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FICTITIOUS CONTROLS AND APPROXIMATION OF AN OPTIMAL CONTROL PROBLEM FOR PERONA-MALIK EQUATION

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Abstract. We discuss the existence of solutions to an optimal control problem for the Cauchy-Neumann boundary value problem for the evolutionary Perona-Malik equations. The control variable v is taken as a distributed control. The optimal control problem is to minimize the discrepancy between a given distribution $u_d \in L^2(\Omega)$ and the current system state. We deal with such case of non-linearity when we cannot expect to have a solution of the original boundary value problem for each admissible control. Instead of this we make use of a variant of its approximation using the model with fictitious control in coefficients of the principle elliptic operator. We introduce a special family of regularized optimization problems for linear parabolic equations and show that each of these problems is consistent, well-posed, and their solutions allow to attain (in the limit) an optimal solution of the original problem as the parameter of regularization tends to zero.

Key words: Perona-Malik equation, optimal control problem, fictitious control, control in coefficients, approximation approach..

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1. Introduction

Recently, in the context of time interpolation of satellite multi-spectral images, the following model has been proposed (see [8])

$$u_t - div\left(f\left(|\nabla u|\right)\nabla u\right) + (\nabla u, \boldsymbol{b}) = v \quad \text{in} \quad Q = (0,T)\times\Omega, \tag{1.1}$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \Omega, \tag{1.2}$$

$$\partial_\nu u(t,x) = 0 \quad \text{on} \quad \Sigma = (0,T)\times\partial\Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is a Lipschitz domain, $\boldsymbol{b} \in \mathfrak{B}_{ad}$ and $v \in \mathfrak{V}_{ad}$ are the control functions with

$$\mathfrak{B}_{ad} = \left\{\boldsymbol{b} \in L^\infty(Q)^2 \cap BV(Q)^2 \ : \ \|\boldsymbol{b}\|_{L^\infty(Q)^2} \le \kappa\right\}, \tag{1.4}$$

$$\mathfrak{V}_{ad} = \left\{v \in L^2(0,T;L^2(\Omega))\right\}, \tag{1.5}$$

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∂_ν stands for the outward normal derivative, $f \in C^{1,1}(\mathbb{R}_+)$ is a non-increasing real function such that $f(s) \to 0$ when $s \to +\infty$ and $f(s) \to 1$ when $s \to +0$. In particular,

$$f\left(|\nabla u|\right) = \frac{1}{1 + |\nabla u|^2}. \tag{1.6}$$

In fact, the Cauchy-Neumann problem (1.1)–(1.3) can be viewed as some improvement of the Perona-Malik model [23] that was proposed in order to avoid the blurring in images and to reduce the diffusivity at those locations which have a larger likelihood to be edges. This likelihood is measured by $|\nabla u|^2$.

However, the indicated problem is ill-posed due to the degenerate behavior of the multiplayer $f(|\nabla u|)$, $f(|\nabla u|) \longrightarrow 0$ as the gradient $|\nabla u|$ tends to infinity. So, equation (1.1) acts like a standard convection-diffusion equation inside the regions where the magnitude of the gradient of u is weak, whereas at those points where the magnitude of the gradient is large enough, the diffusion is 'stopped'.

Moreover, it can be shown that the equation (1.1), as an example of the nonlinear equation of the porous medium type, combines forward-backforward diffusion flow with the convection (or drift) of the function u in accordance with the velocity field b. In particular, the operator $div\left(f\left(|\nabla u|\right)\nabla u\right)$ implies the forward diffusion in the regions where the squared gradient magnitude of the function u is less than 1, whereas the backward diffusion appears in the area where absolute values of the gradient are larger than 1.

Thus, the model (1.1) is an ill-posed problem from the mathematical point of view and can produce many unexpected phenomena (see [13]). In particular, we have no results of existence and consistency of the initial-boundary value problem (1.1)–(1.3). To overcome this problem, many authors have been looking for some regularizations of the equation (1.1) which inherit its usefulness in image restoration but have better mathematical behavior (see, for instance, [1,3,7,14,15, 21] and the references therein). In order to guarantee the existence and uniqueness of solution to the initial-boundary value problem (1.1)–(1.3), the authors in [8] proposed to specify the equation (1.1) as follows

$$u_t - div\left(K(t,x)\nabla u\right) + (\nabla u, \boldsymbol{b}) = v \quad \text{in} \;\; Q = (0,T)\times\Omega \tag{1.7}$$

with $K(t,x) = f\left(|\nabla Y^*_\sigma|\right)$, where $\nabla Y^*_\sigma = \nabla G_\sigma * Y^*$ is the spatially regularized gradient of Y^*, G_σ denotes the two-dimensional Gaussian filter kernel,

$$G_\sigma(x) = \frac{1}{2\pi\sigma^2} e^{-\frac{|x|^2}{2\sigma^2}}, \quad x \in \mathbb{R}^2,$$

$$\left(\nabla G_\sigma * Y^*\right)(x) := \int_\Omega \nabla G_\sigma(x-y) Y^*(y)\, dy, \quad \forall\, x \in \Omega,$$

and $Y^* \in C([0,T]; L^2(\Omega))$ is a special function which describes the simplest model of image evolution over the interval $[0,T]$, and this function is defined as a solution of the following optimization problem

$$\int_\Omega \Big[\Big(\frac{\partial Y}{\partial t}\Big|_{t=(T_0+T_1)/2} - div\left(f\left(|\nabla Y_\sigma|\right)\nabla Y\right)\big|_{t=(T_0+T_1)/2} + \Big(\nabla Y\big|_{t=(T_0+T_1)/2}, \boldsymbol{b}\Big) - v\Big)^2\Big]\,dx + \int_\Omega \left[\lambda_1^2|\nabla v|^2 + \lambda_2^2\left(|\nabla \boldsymbol{b}_1|^2 + |\nabla \boldsymbol{b}_2|^2\right)\right]\,dx \to \inf_{\substack{v\in H^1(\Omega)\\ \boldsymbol{b}\in H^1(\Omega;\mathbb{R}^2)}}. \quad (1.8)$$

However, it is well-known that the Perona–Malik model with the spatially regularized gradient has several serious practical and theoretical difficulties. The first one is that the spatial regularization of gradient in the form $f\left(|\nabla G_\sigma * u|\right)$ leads to the loss of accuracy in the case when the signal is noisy, with white noise, for instance [7]. Then the noise introduces very large, in theory unbounded, oscillations of the gradient ∇u. As a result, the conditional smoothing introduced by the model will not help, since all these noise edges will be kept.

The second drawback of the Perona–Malik model with the regularized gradient (see also the model (1.7), (1.2), (1.3)) is the fact that the space-invariant Gaussian smoothing inside the divergent term tends to push the edges in u away from their original locations. We refer to [26] where this issue is studied in details. This effect, known as edge dislocation, can be detrimental especially in the context of the boundary detection problem and its application to the remote sensing and monitoring.

In view of this, our prime interest in this paper is to study the equation (1.1) and the corresponding PDE-constrained optimization problem without the space-invariant Gaussian smoothing inside the divergent term. With that in mind we consider the following optimal control problem

$$(\mathcal{R})\quad \text{Minimize } J(v,u) = \int_{Q_T}\left|D\left(\frac{1}{1+|\nabla u|^2}\right)\right| + \frac{1}{2}\int_\Omega |u(T)-u_d|^2\,dx + \frac{\lambda}{2}\int_0^T\int_\Omega |\nabla u|^2\,dxdt + \frac{\gamma}{2}\int_0^T\int_\omega |v|^2\,dxdt \quad (1.9)$$

subject to the constraints

$$u_t - div\left(\frac{\nabla u}{1+|\nabla u|^2}\right) = v\chi_\omega \quad \text{in } \ Q_T := (0,T)\times\Omega, \quad (1.10)$$

$$\partial_\nu u = 0 \quad \text{on } \ (0,T)\times\partial\Omega, \quad (1.11)$$

$$u(0,\cdot) = u_0 \quad \text{in } \ \Omega, \quad (1.12)$$

$$v \in \mathfrak{V}_{ad} := L^2(0,T;L^2(\omega)), \quad (1.13)$$

where $T>0$, Ω is a bounded open subset of $\mathbb{R}^N$ with a Lipschitz boundary, $N\ge 2$, ω is an open nonempty subset of Ω, $\chi_\omega = \left\{\begin{array}{l} 1,\ x\in\omega, \\ 0,\ x\in\Omega\setminus\omega \end{array}\right\}$ is the characteristic function of the set ω, ∂_ν stands for the outward normal derivative, $u_0, u_d \in L^2(\Omega)$ are given functions, λ, γ are given positive constants, and $v : \omega \to \mathbb{R}$ is a control.

Let us mention that control problems for the non-smoothed Perona-Malik equation have received very little attention in the literature. Formulating the control problem (1.9)–(1.13) for the nonlinear equation of the porous medium type is mainly motivated by the observation that this statement can be successfully applied to the image processing, in particular, to the reduction of mixture of Gaussian and impulse noise with keeping safe the image contours and texture (see, for instance, [2] and the references therein). On the other hand, the novelty of this problem is that we involve into optimization the nonlinear equation with rather special type (non-convex and non-coercive) of non-linearity. Because of this the situation is even more delicate since (1.10) is not well-posed for the given type of non-linearity.

As was mentioned before, the operator $div\,(f\,(|\nabla u|)\,\nabla u)$ with a function f given by (1.6) provides an example of a non-linear operator in divergence form with a so-called degenerate nonlinearity. Moreover, since the function $\mathbb{R}^N \ni s \mapsto \frac{s}{1+|s|^2} \in \mathbb{R}^N$ is neither monotone nor coercive, we have no existence result for the initial-boundary value problem (IBVP) (1.10)–(1.12) and its uniqueness. With that in mind, we say that (v, u) is a feasible pair to the problem (1.9)–(1.13) if

$$v \in \mathfrak{V}_{ad} := L^2(0,T; L^2(\omega)), \quad u \in L^2(0,T; H^1(\Omega)), \quad J(v,u) < +\infty, \tag{1.14}$$

and the following integral identity

$$\int_0^T \int_\Omega \left(-u \frac{\partial \varphi}{\partial t} + \frac{(\nabla u, \nabla \varphi)}{1 + |\nabla u|^2} \right) dxdt = \int_0^T \int_\omega v\varphi \, dxdt + \int_\Omega u_0(x)\varphi(0,x)\, dx \tag{1.15}$$

holds for any function $\varphi \in \Phi$, where

$$\Phi = \left\{ \varphi \in C^1(\overline{Q_T}) \ : \ \varphi(T,\cdot) = 0 \text{ in } \Omega \text{ and } \partial_\nu \varphi = 0 \text{ on } \ (0,T) \times \partial\Omega \right\}.$$

In order to find out in what sense the solution takes the initial value $u(0,\cdot) = u_0$, we give the following result.

Proposition 1.1. Let (v, u) be a feasible pair to the problem (1.9)–(1.13). Then, for any $\eta \in C_0^\infty(\Omega)$, the scalar function $h(t) = \displaystyle\int_\Omega u(t,x)\eta(x)\, dx$ belongs to $W^{1,1}(0,T)$ and $h(0) = \displaystyle\int_\Omega u_0(x)\eta(x)\, dx$.

