JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **30**, Issue **1**, June **2022**, pp. 98–121, DOI 10.15421/142205

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

INITIAL-BOUNDARY VALUE PROBLEMS FOR ANISOTROPIC PARABOLIC EQUATIONS WITH VARIABLE EXPONENTS OF THE NONLINEARITY IN UNBOUNDED DOMAINS WITH CONDITIONS AT INFINITY

Mykola Bokalo*

Abstract. We deal with the initial-boundary value problems with some restrictions at infinity for linear and nonlinear anisotropic parabolic second-order equations in unbounded domains with respect to the spatial variables. The weak solutions of our problem in Lebesgue and Sobolev spaces with variable exponents is considered. We prove theorems on the existence and uniqueness of the weak solutions using the method based on Saint-Venant principle, and the monotonicity method. Moreover, we obtain estimate of the weak solutions.

Key words: parabolic equation, variable exponent of nonlinearity, Lebesgue space with variable exponent, Sobolev space with variable exponent, unbounded domain, Saint-Venant principle, monotonicity method.

2010 Mathematics Subject Classification: 35D30, 35K10, 35K20, 35K55.

Communicated by Prof. O. Kupenko

1. Introduction

Let n be a natural number, and $\mathbb{R}^n$ be the linear space of ordered collections $x = (x_1, ..., x_n)$ of real numbers with a norm $|x| := (|x_1|^2 + ... + |x_n|^2)^{1/2}$. Suppose that Ω is an unbounded domain in $\mathbb{R}^n$, and $\partial\Omega$ (boundary of the domain Ω) is a piecewise-smooth surface. Let $\nu = (\nu_1, ..., \nu_n)$ be a outward-pointing normal unit vector on $\partial\Omega$. Suppose $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where Γ_0 is a closure of an open set on $\partial\Omega$ (in particular, $\Gamma_0 = \emptyset$ or $\Gamma_0 = \partial\Omega$), $\Gamma_1 := \partial\Omega \setminus \Gamma_0$. Put $Q := \Omega \times (0, T)$, $\Sigma_0 := \Gamma_0 \times (0, T)$, $\Sigma_1 := \Gamma_1 \times (0, T)$, where $T > 0$. Denote by $Bd(\Omega)$ the set of all bounded subdomains of Ω.

We consider the problem: *to find the function* $u : \overline{Q} \to \mathbb{R}$ *that satisfies (in some sense) the parabolic equation*

$$u_t - \sum_{i=1}^{n} \frac{d}{dx_i} a_i(x, t, u, \nabla u) + a_0(x, t, u, \nabla u) = f(x, t), \quad (x, t) \in Q\,, \tag{1.1}$$

*Department of Mathematical Statistics and Differential Equations, Ivan Franko National University of Lviv, 1, Universytetska St., Lviv, 79000, Ukraine, mm.bokalo@gmail.com

the boundary conditions

$$u\Big|_{\Sigma_0} = 0, \qquad \frac{\partial u}{\partial \nu_a}\Big|_{\Sigma_1} = 0, \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x)\,, \quad x \in \Omega, \tag{1.3}$$

where $a_i : Q \times \mathbb{R}^{1+n} \to \mathbb{R}$, $i = \overline{0,n}$, $f : Q \to \mathbb{R}$, $u_0 : \Omega \to \mathbb{R}$ *are given real-valued functions,* $\dfrac{\partial u(x,t)}{\partial \nu_a} := \sum_{i=1}^n a_i(x,t,u,\nabla u)\,\nu_i(x)$ *is an exterior conormal derivative of* u *in point* $(x,t) \in \Sigma_1$.

Remark 1.1. An simpler example of the equations of type (1.1) considered here is

$$u_t - \sum_{i,j=1}^n (\widehat{a}_{ij}(x,t)u_{x_j})_{x_i} + \sum_{j=1}^n \widehat{a}_j(x,t)u_{x_j} + \widehat{a}_0(x,t)u = f(x,t), \quad (x,t) \in Q, \tag{1.4}$$

where $\widehat{a}_{ij} = \widehat{a}_{ji} \in L_\infty(Q)$, $i,j = \overline{1,n}$, are functions such that for a.e. $(x,t) \in Q$ we have

$$\sum_{i,j=1}^n \widehat{a}_{ij}(x,t)\eta_i\eta_j \geqslant \omega \sum_{l=1}^n |\eta_l|^2, \quad (\eta_1,...,\eta_n) \in \mathbb{R}^n, \qquad \omega = \text{const} > 0,$$

and $\widehat{a}_j \in L_\infty(Q)$, $j = \overline{0,n}$, $f : Q \to \mathbb{R}$ is such that $f \in L_2(\Omega' \times (0,T))$ for all $\Omega' \in Bd(\Omega)$.

In Remark 3.4, we have given additional conditions for the coefficients of equation (1.4), which together with those indicated here guarantee the existence and uniqueness of a weak solution of problem (1.4), (1.2), (1.3) in some class of functions, which have corresponding behavior at infinity. □

Remark 1.2. An more complex example of the equations of type (1.1) considered here is

$$u_t - \sum_{i,j=1}^k (\widehat{a}_{ij}(x,t)u_{x_j})_{x_i} - \sum_{i=k+1}^n (\widehat{a}_i(x,t)|u_{x_i}|^{p_i(x)-2}u_{x_i})_{x_i} + \widehat{a}_0(x,t)u = f(x,t), \tag{1.5}$$

$(x,t) \in Q$, where $k \in \{1,...,n-1\}$ and Ω such that $\Omega \cap \{x = (x_1,...,x_k,x_{k+1},...,x_n) \in \mathbb{R}^n \,\big|\, |x_1|^2+...+|x_k|^2 < \tau^2\}$ is bounded for each $\tau > 0$, for example, $\Omega = \Omega_1 \times \Omega_2$, Ω_1 is an unbounded domain in space $\{(x_1,...,x_k) \,\big|\, x_1,...,x_k \in \mathbb{R}\}$, and Ω_2 is a bounded domain in space $\{(x_{k+1},...,x_n) \,\big|\, x_{k+1},...,x_n \in \mathbb{R}\}$. Also we suppose that 1) $\widehat{a}_{ij} = \widehat{a}_{ji} \in L_\infty(Q)$, $i,j = \overline{1,k}$, are functions such that for a.e. $(x,t) \in Q$ a quadratic form $\sum_{i,j=1}^k \widehat{a}_{ij}(x,t)\eta_i\eta_j$, $(\eta_1,...,\eta_k) \in \mathbb{R}^k$, is positive, 2) for every $i \in$

$\{0,k+1,...,n\}$ a function $\widehat{a}_i : Q \to \mathbb{R}$ is measurable, and $0 < \operatorname{ess\,inf}_{\Omega'\times(0,T)} \widehat{a}_i \leqslant \operatorname{ess\,sup}_{\Omega'\times(0,T)} \widehat{a}_i < +\infty$ for all $\Omega' \in Bd(\Omega)$, 3) for every $i \in \{k+1,...,n\}$ a function $p_i : \Omega \to \mathbb{R}$ is measurable, and $1 < \operatorname{ess\,inf}_{\Omega'} p_i \leqslant \operatorname{ess\,sup}_{\Omega'} p_i < +\infty$ for all $\Omega' \in Bd(\Omega)$ (the functions $p_i, i = \overline{k+1,n}$, are called *exponents of the nonlinearity*).

In remark 3.5, we have given additional conditions for the coefficients of equation (1.5), which together with those indicated here guarantee the existence and uniqueness of a weak solution of problem (1.5), (1.2), (1.3) in some class of functions, which have corresponding behavior at infinity. □

Initial-boundary value problems for parabolic equations in unbounded domains with respect to the spatial variables were studied by many authors. As is well known, to guarantee the uniqueness of the solution of the initial-boundary value problems for linear parabolic equations in unbounded domains (in particular, these problems can be described by (1.4), (1.2), (1.3)) we need some restrictions on solution's behavior as $|x| \to +\infty$ (for example, solution's growth restriction as $|x| \to +\infty$, or belonging of solution to some functional spaces). Since the uniqueness of solution is the determining condition to the well-posedness of problems for evolutionary equations, then it is naturally to formulate the initial-boundary value problem for equation (1.1) in the following form: to find the solution of this equation that satisfies conditions (1.2), (1.3), and some restrictions on its behavior as $|x| \to +\infty$. Firstly this was obtained in [1]. There it was shown that the classical solution of the Cauchy problem for heat equation

$$u_t - \Delta u = 0, \quad (x,t) \in \mathbb{R}^n \times (0,T], \qquad u|_{t=0} = u_0(x),\ x \in \mathbb{R}^n, \tag{1.6}$$

is a unique in the class of the functions such that

$$|u(x,t)| \leqslant A\, e^{a|x|^2} \quad \text{for all} \quad (x,t) \in \mathbb{R}^n \times [0,T], \tag{1.7}$$

where constants a, A are depending on u, while restriction (1.7) is an essential condition for the uniqueness of the solution of the problem. Or rather, in [1], [2] was proved that problem (1.6) with $u_0 \equiv 0$ has a nontrivial solution with growth $Ae^{a|x|^{2+\varepsilon}}$ as $|x| \to +\infty$ for $\varepsilon > 0$. Remark that restriction (1.7) can be interpreted as an analog of the boundary condition at infinity. Similar results for weak solutions of linear parabolic equations from a wide class were obtained in [3], and to substantiate these results used an analogue of the principle of Saint-Venant known in mechanics. The similar situation is with nonlinear parabolic equations from certain classes (see [4–9], etc).

Note that we need some restrictions on the data-in behavior as $|x| \to +\infty$ to solvability of the initial-boundary value problems for parabolic equations considered above. In particular, in the paper [1] it was shown that a classical solution of a problem (1.6), (1.7) exists if u_0 satisfies the condition: $|u_0(x)| \leqslant B\, e^{b|x|^2}$ for all $x \in \mathbb{R}^n$, where b, B are any constants.

