{I^*}\right)^{\alpha}\right)\right),$$

$$\Omega_1 = (0;\xi)\,,\Omega_2 = (\xi;l)\,,0<\xi<l;\Omega=\Omega_1\cup\Omega_2;$$

$T>0$ is the specified time duration; $h_0\left(x\right)$ is a known function; a is the soil compressibility coefficient; $n,\ \ n_\omega$ are the porosity of soil and inclusion material, respectively; $e=\frac{n}{1-n}$ is the soil void ratio; γ is the specific weight of the pore fluid; h is the pressure; $k,\ \ k_\omega$ are the filtration coefficients of the main soil and soil inclusion, respectively; u is the filtration rate which is determined according to (2.2); $u^{\pm}$ are the filtration rates at $x=\xi-0$ and $x=\xi+0$, respectively; $[h]=h^+-h^-$ is the pressure jump at the thin inclusion.

We shall show that k^* is always positive. Let us turn to the starting relations from [9] from which the generalized Darcy's law (2.1) is derived. Particularly, it is based on ([9, formula (1.20)])

$$\frac{dq}{di} = k\left(1-e^{-i\left(\frac{i}{I^*}\right)^{\alpha}}\right) \tag{4.6}$$

from which we have [9, formula (1.21)]

$$q = k\left(i-\frac{I}{\gamma\left(\frac{1}{\alpha}\right)}\gamma\left(\frac{1}{\alpha},\left(\frac{i}{I^*}\right)^{\alpha}\right)\right). \tag{4.7}$$

From (4.7), $q|_{i=0}=0$, from (4.6), $\left.\frac{dq}{di}\right|_{i>0}>0$. Therefore, the function q as the function of the absolute value of i is increasing for $i\in(0;+\infty)$, and is equal to zero for $i=0$. That is, $q|_{i>0}>0$. It follows that

$$i-\frac{I}{\gamma\left(\frac{1}{\alpha}\right)}\gamma\left(\frac{1}{\alpha},\left(\frac{i}{I^*}\right)^{\alpha}\right)>0$$

or

$$1-\frac{I}{i\gamma\left(\frac{1}{\alpha}\right)}\gamma\left(\frac{1}{\alpha},\left(\frac{i}{I^*}\right)^{\alpha}\right)>0.$$

Then it further reinforces the above inequality that when $i>0$

$$1-\frac{I}{(i+\varepsilon^2)\gamma\left(\frac{1}{\alpha}\right)}\gamma\left(\frac{1}{\alpha},\left(\frac{i}{I^*}\right)^{\alpha}\right)>0.$$

Since $\gamma\left(\frac{1}{\alpha},0\right)=0$, then for $i=0$ from the formula for $k^*(h,I)$ we obtain $k^*(h,I)=k(h)>0$. Therefore

$$k^*(h,I)>0,\ i\in[0;+\infty)$$

and similarly

$$k^*_{\omega}(h,I_{\omega})>0,\ i\in[0;+\infty).$$

Similarly to [6] we introduce the following notation: $Q_T=\Omega\times(0;T]$, $Q_T^1=\Omega_1\times(0;T]$, $Q_T^2=\Omega_2\times(0;T]$.

Assume that the function $h_0(x)$ is continuous on each of the closures $\overline{\Omega}_1$, $\overline{\Omega}_2$. Also with respect to the coefficients k^*, k^*_{ω}, assume that

1)

$$0<k^*_{min}\le k^*(s_1,s_2)\le k^*_{max}<\infty,$$

$$0<k^*_{\omega,min}\le k^*_{\omega}(s_1,s_2)\le k^*_{\omega,max},$$

$\forall s_1\in(-\infty;+\infty)$, $\forall s_2\in[0;+\infty)$; k^*_{min}, k^*_{max}, $k^*_{\omega,min}$, $k^*_{\omega,max}$ are positive constants;

2)

$$\left|k^*\left(p_1,s_1\right)-k^*\left(p_2,s_2\right)\right|\le k^*_L\left|p_1-p_2\right|,\ 0<k^*_L<\infty;$$

$$\left|k^*_{\omega}\left(p_1,s_1\right)-k^*_{\omega}\left(p_2,s_2\right)\right|\le k^*_{\omega,L}\left|p_1-p_2\right|,\ 0<k^*_{\omega,L}<\infty.$$

Also, the function $k^*=k^*(h,I)$ must be continuous on $\overline{\Omega}_1$, $\overline{\Omega}_2$ and continuously differentiated on Ω_1, Ω_2.

Definition 4.1. The classical solution of the initial-boundary value problem (4.1)–(4.5) which allows a discontinuity of the first kind at the point $x=\xi$ is a function $h(x,t)\in\Psi$ that satisfies $\forall(x,t)\in\overline{Q}_T$ equation (4.1) and the initial condition (4.4).