Proof. We set $\varphi(t,x) = \eta(x)\zeta(t)$ where $\zeta(\cdot)$ is a smooth function on $[0,T]$ and $\zeta(T) = 0$. Then it is clear that $\varphi \in \Phi$ and, therefore, the integral identity (1.15) yields the equality

$$\int_0^T \Bigg[-h(t)\zeta'(t) + \underbrace{\left(\int_\Omega \frac{(\nabla u, \nabla \eta)}{1 + |\nabla u|^2}\, dx - \int_\omega \eta v \, dx \right)}_{H(t)} \zeta(t) \Bigg] dt = \underbrace{\left(\int_\Omega u_0 \eta \, dx \right)}_{k} \zeta(0). \tag{1.16}$$

Since $h \in L^1(0,T)$ and $H \in L^1(0,T)$, it follows from (1.16) that $h \in W^{1,1}(0,T)$, i.e., the function $h(t)$ is absolutely continuous on $[0,T]$. Moreover, from (1.16) we deduce that $h(0) = k$. □

For further convenience we denote the set of all feasible solutions to the problem (1.9)–(1.13) by Ξ. Because of the degenerate behavior of multiplier $f(|\nabla u|)$, the structure of the set Ξ and its main topological properties are unknown in general.

The main focus in this paper consists in providing an approximation framework which in spite of the technical difficulties leads to an implementable scheme, namely, to the so-called indirect approach proving the existence of optimal solutions and giving the procedure of their efficient approximation. With that in mind, we show that the original optimal control problem (1.9)–(1.13) can be approximated efficiently by a special family of optimal control problems for linear parabolic equations with the fictitious BV-control in the principle part of elliptic operator $div\,(\rho\nabla u)$. In spite of the fact that the concept of fictitious controls is not new in the literature, in this paper we utilize it in a new manner combining it with the pointwise convergence of the gradients of solutions to some parabolic equations.

The paper is organized as follows. In the next section, we give some preliminaries and notions that will be needed in the sequel. Section 3 contains a few technical results concerning the almost everywhere convergence of the gradients of solutions to linear parabolic equations with BV-coefficients in the main part of the elliptic operator. These results were obtained in the spirit of Bocardo and Murat approach (see Theorems 4.1 and 4.3 in [6]). In Section 4 we give a precise statement of the fictitious optimal control problems for linear parabolic equations with the constrained BV-controls in the coefficients. We also discuss in this section the existence issues for the proposed control problems. The announced approximation framework is the subject of Section 5, where we provide an asymptotic analysis of a family of approximated optimal control problems and show that some optimal pairs to the original problem (1.9)–(1.13) can be attained (in an appropriate topology) by optimal solutions to the approximated problems.

2. Preliminaries and Basic Definitions

We begin with some notation. For vectors $\xi \in \mathbb{R}^N$ and $\eta \in \mathbb{R}^N$, $(\xi,\eta) = \xi^t\eta$ denotes the standard vector inner product in $\mathbb{R}^N$, where ^{t} denotes the transpose operator. The norm $|\xi|$ is the Euclidean norm given by $|\xi| = \sqrt{(\xi,\xi)}$.

Let Ω be a given bounded open subset of R^N $(N \geq 2)$ with a sufficiently smooth boundary. We suppose that the unit outward normal $\nu = \nu(x)$ is well-defined for $\mathcal{H}^{N-1}$-a.a. $x \in \partial\Omega$, where a.a. it means here with respect to the $(N-1)$-dimensional Hausdorff measure $\mathcal{H}^{N-1}$. For any subset $D \subset \Omega$ we denote by $|D|$ its N-dimensional Lebesgue measure $\mathcal{L}^N(D)$. For a subset $D \subseteq \Omega$ let $\overline{D}$ denote its closure and ∂D its boundary. We define the characteristic function χ_D

of D by

$$\chi_D(x) := \begin{cases} 1, & \text{for } x \in D, \\ 0, & \text{otherwise.} \end{cases}$$

Let X denote a real Banach space with norm $\|\cdot\|_X$, and let X' be its dual. Let $\langle\cdot,\cdot\rangle_{X';X}$ be the duality form on $X' \times X$. By $\rightharpoonup$ and $\overset{*}{\rightharpoonup}$ we denote the weak and weak* convergence in normed spaces, respectively.

For given $1 \le p \le +\infty$, the space $L^p(\Omega;\mathbb{R}^N)$ is defined by

$$L^p(\Omega;\mathbb{R}^N) = \left\{ f : \Omega \to \mathbb{R}^N \ : \ \|f\|_{L^p(\Omega;\mathbb{R}^N)} < +\infty \right\},$$

where $\|f\|_{L^p(\Omega;\mathbb{R}^N)} = \left(\int_\Omega |f(x)|^p\, dx\right)^{1/p}$ for $1 \le p < +\infty$. The inner product of two functions f and g in $L^p(\Omega;\mathbb{R}^N)$ with $p \in [1,\infty)$ is given by

$$(f,g)_{L^p(\Omega;\mathbb{R}^N)} = \int_\Omega (f(x), g(x))\ dx = \int_\Omega \sum_{k=1}^N f_k(x) g_k(x)\, dx.$$

We denote by $C_c^\infty(\mathbb{R}^N)$ a locally convex space of all infinitely differentiable functions with compact support. We recall here some functional spaces that will be used throughout this paper. We define the Banach space $H^1(\Omega)$ as the closure of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|y\|_{H^1(\Omega)} = \left(\int_\Omega \left(y^2 + |\nabla y|^2\right)\, dx \right)^{1/2}.$$

We denote by $\left(H^1(\Omega)\right)'$ the dual space of $H^1(\Omega)$. We also set $H^1(\Omega;\partial\Omega) = \left\{u \in H^1(\Omega) \ : \ \frac{\partial u}{\partial \nu} = 0\right\}$.

Let $k > 0$. In what follows, we will often use composition of functions in Sobolev space $H^1(\Omega)$ with the Lipschitz continuous function

$$T_k(s) = \max\{-k, \min\{s,k\}\}.$$

We recall the well-know result on Sobolev spaces about composition with regular functions.

Theorem 2.1. *Let $G : \mathbb{R} \to \mathbb{R}$ be a Lipschitz continuous function such that $G(0) = 0$. If u belongs to $H^1(\Omega)$, then $G(u)$ belongs to $H^1(\Omega)$ as well, and*

$$\nabla G(u) = G'(u)\nabla u \quad \textit{almost everywhere in } \Omega.$$

As a result, we have

$$\nabla T_k(u) = \nabla u \chi_D\{|u| \le k\} \quad \text{almost everywhere in } \Omega. \tag{2.1}$$

Weak and Strong Convergence in $L^1(\Omega)$. Throughout the paper we will often use the concepts of the weak and strong convergence in $L^1(\Omega)$. Hereinafter, ε

denotes a small parameter which varies within a strictly decreasing sequence of positive numbers converging to 0. When we write $\varepsilon > 0$, we consider only the elements of this sequence, in the case $\varepsilon \geq 0$ we also consider its limit $\varepsilon = 0$. Let $\{a_\varepsilon\}_{\varepsilon>0}$ be a sequence in $L^1(\Omega)$. We recall that $\{a_\varepsilon\}_{\varepsilon>0}$ is called equi-integrable if for any $\delta > 0$ there is $\tau = \tau(\delta)$ such that $\int_S |a_\varepsilon|\,dx < \delta$ for all a_ε and for every measurable subset $S \subset \Omega$ of Lebesgue measure $|S| < \tau$. A sufficient condition for the sequence $\{a_\varepsilon\}_{\varepsilon>0}$ to be equi-integrable is that there exists a constant $C > 0$ such that

$$\sup_{\varepsilon>0} \int_\Omega |a_\varepsilon|^{1+\theta}\,dx \leq C \tag{2.2}$$

for some $\theta > 0$.

Theorem 2.2 (Dunford–Pettis). *Let $\{a_\varepsilon\}_{\varepsilon>0}$ be a sequence in $L^1(\Omega)$. Then this sequence is relatively compact with respect to the weak convergence in $L^1(\Omega)$ if and only if $\{a_\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $L^1(\Omega)$, i.e., $\sup_{\varepsilon>0} \|u_\varepsilon\|_{L^1(\Omega)} < +\infty$, and $\{a_\varepsilon\}_{\varepsilon>0}$ is equi-integrable.*

Theorem 2.3 (Lebesgue–Vitali). *If a sequence $\{a_\varepsilon\}_{\varepsilon>0} \subset L^1(\Omega)$ is equi-integrable and there exists a function $a \in L^1(\Omega)$ such that $a_\varepsilon(x) \to a(x)$ almost everywhere in Ω then $a_\varepsilon \to a$ in $L^1(\Omega)$.*

A typical application of VitaliвЂ™s theorem is provided by the next simple lemma.

Lemma 2.1. *Let $\{a_\varepsilon\}_{\varepsilon>0}$ be a sequence in $L^1(\Omega)$ such that $a_\varepsilon(x) \to a(x)$ almost everywhere in Ω, and this sequence is uniformly bounded in $L^p(\Omega)$ for some $p > 1$. Then*

$$a_\varepsilon \to a \quad \text{in } L^r(\Omega) \text{ for all } 1 \leq r < p. \tag{2.3}$$

The next lemma is useful in many applications.

Lemma 2.2. *Let $\{a_\varepsilon\}_{\varepsilon>0}$, $\{b_\varepsilon\}_{\varepsilon>0}$, a, and b be a measurable functions such that*

$$a_\varepsilon(x) \to a(x) \quad \text{a.e. in } \Omega, \quad \sup_{\varepsilon>0} \|a_\varepsilon\|_{L^\infty(\Omega)} < \infty, \tag{2.4}$$

$$b_\varepsilon \rightharpoonup b \quad \text{in } L^1(\Omega). \tag{2.5}$$

Then

$$ab \in L^1(\Omega) \quad \text{and} \quad a_\varepsilon b_\varepsilon \rightharpoonup ab \quad \text{in } L^1(\Omega). \tag{2.6}$$

Functions with Bounded Variation. Let $f : \Omega \to \mathbb{R}$ be a function of $L^1(\Omega)$. Define

$$\int_\Omega |Df| = \sup\Big\{ \int_\Omega f\, div\,\varphi\,dx \ : \ \varphi = (\varphi_1, \ldots, \varphi_N) \in C_0^1(\Omega; \mathbb{R}^N),\ |\varphi(x)| \leq 1 \text{ for } x \in \Omega \Big\},$$

where $div\,\varphi = \sum_{i=1}^{N} \frac{\partial \varphi_i}{\partial x_i}$. According to the Radon-Nikodym theorem, if $\int_\Omega |Df| < +\infty$ then the distribution Df is a measure and there exist a vector-valued function $\nabla f \in [L^1(\Omega)]^N$ and a measure $D_s f$, singular with respect to the N-dimensional Lebesgue measure $\mathcal{L}^N \lfloor \Omega$ restricted to Ω, such that

$$Df = \nabla f \mathcal{L}^N \lfloor \Omega + D_s f.$$

Definition 2.1. A function $f \in L^1(\Omega)$ is said to have a bounded variation in Ω if $\int_\Omega |Df| < +\infty$. By $BV(\Omega)$ we denote the space of all functions in $L^1(\Omega)$ with bounded variation.

Under the norm $\|f\|_{BV(\Omega)} = \|f\|_{L^1(\Omega)} + \int_\Omega |Df|$, $BV(\Omega)$ is a Banach space. The following compactness result for BV-functions is well-known:

Proposition 2.1. The uniformly bounded sets in BV-norm are relatively compact in $L^1(\Omega)$.

Definition 2.2. A sequence $\{f_k\}_{k=1}^{\infty} \subset BV(\Omega)$ weakly-$*$ converges to some $f \in BV(\Omega)$, and we write $f_k \overset{*}{\rightharpoonup} f$ if and only if the two following conditions hold: $f_k \to f$ strongly in $L^1(\Omega)$, and $Df_k \rightharpoonup Df$ weakly-$*$ in $\mathcal{M}(\Omega; \mathbb{R}^N)$, where $\mathcal{M}(\Omega; \mathbb{R}^N)$ stands for the space of all vector-valued Borel measures which is, according to the Riesz theory, the dual of the space $C(\Omega; \mathbb{R}^N)$ of all continuous vector-valued functions φ vanishing at infinity.