However, there are nonlinear parabolic equations for which the corresponding initial-boundary value problems are unique solvable without any conditions at infinity. First result was proved in [10] for equation (1.5) with $p_0 = \text{const} > 2$, and $p_{k+1} = \ldots = p_n = 2$. Similar results were obtained for nonlinear parabolic equations in [10–20], etc.

Nonlinear differential equations with variable exponents of the nonlinearity (for example, equation (1.5)) appear as mathematical models in various physical processes. In particular, these equations describe electroreological substance flows, image recovering processes, electric current in the conductor with changing temperature field (see [21]). Nonlinear differential equations with variable exponents of the nonlinearity were intensively studied in [22–29], etc. The corresponding generalizations of Lebesgue and Sobolev spaces (see [30]) were used in these investigations.

In this work we consider a class of second order parabolic equations in unbounded domains with respect to the spatial variables, which require for the correct formulation of the initial-boundary value problems of setting conditions for the behavior of the solution at infinity. This class contains both linear (see, for example, (1.4)) and nonlinear equations with variable exponents of the nonlinearity (see, for example, (1.5))). Here we complement and generalize results for linear (see, for example, [3]), and nonlinear parabolic equations with constant exponents of the nonlinearity (see, for example, [6]). As we know from the available sources, nonlinear parabolic equations with variable exponents of the nonlinearity were not previously investigated in the context of the problem under consideration. In our researches, we use an analog of the well-known in mechanics Saint-Venant principle. It was developed in [3, 6, 31, 32], and others. Moreover, to prove the solvability of our problem we use the method of exhaustion for unbounded domains, and the monotonicity method [33].

The article is organized as follows. In Section 2, we describe functional spaces which are used in the sequel. In Section 3, we set the researched problem and formulate the main results. Section 4 contains auxiliary statements that are used in the next section. Finally, Section 5 is devoted to substantiation of the main results.

2. Main notation

Firstly, we introduce some functional spaces. Let $r : \Omega \to \mathbb{R}$ be a measurable function, $r(x) \geqslant 1$ for almost every (a.e.) $x \in \Omega$, and $\operatorname{ess\,sup}_{x\in\Omega'} r(x) < \infty$ for any $\Omega' \in Bd(\Omega)$. For any $\Omega' \in Bd(\Omega)$ we denote by $L_{r(\cdot)}(\Omega')$ the linear space of (classes of) measurable functions $v : \Omega' \to \mathbb{R}$ such that $\rho_{\Omega',r}(v) := \int_{\Omega'} |v(x)|^{r(x)}\, dx < \infty$. This is the Banach space with a norm

$$\|v\|_{L_{r(\cdot)}(\Omega')} := \inf\{\lambda > 0 \;\mid\; \rho_{\Omega',r}(v/\lambda) \leqslant 1\}.$$

Space $L_{r(\cdot)}(\Omega')$ is called *the Lebesgue space with variable exponent* or *generalized Lebesgue space* (see, for example, [30]). If $r(x) > 1$ for a.e. $x \in \Omega$, put by definition $r'(x) := r(x)/(r(x)-1)$ for a.e. $x \in \Omega$. As is well known, the dual space $(L_{r(\cdot)}(\Omega'))'$ can be identified with $L_{r'(\cdot)}(\Omega')$ under the condition $\operatorname{ess\,inf}_{x\in\Omega'} r(x) > 1$. Note also that in the case $r(x) = r = \text{const} \geqslant 1$ for a.e. $x \in \Omega' \in Bd(\Omega)$ we have $L_{r(\cdot)}(\Omega') = L_r(\Omega')$, and $\|\cdot\|_{L_{r(\cdot)}(\Omega')} = \|\cdot\|_{L_r(\Omega')}$.

Denote by $L_{r(\cdot),\,\mathrm{loc}}(\overline{\Omega})$ the linear space of (classes of) measurable functions $v:\Omega\to\mathbb{R}$ such that their restrictions $v|_{\Omega'}$ belong to the space $L_{r(\cdot)}(\Omega')$ for any set $\Omega'\in Bd(\Omega)$. This space with a family of seminorms $\{\|\cdot\|_{L_{r(\cdot)}(\Omega')}\,|\,\Omega'\in Bd(\Omega)\}$ is complete locally convex. Then a sequence $\{v_l\}_{l=1}^{\infty}$ converges to v in $L_{r(\cdot),\,\mathrm{loc}}(\overline{\Omega})$ *strongly* (correspondly, *weakly*), if for any domain $\Omega'\in Bd(\Omega)$ the sequence $\{v_l|_{\Omega'}\}_{l=1}^{\infty}$ converges to $v|_{\Omega'}$ in $L_{r(\cdot)}(\Omega')$ *strongly* (correspondly, *weakly*). As above, we introduce the space $L_{r(\cdot)}(Q')$, where $Q'=\Omega'\times(0,T)$, $\Omega'\in Bd(\Omega)$, by using the functional $\rho_{Q',r}(w) := \iint_{Q'} |w(x,t)|^{r(x)}\,dxdt$ instead of $\rho_{\Omega',r}(v)$. Then we define a complete locally convex space $L_{r(\cdot),\,\mathrm{loc}}(\overline{Q})$ along with a family of seminorms $\{\|\cdot\|_{L_{r(\cdot)}(\Omega'\times(0,T))}\mid \Omega'\in Bd(\Omega)\}$.

Let the following condition holds:

$(\mathbf{P})$ $p=(p_0,p_1,\dots,p_n):\Omega\to\mathbb{R}^{1+n}$ *is a vector-valued function such that for every* $i\in\{0,1,\dots,n\}$ *the function* $p_i:\Omega\to\mathbb{R}$ *is measurable, and for any* $\Omega'\in Bd(\Omega)$ *we have* $1<\operatorname{ess\,inf}_{\Omega'} p_i\leqslant \operatorname{ess\,sup}_{\Omega'} p_i<+\infty$.

Let $p'=(p_0',p_1',\dots,p_n')$ be the vector-valued function such that $\frac{1}{p_i(x)}+\frac{1}{p_i'(x)}=1$ for a.e. $x\in\Omega$, $i=\overline{0,n}$. Obviously, the function p' satisfies condition $(\mathbf{P})$ with p_i' instead of p_i, $i=\overline{0,n}$.

For any domain $\Omega'\in Bd(\Omega)$ we define the space

$$W^1_{p(\cdot)}(\Omega') := \{v\in L_{p_0(\cdot)}(\Omega')\mid v_{x_i}\in L_{p_i(\cdot)}(\Omega'),\ i=\overline{1,n}\}.$$

This is the Banach space with the norm

$$\|v\|_{W^1_{p(\cdot)}(\Omega')} := \|v\|_{L_{p_0(\cdot)}(\Omega')} + \sum_{i=1}^{n}\|v_{x_i}\|_{L_{p_i(\cdot)}(\Omega')}.$$

Space $W^1_{p(\cdot)}(\Omega')$ is called the *Sobolev space with variable exponent* or *generalized Sobolev space* (see, for example, [30]). Denote by $W^1_{p(\cdot),\,\mathrm{loc}}(\overline{\Omega})$ the complete locally convex space of (classes of) functions $v\in L_{p_0(\cdot),\,\mathrm{loc}}(\overline{\Omega})$ such that $v_{x_i}\in L_{p_i(\cdot),\,\mathrm{loc}}(\overline{\Omega})$, $i=\overline{1,n}$, along with a family of seminorms $\big\{\|v\|_{W^1_{p(\cdot)}(\Omega')}\mid \Omega'\in Bd(\Omega)\big\}$. Let $\widetilde{W}^1_{p(\cdot),\,\mathrm{loc}}(\overline{\Omega})$ be the closure of the set $\widetilde{C}^1(\overline{\Omega}) := \{v\in C^1(\overline{\Omega})\mid v|_{\Gamma_0}=0\}$ in space $W^1_{p(\cdot),\,\mathrm{loc}}(\overline{\Omega})$. By $\widetilde{W}^1_{p(\cdot),\mathrm{c}}(\Omega)$ we denote a subspace of $\widetilde{W}^1_{p(\cdot),\,\mathrm{loc}}(\overline{\Omega})$ consisting of functions with bounded supports.

For the domain $Q'=\Omega'\times(0,T)$, where $\Omega'\in Bd(\Omega)$, we put

$$W^{1,0}_{p(\cdot)}(Q') := \{w\in L_{p_0(\cdot)}(Q')\ \mid\ w_{x_i}\in L_{p_i(\cdot)}(Q'),\ \ i=\overline{1,n}\}.$$

This is the Banach space with the norm

$$\|w\|_{W^{1,0}_{p(\cdot)}(Q')} := \|w\|_{L_{p_0(\cdot)}(Q')} + \sum_{i=1}^{n}\|w_{x_i}\|_{L_{p_i(\cdot)}(Q')}.$$

Denote by $W^{1,0}_{p(\cdot),\,\mathrm{loc}}(\overline{Q})$ the complete locally convex space of (classes of) functions $w \in L_{p_0(\cdot),\,\mathrm{loc}}(\overline{Q})$ such that $w_{x_i} \in L_{p_i(\cdot),\,\mathrm{loc}}(\overline{Q})$, $i = \overline{1,n}$, along with a family of seminorms $\big\{\|w\|_{W^{1,0}_{p(\cdot)}(\Omega'\times(0,T))}\big|\ \Omega' \in Bd(\Omega)\big\}$. By $\widetilde{W}^{1,0}_{p(\cdot),\,\mathrm{loc}}(\overline{Q})$ we denote a subspace of functions $w \in W^{1,0}_{p(\cdot),\,\mathrm{loc}}(\overline{Q})$ such that $w(\cdot,t)$ belongs to $\widetilde{W}^{1}_{p(\cdot),\,\mathrm{loc}}(\overline{\Omega})$ for a.e. $t \in (0,T)$.