Here, Ψ is a set of functions $\psi(x,t)$ which, together with $\frac{\partial\psi}{\partial x}$, are continuous on each of the closures $\overline{Q}^1_T$, $\overline{Q}^2_T$, have bounded continuous partial derivatives $\frac{\partial\psi}{\partial t}$, $\frac{\partial^2\psi}{\partial x^2}$ on Q^1_T, Q^2_T, and satisfy conditions (4.2), (4.3), (4.5).

For further calculations we will note one more aspect. The conjugation condition with non-ideal contact (see [16, page 291, formula (7.4)]) which can be called classic

$$\left.\left(\varkappa\left(x,u\right)\frac{\partial u}{\partial x}\right)\right|_{x=\xi}=r\left[u\right]$$

includes r, a known constant, and $0 < r_0 \le r < \infty$. Consider condition (4.5). When the second of conditions 1) for the coefficient in the right part of the conjugation condition (4.5)

$$\left(-k^*(h,I)\frac{\partial h}{\partial x}\right)^{\pm}\Bigg|_{x=\xi} = -\frac{[h]}{\displaystyle\int_0^d \frac{dx}{k^*_\omega(h,I_\omega)}}$$

is satisfied, we have

$$\frac{d}{k^*_{\omega,max}} \le \int_0^d \frac{dx}{k^*_\omega(h,I_\omega)} \le \frac{d}{k^*_{\omega,min}}.$$

Further,

$$k^*_{\omega,min}\frac{[h]}{d} \le \frac{[h]}{\displaystyle\int_0^d \frac{dx}{k^*_\omega(h,I_\omega)}} \le k^*_{\omega,max}\frac{[h]}{d}.$$

Thus, in the case of a modified conjugation condition (4.5), we have

$$0 < \frac{k^*_{\omega,min}}{d} \le r \le \frac{k^*_{\omega,max}}{d} < \infty. \tag{4.8}$$

Estimate (4.8) allows us to generalize the theorems proved in [6, 16] for problems with the classical conjugation condition with non-ideal contact, for the case of a modified conjugation condition (4.5).

5. Generalized solution of problem (4.1)-(4.5)

Similarly to [6] let H_0 be the space of functions $s(x)$ that in each of the regions Ω_i belong to the Sobolev space $W_2^1(\Omega_i)$, $i = 1, 2$, and they acquire zero values at the ends of the segment $[0; l]$ where the function $h(x,t)$ is set the boundary conditions of the first kind.

Let $h(x,t) \in \Psi$ be the classical solution of the initial-boundary value problem (4.1)–(4.5). Take $s(x) \in H_0$. Multiply equation (4.1) and initial condition (4.4) by $s(x)$. Integrating them on the segment $[0; l]$ and taking into account the conjugation conditions (4.5), we obtain

$$\int_0^l \frac{\gamma a}{1+e}\frac{\partial h}{\partial t}s(x)\,dx + \int_0^l k^*(h,I)\frac{\partial h}{\partial x}\frac{ds}{dx}dx + \frac{[h]\,[s]}{\displaystyle\int_0^d \frac{dx}{k^*_\omega(h,I_\omega)}} = 0, \tag{5.1}$$

$$\int_0^l h(x,0)\,s(x)\,dx = \int_0^l h_0(x)\,s(x)\,dx. \tag{5.2}$$

Thus, if $h(x,t) \in \Psi$ is a classical solution of the initial-boundary value problem (4.1)-(4.5), then $h(x,t)$ is a solution of problem (5.1), (5.2) in a weak formulation.

Let H be the space of functions $v\,(x,t)$ that are square-integrable together with their first derivatives $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial x}$ on each of the intervals $(0;\xi)$, $(\xi;l)$, $\forall t\in(0\,;T]$, $T>0$, and they satisfy the same boundary conditions of the first kind as the function $h\,(x,t)$.

Definition 5.1. Function $h\,(x,t)\in H$ that for any $s\,(x)\in H_0$ satisfies integral relations (5.1), (5.2) is called a generalized solution of the initial-boundary value problem (4.1)–(4.5).

An approximate generalized solution of the initial-boundary value problem (4.1)-(4.5) will be sought in the form

$$\hat{h}\,(x,t)=\sum_{i=1}^{N}h_i\,(t)\,\varphi_i\,(x), \tag{5.3}$$

where $\{\varphi_i(x)\}_{i=1}^{N}$ is the basis of finite-dimensional subspace $M_0\subset H_0$; $h_i\,(t)$, $i=\overline{1,N}$ are unknown coefficients that depend only on time.

The set of functions that can be represented in the form (5.3) generate a finite-dimensional subspace $M_1\subset H_1$.