In the proposition below we give a compactness result related to this convergence, together with the lower semicontinuity property (see [4]):

Proposition 2.2. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence in $BV(\Omega)$ strongly converging to some f in $L^1(\Omega)$ and satisfying $\sup_{k\in\mathbb{N}} \int_\Omega |Df_k| < +\infty$. Then

(i) $f \in BV(\Omega)$ and $\int_\Omega |Df| \leq \liminf_{k\to\infty} \int_\Omega |Df_k|$;

(ii) $f_k \overset{*}{\rightharpoonup} f$ in $BV(\Omega)$.

The following embedding results for BV-function is useful in many applications (see [5, p.378]).

Proposition 2.3. Let Ω be an open bounded subset of $\mathbb{R}^N$ with a Lipschitz boundary. Then the embedding $BV(\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega)$ is continuous and the embeddings $BV(\Omega) \hookrightarrow L^p(\Omega)$ are compact for all p such that $1 \leq p < \frac{N}{N-1}$. Moreover, there exists a constant $C_{em} > 0$ which depends only on Ω and p such that for all u in $BV(\Omega)$,

$$\left(\int_\Omega |u|^p \, dx \right)^{1/p} \leq C_{em} \|u\|_{BV(\Omega)}, \quad \forall\, p \in \left[1, \frac{N}{N-1}\right].$$

3. Some Auxiliaries

In this section we give a few technical results that can be viewed as some specification of the well-known results of Bocardo and Murat (see Theorems 4.1 and 4.3 in [6]).

Proposition 3.1. Let $\{u_k\}_{k\in\mathbb{N}}$ be a weakly convergent sequence in $L^2(0,T;H^1(\Omega))$, and

$$u_k \rightharpoonup u \quad \text{weakly in} \;\; L^2(0,T;H^1(\Omega)). \tag{3.1}$$

Assume that

$$\frac{\partial u_k}{\partial t} = h_k \quad \text{in} \;\; \mathcal{D}'((0,T)\times\Omega) \quad \forall\, k\in\mathbb{N}, \tag{3.2}$$

where $\{h_k\}_{k\in\mathbb{N}}$ is a bounded sequence in $L^2(0,T;H^{-1}(\Omega))$. Then

$$u_k \to u \quad \text{strongly in} \;\; L^2_{loc}(0,T;L^2_{loc}(\Omega)). \tag{3.3}$$

Proof. For arbitrary test functions $\psi\in C_0^\infty(\Omega)$ and $\eta\in C_0^\infty(0,T)$, we set

$$\phi(t,x)=\eta(t)\psi(x), \quad z_k=\phi u_k, \quad \alpha_k=\phi h_k+\frac{\partial\phi}{\partial t}u_k.$$

Then, in view of the dense embeddings $H^1(\Omega)\subset L^2(\Omega)\subset H^{-1}(\Omega)$, we see that, for any bounded open subset S such that $\operatorname{supp}(\psi)\subset S\subset\Omega$,

$$\begin{aligned} z_k(t,\cdot)\in H_0^1(S) \text{ and } \frac{\partial\phi(t,\cdot)}{\partial t}u_k(t,\cdot)\in H^{-1}(S) \;\; \text{a.e.} \;\; t\in(0,T),\\ \frac{\partial z_k}{\partial t}=\alpha_k \quad \text{in} \;\; \mathcal{D}'((0,T)\times S), \quad \forall\, k\in\mathbb{N},\\ \sup_{k\in\mathbb{N}}\|z_k\|_{L^2(0,T;H_0^1(S))}\le C, \; \sup_{k\in\mathbb{N}}\|\alpha_k\|_{L^2(0,T;H^{-1}(S))}\le C \text{ with some } \; C>0. \end{aligned} \tag{3.4}$$

Moreover, all these functions have their support included in the same compact subset of $(0,T)\times S$.

Since the embeddings $H_0^1(S)\hookrightarrow L^2(S)$ and $L^2(S)\hookrightarrow H^{-1}(S)$ are compact, the brilliant Aubin's Lemma (see [24, Section 8, Corollary 4]) and conditions (3.4) ensure that the sequence $\{z_k\}_{k\in\mathbb{N}}$ is compact in $L^2(0,T;L^2(S))$. This implies (3.3). □

Proposition 3.2. Let $\varepsilon\in(0,1)$ and $K\in(0,\infty)$ be given values. Assume that the sequences

$$\begin{aligned} \{u_k\}_{k=1}^\infty\subset L^2(0,T;H^1(\Omega)), \quad \{v_k\}_{k=1}^\infty\subset L^2(0,T;L^2(\Omega)),\\ \text{and} \quad \{\rho_k\}_{k=1}^\infty\subset BV(Q_T)\cap L^\infty(Q_T) \end{aligned} \tag{3.5}$$

are bounded and such that

$$u_k \rightharpoonup u \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{3.6}$$

$$v_k \rightharpoonup v \text{ weakly in } L^2(0,T;L^2(\Omega)), \tag{3.7}$$

$$\rho_k \rightharpoonup \rho \text{ weakly-}* \text{ in } BV(Q_T) \text{ and a.e. in } Q_T, \tag{3.8}$$

$$\rho_k \geq \varepsilon \quad \text{a.e. in } \; Q_T, \quad \forall\, k \in \mathbb{N}, \tag{3.9}$$

$$\frac{\partial u_k}{\partial t} - div\,(\rho_k \nabla u_k) = v_k \quad \text{in } \; \mathcal{D}'(Q_T), \quad \forall\, k \in \mathbb{N}. \tag{3.10}$$

Then

$$\nabla T_K(u_k) \to \nabla T_K(u) \text{ strongly in } L^2_{loc}(0,T;L^2_{loc}(\Omega))^N, \tag{3.11}$$

where $T_K : \mathbb{R} \to \mathbb{R}$ is the truncation at height K.

Proof. Let us denote the duality pairing between

$$L^2(0,T;H^{-1}(\Omega)) \quad \text{and} \quad L^2(0,T;H^1_0(\Omega))$$

by $< \cdot, \cdot >_{Q_T}$. We also set $S_K(u) = \int_0^u T_K(s)\,ds$. Then, using the trick with approximation by convolution, it is easy to show that:

For any $\phi \in C_0^\infty(0,T;C_0^\infty(\Omega))$ and any $u \in L^2(0,T;H^1(\Omega))$ with $\frac{\partial u}{\partial t} \in L^2(0,T;H^{-1}(\Omega))$, we have

$$\left\langle \frac{\partial u}{\partial t}, \phi T_K(u) \right\rangle_{Q_T} = - \iint_{Q_T} \frac{\partial \phi}{\partial t} S_K(u)\,dxdt. \tag{3.12}$$

With an arbitrary compact subset $A \subset Q_T = (0,T)\times\Omega$ we associate a function $\phi_A \in C_0^\infty(0,T;C_0^\infty(\Omega))$ such that $0 \leq \phi_A(t,x) \leq 1$ in Q_T and $\phi_A(t,x) = 1$ on A. Then using in (3.10) the test function

$$z_k = [T_K(u_k) - T_K(u)]\,\phi_A,$$

we obtain

$$\left\langle \frac{\partial u_k}{\partial t}, \phi_A T_K(u_k) \right\rangle_{Q_T} \overset{\text{by (3.12)}}{=} - \iint_{Q_T} \frac{\partial \phi_A}{\partial t} S_K(u_k)\,dxdt$$

and, therefore, (3.10) yields

$$\begin{aligned} - \iint_{Q_T} \frac{\partial \phi_A}{\partial t} S_K(u_k)\,dxdt &- \left\langle \frac{\partial u_k}{\partial t}, \phi_A T_K(u) \right\rangle_{Q_T} \\ &+ \iint_{Q_T} \phi_A \rho_k \left(\nabla u_k, \nabla T_K(u_k) - \nabla T_K(u)\right)\,dxdt \\ &+ \iint_{Q_T} [T_K(u_k) - T_K(u)]\,\rho_k \left(\nabla u_k, \nabla \phi_A\right)\,dxdt \\ &= \int_0^T \langle v_k, [T_K(u_k) - T_K(u)]\,\phi_A \rangle_{H^{-1}(\Omega),H^1_0(\Omega)}\,dt. \end{aligned} \tag{3.13}$$

As follows from the initial assumptions (3.5)–(3.8), the sequence $\{h_k\}_{k\in\mathbb{N}}$ with

$$h_k = div\,(\rho_k \nabla u_k) + v_k$$

is bounded in $L^2(0,T;H^{-1}(\Omega))$. Then, Proposition 3.1 implies that, up to a subsequence, the following assertion holds

$$\begin{gathered} T_K(u_k) - T_K(u) \rightharpoonup 0 \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \\ T_K(u_k) - T_K(u) \to 0 \quad \text{strongly in } L^2_{loc}(Q_T), \text{ and a.e. in } Q_T. \end{gathered} \tag{3.14}$$

Therefore, the last term in (3.13) tends to zero as $k \to \infty$.

Moreover, using the fact that $\rho_k(x) - \rho(x) \to 0$ a.e. in Q_T and the sequence $\{(\nabla u_k, \nabla \phi_A)\}_{k\in\mathbb{N}}$ is bounded in $L^2(Q_T)$, we deduce that

$$\iint_{Q_T} [T_K(u_k) - T_K(u)]\, \rho_k\, (\nabla u_k, \nabla \phi_A)\; dxdt \to 0 \quad \text{as } k \to \infty.$$

Since

$$\frac{\partial}{\partial t} S_K(u_k) = T_K(u_k) \frac{\partial u_k}{\partial t} \quad \text{in } \ \mathcal{D}'((0,T)\times\Omega) \quad \forall\, k \in \mathbb{N},$$

it follows from Proposition 3.1 that $S_K(u_k) \to S_K(u)$ strongly in $L^2_{loc}(Q_T)$, which yields

$$\lim_{k\to\infty} \iint_{Q_T} \frac{\partial \phi_A}{\partial t} S_K(u_k)\, dxdt = \iint_{Q_T} \frac{\partial \phi_A}{\partial t} S_K(u)\, dxdt.$$

As for the second term in (3.13), we see that $\phi_A T_K(u) \in L^2(0,T;H^1_0(\Omega))$ and $\frac{\partial u_k}{\partial t}$ is a bounded term in $L^2(0,T;H^{-1}(\Omega))$. Hence,

$$\left\langle \frac{\partial u_k}{\partial t}, \phi_A T_K(u) \right\rangle_{Q_T} \to \left\langle \frac{\partial u}{\partial t}, \phi_A T_K(u) \right\rangle_{Q_T} \overset{\text{by (3.12)}}{=} \iint_{Q_T} \frac{\partial \phi_A}{\partial t} S_K(u)\, dxdt$$

as $k \to \infty$.