By definition, put

$$C([0,T];L_{2,\mathrm{loc}}(\overline{\Omega})) := \{w : [0,T] \to L_{2,\mathrm{loc}}(\overline{\Omega}) \;\big|\; w \in C([0,T];L_2(\Omega'))\ \forall\,\Omega' \in Bd(\Omega)\}.$$

This space with the family of seminorms

$$\{\|w\|_{C([0,T];L_2(\Omega'))} := \max_{t\in[0,T]} \|w(\cdot,t)\|_{L_2(\Omega')}\ |\Omega' \in Bd(\Omega)\}$$

is complete locally convex.

Denote by

$$\mathbb{U}_{p,\mathrm{loc}}(\overline{Q}) := \widetilde{W}^{1,0}_{p(\cdot),\mathrm{loc}}(\overline{Q}) \cap C([0,T];L_{2,\mathrm{loc}}(\overline{\Omega})).$$

This space is complete locally convex along with a family of seminorms $\{\|w\|_{W^{1,0}_{p(\cdot)}(\Omega'\times(0,T))} + \|w\|_{C([0,T];L_2(\Omega'))}\ |\ \Omega' \in Bd(\Omega)\}$.

Finally, let $C^1_c(0,T) \subset C^1(0,T)$ be a set of functions with compact supports on $(0,T)$.

3. Statement of the problem and formulation of main results

We will consider weak solutions of the problem (1.1) – (1.3). To define them, we introduce corresponding data-in classes.

Let $p = (p_0, p_1, \ldots, p_n)$ be a vector-valued function that satisfies condition $(\mathbf{P})$. By $\mathbb{A}_p$ we denote all ordered collections $(a_0, a_1, \ldots, a_n)$ of the real functions satisfying the following conditions:

$(\mathbf{A}_1)$ for every $i \in \{0,1,\ldots,n\}$, function $a_i(x,t,\rho,\xi)$, $(x,t,\rho,\xi) \in Q \times \mathbb{R}^{1+n}$, is a Carathéodory, i.e., function $a_i(x,t,\cdot,\cdot) : \mathbb{R}^{1+n} \to \mathbb{R}$ is a continuous for a.e. $(x,t) \in Q$, and function $a_i(\cdot,\cdot,\rho,\xi) : Q \to \mathbb{R}$ is a measurable for every $(\rho,\xi) \in \mathbb{R}^{1+n}$; in addition, $a_i(x,t,0,0) = 0$ for a.e. $(x,t) \in Q, i = \overline{0,n}$;

$(\mathbf{A}_2)$ for every $i \in \{0,1,\ldots,n\}$, for a.e. $(x,t) \in Q$, and for every $(\rho,\xi) \in \mathbb{R}^{1+n}$ the following inequality holds

$$|a_i(x,t,\rho,\xi)| \leqslant h_{i,1}(x,t)\big(|\rho|^{p_0(x)/p_i'(x)} + \sum_{j=1}^{n} |\xi_j|^{p_j(x)/p_i'(x)}\big) + h_{i,2}(x,t),$$

where $h_{i,1} \in L_{\infty,\,\mathrm{loc}}(\overline{Q})$, $h_{i,2} \in L_{p'_i(\cdot),\,\mathrm{loc}}(\overline{Q})$.

Now we give a definition of a weak solution of problem (1.1) – (1.3). We assume that p satisfies condition (**P**), $(a_0, a_1, ..., a_n) \in \mathbb{A}_p$, $f \in L_{2,\mathrm{loc}}(\overline{Q})$, $u_0 \in L_{2,\mathrm{loc}}(\overline{\Omega})$.

Definition 3.1. A weak solution of problem (1.1) — (1.3) is called a function $u \in \mathbb{U}_{p,\mathrm{loc}}(\overline{Q})$ that satisfies (in the sense of space $C([0,T]; L_{2,\mathrm{loc}}(\overline{\Omega}))$) the initial condition

$$u(\cdot, 0) = u_0(\cdot) \quad \text{a.e. on} \quad \Omega, \tag{3.1}$$

and the integral identity

$$\iint_Q \big[-u\psi\varphi' + \sum_{i=1}^{n} a_i(x,t,u,\nabla u)\psi_{x_i}\varphi + a_0(x,t,u,\nabla u)\psi\varphi\big]\, dxdt$$
$$= \iint_Q f\psi\varphi\, dxdt \quad \forall\, \psi \in \widetilde{W}^1_{p(\cdot),c}(\Omega),\ \forall\, \varphi \in C^1_c(0,T). \tag{3.2}$$

Suppose $0 \in \Omega$. Let $k \in \{1, \dots, n\}$ be a number such that for any $\tau > 0$ the set $\widetilde{\Omega}_\tau := \Omega \cap \{x \in \mathbb{R}^n \mid |x_1|^2 + \ldots + |x_k|^2 < \tau^2\}$ is bounded. For any $\tau > 0$ we denote by Ω_τ a connected component of the set $\widetilde{\Omega}_\tau$ that contains 0. For any $\tau > 0$ put $Q_\tau := \Omega_\tau \times (0,T)$. Obviously, $\Omega = \bigcup_{\tau>0} \Omega_\tau$, $Q = \bigcup_{\tau>0} Q_\tau$.

The choice of value k depends on the geometry of the domain Ω (up to the numbering of variables $x_1, ..., x_n$). Obviously, in the general case we can take $k = n$, and, in this case, the class of equations considered below will consist of generalizations of equation (1.4), or rather, of almost linear equations. But in the case of $k < n$ the class of equations to which the following results apply is wider than in the case of $k = n$, and the smaller the value of k the wider the class of these equations (to confirm this, see (1.5)).

Let us illustrate possibilities of the value's k considered two examples.

Example 3.1. Assume $\Omega = \Omega_1 \times \Omega_2$, where Ω_1 is an unbounded domain in $\mathbb{R}^l := \{(x_1, \dots, x_l) \mid x_i \in \mathbb{R},\ i = \overline{1,l}\}$ for some $l \in \{1, \dots, n-1\}$, Ω_2 is a bounded domain in $\mathbb{R}^{n-l} := \{(x_{l+1}, \dots, x_n) \mid x_i \in \mathbb{R},\ i = \overline{l+1,n}\}$, and $0 \in \Omega$. Then we can take arbitrary $k \in \{l, \dots, n\}$. If $k = l$, then $\Omega_\tau = \Omega_{1,\tau} \times \Omega_2$ for any $\tau > 0$, where $\Omega_{1,\tau}$ is a connected component of the set $\Omega_1 \cap \{(x_1, \dots, x_l) \in \mathbb{R}^l \mid |x_1|^2 + \ldots + |x_l|^2 < \tau^2\}$ such that $0 \in \Omega_{1,R}$. □

Example 3.2. Suppose

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid -\infty < x_1 < +\infty, \quad -\phi_1(x_1) < x_2 < \phi_2(x_1)\},$$

where for each $m \in \{1, 2\}$ a function ϕ_m is continuous on $\mathbb{R}$, and $\phi_m(s) > 0$ for all $s \in \mathbb{R}$. Then we can take either $k = 1$ or $k = 2$. In case $k = 1$, we have $\Omega_\tau = \big\{(x_1, x_2) \in \mathbb{R}^2 \,\big|\, |x_1| < \tau, \quad -\phi_1(x_1) < x_2 < \phi_2(x_1)\big\}$ for any $\tau > 0$. If $k = 2$, then

$$\Omega_\tau = \big\{(x_1, x_2) \in \mathbb{R}^2 \,\big|\, |x_1| < \tau,$$
$$-\min\{\phi_1(x_1), \sqrt{\tau^2 - |x_1|^2}\} < x_2 < \min\{\phi_2(x_1), \sqrt{\tau^2 - |x_1|^2}\big\}$$

for any $\tau > 0$. □

By definition, put

$$\Gamma_{j,\tau} := \Gamma_j \cap \partial\Omega_\tau,\ j = 0,1, \quad \Gamma_{*,\tau} := \Omega \cap \partial\Omega_\tau,$$

$$\Sigma_{j,\tau} := \Gamma_{j,\tau} \times (0,T),\ j = 0,1, \quad \Sigma_{*,\tau} := \Gamma_{*,\tau} \times (0,T), \quad \tau > 0.$$

We will use a notation

$$\nabla_k v := (v_{x_1}, \ldots, v_{x_k}), \quad |\nabla_k v| := (|v_{x_1}|^2 + \ldots + |v_{x_k}|^2)^{1/2}.$$

Everywhere further we will consider that is carried out the following condition:

($\mathbf{P}^*$) *$p = (p_0, p_1, \ldots, p_n) : \Omega \to \mathbb{R}^{1+n}$ satisfy condition* ($\mathbf{P}$)*, and $p_0(x) = p_1(x) = \ldots = p_k(x) = 2$ for a.e. $x \in \Omega$.*

Suppose $\mathbb{A}_p^*$ is a subset of $\mathbb{A}_p$, which every element satisfies conditions ($\mathbf{A}_1$), ($\mathbf{A}_2$), and the following condition:

($\mathbf{A}_3$) for a.e. $(x,t) \in Q$ and for every (ρ_1, ξ^1), $(\rho_2, \xi^2) \in \mathbb{R}^{1+n}$, we have

$$\sum_{i=1}^{k} |a_i(x,t,\rho_1,\xi^1) - a_i(x,t,\rho_2,\xi^2)| \leqslant g_1(x,t)|\xi'^1 - \xi'^2| + g_2(x,t)|\rho_1 - \rho_2|, \tag{3.3}$$

$$\sum_{i=1}^{n} (a_i(x,t,\rho_1,\xi^1) - a_i(x,t,\rho_2,\xi^2))(\xi_i^1 - \xi_i^2) + (a_0(x,t,\rho_1,\xi^1) - a_0(x,t,\rho_2,\xi^2))(\rho_1 - \rho_2) \geqslant q_1(x,t)|\xi'^1 - \xi'^2|^2 + q_2(x,t)|\rho_1 - \rho_2|^2, \tag{3.4}$$

where $\xi'^j := (\xi_1^j, \ldots, \xi_k^j)$, $|\xi'^j| := (|\xi_1^j|^2 + \ldots + |\xi_k^j|^2)^{1/2}$, $j \in \{1,2\}$, and g_1, g_2, q_1, $q_2 : \overline{Q} \to \mathbb{R}$ are continuous functions on $\overline{Q}$ that satisfy the following conditions:

• $g_1(x,t) > 0$, $g_2(x,t) \geqslant 0$, $q_1(x,t) > 0$ for all $(x,t) \in \overline{Q}$, $\inf_{\overline{Q}} q_2 > -\infty$;

• there exist a real number μ, and continuous functions d_1, d_2, λ defined on $[1,+\infty)$ such that

$$q_2(x,t) + \mu > 0 \quad \text{for all } (x,t) \in \overline{Q}, \tag{3.5}$$

$$\text{for all } \ \tau \geqslant 1: \quad d_1(\tau) \geqslant \max_{\Sigma_{*,\tau}} \frac{g_1}{\sqrt{q_1}}, \quad d_2(\tau) \geqslant \max_{\Sigma_{*,\tau}} g_2, \tag{3.6}$$

$$\text{for all } \ \tau \geqslant 1: \quad -\mu < \lambda(\tau) \leqslant \inf_{t,v} \frac{\displaystyle\int_{\Gamma_{*,\tau}} \left[q_1|\nabla_k v|^2 + q_2|v|^2\right] d\Gamma}{\displaystyle\int_{\Gamma_{*,\tau}} |v|^2\, d\Gamma}, \tag{3.7}$$

where the infimum is taken over all numbers $t \in [0,T]$, and all functions v that are continuously differentiable in the neighborhood of $\overline{\Gamma_{*,\tau}}$, and $v = 0$

on $\partial\Gamma_{*,\tau} \cap \Gamma_0$ (in particular, $-\mu < \lambda(\tau) \leqslant \min_{\overline{\Sigma_{*,\tau}}} q_2$),
while

$$\int_1^{+\infty} \frac{d\tau}{A_\mu(\tau)} = +\infty, \tag{3.8}$$

where

$$A_\mu(\tau) := \frac{d_1(\tau)}{\sqrt{\lambda(\tau)+\mu}} + \frac{d_2(\tau)}{\lambda(\tau)+\mu}, \quad \tau \geqslant 1. \tag{3.9}$$

Remark 3.1. If $\sup_{\overline{Q}} \dfrac{g_1}{\sqrt{q_1}} < +\infty$, $\sup_{\overline{Q}} g_2 < +\infty$, then functions d_1, d_2, λ can be chosen as constants. Namely, $d_1(\tau) := d_{1,0}$, $d_2(\tau) := d_{2,0}$, $\lambda(\tau) := \lambda_0$ for all $\tau \geqslant 1$, where $d_{1,0}$, $d_{2,0}$, λ_0 are constants such that

$$d_{1,0} \geqslant \sup_{\overline{Q}} \frac{g_1}{\sqrt{q_1}}, \quad d_{2,0} \geqslant \sup_{\overline{Q}} g_2, \quad \lambda_0 \leqslant \inf_{\overline{Q}} q_2.$$

Then we can take μ such that $\lambda_0 > -\mu$, and

$$A_\mu(\tau) = A_{\mu,0} := \frac{d_{1,0}}{\sqrt{\lambda_0+\mu}} + \frac{d_{2,0}}{\lambda_0+\mu} \quad \text{for all} \quad \tau \geqslant 1.$$

□

Suppose $\mathbb{A}_p^{**}$, in the case of $k < n$, is a subset of $\mathbb{A}_p^*$, which arbitrary element satisfies the following condition:

($\mathbf{A}_4$) for a.e. $(x,t) \in Q$ and for every $(\rho,\xi) \in \mathbb{R}^{1+n}$, we have

$$\sum_{i=0}^{n} a_i(x,t,\rho,\xi)\xi_i + a_0(x,t,\rho,\xi)\rho \geqslant q_3(x,t) \sum_{i=k+1}^{n} |\xi_i|^{p_i(x)} - q_4(x,t)|\rho|^2 - h(x,t), \tag{3.10}$$

where $q_3, q_4 \in C(\overline{Q})$, $q_3(x,t) > 0$ for all $(x,t) \in \overline{Q}$, $0 \leqslant \sup_{\overline{Q}} q_4 < +\infty$, $h \in L_{1,\mathrm{loc}}(\overline{Q})$, $h \geqslant 0$ a.e. on Q.

In the case of $k = n$ we will assume that $\mathbb{A}_p^{**} := \mathbb{A}_p^*$.

It is easy to prove that the initial problem

$$\frac{d\tau}{d\alpha} = A_\mu(\tau), \quad \tau(0) = 1 \tag{3.11}$$

has a unique solution $\tau(\alpha)$, $\alpha \in [0,+\infty)$, and this solution is determined by the equality

$$\int_1^{\tau(\alpha)} \frac{ds}{A_\mu(s)} = \alpha, \quad \alpha \geqslant 0. \tag{3.12}$$

From this and (3.8) it follows that

$$\tau(\alpha) \to +\infty \quad \text{as} \quad \alpha \to +\infty. \tag{3.13}$$

Suppose $\tau(\alpha)$, $\alpha \in [0,+\infty)$, is a solution of problem (3.11), and put

$$\Omega^\alpha := \Omega_{\tau(\alpha)}, \quad \Gamma_j^\alpha := \Gamma_{j,\tau(\alpha)},\ j=0,1, \quad \Gamma_*^\alpha := \Gamma_{*,\tau(\alpha)},$$
$$Q^\alpha := Q_{\tau(\alpha)}, \quad \Sigma_j^\alpha := \Sigma_{j,\tau(\alpha)},\ j=0,1, \quad \Sigma_*^\alpha := \Sigma_{*,\tau(\alpha)}.$$

Note that in view of (3.13) we have $\Omega = \bigcup_{\alpha>0}\Omega^\alpha,\ Q = \bigcup_{\alpha>0} Q^\alpha$.

Let $\{\Lambda_m\}_{m=1}^\infty$ be a sequence of real numbers such that for all $m\in\mathbb{N}$ we have

$$-\mu < \Lambda_m \leqslant \inf_{t,v} \frac{\displaystyle\int_{\Omega^m} \left[q_1|\nabla_k v|^2 + q_2|v|^2\right] dx}{\displaystyle\int_{\Omega^m} |v|^2\, dx}, \tag{3.14}$$

where the infimum is taken over all numbers $t\in[0,T]$, and functions $v \in C^1(\overline{\Omega^m})$ such that $v=0$ on $\partial\Omega^m \setminus \Gamma_1^m$ (in particular, $-\mu < \Lambda_m \leqslant \min_{\overline{Q^m}} q_2$).

Denote

$$\begin{aligned} E_{k,\mu}(w) &:= q_1|\nabla_k w|^2 + (q_2+\mu)|w|^2,\\ \langle w\rangle_\alpha &:= \left(\iint_{Q^\alpha} E_{k,\mu}(w) e^{-2\mu t}\, dxdt\right)^{1/2}, \quad \alpha \geqslant 0. \end{aligned} \tag{3.15}$$

Now we formulate our main results.

Theorem 3.1 (a uniqueness of the solution)**.** *Let p satisfies condition* ($\mathbf{P}^*$)*, $f\in L_{2,\mathrm{loc}}(\overline{Q})$, $u_0 \in L_{2,\mathrm{loc}}(\overline{\Omega})$, $(a_0,a_1,\dots,a_n)\in\mathbb{A}_p^*$. Then problem (1.1) – (1.3) has at most one weak solution such that*

$$e^{-R/2}\,\langle u\rangle_R \to 0 \quad \text{as} \quad R\to+\infty \tag{3.16}$$

(an analog of the boundary condition at infinity), where $\langle\cdot\rangle_R$ defined in (3.15)*.*

Remark 3.2. Assertion (3.16) is equivalent to the condition

$$e^{-\int_1^r \frac{ds}{A_\mu(s)}} \iint_{Q_r} \left[q_1|\nabla_k u|^2 + (q_2+\mu)|u|^2\right] dxdt \to 0 \quad \text{as} \quad r\to+\infty. \tag{3.17}$$

It follows from (3.12), if to remark that $Q^R = Q_r$, if $R = \int_1^r \frac{ds}{A_\mu(s)}$. □

Theorem 3.2 (an existence of the solution)**.** *Let p satisfies condition* ($\mathbf{P}^*$)*, $f\in L_{2,\mathrm{loc}}(\overline{Q})$, $u_0\in L_{2,\mathrm{loc}}(\overline{\Omega})$, $(a_0,a_1,\dots,a_n)\in\mathbb{A}_p^{**}$. Also suppose for some number $\varkappa\in(0,1)$ the following inequality holds*

$$(\Lambda_m+\mu)^{-1}\iint_{Q^m} |f|^2 e^{-2\mu t}\, dxdt + \int_{\Omega^m} |u_0|^2\, dx \leqslant C_1\, e^{(1-\varkappa)m} \quad \forall\, m\in\mathbb{N}, \tag{3.18}$$

where $C_1>0$ is a constant.