Definition 5.2. An approximate generalized solution of the initial-boundary value problem (4.1)–(4.5) is a function $\hat{h}(x,t)\in M_1$ that for an arbitrary function $S(x)\in M_0$ satisfies integral relations

$$\int_0^l\frac{\gamma a}{1+e}\frac{\partial\hat{h}}{\partial t}S\,(x)\,dx+\int_0^l k^*(\hat{h},I)\frac{\partial\hat{h}}{\partial x}\frac{dS}{dx}dx+\frac{\left[\hat{h}\right]\,[S]}{\displaystyle\int_0^d\frac{dx}{k_\omega^*(\hat{h},I_\omega)}}=0, \tag{5.4}$$

$$\int_0^l\hat{h}\,(x,0)\,S\,(x)\,dx=\int_0^l h_0\,(x)\,S\,(x)\,dx. \tag{5.5}$$

Next, from the weak formulation of (5.4), (5.5) we obtain (assuming the function $S(x)$ equal to each basis function $\varphi_i(x)$, $i=\overline{1,N}$) the Cauchy problem for a system of nonlinear differential equations

$$\mathbf{M}\,(\mathbf{H})\,\frac{d\mathbf{H}}{dt}+\mathbf{L}\,(\mathbf{H})\,\mathbf{H}\,(\boldsymbol{t})=\mathbf{0}, \tag{5.6}$$

$$\widetilde{\mathbf{M}}\mathbf{H}^{(\mathbf{0})}=\widetilde{\mathbf{F}}, \tag{5.7}$$

where

$$\widetilde{\mathbf{F}}=\left(\tilde{f}_i\right)_{i=1}^{N},\quad \widetilde{\mathbf{M}}=(\tilde{m}_{ij})_{i,j=1}^{N},\quad \tilde{m}_{ij}=\int_0^l\varphi_i\varphi_j dx,\quad \mathbf{M}=(m_{ij})_{i,j=1}^{N},$$
$$\mathbf{L}=(l_{ij})_{i,j=1}^{N},\quad \mathbf{H}=(h_i\,(t))_{i=1}^{N},\quad \mathbf{H}^{(\mathbf{0})}=(h_i\,(0))_{i=1}^{N},$$

$$m_{ij} = \int_0^l \frac{\gamma a}{1+e} \varphi_i \varphi_j dx,$$

$$l_{ij} = \int_0^l k^*(\hat{h}, I) \frac{d\varphi_i}{dx} \frac{d\varphi_j}{dx} dx + \frac{[\varphi_i]\,[\varphi_j]}{\int_0^d \frac{dx}{k_\omega^*(\hat{h}, I_\omega)}}.$$

The square matrix $\mathbf{M}(\mathbf{H})$ is symmetric and positive definite because $\frac{\gamma a}{1+e} > 0$, $\forall (x,t) \in \overline{Q}_T$. Given the proven positivity of the coefficients k^*, k_ω^*, as well as the assumptions 1) made regarding their limitations, the matrix $\mathbf{L}(\mathbf{H})$ will also be symmetric and positively definite [6, page 417]. Next, similarly to [6, problem (3.14) of Chapter 8], we write the system (5.7) in the form

$$\frac{d\mathbf{H}}{dt} = \mathbf{\Phi}(\mathbf{H}), \tag{5.8}$$

where $\mathbf{\Phi}(\mathbf{H}) = -\mathbf{M}^{-1}\mathbf{L}(\mathbf{H})\,\mathbf{H}(\boldsymbol{t})$**.** The functions $\mathbf{\Phi}(\mathbf{H})$, $\partial\mathbf{\Phi}/\partial\mathbf{H}$ are continuous. Thus, there is a single approximate generalized solution $\hat{h}(x,t) \in M_1$ of the initial-boundary value problem (4.1)–(4.5).

We will introduce the following norms [6, page 380]:

$$\|u\|_{L_2}^2 = \int_0^l u^2(x,t)dx,$$

$$\|u\|_{H_0^1}^2 = \left\| \frac{\partial u}{\partial x} \right\|_{L_2}^2,$$

$$\|u\|_{L_2 \times L_2}^2 = \|u\|_{L_2(Q_T)}^2 = \int_0^T \int_0^l u^2 dxdt,$$

$$\|u\|_{H_0^1 \times L_2}^2 = \int_0^T \|u\|_{H_0^1}^2 dt = \int_0^T \int_0^l \left(\frac{\partial u}{\partial x} \right)^2 dxdt,$$

$$\|u\|_{L_2 \times L_\infty} = \sup_{t \in (0,T]} \|u(\cdot,t)\|_{L_2},$$

$$\|\nabla_x u\|_{L_\infty \times L_\infty} = \sup_{(x,t) \in Q_T} \left| \frac{\partial u(x,t)}{\partial x} \right|,$$

$$\|u\|_{W_2^1 \times L_2}^2 = \int_0^T \int_0^l \left(u^2 + \left(\frac{\partial u}{\partial x} \right)^2 \right) dxdt,$$

$$\|[u]\|_{L_2}^2 = \int_0^T [u]^2 dt = \int_0^T (u(\xi+0,t) - u(\xi-0,t))^2 dt.$$

Similarly to [6, page 380, Theorem 1] we can prove the following result.