Thus, we have shown that

$$\lim_{k\to\infty} \iint_{Q_T} \phi_A \rho_k\, (\nabla u_k, \nabla T_K(u_k) - \nabla T_K(u))\; dxdt = 0. \tag{3.15}$$

Taking this fact into account, we observe that

$$\begin{aligned}
&\iint_{Q_T} \phi_A \rho \big|\nabla T_K(u_k) - \nabla T_K(u)\big|^2 \, dxdt \\
&\qquad = \iint_{Q_T} \phi_A \left(\rho - \rho_k\right) \left(\nabla T_K(u_k), \nabla T_K(u_k) - \nabla T_K(u)\right) \, dxdt \\
&\qquad\quad + \iint_{Q_T} \phi_A \left(\rho_k \nabla T_K(u_k) - \rho \nabla T_K(u), \nabla T_K(u_k) - \nabla T_K(u)\right) \, dxdt \\
&\qquad = \iint_{Q_T} \phi_A \left(\rho - \rho_k\right) \left(\nabla T_K(u_k), \nabla T_K(u_k) - \nabla T_K(u)\right) \, dxdt \\
&\qquad\quad + \iint_{Q_T} \phi_A \left(\rho_k \nabla T_K(u_k), \nabla T_K(u_k) - \nabla T_K(u)\right) \, dxdt \\
&\qquad\quad - \iint_{Q_T} \phi_A \left(\rho \nabla T_K(u), \nabla T_K(u_k) - \nabla T_K(u)\right) \, dxdt \\
&\qquad = \iint_{Q_T} \phi_A \left(\rho - \rho_k\right) \left(\nabla T_K(u_k), \nabla T_K(u_k) - \nabla T_K(u)\right) \, dxdt \\
&\qquad\quad + \iint_{Q_T} \phi_A \left(\rho_k \nabla u_k, \nabla T_K(u_k) - \nabla T_K(u)\right) \, dxdt \\
&\qquad\quad - \iint_{Q_T} \phi_A \left(\rho_k \nabla u, \nabla T_K(u_k) - \nabla T_K(u)\right) \chi_{\Lambda_k} \, dxdt \\
&\qquad\quad - \iint_{Q_T} \phi_A \left(\rho \nabla T_K(u), \nabla T_K(u_k) - \nabla T_K(u)\right) \, dxdt, \qquad (3.16)
\end{aligned}$$

where χ_{Λ_K} stands for the characteristic function of the set

$$\Lambda_k := \left\{(t,x) \in Q_T \;:\; |u_k(t,x)| > K\right\}.$$

In view of (3.8), (3.14), and (3.15), we have:

$$\iint_{Q_T} \phi_A \left(\rho - \rho_k\right) \left(\nabla T_K(u_k), \nabla T_K(u_k) - \nabla T_K(u)\right) \, dxdt \overset{\text{by Lemma 2.2}}{\to} 0,$$

$$\iint_{Q_T} \phi_A \left(\rho_k \nabla u_k, \nabla T_K(u_k) - \nabla T_K(u)\right) \, dxdt \overset{\text{by (3.15)}}{\to} 0,$$

$$\iint_{Q_T} \phi_A \left(\rho \nabla T_K(u), \nabla T_K(u_k) - \nabla T_K(u)\right) \, dxdt \overset{\text{by (3.14)}}{\to} 0.$$

As a result, it follows from (3.16) that

$$\begin{aligned}
&\lim_{k\to\infty} \iint_{Q_T} \phi_A \rho \big|\nabla T_K(u_k) - \nabla T_K(u)\big|^2 \, dxdt \\
&\qquad\qquad = - \lim_{k\to\infty} \iint_{Q_T} \phi_A \left(\rho_k \nabla u, \nabla T_K(u_k) - \nabla T_K(u)\right) \chi_{\Lambda_k} \, dxdt.
\end{aligned}$$

Utilizing the fact that $\chi_{\Lambda_K}\nabla T_K(u_k)=0$ almost everywhere in Q_T, we see that

$$\lim_{k\to\infty}\iint_{Q_T}\phi_A\rho\big|\nabla T_K(u_k)-\nabla T_K(u)\big|^2\,dxdt = \lim_{k\to\infty}\iint_{Q_T}\phi_A\left(\rho_k\nabla u,\nabla T_K(u)\right)\chi_{\Lambda_k}\,dxdt.$$

Moreover, in view of the weak convergence (3.6) and the Lebesgue dominated Theorem, we have

$$\phi_A\nabla T_K(u)\chi_{\Lambda_k}\to 0\quad\text{strongly in } L^2(Q_T)^N.$$

Hence,

$$0=\lim_{k\to\infty}\iint_{Q_T}\phi_A\rho\big|\nabla T_K(u_k)-\nabla T_K(u)\big|^2\,dxdt\geq\varepsilon\left\|T_K(u_k)-\nabla T_K(u)\right\|^2,$$

and we arrive at the announced convergence (3.11). □

In fact, the main result of Proposition 3.2 can be specified as follows.

Theorem 3.1. *Let $\varepsilon\in(0,1)$ be a given value and let*

$$\begin{array}{c}\{u_k\}_{k=1}^\infty\subset L^2(0,T;H^1(\Omega)),\quad \{v_k\}_{k=1}^\infty\subset L^2(0,T;L^2(\Omega)),\\ \textit{and}\quad \{\rho_k\}_{k=1}^\infty\subset BV(Q_T)\cap L^\infty(Q_T)\end{array}\tag{3.17}$$

be bounded sequences satisfying conditions (3.6)–(3.10)*. Then*

$$\nabla u_k\to\nabla u \textit{ strongly in } L^q(0,T;L^q(\Omega))^N \textit{ for any } q\in[1,2).\tag{3.18}$$

Proof. We fix an arbitrary compact subset $A\subset Q_T=(0,T)\times\Omega$ and associate with it a smooth function $\phi_A\in C_0^\infty(0,T;C_0^\infty(\Omega))$ such that $0\leq\phi_A(t,x)\leq 1$ in Q_T and $\phi_A(t,x)=1$ on A. In accordance with the initial assumptions, the functions $\{v_k\}_{k=1}^\infty$ and v belong to the space $L^2(0,T;H^{-1}(\Omega))$. Hence,

$$\frac{\partial u}{\partial t}\in L^2(0,T;H^{-1}(\Omega))\quad\text{and}\quad\frac{\partial u_k}{\partial t}\in L^2(0,T;H^{-1}(\Omega)),\quad\forall\,k\in\mathbb{N}.$$

Therefore, in order to perform the usual integration by parts in the variational equality (3.10), we can use for this $T_K(u_k-u)\phi_A$ as a test function. Taking into account the representation (3.12) and using the fact that

$$\frac{\partial\left(u_k-u\right)}{\partial t}-div\left(\rho_k\nabla u_k-\rho\nabla u\right)=v_k-v\quad\text{in }\ \mathcal{D}'(Q_T),\quad\forall\,k\in\mathbb{N},$$

we obtain

$$\begin{aligned}-\iint_{Q_T}\frac{\partial\phi_A}{\partial t}S_K(u_k-u)\,dxdt&+\iint_{Q_T}\phi_A\left(\rho_k\nabla u_k-\rho\nabla u,\nabla T_K(u_k-u)\right)dxdt\\&+\iint_{Q_T}\left[T_K(u_k-u)\right]\left(\rho_k\nabla u_k-\rho\nabla u,\nabla\phi_A\right)dxdt\\&=\int_0^T\left\langle v_k-v,\left[T_K(u_k-u)\right]\phi_A\right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,dt.\end{aligned}\tag{3.19}$$

Due to Proposition 3.1, we have

$$T_K(u_k - u) \rightharpoonup 0 \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{3.20}$$

$$T_K(u_k - u) \to 0 \ \text{ strongly in } L^2_{loc}(Q_T), \text{ and a.e. in } Q_T, \tag{3.21}$$

$$S_K(u_k - u) \to 0 \text{ strongly in } L^2_{loc}(Q_T). \tag{3.22}$$

Then, in view of (3.7), the first and last terms in (3.19) tend to zero as $k \to \infty$. Moreover, using the fact that $\{\rho_k\}_{k=1}^{\infty} \subset L^\infty(Q_T)$, $\rho_k(x) - \rho(x) \to 0$ a.e. in Q_T, and the sequence $\{(\nabla u_k - \nabla u, \nabla \phi_A)\}_{k\in\mathbb{N}}$ is bounded in $L^2(Q_T)$, by the Lebesgue dominated theorem we deduce that

$$\begin{aligned}\iint_{Q_T} &\left[T_K(u_k-u)\right](\rho_k \nabla u_k - \rho \nabla u, \nabla \phi_A)\,dxdt \\ &= \iint_{Q_T} [T_K(u_k-u)]\,\rho_k\,(\nabla u_k - \nabla u, \nabla\phi_A)\,dxdt \\ &\quad + \iint_{Q_T} [T_K(u_k-u)]\,(\rho_k-\rho)\,(\nabla u, \nabla \phi_A)\,dxdt \to 0 \quad \text{as } k\to\infty. \end{aligned}\tag{3.23}$$

Thus, passing to the limit in (3.19) when k tends to infinity, we obtain

$$\begin{aligned}\lim_{k\to\infty} \iint_{Q_T} &\phi_A\,(\rho_k\nabla u_k - \rho\nabla u, \nabla T_K(u_k-u))\,dxdt \\ &= \lim_{k\to\infty}\iint_{Q_T} \phi_A \rho\,(\nabla u_k - \nabla u, \nabla T_K(u_k-u))\,dxdt \\ &\quad + \lim_{k\to\infty}\iint_{Q_T} \phi_A(\rho_k-\rho)\,(\nabla u_k, \nabla T_K(u_k-u))\,dxdt \\ &= \lim_{k\to\infty}\iint_{Q_T} \phi_A\rho\,(\nabla(u_k-u), \nabla T_K(u_k-u))\,dxdt = 0, \end{aligned}\tag{3.24}$$

where

$$\lim_{k\to\infty}\iint_{Q_T} \phi_A(\rho_k-\rho)\,(\nabla u_k, \nabla T_K(u_k-u))\,dxdt = 0$$

by Lemma 2.2. Setting

$$E_k := \phi_A\rho|\nabla(u_k-u)|^2 \quad \text{ in } Q_T$$

and splitting the set A onto

$$\begin{aligned} B_k^K &= \{(t,x)\in A \ : |u_k(t,x)-u(t,x)| \le K\}, \\ G_k^K &= \{(t,x)\in A \ : |u_k(t,x)-u(t,x)| > K\}, \end{aligned}$$

we see that

$$\begin{aligned}\iint_A E_k^\theta\,dxdt &= \iint_{B_k^K} E_k^\theta\,dxdt + \iint_{G_k^K} E_k^\theta\,dxdt \\ &\le \left(\iint_{B_k^K} E_k\,dxdt\right)^\theta |B_k^K|^{1-\theta} + \left(\iint_{G_k^K} E_k\,dxdt\right)^\theta |G_k^K|^{1-\theta}\end{aligned}$$

by Hólder inequality with some $\theta \in (0,1)$. Since, for K fixed, we have $|G_k^K| \to 0$ as $k \to \infty$, and since the sequence $\{\rho\nabla(u_k - u)\}_{k=1}^{\infty}$ is bounded in $L^2(0,T;L^2(\Omega)^N)$, it follows that $\sup_{k\in\mathbb{N}} \|E_k\|_{L^1(Q_T)} < \infty$, and, therefore,

$$\lim_{k\to\infty} \left(\iint_{G_k^K} E_k \, dxdt \right)^{\theta} |G_k^K|^{1-\theta} = 0.$$

Hence,

$$\begin{aligned} 0 &\le \lim_{k\to\infty} \iint_A E_k^{\theta} \, dxdt \\ &\le \lim_{k\to\infty} \left[\left(\iint_{B_k^K} E_k \, dxdt \right)^{\theta} |B_k^K|^{1-\theta} \right] \\ &= \left(\lim_{k\to\infty} \iint_{Q_T} \phi_A \rho \left(\nabla(u_k - u), \nabla T_K(u_k - u) \right) \, dxdt \right)^{\theta} \\ &\quad \times \lim_{k\to\infty} |B_k^K|^{1-\theta} \overset{\text{by } (3.24)}{=} 0. \end{aligned} \tag{3.25}$$

As a result, we deduce from (3.25) that $E_k^{\theta} \to 0$ strongly in $L^1(A)$. So, using a sequence of compact sets $A \subset Q_T$, there exists a subsequence of $\{E_k\}_{k\in\mathbb{N}}$ such that

$$E_{k_n}(t,x) \to 0 \quad \text{for almost each } (t,x) \in Q_T.$$

Then the estimate (3.9) implies that

$$\nabla u_{k_n}(t,x) \to \nabla u(t,x) \quad \text{for almost each } (t,x) \in Q_T \text{ as } n \to \infty.$$