Then there exists a weak solution of problem (1.1) – (1.3) satisfying condition (3.16). Moreover, for this solution the following estimate is fulfilled:

$$\langle u\rangle_m \leqslant C_2\, e^{(1-\varkappa)m/2} \quad \forall\, m\in\mathbb{N}, \tag{3.19}$$

where $C_2 := [(2+e^{1/2}-e^{-\varkappa/2})/(1-e^{-\varkappa/2})]\sqrt{C_1}$, $\langle\cdot\rangle_m$ defined in (3.15)*.*

Remark 3.3. Estimate (3.19) is equivalent to the estimate

$$\iint_{Q_r} \big[q_1|\nabla_k u|^2 + (q_2+\mu)|u|^2\big] e^{-2\mu t}\, dxdt \leqslant C_3\, e^{(1-\varkappa)\int_1^r \frac{ds}{A_\mu(s)}} \quad \forall\, r \geqslant 1, \quad (3.20)$$

where $C_3 > 0$ is a constant depending only on $\varkappa$ and C_1.

The statement is substantiated in the same way as (3.17). □

Remark 3.4. For equation (1.4) the conditions of Theorems 1 and 2 are satisfied if functions $\widehat{a}_{ij}$, $i,j=\overline{1,n}$, $\widehat{a}_i$, $i=\overline{0,n}$, are as in Remark 1.1, and for a.e. $(x,t)\in Q$ following hold

$$g_1(x,t) \geqslant \sum_{i=1}^n \Big(\sum_{j=1}^n |\widehat{a}_{ij}(x,t)|^2\Big)^{1/2}, \quad g_2(x,t)=0,$$

$$q_1(x,t)=\omega/2, \quad q_2(x,t) \leqslant \Big(\widehat{a}_0(x,t) - \frac{1}{2\omega}\sum_{i=1}^n |\widehat{a}_i(x,t)|^2\Big),$$

where g_1, g_2, q_1, q_2 are as in $(\mathbf{A}_3)$ with $\mu=0$, and f, u_0 satisfy (3.18). □

Remark 3.5. For equation (1.5) the conditions of Theorems 1 and 2 are satisfied if functions $\widehat{a}_{ij}$, $i,j=\overline{1,k}$, $\widehat{a}_i$, $i=\overline{k+1,n}$, $\widehat{a}_0$ are as in Remark 1.2, and for a.e. $(x,t)\in Q$ following inequalities hold

$$\sqrt{k}\sum_{i=1}^k \max_{j\in\{1,\dots,k\}} \big|\widehat{a}_{ij}(x,t)\big| \leqslant g_1(x,t),$$

$$\sum_{i,j=1}^k \widehat{a}_{ij}(x,t)\eta_i\eta_j \geqslant q_1(x,t)\sum_{i=1}^k |\eta_i|^2 \quad \forall\,(\eta_1,\dots,\eta_k)\in\mathbb{R}^k,$$

$$\widehat{a}_0(x,t) \geqslant q_2(x,t), \qquad \min_{i\in\{k+1,\dots,n\}} \widehat{a}_i(x,t) \geqslant q_3(x,t),$$

where g_1, q_1, q_2, q_3 are as in $(\mathbf{A}_3)$, $(\mathbf{A}_4)$ together with $g_2=0$, $q_4=0$, $\mu=0$, and f, u_0 satisfy (3.18). □

4. Auxiliary statements

Here we give some auxiliary results which will be used in Section 5. We denote

$$a_i(v)(x,t) := a_i(x,t,v(x,t),\nabla v(x,t)), \quad (x,t)\in Q, \quad i=\overline{0,n}, \qquad (4.1)$$

$$\partial_0 v = v, \quad \partial_i v = \partial_i v, \quad i=\overline{1,n}. \qquad (4.2)$$

Recall that $\mathrm{Lip}(\overline{\Omega})$ is the linear space of Lipschitz continuous functions on $\overline{\Omega}$.

Lemma 4.1 (Lemma 1, [24]). *Suppose p satisfies condition* $(\mathbf{P})$, $R > 0$ *is an arbitrary fixed number, and a function* $w \in \widetilde{W}^{1,0}_{p(\cdot),\mathrm{loc}}(\overline{Q})$ *satisfies the integral identity*

$$\iint_{Q_R} \left[-w\psi\varphi' + \sum_{i=0}^{n} g_i \partial_i \psi\varphi \right] dxdt = 0 \\ \forall\, \psi \in \widetilde{W}^{1}_{p(\cdot),\mathrm{c}}(\Omega),\ supp\,\psi \subset \overline{\Omega_R},\ \forall\, \varphi \in C^1_c(0,T), \tag{4.3}$$

where $g_i \in L_{p_i'(\cdot),\mathrm{loc}}(\overline{Q})$, $i = \overline{0,n}$, *are given functions.*

Then for arbitrary function $\zeta \in \mathrm{Lip}(\overline{\Omega})$, $supp\,\zeta \subset \overline{\Omega_R}$, $\zeta \geqslant 0$ *we have* $\sqrt{\zeta} w \in C([0,T]; L_2(\Omega_R))$ *(hence,* $w \in C([0,T]; L_2(\Omega_{R'}))$ *for every* $R' \in (0,R)$*). Moreover, for arbitrary functions* $\theta \in C^1([0,T])$*, and for any numbers* $t_1, t_2 \in [0,T]$ $(t_1 < t_2)$ *the following equality holds*

$$\frac{1}{2}\big[\theta(t) \int_{\Omega_R} |w(x,t)|^2 \zeta(x)\, dx\big]\Big|_{t=t_1}^{t=t_2} - \frac{1}{2}\int_{t_1}^{t_2}\int_{\Omega_R} |w|^2 \zeta\, \theta'\, dxdt \\ + \int_{t_1}^{t_2}\int_{\Omega_R} \big[\sum_{i=0}^{n} g_i \partial_i (w\zeta)\big] \theta\, dxdt = 0. \tag{4.4}$$

If, in addition, it is known that $w|_{\Gamma_{*,R}\times(0,T)} = 0$*, then* $w \in C([0,T]; L_2(\Omega_R))$*, and we can take* $\zeta(x) = 1$, $x \in \overline{\Omega}$*, in (4.4).*

Lemma 4.2 (an analog of Saint-Venant principle). *Assume p satisfies condition* $(\mathbf{P}^*)$, $(a_0, a_1, \dots, a_n) \in \mathbb{A}^*_p$, $f \in L_{2,\mathrm{loc}}(\overline{Q})$, $u_0 \in L_{2,\mathrm{loc}}(\overline{\Omega})$. *Suppose* $R > 0$ *is an arbitrary number, and* $u_1, u_2 \in \mathbb{U}_{p,\mathrm{loc}}(\overline{Q})$ *such that for each* $l \in \{1,2\}$ *we have*

$$u_l(\cdot,0) = u_0(\cdot) \quad a.e.\ on\ \ \Omega^R, \tag{4.5}$$

and

$$\iint_{Q^R} \big[-u_l\psi\varphi' + \sum_{i=0}^{n} a_i(u_l)\partial_i\psi\varphi\big]\, dxdt = \iint_{Q^R} f\psi\varphi\, dxdt, \\ \forall\, \psi \in \widetilde{W}^{1}_{p(\cdot),c}(\Omega),\ supp\,\psi \subset \overline{\Omega^R},\ \ \forall\, \varphi \in C^1_c(0,T). \tag{4.6}$$

Then for every R_1, R_2, $0 < R_1 < R_2 \leqslant R$*, the following inequality holds*

$$\langle u_1 - u_2\rangle_{R_1} \leqslant e^{(R_1-R_2)/2}\, \langle u_1 - u_2\rangle_{R_2}. \tag{4.7}$$

Remark 4.1. The inequality of type (4.7) has been obtained in [3] for weak solutions from $W^{1,1}_{2,\mathrm{loc}}$ to linear parabolic equations, and in [6], [31], [32] and other works for weak solutions from $W^{1,0}_{2,\mathrm{loc}}$ to quasilinear parabolic equations with constant nonlinearty exponents. This inequality is an analog of the well-known in elasticity theory Saint-Venant principle. □

The proof of Lemma 2. For an arbitrary $x \in \mathbb{R}^n$ we set $x = (x', x'')$, where $x' = (x_1, ..., x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, ..., x_n) \in \mathbb{R}^{n-k}$. Let $|x'| = (|x_1|^2 + \ldots + |x_k|^2)^{1/2}$. For any $\delta \in (0, 1)$, $\tau \in [1, +\infty)$, $x' \in \mathbb{R}^k$ we denote

$$\psi_\delta(x', \tau) := \begin{cases} 1, & \text{if } |x'| \leqslant \tau - \delta, \\ (\tau - |x'|)/\delta, & \text{if } \tau - \delta < |x'| < \tau, \\ 0, & \text{if } |x'| \geqslant \tau. \end{cases}$$

Obviously, for every $i \in \{1, \ldots, k\}$ we have $\partial_i \psi_\delta(x', \tau) := 0$ if $|x'| < \tau - \delta$ or $|x'| > \tau$, and

$$\partial_i \psi_\delta(x', \tau) = -\frac{x_i}{\delta |x'|} \quad \text{if } \ \tau - \delta < |x'| < \tau. \tag{4.8}$$

By definition, put $w := u_1 - u_2$. Let $\delta \in (0, 1)$, $\tau \in (1, \tau(R))$ be arbitrary fixed. We subtract the integral identity (4.6) for $l = 2$ from this identity for $l = 1$. Applying Lemma 1 to their difference with $t_1 = 0$, $t_2 = T$, $\theta(t) := e^{-2\mu t}$, $t \in \mathbb{R}$, $\zeta(x) := \psi_\delta(x', \tau)$, $x = (x', x'') \in \overline{\Omega}$, we obtain

$$\frac{1}{2}\Big[e^{-2\mu t} \int_{\Omega_\tau} |w(x,t)|^2 \psi_\delta(x', \tau)\, dx\Big]\Big|_{t=0}^{t=T} + \mu \iint_{Q_\tau} |w|^2 \psi_\delta e^{-2\mu t}\, dxdt$$
$$+ \iint_{Q_\tau} \Big[\sum_{i=0}^{n} (a_i(u_1) - a_i(u_2)) \partial_i w \psi_\delta\Big] e^{-2\mu t}\, dxdt$$
$$= - \iint_{Q_\tau} \Big[\sum_{i=1}^{k} (a_i(u_1) - a_i(u_2)) w \partial_i \psi_\delta\Big] e^{-2\mu t}\, dxdt. \tag{4.9}$$