Theorem 5.1. *Let $h(x,t)$ be a classical solution of the initial-boundary value problem (4.1)–(4.5), and $\hat{h}(x,t)$ be a generalized solution of this problem from space M_1. Then, under the conditions 1), 2) imposed on k^*, k_ω^*, taking into account*

(4.8), *there are such positive constants* c, δ_1, δ_2, *that for an arbitrary function* $\tilde{h}(x,t) \in M_1$ *the following inequality holds:*

$$\left\|h-\hat{h}\right\|^2_{L_2\times L_\infty} + \delta_1\left\|h-\hat{h}\right\|^2_{H^1_0\times L_2} + \delta_2\left\|\left[h-\hat{h}\right]\right\|^2_{L_2} \leq c\Big\{\left\|h-\tilde{h}\right\|^2_{L_2\times L_\infty} + \left\|h-\tilde{h}\right\|^2_{H^1_0\times L_2} + \left\|\left[h-\tilde{h}\right]\right\|^2_{L_2} + \left\|\frac{\partial(h-\tilde{h})}{\partial t}\right\|^2_{L_2\times L_2}\Big\}, \quad \forall\tilde{h}\in M_1. \quad (5.9)$$

Dependence (5.9) is used in estimating the accuracy of the finite element method.

6. Finite element method

We will cover the closure $\Omega = \overline{\Omega}_1 \cup \overline{\Omega}_2$ with a finite element grid with the total number of nodes N. The point $x = \xi$ should be double numbered, the node on the left $x = \xi - 0$ and the node on the right $x = \xi + 0$. Let in (4.8) $\varphi_i(x)$ be the basis functions of the finite element method which allow a discontinuity of the first kind at the point $x = \xi$ and are polynomials of m-th degree. Then the space of functions $\hat{h}(x,t)$ of the form (5.3) with the specified basic functions is denoted H^N_m.

Theorem 6.1. *Let the classical solution* $h(x,t)$ *of the boundary value problem* (4.1)–(4.5) *have partial derivatives* $\frac{\partial^{m+1}(\cdot)}{\partial x^{m+1}}$, $\frac{\partial^{m+2}(\cdot)}{\partial x^m \partial t}$ *limited on* Q^i_T, $i = 1,2$. *Then the approximate generalized solution* $\hat{h}(x,t) \in H^N_m$ *has an estimate*

$$\left\|h-\hat{h}\right\|_{W^1_2\times L_2} \leq c\cdot h^m_{max},$$

where m *is the degree of FEM polynomials,* $c = const > 0$,

$$h_{max} = \max_{i=\overline{0,N-1}} (x_{i+1}-x_i),$$

$[x_{i+1}; x_i]$ *are finite elements.*

Proof. The validity of the theorem follows from the estimate (5.9) of the previous theorem taking into account the interpolation estimates [6, page 387, Theorem 2]. □

7. Time discretization methods

Problem (5.6), (5.7) is a Cauchy problem for a system of nonlinear differential equations of the first order. Finding its solution also requires the use of appropriate discretization methods. The application of the Crank-Nicolson method is subr-stantiated in [15]:

$$\mathbf{M}\left(\frac{1}{2}\left(\mathbf{H}^{(j+1)}+\mathbf{H}^{(j)}\right)\right)\frac{\mathbf{H}^{(j+1)}-\mathbf{H}^{(j)}}{\tau}$$
$$+\mathbf{L}\left(\frac{1}{2}\left(\mathbf{H}^{(j+1)}+\mathbf{H}^{(j)}\right)\right)\cdot\frac{1}{2}\left(\mathbf{H}^{(j+1)}+\mathbf{H}^{(j)}\right)=\mathbf{0},\quad j=0,1,2,...,m_\tau-1.$$

Here the time segment $[0,T]$ is divided into m_τ equal parts with the step $\tau=\frac{T}{m_\tau}$; $\mathbf{H}^{(j)}$ is the approximate solution of the Cauchy problem (5.6), (5.7) for $t=j\tau$. Let also introduce the following notation: h_j is the classical solution of the initial-boundary value problem (4.1)–(4.5) for $t=j\tau$; $\hat{h}_j$ is the approximate generalized solution of the initial-boundary value problem (4.1)-(4.5) for $t=j\tau$; $\phi_{j+1/2}=\frac{1}{2}(\phi_{j+1}+\phi_j)$; $z_j=h_j-\hat{h}_j$.

Given (4.8), similarly to Theorem 5 [6, Chapter 8] it is also valid the following result.