To conclude the proof, it remains to notice that since the sequence $\{\nabla u_k\}_{k=1}^{\infty}$ is bounded in the space $L^2(0,T;L^2(\Omega)^N)$, it follows from Vitaly's theorem (see Lemma 2.1) that

$$\nabla u_k \to \nabla u \quad \text{strongly in } L^q(Q_T).$$

□

4. Regularization of the Original Optimal Control Problem

We introduce the following family of approximating control problems

$$(\mathcal{R}_\varepsilon) \quad \text{Minimize } J_\varepsilon(\rho, v, u) = \frac{1}{2} \int_\Omega |u(T) - u_d|^2 \, dx + \frac{\lambda}{2} \int_0^T \int_\Omega |\nabla u|^2 \, dxdt$$
$$+ \frac{\gamma}{2} \int_0^T \int_\omega |v|^2 \, dxdt + \int_{Q_T} |D\rho| + \frac{1}{\varepsilon} \int_0^T \int_\Omega |\rho - \frac{1}{1 + |\nabla u|^2}|^2 \, dxdt \tag{4.1}$$

subject to the constraints

$$u_t - div\,(\rho \nabla u) = v\chi_\omega \quad \text{in} \quad Q_T := (0,T)\times\Omega, \tag{4.2}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \tag{4.3}$$

$$u(0,\cdot) = u_0 \quad \text{in} \quad \Omega, \tag{4.4}$$

$$v \in \mathfrak{V}_{ad} := L^2(0,T;L^2(\omega)), \tag{4.5}$$

$$\rho \in \mathfrak{R}_{ad} := \{h \in BV(Q_T)\cap L^\infty(Q_T) \;:\; 0 \le h(t,x) \le 1 \text{ a.e. in } \; Q_T\}. \tag{4.6}$$

We say that a tuple (ρ, v, u) is a feasible solution to the problem (4.1)–(4.6) if

$$\rho \in \mathfrak{R}_{ad}, \quad v \in \mathfrak{V}_{ad}, \quad u \in L^2(0,T;H^1(\Omega)), \tag{4.7}$$

$$\rho(t,x) \ge \max\left\{\frac{\varepsilon^2}{1+\varepsilon^2}, \frac{1}{1+|\nabla u(t,x)|^2}\right\} \text{ a.e. in } \; Q_T, \tag{4.8}$$

and this triplet satisfies the following integral identity

$$\int_0^T\int_\Omega \left(-\varphi_t u + \rho\,(\nabla u, \nabla\varphi)\right)\,dxdt = \int_0^T\int_\omega v\varphi\,dxdt + \int_\Omega u_0(x)\varphi(0,x)\,dx \tag{4.9}$$

for each $\varphi \in \Psi$, where

$$\Psi = \left\{\varphi \in C^1(\overline{Q_T}) \;:\; \varphi(T,\cdot) = 0 \text{ in } \Omega \text{ and } \partial_\nu\varphi = 0 \text{ on } \; (0,T)\times\partial\Omega\right\}.$$

The set of all feasible solution is denoted by Ξ_ε.

Remark 4.1. Let us show that $\Xi_\varepsilon \ne \emptyset$ for each $\varepsilon > 0$. Indeed, taking $z = e^{-\alpha t}u$, we obtain the following IBVP for z:

$$z_t + \alpha z - div\,\widehat{A} = e^{-\alpha t}v\chi_\omega, \quad z\Big|_{i=0} = u_0, \tag{4.10}$$

where the vector function $\widehat{A} = \rho e^{\alpha t}\nabla z$ possesses the following monotonicity, coercivity, and boundedness conditions

$$\left(\widehat{A}(t,x,\xi) - \widehat{A}(t,x,\eta), \xi - \eta\right) \ge 0,$$

$$\left(\widehat{A}(t,x,\xi),\xi\right) \ge \frac{\varepsilon^2}{1+\varepsilon^2}|\xi|^2, \left(\widehat{A}(t,x,\xi),\xi\right) \le e^{\alpha T}|\xi|^2,$$

and the operator $Bz = \alpha z - div\,\widehat{A}$ is coercive in the space $L^2(0,T;H^1(\Omega))$, i.e.

$$\begin{aligned}\langle Bz, z\rangle_{L^2\left(0,T;(H^1(\Omega))^*\right);L^2(0,T;H^1(\Omega))} &\ge \alpha\|z\|^2_{L^2(Q_T)} + \frac{\varepsilon^2}{1+\varepsilon^2}\|\nabla z\|^2_{L^2(Q_T;\mathbb{R}^N)}\\ &\ge c_0\|z\|^2_{L^2(0,T;H^1(\Omega))}.\end{aligned}$$

Hence, the problem (4.10) has a unique solution for each $v \in \mathfrak{V}_{ad}$ [20]. As for the original IBVP, the same result follows by multiplying of z by $e^{\alpha t}$. Moreover, in this case the integral identity (4.9) holds for any function $\varphi \in \Psi$ and the energy equality

$$\int_\Omega u^2(t,x)\,dx + 2\int_0^t \int_\Omega \rho |\nabla u|^2\,dxdt = 2\int_0^t \int_\omega vu\,dxdt + \int_\Omega u_0^2\,dx, \quad 0 \le t \le T, \quad (4.11)$$

is valid.

Our next step deals with the study of topological properties of the set of feasible solutions Ξ_ε to the problem (4.1)–(4.6).

Definition 4.1. A sequence $\{(\rho_k, v_k, u_k) \in \Xi_\varepsilon\}_{k\in\mathbb{N}}$ is called bounded if

$$\sup_{k\in\mathbb{N}} \left[\|\rho_k\|_{BV(Q_T)} + \|v_k\|_{L^2(0,T;L^2(\omega))} + \|u_k\|_{L^2(0,T;H^1(\Omega))}\right] < +\infty.$$

Definition 4.2. We say that a bounded sequence $\{(\rho_k, v_k, u_k) \in \Xi_\varepsilon\}_{k\in\mathbb{N}}$ of feasible solutions τ-converges to a triplet

$$(\rho, v, u) \in BV(Q_T) \times L^2(0,T;L^2(\omega)) \times L^2(0,T;H^1(\Omega))$$

if conditions

$$u_k \rightharpoonup u \text{ weakly in } L^2(0,T;H^1(\Omega)), \quad (4.12)$$

$$v_k \rightharpoonup v \text{ weakly in } L^2(0,T;L^2(\omega)), \quad (4.13)$$

$$\rho_k \rightharpoonup \rho \text{ weakly-}* \text{ in } BV(Q_T) \text{ and a.e. in } Q_T \quad (4.14)$$

hold true.

Remark 4.2. As follows from Theorem 3.1, if $\{(\rho_k, v_k, u_k) \in \Xi_\varepsilon\}_{k\in\mathbb{N}}$ is a τ-convergent sequence of feasible solutions and $(\rho_k, v_k, u_k) \xrightarrow{\tau} (\rho, v, u)$, then $\nabla u_k \to \nabla u$ strongly in $L^q(0,T;L^q(\Omega))^N$ for any $q \in [1,2)$ and, passing to a subsequence if necessary, we can assert that $\nabla u_k(t,x) \to \nabla u(t,x)$ a.e. in $Q_T = (0,T)\times\Omega$.

Remark 4.3. As immediately follows from (4.9), if (ρ, v, u) is a feasible solution to the problem (4.1)–(4.6), then the equality

$$\frac{\partial u_k}{\partial t} - div\,(\rho_k \nabla u_k) = \chi_\omega v_k \quad \text{in} \quad \mathcal{D}'(Q_T)$$

holds in the sense of distributions for each $k \in \mathbb{N}$. Moreover, if a sequence $\{(\rho_k, v_k, u_k) \in \Xi_\varepsilon\}_{k\in\mathbb{N}}$ is bounded in the sense of Definition 4.1, then $div\,(\rho_k\nabla u_k) + \chi_\omega v_k \in L^2(0,T;H^{-1}(\Omega))$. Therefore, $u_k \in C([0,T];L^2(\Omega))$ for all $k \in \mathbb{N}$ (see [25, Proposition III.1.2]) and due to J.L. Lions [22, Chapitre 1, Theorem 5.1] (we refer also to [24] for some generalizations), the Banach space

$$W = \left\{\varphi \ : \varphi \in L^2(0,T;H^1(\Omega)), \frac{\partial \varphi}{\partial t} \in L^2(0,T;H^{-1}(\Omega))\right\}$$

with the norm of the graph

$$\|\varphi\|_W = \|\varphi\|_{L^2(0,T;H^1(\Omega))} + \left\| \frac{\partial \varphi}{\partial t} \right\|_{L^2(0,T;H^{-1}(\Omega))},$$

is compactly embedded into $L^2(0,T;L^2(\Omega))$.

Thus, the first term in the objective functional (4.1) is well defined onto the set of feasible solutions. So, if $\{u_k\}_{k\in\mathbb{N}}$ is a bounded sequence in W and $u_k \rightharpoonup u$ weakly in $L^2(0,T;H^1(\Omega))$, then $u_k \to u$ strongly in $L^2(0,T;L^2(\Omega))$ and, as a consequence, $u_k(T,\cdot) \to u(T,\cdot)$ strongly in $L^2(\Omega)$.

Before proceeding further, we establish the following important property.

Proposition 4.1. For every $\varepsilon \in (0,1)$ the set Ξ_ε is sequentially closed with respect to the τ-convergence.

Proof. Let $\{(\rho_k, v_k, u_k)\}_{k\in\mathbb{N}} \subset \Xi_\varepsilon$ be a τ-convergent sequence of feasible solutions to the optimal control problem (4.1)–(4.6). Let (ρ, v, u) be its τ-limit. Our aim is to show that $(\rho, v, u) \in \Xi_\varepsilon$.

Since the inclusions $\chi_\omega v \in \mathfrak{V}_{ad} := L^2(0,T;L^2(\Omega))$ and $u \in L^2(0,T;H^1(\Omega))$ are obvious, let us show that the condition (3.9) is valid for some $\varepsilon > 0$. Indeed, in view of Remark 4.2, we can suppose that, up to a subsequence,

$$u_k(t,x) \to u(t,x) \quad \text{and} \quad \frac{1}{1+|\nabla u_k(t,x)|^2} \to \frac{1}{1+|\nabla u(t,x)|^2} \text{ a.e. in } Q_T.$$

Hence, in view of the definition of τ-convergence, the limit passage in the relation

$$\rho_k(t,x) \geq \max\left\{\frac{\varepsilon^2}{1+\varepsilon^2}, \frac{1}{1+|\nabla u_k(t,x)|^2}\right\} \text{ a.e. in } Q_T$$

immediately leads us to the inequality (3.9) with $\widehat{\varepsilon} = \frac{\varepsilon^2}{1+\varepsilon^2}$. As for the inclusion $\rho \in \mathfrak{R}_{ad}$, it is a direct consequence of the weak-$*$ compactness of bounded set $\mathfrak{R}_{ad}$ in $BV(Q_T)$.

It remains to show that the limit triplet (ρ, v, u) is related by the integral identity (4.9). To do so, it is enough to fix an arbitrary test function $\varphi \in \Psi$ and pass to the limit in relation

$$\int_0^T \int_\Omega \left(-\varphi_t u_k + \rho_k \left(\nabla u_k, \nabla \varphi\right)\right) dxdt = \int_0^T \int_\omega v_k \varphi \, dxdt + \int_\Omega u_0(x)\varphi(0,x)\, dx. \quad (4.15)$$

Since $\rho_k \nabla u_k \to \rho \nabla u$ strongly in $L^q(Q_T)$ for $q \in [1,2)$ by Lemma 2.1, it follows that the limit passage in (4.15) leads to the integral identity (4.9). Thus, (ρ, v, u) is a feasible solution to optimal control problem (4.1)–(4.6). $\square$

We are now in a position to state the existence of optimal solutions to the problem (4.1)–(4.6).