Let $\nabla_k w := (\partial_1 w, \ldots, \partial_k w)$, $|\nabla_k w| := (\sum_{i=1}^k |\partial_i w|^2)^{1/2}$. In view of (3.3) we have

$$\sum_{i=1}^{k} |a_i(u_1) - a_i(u_2)| \leqslant g_1 |\nabla_k w| + g_2 |w| \quad \text{a. e. on } Q. \tag{4.10}$$

From (4.9), taking into account (3.4), (4.5), (4.8), and (4.10), we deduce

$$\iint_{Q_\tau} \big[q_1 |\nabla_k w|^2 + (q_2 + \mu)|w|^2\big] \psi_\delta e^{-2\mu t}\, dxdt$$
$$\leqslant \frac{1}{\delta} \iint_{Q_\tau \setminus Q_{\tau-\delta}} \big[g_1 |\nabla_k w| + g_2 |w|\big] |w| e^{-2\mu t}\, dxdt. \tag{4.11}$$

Note that for an arbitrary function $P \in L_{1,\mathrm{loc}}(\overline{Q})$ we have

$$\iint_{Q_\tau \setminus Q_{\tau-\delta}} P(x,t)\, dxdt = \int_{\tau-\delta}^{\tau} \Big(\iint_{\Sigma_{*,\sigma}} P(x,t)\, d\Gamma\, dt\Big)\, d\sigma, \quad \tau > 0.$$

Using the latter assertion, we pass to the limit in (4.11) as $\delta \to 0+$. So, we get

$$\iint_{Q_\tau} \big[q_1|\nabla_k w|^2 + (q_2+\mu)|w|^2\big] e^{-2\mu t}\,dxdt \leqslant \iint_{\Sigma_{*,\tau}} \big[g_1|\nabla_k w| + g_2|w|\big]|w| e^{-2\mu t}\,d\Gamma\,dt \quad \text{for a.e. } \tau \in (0,\tau(R)). \tag{4.12}$$

From Cauchy-Bunyakovsky-Schvartz inequality it follows that for a.e. $\tau \in (0,\tau(R))$

$$\iint_{\Sigma_{*,\tau}} \big[g_1|\nabla_k w| + g_2|w|\big]|w| e^{-2\mu t}\,d\Gamma\,dt \leqslant \Big(\iint_{\Sigma_{*,\tau}} |g_1|^2|\nabla_k w|^2 e^{-2\mu t}\,d\Gamma\,dt\Big)^{1/2} \times \Big(\iint_{\Sigma_{*,\tau}} |w|^2 e^{-2\mu t}\,d\Gamma\,dt\Big)^{1/2} + \iint_{\Sigma_{*,\tau}} g_2|w|^2 e^{-2\mu t}\,d\Gamma\,dt. \tag{4.13}$$

By virtue of (3.6) and (3.7), we obtain for a.e. $\tau \in (0,\tau(R))$ and for a.e. $t \in (0,T)$

$$\int_{\Gamma_{*,\tau}} |g_1|^2|\nabla_k w|^2\,d\Gamma \leqslant \int_{\Gamma_{*,\tau}} [|g_1|^2/q_1] q_1|\nabla_k w|^2\,d\Gamma \leqslant (d_1(\tau))^2 \int_{\Gamma_{*,\tau}} \big[q_1|\nabla_k w|^2 + (q_2+\mu)|w|^2\big]\,d\Gamma, \tag{4.14}$$

$$\int_{\Gamma_{*,\tau}} |w|^2\,d\Gamma \leqslant \int_{\Gamma_{*,\tau}} \big[q_1|\nabla_k w|^2 + (q_2+\mu)|w|^2\big]\,d\Gamma \Big/ \Big[\int_{\Gamma_{*,\tau}} \big[q_1|\nabla_k w|^2 + (q_2+\mu)|w|^2\big]\,d\Gamma \Big/ \int_{\Gamma_{*,\tau}} |w|^2\,d\Gamma\Big] \leqslant (\lambda(\tau)+\mu)^{-1} \int_{\Gamma_{*,\tau}} \big[q_1|\nabla_k w|^2 + (q_2+\mu)|w|^2\big]\,d\Gamma, \tag{4.15}$$

$$\int_{\Gamma_{*,\tau}} g_2|w|^2\,d\Gamma \leqslant d_2(\tau) \int_{\Gamma_{*,\tau}} |w|^2\,d\Gamma \leqslant d_2(\tau)(\lambda(\tau)+\mu)^{-1} \int_{\Gamma_{*,\tau}} \big[q_1|\nabla_k w|^2 + (q_2+\mu)|w|^2\big]\,d\Gamma. \tag{4.16}$$

From (4.12), taking into account (4.13) – (4.16), we infer

$$\iint_{Q_\tau} \big[q_1|\nabla_k w|^2 + (q_2+\mu)|w|^2\big] e^{-2\mu t}\,dxdt \leqslant \big[d_1(\tau)(\lambda(\tau)+\mu)^{-1/2} + d_2(\tau)(\lambda(\tau)+\mu)^{-1}\big] \times \iint_{\Sigma_{*,\tau}} \big[q_1|\nabla_k w|^2 + (q_2+\mu)|w|^2\big] e^{-2\mu t}\,d\Gamma\,dt. \tag{4.17}$$

In view of (3.9), (3.15), and (4.17) we establish for a.e. $\tau \in (0, \tau(R))$

$$\iint_{Q_\tau} E_{k,\mu}(w) e^{-2\mu t}\, dxdt \leqslant A_\mu(\tau) \iint_{\Sigma_{*,\tau}} E_{k,\mu}(w) e^{-2\mu t}\, d\Gamma\, dt. \tag{4.18}$$

Denote

$$F(\tau) := \iint_{Q_\tau} E_{k,\mu}(w) e^{-2\mu t}\, dxdt \equiv \int_0^\tau \Big(\int_{\Sigma_{*,\sigma}} E_{k,\mu}(w) e^{-2\mu t}\, d\Gamma\, dt \Big)\, d\sigma, \tag{4.19}$$

for all $\tau \in [1, \tau(R)]$. Then for a.e. $\tau \in (1, \tau(R))$

$$\iint_{\Sigma_{*,\tau}} E_{k,\mu}(w) e^{-2\mu t}\, d\Gamma\, dt = \frac{d}{d\tau} \int_0^\tau \Big(\int_{\Sigma_{*,\sigma}} E_{k,\mu}(w) e^{-2\mu t}\, d\Gamma\, dt \Big)\, d\sigma = \frac{dF(\tau)}{d\tau}. \tag{4.20}$$

From (4.18), using (4.19), and (4.20), we obtain

$$F(\tau) \leqslant A_\mu(\tau) \frac{dF(\tau)}{d\tau} \quad \text{for a.e. } \ \tau \in [1, \tau(R)]. \tag{4.21}$$

Suppose $\tau = \tau(\alpha)$, $\alpha \in [0, +\infty)$, is a solution of problem (3.11), and R_1, R_2 are arbitrary real numbers such that $0 < R_1 < R_2 \leqslant R$. In view of (3.11) and (4.21) we get

$$F(\tau(\alpha)) \leqslant \frac{dF(\tau(\alpha))}{d\tau} \frac{d\tau(\alpha)}{d\alpha}, \quad \alpha \in [R_1, R_2].$$

It follows that

$$0 \leqslant \frac{dF(\tau(\alpha))}{d\alpha} - F(\tau(\alpha)), \quad \alpha \in [R_1, R_2]. \tag{4.22}$$

Multiplying (4.22) by $e^{-\alpha}$, we deduce $0 \leqslant \dfrac{d}{d\alpha}\Big(e^{-\alpha} F(\tau(\alpha))\Big)$, $\alpha \in [R_1, R_2]$. Integrating the latter inequality in α from R_1 to R_2, we infer

$$F(\tau(R_1)) \leqslant e^{R_1 - R_2} F(\tau(R_2)). \tag{4.23}$$

From (4.23), taking into account $\langle w \rangle_\alpha = \sqrt{F(\tau(\alpha))}$, we imply (4.7). □

5. Proofs of the main results

The proof of Theorem 1. Let us show that problem (1.1) – (1.3) has no more than one weak solution. Assume the opposite. Let u_1 and u_2 be different weak solutions of problem (1.1) – (1.3), which satisfy condition (3.16). It is clear that for arbitrary $R > 0$ a functional $\langle \cdot \rangle_R$ is a seminorm in space $\mathbb{U}_{p,\mathrm{loc}}(\overline{Q})$. From this fact and (3.16) we deduce

$$e^{-R/2} \langle u_1 - u_2 \rangle_R \leqslant e^{-R/2} (\langle u_1 \rangle_R + \langle u_2 \rangle_R) = e^{-R/2} \langle u_1 \rangle_R + e^{-R/2} \langle u_2 \rangle_R = \beta(R),$$

where $\beta(R) \to 0$ as $R \to +\infty$. Using this assertion and Lemma 2 (see (4.7)) for arbitrary R_1, R_2 such that $R_1 < R_2$, we obtain the estimate

$$\langle u_1 - u_2 \rangle_{R_1} \leqslant e^{(R_1 - R_2)/2} \langle u_1 - u_2 \rangle_{R_2} = e^{R_1/2} \beta(R_2). \tag{5.1}$$

We fix R_1, and tend R_2 to $+\infty$. From (5.1) it follows that $\langle u_1 - u_2 \rangle_{R_1} = 0$. Thus $u_1 = u_2$ almost everywhere on Q^{R_1}. As R_1 is arbitrary, we get $u_1 = u_2$ almost everywhere on Q. This contradiction proves Theorem 1. □

The proof of Theorem 2. The proof is in four steps.
Step 1 (the solution's approximations). Let $\alpha > 0$ be an arbitrary number. By $\widehat{W}^1_{p(\cdot)}(\Omega^\alpha)$ define the closure of space $\{v \in C^1(\overline{\Omega^\alpha}) \mid v\big|_{\partial\Omega^\alpha \setminus \Gamma_1^\alpha} = 0\}$ in $W^1_{p(\cdot)}(\Omega^\alpha)$. By $\widehat{W}^{1,0}_{p(\cdot)}(Q^\alpha)$ denote a space of functions $w \in W^{1,0}_{p(\cdot)}(Q^\alpha)$ such that, for a.e. $t \in (0,T)$, $w(\cdot,t)$ belongs to $\widehat{W}^1_{p(\cdot)}(\Omega^\alpha)$. We set $\widehat{\mathbb{U}}_p(Q^\alpha) := \widehat{W}^{1,0}_{p(\cdot)}(Q^\alpha) \cap C([0,T]; L_2(\Omega^\alpha))$.