Theorem 7.1. *Let $h(x,t)$ be a classical solution of the initial-boundary value problem* (4.1)–(4.5)*. Let the functions $\frac{\partial h}{\partial t}$, $\frac{\partial h}{\partial x}$ be twice continuously differentiable over time on $\overline{Q}_T^i$, $i=1,2$. Let also assume that the derivatives $\frac{\partial^3 h}{\partial t^3}$, $\frac{\partial^3 h}{\partial t^2\partial x}$ are uniformly limited in modulus by a constant c_1, $\forall(x,t)\in\overline{Q}_T$. If conditions 1), 2) are satisfied, then there are positive constants c, δ_1, r_0, τ_0, that depend on the constants of conditions 1), 2), as well as T, l, such that for $\forall\tau\leq\tau_0$ the classical solution $h(x,t)$ and the approximate generalized solution obtained using the Crank-Nicolson method, $\hat{h}(x,t)\in M_1$, of the problems* (4.1)–(4.5) *and* (5.6), (5.7)*, respectively, satisfy the inequality*

$$\|z_{m_\tau}\|^2_{L_2}+\delta_1\sum_{j=0}^{m_\tau-1}\left\|z_{j+1/2}\right\|^2_{H_0^1}\tau+r_0\sum_{j=0}^{m_\tau-1}\left[z_{j+1/2}\right]^2\tau$$
$$\leq c\left(\sum_{j=0}^{m_\tau-1}\left\|(h-\tilde{h})_{j+1/2}\right\|^2_{H_0^1}\tau++\sum_{j=1}^{m_\tau-1}\left\|\frac{(h-\tilde{h})_{j+1/2}-(h-\tilde{h})_{j-1/2}}{\tau}\right\|^2_{L_2}\tau\right.$$
$$+\sum_{j=0}^{m_\tau-1}\left[(h-\tilde{h})_{j+1/2}\right]^2\tau+\left\|(h-\tilde{h})_0\right\|^2_{L_2}+\left\|(h-\tilde{h})_{m_\tau-1/2}\right\|^2_{L_2}$$
$$\left.+\left\|(h-\tilde{h})_{1/2}\right\|^2_{L_2}+O(\tau^4)\right),\ \forall\tilde{h}\in M_1.\quad(7.1)$$

Similarly to Theorem 6 [6, Chapter 8] and taking into account estimate (7.1), we have:

Theorem 7.2. *Let the classical solution $h(x,t)$ of the problem* (4.1)–(4.5) *satisfy the conditions of Theorem 7.1. Then for the errors z of the approximate generalized* *solution* $\hat{h}(x,t) \in H_m^N$ *of the problem* (5.4), (5.5) *obtained using the Crank-Nicolson method, the following estimate is valid:*

$$\|z_{m_\tau}\|_{L_2}^2 + \delta_1 \tau \sum_{j=0}^{m_\tau - 1} \left\|z_{j+1/2}(h)\right\|_{H_0^1}^2 \le c \cdot \left(h_{max}^{2m} + \tau^4\right).$$

However, the practical implementation of the Crank-Nicolson method for the nonlinear Cauchy problem (5.6), (5.7) requires the use of iterations. Instead of the Crank-Nicolson method, one can use the predictor-corrector method [16], which for the system of equations (5.6) has the following form:

$$\mathbf{M}\left(\mathbf{H}^{(j)}\right)\frac{\mathbf{W}^{(j+1)} - \mathbf{H}^{(j)}}{\tau} + \mathbf{L}\left(\mathbf{H}^{(j)}\right)\frac{1}{2}\left(\mathbf{W}^{(j+1)} + \mathbf{H}^{(j)}\right) = \mathbf{0},$$

$$\mathbf{M}\left(\frac{1}{2}\left(\mathbf{W}^{(j+1)} + \mathbf{H}^{(j)}\right)\right)\frac{\mathbf{H}^{(j+1)} - \mathbf{H}^{(j)}}{\tau} +$$

$$+\mathbf{L}\left(\frac{1}{2}\left(\mathbf{W}^{(j+1)} + \mathbf{H}^{(j)}\right)\right) \cdot \frac{1}{2}\left(\mathbf{H}^{(j+1)} + \mathbf{H}^{(j)}\right) = \mathbf{0},\ j = 0, 1, 2, ..., m_\tau - 1,$$

where $\mathbf{W}^{(j+1)}$ are auxiliary vector functions.