Theorem 4.1. *Let $u_d \in L^\infty(\Omega)$ be a given function, and let λ and γ be given constants. Then, for each $\varepsilon \in (0,1)$, the optimal control problem (4.1)–(4.6) admits at least one solution $(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon) \in \Xi_\varepsilon$.*

Proof. Let $\varepsilon \in (0,1)$ be a fixed value. Then, as it was indicated in Remark 4.1, the optimal control problem (4.1)–(4.6) is consistent, that is, $\Xi_\varepsilon \neq \emptyset$.

Let $\{(\rho_k, v_k, u_k) \in \Xi_\varepsilon\}_{k\in\mathbb{N}}$ be a minimizing sequence to the problem (4.1)–(4.6). Then the relation

$$\inf_{(\rho,v,u)\in\Xi_\varepsilon} J_\varepsilon(\rho,v,u) = \lim_{k\to\infty}\Big[\frac{1}{2}\int_\Omega |u_k(T)-u_d|^2\,dx + \frac{\lambda}{2}\int_0^T\int_\Omega |\nabla u_k|^2\,dxdt$$
$$+\frac{\gamma}{2}\int_0^T\int_\omega |v_k|^2\,dxdt \quad + \int_{Q_T}|D\rho_k| + \frac{1}{\varepsilon}\int_0^T\int_\Omega |\rho_k - \frac{1}{1+|\nabla u_k|^2}|^2\,dxdt\Big] < +\infty$$

and definition of the set $\mathfrak{R}_{ad}$ imply existence of a constant $C>0$ such that

$$\begin{aligned} \sup_{k\in\mathbb{N}}\|\nabla u_k\|_{L^2(0,T;L^2(\Omega)^N)} &\le C,\\ \sup_{k\in\mathbb{N}}\|v_k\|_{L^2(0,T;L^2(\omega))} &\le C,\\ \text{and}\sup_{k\in\mathbb{N}}\|\rho_k\|_{BV(Q_T)} &\le C. \end{aligned} \tag{4.16}$$

Then, from the energy equality (4.11), we deduce that

$$\begin{aligned}\int_0^T\int_\Omega u_k^2(t,x)\,dxdt &\le 2T\int_0^T\int_\omega v_k u_k\,dxdt + T\int_\Omega u_0^2\,dx\\ &\le 2T^2\int_0^T\int_\omega v_k^2\,dxdt + \frac{1}{2}\int_0^T\int_\Omega u_k^2\,dxdt + T\int_\Omega u_0^2\,dx.\end{aligned}$$

Hence,

$$\sup_{k\in\mathbb{N}}\|u_k\|_{L^2(0,T;L^2(\Omega))} \le 4T^2C^2 + 2T\|u_0\|^2_{L^2(\Omega)}.$$

Utilizing this fact together with (4.16), we see that $\{(\rho_k, v_k, u_k) \in \Xi_\varepsilon\}_{k\in\mathbb{N}}$ is a bounded sequence in the sense of Definition 4.1. As a result, there exist functions $\rho^0_\varepsilon \in BV(Q_T)$, $v^0_\varepsilon \in L^2(0,T;L^2(\omega))$, and $u^0_\varepsilon \in L^2(0,T;H^1(\Omega))$ such that, up to a subsequence, $(\rho_k, v_k, u_k) \xrightarrow{\tau} (\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon)$ as $k\to\infty$. Since the set Ξ_ε is sequentially closed with respect to the τ-convergence (see Proposition 4.1), it follows that the τ-limit tuple $(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon)$ is a feasible solution to optimal control problem (4.1)–(4.6) (i.e., $(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon) \in \Xi_\varepsilon$). To conclude the proof, we observe that $\nabla u_k(t,x) \to \nabla u^0_\varepsilon(t,x)$ a.e. in Q_T (see Remark 4.2) and, therefore,

$$\rho_k(t,x) - \frac{1}{1+|\nabla u_k(t,x)|^2} \to \rho^0_\varepsilon(t,x) - \frac{1}{1+|\nabla u^0_\varepsilon(t,x)|^2} \text{ a.e. in } Q_T.$$

Since

$$\left\|\rho_k - \frac{1}{1+|\nabla u_k|^2}\right\|_{L^\infty(Q_T)} \le 2 \text{ for all } k \in \mathbb{N},$$

it follows that the sequence $\left\{\rho_k - \dfrac{1}{1+|\nabla u_k|^2}\right\}_{k\in\mathbb{N}}$ is equi-integrable. Hence, Vitaly's theorem implies that

$$\rho_k - \frac{1}{1+|\nabla u_k|^2} \to \rho_\varepsilon^0 - \frac{1}{1+|\nabla u_\varepsilon^0|^2} \quad \text{strongly in } L^2(Q_T) \tag{4.17}$$

(see Lemma 2.1). Taking this fact into account and observing that

$$\begin{aligned}
\liminf_{k\to\infty} \int_0^T \int_\Omega |\rho_k - \frac{1}{1+|\nabla u_k|^2}|^2\,dxdt &\overset{\text{by (4.17)}}{=} \int_0^T \int_\Omega |\rho_\varepsilon^0 - \frac{1}{1+|\nabla u_\varepsilon^0|^2}|^2\,dxdt,\\
\lim_{k\to\infty} \int_\Omega |u_k(T) - u_d|^2\,dx &\overset{\text{by Remark (4.3)}}{\ge} \int_\Omega |u_\varepsilon^0(T) - u_d|^2\,dx,\\
\lim_{k\to\infty} \int_0^T \int_\Omega |\nabla u_k|^2\,dxdt &\overset{\text{by (4.12)}}{=} \int_0^T \int_\Omega |\nabla u_\varepsilon^0|^2\,dxdt,\\
\liminf_{k\to\infty} \int_0^T \int_\omega |v_k|^2\,dxdt &\overset{\text{by (4.13)}}{\ge} \int_0^T \int_\Omega |v_\varepsilon^0|^2\,dxdt,\\
\liminf_{k\to\infty} \int_{Q_T} |D\rho_k| &\overset{\text{by (4.14)}}{\ge} \int_{Q_T} |D\rho_\varepsilon^0|,
\end{aligned}$$

we see that the cost functional J_ε is sequentially lower τ-semicontinuous. Thus

$$J_\varepsilon(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \le \liminf_{k\to\infty} J_\varepsilon(\rho_k, v_k, u_k) \le \lim_{k\to\infty} J_\varepsilon(\rho_k, v_k, u_k) = \inf_{(\rho,v,u)\in\Xi_\varepsilon} J_\varepsilon(\rho, v, u),$$

and, therefore, $(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0)$ is an optimal triplet. □

5. Asymptotic Analysis of the Approximated OCP $(\mathcal{R}_\varepsilon)$

The main goal of this section is to show that the original OCP $(\mathcal{R})$ is solvable and some solutions can be attained (in an appropriate topology) by optimal solutions to the approximated problems $(\mathcal{R}_\varepsilon)$. With that in mind, we make use of the concept of variational convergence of constrained minimization problems (see [9, 17, 18]) and study the asymptotic behavior of a family of OCPs $(\mathcal{R}_\varepsilon)$ as $\varepsilon \to 0$.

Before proceeding further, we adopt the following concept.

Definition 5.1. Let

$$\{(\rho_\varepsilon, v_\varepsilon, u_\varepsilon)\}_{\varepsilon>0} \subset BV(Q_T) \times L^2(0,T;L^2(\omega)) \times L^2(0,T;H^1(\Omega))$$

be an arbitrary sequence. We say that this sequence is bounded if

$$\sup_{\varepsilon>0} \left[\|\rho_\varepsilon\|_{BV(Q_T)} + \|v_\varepsilon\|_{L^2(0,T;L^2(\omega))} + \|u_\varepsilon\|_{L^2(0,T;H^1(\Omega))}\right] < +\infty.$$

Definition 5.2. We say that a bounded sequence

$$\{(\rho_\varepsilon, v_\varepsilon, u_\varepsilon)\}_{\varepsilon>0} \subset BV(Q_T) \times L^2(0,T;L^2(\omega)) \times L^2(0,T;H^1(\Omega))$$

is w-convergent as $\varepsilon \to 0$ and $(\rho_\varepsilon, v_\varepsilon, u_\varepsilon) \stackrel{w}{\to} (\rho, v, u)$ if $(\rho_\varepsilon, v_\varepsilon, u_\varepsilon) \stackrel{\tau}{\to} (\rho, v, u)$ as $\varepsilon \to 0$, i.e.,

$$u_\varepsilon \rightharpoonup u \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{5.1}$$

$$v_\varepsilon \rightharpoonup v \text{ weakly in } L^2(0,T;L^2(\omega)), \tag{5.2}$$

$$\rho_\varepsilon \rightharpoonup \rho \text{ weakly-}* \text{ in } BV(Q_T) \text{ and a.e. in } Q_T; \tag{5.3}$$

and $\nabla u_\varepsilon \to \nabla u$ strongly in $L^1(0,T;L^1(\Omega)^N)$.

The following technical result will play a significant role in the sequel.

Lemma 5.1. *Let $\{(\rho_\varepsilon, v_\varepsilon, u_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ be a τ-convergent sequence of feasible solutions to OCPs* (4.1)–(4.6), *and let*

$$(\rho, v, u) \in BV(Q_T) \times L^2(0,T;L^2(\omega)) \times L^2(0,T;H^1(\Omega))$$

be its τ-limit. Then $(\rho_\varepsilon, v_\varepsilon, u_\varepsilon) \stackrel{w}{\to} (\rho, v, u)$ as $\varepsilon \to 0$, and (ρ, v, u) is subjected to the constrains

$$\rho \in \mathfrak{R}_{ad}, \quad v \in \mathfrak{V}_{ad}, \quad u \in L^2(0,T;H^1(\Omega)), \tag{5.4}$$

$$\rho(t,x) \geq \frac{1}{1+|\nabla u(t,x)|^2} \ \textit{a.e. in } \ Q_T, \tag{5.5}$$

$$\int_0^T \int_\Omega \left(-\varphi_t u + \rho\left(\nabla u, \nabla\varphi\right)\right) dxdt = \int_0^T \int_\omega v\varphi \, dxdt + \int_\Omega u_0(x)\varphi(0,x)\, dx, \quad \forall \varphi \in \Psi. \tag{5.6}$$

Proof. Since $\{(\rho_\varepsilon, v_\varepsilon, u_\varepsilon) \in \Xi_\varepsilon\}_{\varepsilon>0}$ is a sequence of feasible solutions, it implies that the equality

$$\int_0^T \int_\Omega \left(-\varphi_t u_\varepsilon + \rho_\varepsilon\left(\nabla u_\varepsilon, \nabla\varphi\right)\right) dxdt = \int_0^T \int_\omega v_\varepsilon\varphi \, dxdt + \int_\Omega u_0(x)\varphi(0,x)\, dx, \quad \forall \varphi \in \Psi \tag{5.7}$$

holds true for all $\varepsilon > 0$. Then the limit passage in (5.7) leads to the relation (5.6). Setting in this relation the test function φ as an element of $C_c^\infty(Q_T) \subset \Psi$, we see that the τ-limit (ρ, v, u) satisfies the equation

$$\frac{\partial u}{\partial t} - div\left(\rho \nabla u\right) = \chi_\omega v$$

in the sense of distributions $\mathcal{D}'(Q_T)$. So, in view of Remark 4.3, we can suppose that, for each $\varepsilon > 0$, we have the equalities

$$\frac{\partial\,(u_\varepsilon - u)}{\partial t} - div\,(\rho_k \nabla u_\varepsilon - \rho\nabla u) = (v_\varepsilon - v)\chi_\omega \quad \text{in} \;\; \mathcal{D}'(Q_T). \tag{5.8}$$