For every $l \in \mathbb{N}$ we consider the problem: *to find the function $u_l \in \widehat{\mathbb{U}}_p(Q^l)$ that satisfies (in the sense of space $C([0,T]; L_2(\Omega^l))$) the initial condition*

$$u_l(\cdot, 0) = u_0(\cdot) \quad \textit{almost everywhere in} \;\; \Omega^l, \tag{5.2}$$

and the integral identity

$$\iint_{Q^l} \Big\{ -u_l \psi \varphi' + \sum_{i=0}^n a_i(u_l) \partial_i \psi \varphi \Big\} \, dxdt = \iint_{Q^l} f \psi \varphi \, dxdt,$$
$$\forall \, \psi \in \widetilde{W}^1_{p(\cdot),c}(\Omega), \;\; \textit{supp}\, \psi \subset \overline{\Omega^l}, \;\; \forall \, \varphi \in C^1_c(0,T). \tag{5.3}$$

To prove the existence of the function $u_l \in \widehat{\mathbb{U}}_p(Q^l)$ we use Faedo-Galerkin method (see, for example, [22]). In view of $(\mathbf{A}_3)$ it is easy to show that the function u_l is a unique.

For every $l \in \mathbb{N}$ the function u_l is extended by zero to Q, and the extension denote by u_l again. Obviously, that $u_l \in \mathbb{U}_{p,\mathrm{loc}}(\overline{Q})$. Now we show that there exists a subsequence of the sequence $\{u_l\}_{l=1}^\infty$ converging to the weak solution of problem (1.1) – (1.3), (3.16) in some sense. We use an approach from [3], [6], and [33].

Step 2 (the convergence of the sequence of solution's approximations). First we estimate $\langle u_l \rangle_l$ for an arbitrary fixed $l \in \mathbb{N}$. From Lemma 1, putting $w = u_l$, $R = l$, $t_1 = 0$, $t_2 = T$, $\theta(t) = e^{-2\mu t}$, $t \in \mathbb{R}$, $\zeta(x) = 1$, $x \in \overline{\Omega}$, and using (5.3) instead of (4.3), we obtain (see (4.1))

$$\frac{1}{2} e^{-2\mu T} \int_{\Omega^l} |u_l(x,T)|^2 \, dx + \iint_{Q^l} \Big[\sum_{i=0}^n a_i(u_l) \, \partial_i u_l + \mu |u_l|^2 \Big] e^{-2\mu t} \, dxdt$$
$$= \iint_{Q^l} f \, u_l \, e^{-2\mu t} \, dxdt + \frac{1}{2} \int_{\Omega^l} |u_0|^2 \, dx. \tag{5.4}$$

From this assertion, taking into account $(\mathbf{A}_1)$ (or rather, the condition $a_i(0) = 0,\ i=\overline{0,n}$), $(\mathbf{A}_3)$ (see (3.4)), and Cauchy inequality:

$$ab \leqslant \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2, \quad a,b\in\mathbb{R},\ \varepsilon>0, \tag{5.5}$$

we infer

$$\iint_{Q^l}\big[q_1|\nabla_k u_l|^2 + (q_2+\mu)|u_l|^2\big]e^{-2\mu t}\,dxdt \\ \leqslant \frac{\varepsilon_1}{2}\iint_{Q^l}|u_l|^2 e^{-2\mu t}\,dxdt + \frac{1}{2\varepsilon_1}\iint_{Q^l}|f|^2e^{-2\mu t}\,dxdt + \frac{1}{2}\int_{\Omega^l}|u_0|^2\,dxdt, \tag{5.6}$$

where $\varepsilon_1>0$ is an arbitrary constant.

We have

$$\begin{aligned}\iint_{Q^l}|u_l|^2e^{-2\mu t}\,dxdt &= \int_0^T e^{-2\mu t}\Big(\int_{\Omega^l}|u_l|^2\,dx\Big)\,dt \\ &= \int_0^T e^{-2\mu t}\Big(\int_{\Omega^l}\big[q_1|\nabla_k u_l|^2 + (q_2+\mu)|u_l|^2\big]\,dx\Big/\Big[\int_{\Omega^l}\big[q_1|\nabla_k u_l|^2 \\ &\quad + (q_2+\mu)|u_l|^2\big]\,dx\Big/\int_{\Omega^l}|u_l|^2\,dx\Big]\Big)\,dt \\ &\leqslant \frac{1}{\Lambda_l+\mu}\iint_{Q^l}\big[q_1|\nabla_k u_l|^2 + (q_2+\mu)|u_l|^2\big]e^{-2\mu t}\,dxdt, \end{aligned} \tag{5.7}$$

where Λ_l is defined in (3.14).

From (5.6) and (5.7), putting $\varepsilon_1 = \Lambda_l+\mu$, we get

$$\iint_{Q^l}E_{k,\mu}(u_l)e^{-2\mu t}\,dxdt \leqslant (\Lambda_l+\mu)^{-1}\iint_{Q^l}|f|^2e^{-2\mu t}\,dxdt + \int_{\Omega^l}|u_0|^2\,dx.$$

The latter inequality and (3.18) imply the estimate

$$\langle u_l\rangle_l \leqslant \sqrt{C_1}\,e^{(1-\varkappa)l/2}, \quad l\in\mathbb{N}. \tag{5.8}$$

Let $m\in\mathbb{N}$ be an arbitrary fixed number, and let $l,r\in\mathbb{N}$ be arbitrary numbers, while $l\geqslant m$. We have

$$\langle\, u_{l+r}-u_l\rangle_m \leqslant \sum_{i=0}^{r-1}\langle\, u_{l+i+1}-u_{l+i}\rangle_m. \tag{5.9}$$

For each $i\in\{0,\dots,r-1\}$ and the functions u_{l+i+1}, u_{l+i}, using Lemma 2 with $R=l+i$, we obtain

$$\langle\, u_{l+i+1}-u_{l+i}\rangle_m \leqslant e^{-1/2}\langle\, u_{l+i+1}-u_{l+i}\rangle_{m+1} \leqslant \dots \\ \leqslant e^{-(l+i-m)/2}\langle\, u_{l+i+1}-u_{l+i}\rangle_{l+i}. \tag{5.10}$$

In view of (5.8), we have

$$\begin{aligned}\langle u_{l+i+1} - u_{l+i}\rangle_{l+i} &\leqslant \langle u_{l+i+1}\rangle_{l+i+1} + \langle u_{l+i}\rangle_{l+i}\\ &\leqslant \sqrt{C_1}\big[e^{(1-\varkappa)(l+i+1)/2} + e^{(1-\varkappa)(l+i)/2}\big]\\ &\leqslant \sqrt{C_1}\big[e^{1/2}+1\big]e^{(1-\varkappa)(l+i)/2} = C_4\, e^{(1-\varkappa)(l+i)/2}, \end{aligned}\qquad (5.11)$$

where $C_4 := \sqrt{C_1}\big(e^{1/2}+1\big)$.

Using (5.9) – (5.11), we find

$$\begin{aligned}\langle u_{l+r} - u_l\rangle_m &\leqslant C_4 \sum_{i=0}^{r-1} e^{-(l+i-m)/2}\, e^{(1-\varkappa)(l+i)/2}\\ &\leqslant C_4 e^{(m-\varkappa l)/2}\sum_{i=0}^{\infty}(e^{-\varkappa/2})^i \leqslant C_5 e^{(m-\varkappa l)/2}, \end{aligned}\qquad (5.12)$$

where

$$C_5 := C_4/(1-e^{-\varkappa/2}) = \sqrt{C_1}(e^{1/2}+1)/(1-e^{-\varkappa/2}). \qquad (5.13)$$

From (5.12) it follows that $\langle u_{l+r} - u_l\rangle_m \to 0$ as $l \to +\infty$ uniformly by $r \in \mathbb{N}$, that is, $\{\partial_i u_l\}$, $i = \overline{0,k}$, are Cauchy sequences in space $L_2(Q^m)$, where $m \in \mathbb{N}$ is an arbitrary fixed. Hence, there exists a function $u \in L_{2,\mathrm{loc}}(\overline{Q})$ such that $\partial_i u \in L_{2,\mathrm{loc}}(\overline{Q})$, $i = \overline{1,k}$, and

$$\partial_i u_l \underset{l\to\infty}{\longrightarrow} \partial_i u \quad \text{strongly in} \quad L_{2,\mathrm{loc}}(\overline{Q}), \quad i = \overline{0,k}. \qquad (5.14)$$

Taking into account $(\mathbf{A}_3)$ (see (3.3)), from (5.14) we get

$$a_i(u_l) \underset{l\to\infty}{\longrightarrow} a_i(u) \quad \text{strongly in} \quad L_{2,\mathrm{loc}}(\overline{Q}), \quad i = \overline{1,k}. \qquad (5.15)$$