From the view point of the simplicity of practical implementation, a fully implicit linearized difference scheme has proved itself well [8,21,23]. For the system (5.6), it has the form

$$\mathbf{M}\left(\mathbf{H}^{(j)}\right)\frac{\mathbf{H}^{(j+1)} - \mathbf{H}^{(j)}}{\tau} + \mathbf{L}\left(\mathbf{H}^{(j)}\right) \cdot \mathbf{H}^{(j+1)} = \mathbf{0},\ j = 0, 1, 2, ..., m_\tau - 1.$$

8. Results of numerical experiments and their analysis

According to [9, formula (1.24)],

$$I = A\tilde{k}^B,$$

where $A = 4.0 \times 10^{-12}$ and $B = -0.78$ are empirical parameters; $\tilde{k}$ ($\left[\tilde{k}\right] = m^2$) is the soil permeability coefficient, i.e. $k = \frac{\tilde{k}\rho g}{\mu}$, ρ is the density of pore fluid, μ is its viscosity, g is the acceleration of free fall. Since these studies do not yet take into account non-isothermal conditions, the dynamic viscosity of water at constant temperature $25^\circ C$ is used which is

$$\mu = 1.03 \cdot 10^{-8}\ Pa \cdot day.$$

Soil parameters for the numerical experiments were taken from the Hydrus-1D freeware. Specifically, Sandy Clay was considered as the main soil, with $k_0 = 0.0288\ m/day$, $n_0 = 0.38$. Then for the main soil $\tilde{k} = 2.98 \cdot 10^{-14}\ m^2$, and $I =$

0.142. Silty Clay was taken as the inclusion soil, with $k_{0\omega} = 0.0048\ m/day$, $n_{0\omega} = 0.46$, where index "0"denotes the initial values. Then for the inclusion soil $\tilde{k} = 4.97 \cdot 10^{-15}\ m^2$, $I = 0.574$.

The parameter α is also important. According to [9, formula (1.24)] $\alpha \geq 0$. The authors state that the parameter α mainly characterizes the smoothness of the transition from nonlinear to linear part of the curve of dependence $u = u(i)$ for the filtration rate and depends on the distribution of pore sizes in the porous medium. An increase in α means a sharper transition. A larger distribution range of pore sizes means a smoother transition between the linear and nonlinear parts and thus a smaller α value. For instance, for $\alpha \to \infty$ we have from the generalized law (2.1) [1]

$$u = \begin{cases} 0, \quad i \leq I; \\ -k\,(i - I)\,\mathrm{sgn}(i), \quad i \geq I. \end{cases}$$

When $\alpha \to 0$ we obtain the transition to Darcy's linear law.

We used $\alpha = 2$ for the main soil, and $\alpha = 5$ hor the thin inclusion soil in the following numerical experiments.

According to the linear compression dependence for soils,

$$e = -a\sigma + const.$$

Here σ are vertical stresses in the soil skeleton (in one-dimensional case). Further,

$$\frac{\partial e}{\partial t} = -a\frac{\partial \sigma}{\partial t}.$$

Also, according to Terzaghi's effective stress principle [22, 23]

$$\frac{\partial \sigma}{\partial t} = -\gamma\frac{\partial h}{\partial t},$$

and

$$\frac{\partial e}{\partial t} = a\gamma\frac{\partial h}{\partial t}.$$

From the last ratio we obtain

$$\frac{e^{(j+1)} - e^{(j)}}{\tau} = a\gamma\frac{h^{(j+1)} - h^{(j)}}{\tau}, \quad j = 0, 1, 2, ..., m_\tau - 1,$$

or

$$e^{(j+1)} = a\gamma\left(h^{(j+1)} - h^{(j)}\right) + e^{(j)}, \quad j = 0, 1, 2, ..., m_\tau - 1.$$

Obtained ratio was used to determine the variable filtration coefficient in the void ratio according to the Kozeny-Carman equation [7]

$$k = k_0\frac{1 + e_0}{1 + e}\left(\frac{e}{e_0}\right)^3,$$

where k_0, e_0 are the initial values of filtration coefficient and void ratio; k, e are their variable values over time.

In equation (4.1), the soil compressibility coefficient $a = 5.12\times10^{-7}\frac{m^2}{H}$, specific gravity of pore fluid $\gamma_c = 10^4\frac{H}{m^2}$. Initial pressure distribution $h_0(x) = 20\ m$ is corresponding to the application of the respective load to the soil surface. Unobstructed outflow of pore fluid is provided at the upper limit, and there is no drainage at the lower limit.

The model problem considered a soil layer of $l = 25\ m$ thickness. The depth of inclusion $\xi = 10\ m$, and its thickness $d = 0.2\ m$. The x variable step was $0.04\ m$, the time step $\tau = 10\ day$. Piece-square functions were used as FEM basis. The results of numerical experiments are plotted in Figs. 8.2, 8.3.