Therefore, arguing as in the proof of Theorem 3.1, we use for (5.8) the test function $T_K(u_\varepsilon - u)\phi_A$, where A is a compact subset of Q_T, and the function $\phi_A \in C_0^\infty(0,T; C_0^\infty(\Omega))$ is such that $0 \le \phi_A(t,x) \le 1$ in Q_T and $\phi_A(t,x) = 1$ on A. After integration by parts, we obtain

$$\begin{aligned}\iint_{Q_T} \phi_A \rho_\varepsilon \left(\nabla u_\varepsilon - \nabla u, \nabla T_K(u_\varepsilon - u)\right) dxdt &= \iint_{Q_T} \frac{\partial \phi_A}{\partial t} S_K(u_\varepsilon - u)\, dxdt \\ &- \iint_{Q_T} \phi_A(\rho_\varepsilon - \rho)\left(\nabla u, \nabla T_K(u_\varepsilon - u)\right) dxdt \\ &- \iint_{Q_T} \phi_A \rho \left(\nabla u_\varepsilon - \nabla u, \nabla T_K(u_\varepsilon - u)\right) dxdt \\ &- \iint_{Q_T} \phi_A(\rho_\varepsilon - \rho)\left(\nabla u_\varepsilon, \nabla T_K(u_\varepsilon - u)\right) dxdt \\ &+ \int_0^T \langle (v_\varepsilon - v)\chi_\omega, \left[T_K(u_\varepsilon - u)\right]\phi_A \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\, dt. \end{aligned} \tag{5.9}$$

Since, by Proposition 3.1,

$$\begin{aligned} &T_K(u_\varepsilon - u) \rightharpoonup 0 \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \text{ strongly in } L^2_{loc}(Q_T), \text{ and a.e. in } Q_T, \\ &S_K(u_\varepsilon - u) \to 0 \quad \text{strongly in } L^2_{loc}(Q_T) \text{ as } \varepsilon \to 0, \end{aligned}$$

it follows from (5.1)–(5.3) and the Lebesgue dominated theorem that the right hand side of (5.9) tends to zero as $\varepsilon \to 0$. Hence, passing to the limit in (5.9), we deduce:

$$\lim_{\varepsilon \to 0} \iint_{Q_T} \phi_A \rho_\varepsilon \left(\nabla u_\varepsilon - \nabla u, \nabla T_K(u_\varepsilon - u)\right) dxdt = 0. \tag{5.10}$$

Setting

$$E_\varepsilon := \phi_A \rho_\varepsilon |\nabla(u_\varepsilon - u)|^2 \quad \text{in } Q_T$$

and aligning the set A into

$$\begin{aligned} B_\varepsilon &= \left\{(t,x) \in A \;: |u_\varepsilon(t,x) - u(t,x)| \le K\right\}, \\ G_\varepsilon &= \left\{(t,x) \in A \;: |u_\varepsilon(t,x) - u(t,x)| > K\right\}, \end{aligned}$$

we see that

$$\iint_A E_\varepsilon^\theta\, dxdt \le \left(\iint_{B_\varepsilon} E_\varepsilon\, dxdt\right)^\theta |B_\varepsilon|^{1-\theta} + \left(\iint_{G_\varepsilon} E_\varepsilon\, dxdt\right)^\theta |G_\varepsilon|^{1-\theta}$$

by Hölder inequality with some $\theta \in (0,1)$. Since, for K fixed, we have $|G_\varepsilon| \to 0$ as $\varepsilon \to 0$, and the sequence $\{\rho_\varepsilon \nabla(u_\varepsilon - u)\}_{\varepsilon>0}$ is bounded in $L^2(0,T;L^2(\Omega)^N)$, it follows that $\sup_{\varepsilon>0} \|E_\varepsilon\|_{L^1(Q_T)} < \infty$, and, therefore,

$$\lim_{\varepsilon\to 0} \left(\iint_{G_\varepsilon} E_\varepsilon \, dxdt \right)^\theta |G_\varepsilon|^{1-\theta} = 0.$$

Hence,

$$\begin{aligned} 0 &\le \lim_{\varepsilon\to 0} \iint_A E_\varepsilon^\theta \, dxdt \\ &\le \lim_{\varepsilon\to 0} \left[\left(\iint_{B_\varepsilon} E_\varepsilon \, dxdt \right)^\theta |B_\varepsilon|^{1-\theta} \right] \\ &= \left(\lim_{\varepsilon\to 0} \iint_{Q_T} \phi_A \rho \left(\nabla(u_\varepsilon - u), \nabla T_K(u_\varepsilon - u) \right) \, dxdt \right)^\theta \lim_{\varepsilon\to 0} |B_\varepsilon|^{1-\theta} \\ &\overset{\text{by } (3.24)}{=} 0. \end{aligned} \tag{5.11}$$

As a result, we deduce from (5.11) that $E_\varepsilon^\theta \to 0$ strongly in $L^1(A)$. So, using a sequence of compact sets $A \subset Q_T$ converging in an appropriate sense to Q_T, there exists a subsequence of $\{E_\varepsilon\}_{\varepsilon>0}$ (still denoted by the same index) such that

$$E_\varepsilon(t,x) \to 0 \quad \text{for almost each } (t,x) \in Q_T \text{ as } \varepsilon \to 0.$$

Thus,

$$\rho_\varepsilon(t,x) \left| \nabla u_\varepsilon(t,x) - \nabla u(t,x) \right|^2 \to 0 \quad \text{for a.e. } (t,x) \in Q_T \text{ as } \varepsilon_n \to 0. \tag{5.12}$$

Utilizing the fact that $(\rho_\varepsilon, v_\varepsilon, u_\varepsilon) \in \Xi_\varepsilon$ for each $\varepsilon > 0$ and observing that $\dfrac{\varepsilon^2}{1+\varepsilon^2} \to 0$ as $\varepsilon \to 0$, we see that

$$\rho_\varepsilon(t,x) \ge \max\left\{ \frac{\varepsilon^2}{1+\varepsilon^2}, \frac{1}{1+|\nabla u_\varepsilon(t,x)|^2} \right\} \ge \frac{1}{1+|\nabla u_\varepsilon(t,x)|^2} \text{ a.e. in } Q_T \tag{5.13}$$

for $\varepsilon > 0$ small enough. Hence, from (5.13) and (5.11) we deduce:

$$\begin{aligned} 0 &\le \lim_{\varepsilon\to 0} \iint_{Q_T} \frac{1}{1+|\nabla u_\varepsilon|^2} \left| \nabla u_\varepsilon - \nabla u \right|^2 \, dxdt \\ &\le \lim_{\varepsilon\to 0} \iint_{Q_T} \rho_\varepsilon \left| \nabla u_\varepsilon - \nabla u \right|^2 \, dxdt = 0. \end{aligned} \tag{5.14}$$

Since

$$
\begin{aligned}
&\|\nabla u_\varepsilon - \nabla u\|^2_{L^1(0,T;L^1(\Omega)^N)} = \left(\int_0^T \int_\Omega |\nabla u_\varepsilon - \nabla u| \, dxdt\right)^2 \\
&\le \left(\int_0^T \left(\int_\Omega \frac{1}{1+|\nabla u_\varepsilon(x)|^2} \left|\nabla u_\varepsilon - \nabla u\right|^2 \, dx\right)^{1/2} \left(\int_\Omega \left(1+|\nabla u_\varepsilon(x)|^2\right) \, dx\right)^{1/2} dt\right)^2 \\
&\quad \le \int_0^T \int_\Omega \frac{1}{1+|\nabla u_\varepsilon(x)|^2} \left|\nabla u_\varepsilon - \nabla u\right|^2 \, dxdt \int_0^T \int_\Omega \left(1+|\nabla u_\varepsilon(x)|^2\right) \, dx \, dt \\
&\qquad \le \left(|Q_T| + \sup_{\varepsilon>0} \|u_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))}\right) \int_\Omega \frac{1}{1+|\nabla u_\varepsilon(x)|^2} \left|\nabla u_\varepsilon - \nabla u\right|^2 \, dx,
\end{aligned}
$$

it follows from (5.14) that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0} \|\nabla u_\varepsilon - \nabla u\|^2_{L^1(0,T;L^1(\Omega)^N)} \\
&\qquad\qquad \le C \lim_{\varepsilon\to 0} \iint_{Q_T} \frac{1}{1+|\nabla u_\varepsilon|^2} \left|\nabla u_\varepsilon - \nabla u\right|^2 \, dxdt = 0. \quad (5.15)
\end{aligned}
$$

Thus, we can specify the τ-convergence properties (5.1)–(5.3) as follows: in addition to (5.1) $\nabla u_\varepsilon \to \nabla u$ strongly in $L^1(0,T;L^1(\Omega)^N)$, and there exists a subsequence $\{\varepsilon'\}$ such that

$$
\nabla u_{\varepsilon'}(t,x) \to \nabla u(t,x) \quad \text{a.e. in } Q_T. \tag{5.16}
$$

To conclude the proof, it remains to show that

$$
\rho(t,x) \ge \frac{1}{1+|\nabla u(t,x)|^2} \text{ a.e. in } \ Q_T. \tag{5.17}
$$

To do so, it is enough to observe that

$$
\rho_\varepsilon(t,x) \ge \max\left\{\frac{\varepsilon^2}{1+\varepsilon^2}, \frac{1}{1+|\nabla u_\varepsilon(t,x)|^2}\right\} \ge \frac{1}{1+|\nabla u_\varepsilon(t,x)|^2} \text{ a.e. in } \ Q_T \tag{5.18}
$$

for $\varepsilon > 0$ small enough. Using the pointwise convergence (5.16) and (5.3) and passing to the limit in (5.18) as $\varepsilon \to 0$, we arrive to the announced property (5.5). □

Our next step is to discuss the issue related to the existence of solutions to the original optimal control problem (1.9)–(1.13) and their attainability by optimal solutions of the approximated problems $(\mathcal{R}_\varepsilon)$. Before we go on, we assume that the set of feasible solution Ξ to the problem (1.9)–(1.13) is non-empty. In the case when the initial state u_0 is sufficiently smooth and supp $(u_0) \subset \omega$, this assumption can be easily verified. Indeed, let $\varphi \in C^\infty([0,T]; C_c^\infty(\omega))$ be an arbitrary function such that $\varphi(0,x) = u_0(x)$ in Ω. Then it is easy to check that the pair

$$
(v,u) := \left(\left[\varphi_t - div\left(\frac{\nabla\varphi}{1+|\nabla\varphi|^2}\right)\right] \Big\lfloor_{x\in\omega}, \varphi\right)
$$

belongs to the set Ξ. Thus, $\Xi \neq \emptyset$.

We begin with the following result that can be viewed as a direct consequence of Lemma 5.1 and Theorem 4.1.

Proposition 5.1. Let $u_d \in L^\infty(\Omega)$ be a given function, and λ and γ be given constants. Let $\left\{(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ be a bounded sequence of optimal solutions to the approximated problems (4.1)–(4.6) when the small parameter ε varies within a strictly decreasing sequence of positive numbers converging to zero. Then there is a subsequence of $\left\{(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$, still denoted by the suffix ε, and distributions $\rho^0 \in \mathfrak{R}_{ad} \subset BV(Q_T)$, $v^0 \in \mathfrak{V}_{ad}$, and $u^0 \in L^2(0,T;H^1(\Omega))$ such that they satisfy conditions (5.5)–(5.6), and $(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon) \overset{w}{\to} (\rho^0, v^0, u^0)$ as $\varepsilon \to 0$.

The key point in Proposition 5.1 is the assumption that a given sequence of optimal solutions to the approximated problems (4.1)–(4.6) is bounded. Let us show that this assumption can be omitted if only the original optimal control problem is consistent, i.e. $\Xi \neq \emptyset$.