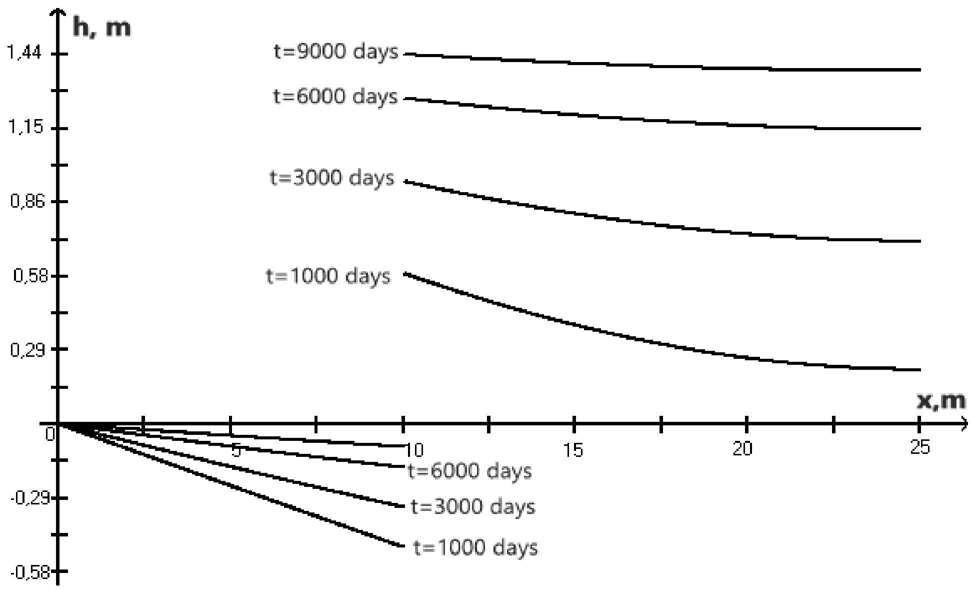

Fig. 8.2. Difference of the distribution of the pressure fields for cases of nonlinear and linear Darcy's laws.

The nonlinearity in Darcy's law has virtually no effect on the distribution of excess pressure during the first 1000 days from the beginning of the study process (Figs. 8.2, 8.3). However, as of the 3000th day, the relative difference in the pressure jumps on the thin inclusion in the linear and nonlinear laws reached 8.4%. Then such relative differences continue to increase and reach 42% on the 9000th day (about 1.5 meters in absolute terms). Thus, the nonlinearity in Darcy's law and the presence of the threshold gradient can introduce significant changes in the distribution of pressures, particularly in the long run. This is important both in terms of natural heterogeneous soils and in terms of hydraulic structures with fine inclusions.

References

1. J. Bear, *Hydraulics of groundwater*, *McGraw-Hill Inc.*, New York, 1979.

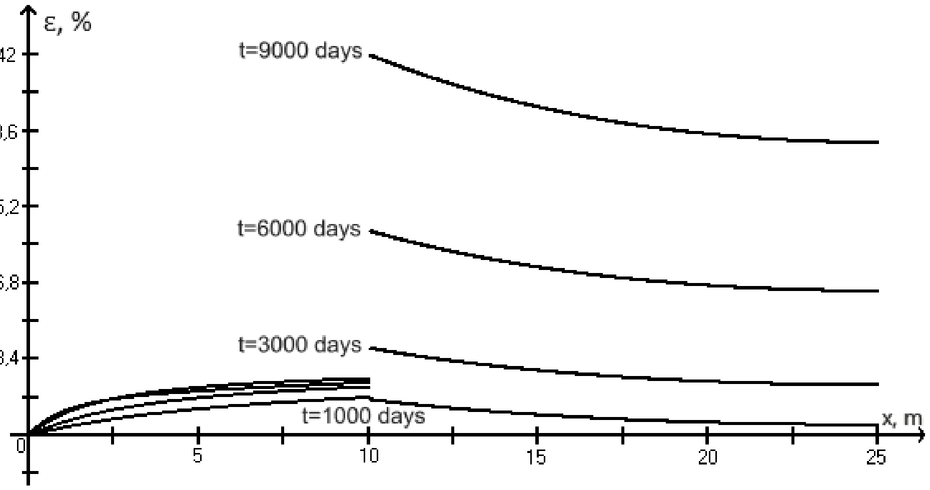

Fig. 8.3. Relative difference of the distribution of the pressure fields for the cases of nonlinear and linear Darcy's laws (in relation to the linear case).

2. V. I. Bilenko, K. V. Bozhonok, S. Yu. Dzyadyk, O. B. Stelya, *Piecewise Polynomial Algorithms for the Analysis of Processes in Inhomogeneous Media*, Cybernetics and Systems Analysis, **54** (4) (2018), 636–642.
3. A. Ya. Bomba, A. P. Safonyk, E. A. Fursachik, *Identification of Mass Transfer Distribution Factor and Its Account for Magnetic Filtration Process Modeling*, Journal of Automation and Information Sciences, **45** (4) (2013), 16–22.
4. A. Ya. Bomba, S. V. Yaroshchak, *Complex approach to modeling of two-phase filtration processes under control conditions*, Journal of Mathematical Sciences, **184** (1) (2012), 56–68.
5. Y. Chui, P. Martyniuk, M. Kuzlo, O. Ulianchuk–Martyniuk, *The conditions of conjugation in the tasks of moisture transfer on a thin clay inclusion taing into account salt solutions and themperature*, Journal of Theoretical and Applied Mechanics(Bulgaria), **49** (1) (2019), 28–38.
6. V. S. Deineka, I. V. Sergienko, V. V. Skopetsky, *Models and methods for solving problems with conjugation conditions*, Naukova dumka (Kiev), 1998.
7. F. M. Francisca, D. A. Glatstein, *Long term hydraulic conductivity of compacted soils permeated with landfill leachate*, Applied Clay Science, **49** (2010), 187–193.
8. V. A. Herus, N. V. Ivanchuk, P. M. Martyniuk, *A System Approach to Mathematical and Computer Modeling of Geomigration Processes Using Freefem++ and Parallelization of Computations*, Cybernetics and Systems Analysis, **54** (2) (2019), 284–294.
9. H.-H. Liu, *Fluid Flow in the Subsurface: History, Generalization and Applications of Physical Laws,Springer, Switzerland*, 2017.
10. L. Chuan-Xun, W. Chang-Jian, L. Meng-Meng, L. Jian-Fei, X. Kang-He, *One-dimensional large-strain consolidation of soft clay with non-Darcian flow and nonlinear compression and non-Darcian flow and nonlinear compression and permeability of soil*, Journal of Central South University, **44** (4) (2017), 967–976.

11. S. I. Lyashko, D. A. Nomirovskii, *The Generalized Solvability and Optimization of Parabolic Systems in Domains with Thin Low-Permeable Inclusions*, Cybernetics and Systems Analysis, **39** (5) (2003), 737–745.
12. P.M. Martyniuk, O.R. Michuta, O.V. Ulianchuk–Martyniuk, M.T. Kuzlo, *Numerical investigation of pressure head jump values on a thin inclusion in one-dimensional non-linear soil moisture transport problem*, Int. J. of Appl. Mathem., **31** (4) (2018), 649–660.
13. O. Michuta, N. Ivanchuk, P. Martyniuk, O. Ostapchuk, *A finite element study of elastic filtration in soils with thin inclusions*, Eastern-European Journal of Enterprise Technologies, **5** (5-107) (2020), 41–48.
14. V. V. Semenov, *Optimization of parabolic systems with conjugation conditions*, *Dop. NANU*, **5** (2003), 66–72.
15. I. V. Sergienko, V. S. Deineka, *Models with conjugation conditions and high-accuracy methods of their discretization*, Cybernetics and Systems Analysis, **36** (1) (2000), 83–101.
16. I. V. Sergienko, V. V. Skopetsky, V. S. Deineka, *Mathematical modeling and research of processes in inhomogeneous environments*, Naukova dumka*(Kiev)*, 1991.
17. I. B. Tymchyshyn, D. A. Nomirovskii, *Generalized Solvability of a Parabolic Model Describing Transfer Processes in Domains with Thin Inclusions, Differential Equations*, **57** (8) (2021), 1053–1062.
18. O. V. Ulianchuk–Martyniuk, *Numerical simulation of the effect of semi-permeable properties of clay on the value of concentration jumps of contaminants in a thin geochemical barrier*, Eurasian Journal of Mathematical and Computer Applications, **8** (1) (2020), 91–104.
19. O. Ulianchuk–Martyniuk, O. Michuta, *Conjugation conditions in the problem of filtering chemical solutions in the case of structural changes to the material and chemical suffusion in the geobarrier*, JP Journal of Heat and Mass Transfer, **19** (1) (2020), 141–154.
20. O. Ulianchuk–Martyniuk, O. Michuta, N. Ivanchuk, *Biocolmatation and the finite element modeling of its influence on changes in the head drop in a geobarrier*, Eastern-European Journal of Enterprise Technologies, **4** (10-106) (2020), 18–26.
21. O. V. Ulianchuk–Martyniuk, O. R. Michuta, N. V. Ivanchuk, *Finite element analysis of the diffusion model of the bioclogging of the geobarrier*, Eurasian Journal Of Mathematical and Computer Applications, **9** (4) (2021), 100–114.
22. A. P. Vlasyuk, P. M. Martynyuk, *Numerical solution of three-dimensional problems of filtration consolidation with regard for the influence of technogenic factors by the method of radial basis functions*, Journal of Mathematical Sciences, **171** (5) (2010), 632–648.
23. A. P. Vlasyuk, P. M. Martynyuk, O. R. Fursovych, *Numerical solution of a one-dimensional problem of filtration consolidation of saline soils in a nonisothermal regime*, Journal of Mathematical Sciences, **160** (4) (2009), 525–535.
24. Xu-dong Zhao, Wen-hui Gong, *Numerical solution of nonlinear large strain consolidation based on non-Darcian flow*, Mathematical Problems in Engineering, **2019)** (2019).
25. Z. Liu, Ya. Xia, M. Shi, J. Zhang, X. Zhu, *Numerical simulation and experiment study on the characteristics of non-Darcian flow and rheological consolidation of saturated clay*, Water, **11** (2019), 1385.

Received 20.04.2022