Proposition 5.2. Assume that $\Xi \neq \emptyset$. Let $\left\{(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon) \in \Xi_\varepsilon\right\}_{\varepsilon>0}$ be a sequence of optimal solutions to the approximated problems (4.1)–(4.6). Then there exists a constant $C > 0$ independent of $\varepsilon > 0$ such that

$$\sup_{\varepsilon>0} \left[\|\rho^0_\varepsilon\|_{BV(Q_T)} + \|v^0_\varepsilon\|_{L^2(0,T;L^2(\omega))} + \|u^0_\varepsilon\|_{L^2(0,T;H^1(\Omega))}\right] \leq C. \tag{5.19}$$

Proof. Let $(\widehat{v}, \widehat{u}) \in \Xi$ be a feasible solution to optimal control problem (1.9)–(1.13). Hence, this pair satisfies conditions (1.14)–(1.15). Setting $\widehat{\rho} := (1+|\nabla \widehat{u}|^2)^{-1}$ in Q_T, we see that

$$0 \leq \widehat{\rho}(t,x) \leq 1 \text{ a.e. in } \ Q_T \quad \text{and} \quad \widehat{\rho} \in BV(Q_T) \cap L^\infty(Q_T),$$

and the pair $(\widehat{\rho}, \widehat{u})$ satisfies inequalities (4.8) for $\varepsilon > 0$ small enough. Hence, $\widehat{\rho} \in \mathfrak{R}_{ad}$ and, as a consequence, we deduce: $(\widehat{\rho}, \widehat{v}, \widehat{u}) \in \Xi_\varepsilon$ for $\varepsilon > 0$ small enough. Therefore,

$$\begin{aligned}\inf_{(\rho,v,u)\in\Xi_\varepsilon} J_\varepsilon(\rho,v,u) &= J_\varepsilon(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon) \leq J_\varepsilon\left(\widehat{\rho}, \widehat{v}, \widehat{u}\right)\\ &= \frac{1}{2}\int_\Omega |\widehat{u}(T) - u_d|^2\,dx + \frac{\lambda}{2}\int_0^T\int_\Omega |\nabla \widehat{u}|^2\,dxdt\\ &\quad + \frac{\gamma}{2}\int_0^T\int_\omega |\widehat{v}|^2\,dxdt + \int_{Q_T}|D\widehat{\rho}| = C < +\infty.\end{aligned}$$

From this and definition of the set $\mathfrak{R}_{ad}$, we deduce that

$$\|\nabla u^0_\varepsilon\|^2_{L^2(0,T;L^2(\Omega)^N)} \leq \frac{2}{\lambda}C, \quad \|v^0_\varepsilon\|^2_{L^2(0,T;L^2(\Omega))} \leq \frac{2}{\gamma}C, \tag{5.20}$$

$$\int_{Q_T} \left|D\rho^0_\varepsilon\right| \leq C, \quad \|\rho^0_\varepsilon\|_{BV(\Omega)} \leq |Q_T| + C, \tag{5.21}$$

$$\int_0^T\int_\Omega \left|\rho^0_\varepsilon - \frac{1}{1+|\nabla u^0_\varepsilon|^2}\right|^2 dxdt \leq C\varepsilon \tag{5.22}$$

for all $\varepsilon > 0$ small enough. Then energy equality (4.11) implies that

$$\int_0^T \int_\Omega \left[u_\varepsilon^0\right]^2 \, dxdt \leq 2T \int_0^T \int_\omega v_\varepsilon^0 u_\varepsilon^0 \, dxdt + T \int_\Omega u_0^2 \, dx$$
$$\leq 2T^2 \int_0^T \int_\omega \left[v_\varepsilon^0\right]^2 \, dxdt + \frac{1}{2} \int_0^T \int_\Omega \left[u_\varepsilon^0\right]^2 \, dxdt + T \int_\Omega u_0^2 \, dx.$$

Therefore,

$$\sup_{\varepsilon > 0} \|u_\varepsilon^0\|_{L^2(0,T;L^2(\Omega))} \leq 8T^2 \frac{C}{\gamma} + 2T\|u_0\|^2_{L^2(\Omega)}. \tag{5.23}$$

Thus, the sequence $\left\{(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \in \Xi_\varepsilon\right\}_{\varepsilon > 0}$ is bounded in

$$BV(Q_T) \times L^2(0,T;L^2(\omega)) \times L^2(0,T;H^1(\Omega)).$$

□

The next step of our analysis is to show that the pair (v^0, u^0) is optimal to the original OCP $(\mathcal{R})$ provided (ρ^0, v^0, u^0) is a cluster tuple of a given sequence of optimal solutions $\left\{(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \in \Xi_\varepsilon\right\}_{\varepsilon > 0}$. To do so, we will utilize some hints from the recent papers [10, 16] where the so-called indirect approach to the existence problem of optimal solutions has been proposed.

Theorem 5.1. *Assume that $\Xi \neq \emptyset$. Let $\left\{(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \in \Xi_\varepsilon\right\}_{\varepsilon > 0}$ be a sequence of optimal solutions to the approximated problems* (4.1)–(4.6)*. Let $(\rho^0, v^0, u^0) \in BV(Q_T) \times L^2(0,T;L^2(\omega)) \times L^2(0,T;H^1(\Omega))$ be a w-cluster tuple (in the sense of Definition 5.2) of a given sequence of optimal solutions Then*

$$(v^0, u^0) \in \Xi, \quad \rho^0(t,x) = \frac{1}{1 + |\nabla u^0(t,x)|^2} \ \text{a.e. in} \ Q_T, \tag{5.24}$$

$$\lim_{\varepsilon \to 0} \inf_{(\rho,v,u) \in \Xi_\varepsilon} J_\varepsilon(\rho, v, u) = \lim_{\varepsilon \to 0} J_\varepsilon(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) = J(v^0, u^0) = \inf_{(v,u) \in \Xi} J(v, u). \tag{5.25}$$

Proof. Arguing as in the proof of Proposition 5.2, it can be shown that there exists a constant $C > 0$ such that estimates (5.20)–(5.23) hold true. Hence, the sequence $\left\{(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \in \Xi_\varepsilon\right\}_{\varepsilon > 0}$ is compact with respect to the τ-convergence. Moreover, in view of Proposition 5.1 and the Lebesgue Dominated Theorem, we can suppose that, up to a subsequence,

$$(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \xrightarrow{w} (\rho^0, v^0, u^0) \tag{5.26}$$

$$\frac{1}{1 + |\nabla u_\varepsilon^0|^2} \to \frac{1}{1 + |\nabla u^0|^2} \quad \text{strongly in } L^2(Q_T) \text{ as } \varepsilon \to 0, \tag{5.27}$$

$$\rho_\varepsilon^0(t,x) - \frac{1}{1 + |\nabla u_\varepsilon^0(t,x)|^2} \to \rho^0(t,x) - \frac{1}{1 + |\nabla u^0(t,x)|^2} \ \text{a.e. in} \ Q_T, \tag{5.28}$$

and $\left(\rho_\varepsilon^0 - \left(1 + |\nabla u_\varepsilon^0|^2\right)^{-1}\right) \in L^\infty(\Omega)$.

Then it follows from Vitaly's theorem (see Lemma 2.1) that

$$\rho_\varepsilon^0 - \left(1 + |\nabla u_\varepsilon^0|^2\right)^{-1} \to \rho^0 - \frac{1}{1 + |\nabla u^0|^2} \text{ strongly in } L^2(\Omega).$$

However, as follows from the third estimate in (5.22), the L^2-limit of the sequence $\left\{\rho_\varepsilon^0 - \frac{1}{1+|\nabla u_\varepsilon^0|^2}\right\}_{\varepsilon>0}$ is equal to zero. Hence, we obtain

$$\rho^0(t,x) = \frac{1}{1 + |\nabla u^0(t,x)|^2} \quad \text{a.e. in } \ Q_T.$$

Thus,

$$(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) \stackrel{w}{\to} \left(\frac{1}{1 + |\nabla u^0|^2}, v^0, u^0\right) \qquad \text{as } \varepsilon \to 0.$$

Taking into account Proposition 5.1, we see that (v^0, u^0) is a feasible solution to the original OCP $(\mathcal{R})$. Moreover, as a direct consequence of the properties (5.27), we have the following estimate

$$\begin{aligned}\liminf_{\varepsilon\to 0} J_\varepsilon(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) &\geq \frac{1}{2}\int_\Omega |u^0(T) - u_d|^2\,dx + \frac{\lambda}{2}\int_0^T\int_\Omega |\nabla u^0|^2\,dxdt \\ &+ \frac{\gamma}{2}\int_0^T\int_\Omega |v^0|^2\,dxdt + \int_{Q_T}\left|D\left(\frac{1}{1 + |\nabla u^0|^2}\right)\right| = J(v^0, u^0). \quad (5.29)\end{aligned}$$

Let us assume for a moment that the pair (v^0, u^0) is not optimal for $(\mathcal{R})$-problem. Then there exists another pair $(v^*, u^*) \in \Xi$ such that

$$J(v^*, u^*) < J(v^0, u^0) < +\infty. \quad (5.30)$$

Setting $\rho^* = \left(1 + |\nabla u^*|^2\right)^{-1}$, we deduce from condition $(v^*, u^*) \in \Xi$ that the tuple (ρ^*, v^*, u^*) is a feasible solution to each approximate problem $(\mathcal{R}_\varepsilon)$, i.e.,

$$(\rho^*, v^*, u^*) \in \Xi_\varepsilon, \quad \forall\, \varepsilon \in (0,1). \quad (5.31)$$

Taking this fact into account, we get

$$\begin{aligned}J(v^0, u^0) &= \frac{1}{2}\int_\Omega |u^0(T) - u_d|^2\,dx + \frac{\lambda}{2}\int_0^T\int_\Omega |\nabla u^0|^2\,dxdt \\ &+ \frac{\gamma}{2}\int_0^T\int_\Omega |v^0|^2\,dxdt + \int_{Q_T}\left|D\left(\frac{1}{1 + |\nabla u^0|^2}\right)\right| \\ &\stackrel{\text{by } (5.29)}{\leq} \liminf_{\varepsilon\to 0} J_\varepsilon(\rho_\varepsilon^0, v_\varepsilon^0, u_\varepsilon^0) = \liminf_{\varepsilon\to 0} \inf_{(\rho,v,u)\in\Xi_\varepsilon} J_\varepsilon(\rho, v, u) \\ &\leq \lim_{\varepsilon\to 0} J_\varepsilon(\rho^*, v^*, u^*) = \frac{1}{2}\int_\Omega |u^*(T) - u_d|^2\,dx + \frac{\lambda}{2}\int_0^T\int_\Omega |\nabla u^*|^2\,dxdt \\ &+ \frac{\gamma}{2}\int_0^T\int_\Omega |v^*|^2\,dxdt + \int_{Q_T}\left|D\left(\frac{1}{1 + |\nabla u^*|^2}\right)\right| \\ &+ \frac{1}{\varepsilon}\int_0^T\int_\Omega \left|\rho^* - \frac{1}{1 + |\nabla u^*\ |^2}\right|^2 dxdt = J(v^*, u^*).\end{aligned}$$

Thus, $J(v^0, u^0) \le J(v^*, u^*)$ and we come into a conflict with condition (5.30). Hence, the limit pair (v^0, u^0) is optimal for the original OCP $(\mathcal{R})$. □

As follows from Theorem 5.1, the optimal solutions to the approximated problems $(\rho^0_\varepsilon, v^0_\varepsilon, u^0_\varepsilon)$ can be considered as a basis for the construction of suboptimal controls to the original problem $(\mathcal{R})$ (for the details we refer to [9, 11, 12, 19])

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