Suppose $m \in \mathbb{N}$ is an arbitrary fixed number, and $l, r \in \mathbb{N}$ are arbitrary numbers such that $l \geqslant m$, $r \geqslant m$. Under the condition $\operatorname{supp}\psi \subset \overline{\Omega^m}$, we subtract the integral identity (5.3) for $l = r$ from this identity for l. Applying Lemma 1 to their difference with $t_1 = 0$, $t_2 = s \in (0,T]$, $\theta(t) := e^{-2\mu t}$, $t \in \mathbb{R}$, $\zeta(x) := \psi_{1/2}(x', \tau(m))$, $x = (x', x'') \in \overline{\Omega}$, we obtain

$$\begin{aligned}&\frac{1}{2}\Big[e^{-2\mu t}\int_{\Omega^m}|u_{lr}(x,t)|^2\psi_{1/2}(x',\tau(m))\,dx\Big]\Big|_{t=0}^{t=s}\\ &\qquad + \int_0^s\!\!\int_{\Omega^m}\Big[\sum_{i=0}^{n}(a_i(u_l)-a_i(u_r))\partial_i u_{lr} + \mu|u_{lr}|^2\Big]\psi_{1/2}e^{-2\mu t}\,dxdt\\ &\quad = -\int_0^s\!\!\int_{\Omega^m}\Big[\sum_{i=1}^{k}(a_i(u_l)-a_i(u_r))u_{lr}\partial_i\psi_{1/2}\Big]e^{-2\mu t}\,dxdt, \end{aligned}\qquad (5.16)$$

where $u_{lr} := u_l - u_r$.

By virtue of $(\mathbf{A}_3)$ and (4.8), (5.2), from (5.16) for all $s \in [0,T]$ we deduce

$$\int_{\Omega^m} |u_{lr}(x,s)|^2 \psi_{1/2}(x',\tau(m))\,dx \leqslant 4e^{2|\mu|T} \int_0^s \int_{\Omega^m} \Big[\sum_{i=1}^k |a_i(u_l)-a_i(u_r)||u_{lr}|\Big]\,dxdt. \quad (5.17)$$

From (5.17), in view of Cauchy-Bunyakovsky-Schvartz inequality, it implies that

$$\max_{t\in[0,T]} \int_{\Omega_{\tau(m)-1/2}} |u_l(x,t)-u_r(x,t)|^2\,dx \leqslant 4e^{2|\mu|T} \sum_{i=1}^k \Big(\iint_{Q^m} |a_i(u_l)-a_i(u_r)|^2\,dxdt\Big)^{1/2} \times \Big(\iint_{Q^m} |u_l-u_r|^2\,dxdt\Big)^{1/2}. \quad (5.18)$$

Using (5.14) and (5.15), from (5.18) we infer that $\{u_l\}$ is the Cauchy sequence in space $C([0,T];L_{2,\mathrm{loc}}(\overline{\Omega}))$. Hence,

$$u \in C([0,T];L_{2,\mathrm{loc}}(\overline{\Omega})) \quad \text{and} \quad u_l \underset{l\to\infty}{\longrightarrow} u \quad \text{in} \quad C([0,T];L_{2,\mathrm{loc}}(\overline{\Omega})). \quad (5.19)$$

Assume $m \in \mathbb{N}$ is an arbitrary fixed number, and $l \in \mathbb{N}$ is an arbitrary number such that $l \geqslant m$. Putting $w = u_l$, $R = \tau(m)$, $t_1 = 0$, $t_2 = T$, $\zeta(x) := \psi_{1/2}(x',\tau(m))$, $x = (x',x'') \in \overline{\Omega}$, $\theta(t) := e^{-2qt}$, $t \in \mathbb{R}$, where $q := \sup_{\overline{Q}} q_4$ (q_4 from condition $(\mathbf{A}_4)$), and using (5.3) instead of (4.3), from Lemma 1 we obtain

$$\frac{1}{2}e^{-2qT} \int_{\Omega^m} |u_l(x,T)|^2 \psi_{1/2}(x',\tau(m))\,dx + \iint_{Q^m} \Big[\sum_{i=0}^n a_i(u_l)\partial_i u_l + q|u_l|^2\Big]\psi_{1/2} e^{-2qt}\,dxdt = -\iint_{Q^m} \sum_{i=1}^k a_i(u_l)u_l\,\partial_i\psi_{1/2} e^{-2qt}\,dxdt + \iint_{Q^m} f u_l \psi_{1/2} e^{-2qt}\,dxdt + \frac{1}{2}\int_{\Omega^m} |u_0|^2 \psi_{1/2}(x',\tau(m))\,dx. \quad (5.20)$$

Estimating the terms of (5.20) with conditions $(\mathbf{A}_1)$, $(\mathbf{A}_3)$ (see (3.3)), $(\mathbf{A}_4)$, (4.8) and Cauchy-Bunyakovsky-Schvartz inequality, we get

$$\iint_{Q_{\tau(m)-1/2}} \Big[q_3 \sum_{i=k+1}^n |\partial_i u_l|^{p_i(x)} + (q-q_4)|u_l|^2\Big] e^{-2qt}\,dxdt \leqslant C_7\Big(\iint_{Q^m}\Big[\sum_{i=0}^k |\partial_i u_l|^2\Big]e^{-2qt}\,dxdt + \iint_{Q^m}\big[|f|^2+h\big]e^{-2qt}\,dxdt + \int_{\Omega^m}|u_0|^2\,dx\Big), \quad (5.21)$$

where constant $C_7 > 0$ is independent of l, but it may be depended on m.

Using (5.14), from (5.21) we obtain

$$\iint_{Q_{\tau(m)-1/2}} \sum_{i=0}^{n} |\partial_i u_l|^{p_i(x)}\, dxdt \leqslant C_8, \quad m, l \in \mathbb{N},\ l \geqslant m, \tag{5.22}$$

where constant $C_8 > 0$ is independent of l, but it may be depended on m.

By virtue of $(\mathbf{A}_2)$, (5.14), (5.22), and discrete Hölder inequality we deduce that for every $i \in \{0, k+1, \dots, n\}$ and arbitrary $m, l \in \mathbb{N}$, $l \geqslant m$,

$$\iint_{Q_{\tau(m)-1/2}} |a_i(u_l)|^{p_i'(x)}\, dxdt \leqslant C_9 \iint_{Q_{\tau(m)-1/2}} \Big[\sum_{j=0}^{n} |\partial_j u_l|^{p_j(x)}\Big]\, dxdt + C_{10} \leqslant C_{11}, \tag{5.23}$$

where positive constants C_9, C_{10}, C_{11} are independent of l, but they may be depended on m.

In view of (5.22), (5.23), and the reflexivity of spaces $L_{p_i(\,\cdot\,)}(Q_\tau)$, $L_{p_i'(\cdot)}(Q_\tau)$, $i = \overline{k+1, n}$, $\tau > 0$, it follows that there exists a subsequence of the sequence $\{u_l\}_{l=1}^{\infty}$ (without loss of generality we use the notation $\{u_l\}_{l=1}^{\infty}$ for this subsequence), and functions $\chi_0 \in L_{2,\,\mathrm{loc}}(\overline{Q})$, $\chi_i \in L_{p'_i(\cdot),\mathrm{loc}}(\overline{Q})$, $i = \overline{k+1, n}$, such that

$$\partial_i u_l \underset{l\to\infty}{\longrightarrow} \partial_i u \quad \text{weakly in } L_{p_i(\cdot),\mathrm{loc}}(\overline{Q}), \quad i = \overline{k+1, n}, \tag{5.24}$$

$$a_0(u_l) \underset{l\to\infty}{\longrightarrow} \chi_0 \quad \text{weakly in } L_{2,\mathrm{loc}}(\overline{Q}), \tag{5.25}$$

$$a_i(u_l) \underset{l\to\infty}{\longrightarrow} \chi_i \quad \text{weakly in } L_{p'_i(\cdot),\mathrm{loc}}(\overline{Q}), \quad i = \overline{k+1, n}. \tag{5.26}$$

Put

$$\chi_i := a_i(u), \quad i = \overline{1, k}. \tag{5.27}$$

Remark that for every $l \in \mathbb{N}$ (see (5.3)) we have the identity

$$\iint_Q \Big[-u_l \psi \varphi' + \sum_{i=0}^{n} a_i(u_l)\partial_i \psi \varphi - f\psi\varphi\Big]\, dxdt = 0,$$
$$\forall\, \psi \in \widetilde{W}^1_{p(\cdot),\mathrm{c}}(\Omega),\ \operatorname{supp}\psi \subset \overline{\Omega^l},\ \forall\, \varphi \in C^1_c(0,T). \tag{5.28}$$

In (5.28) we fix an arbitrary $\psi \in \widetilde{W}^1_{p(\cdot),\mathrm{c}}(\Omega)$, $\varphi \in C^1_c(0,T)$, and pass to the limit as $l \to \infty$, taking into account (5.14), (5.15), (5.25) – (5.27). So, we get

$$\iint_Q \Big[-u\psi\varphi' + \sum_{i=0}^{n} \chi_i \partial_i \psi \varphi - f\psi\varphi\Big]\, dxdt = 0. \tag{5.29}$$

To conclude that u is a weak solution of problem (1.1) – (1.3). It remains to show that the following identity holds

$$\iint_Q \sum_{i=0}^{n} \chi_i \partial_i \psi \varphi\, dxdt = \iint_Q \sum_{i=0}^{n} a_i(u)\partial_i \psi \varphi\, dxdt \quad \forall\, \psi \in \widetilde{W}^1_{p(\cdot),\mathrm{c}}(\Omega),\ \forall\, \varphi \in C^1_c(0,T). \tag{5.30}$$

Indeed, if (5.30) is true, then from this and (5.29) we obtain the integral identity (3.2). In view of (5.14), (5.24) we have $u \in \widetilde{W}^{1,0}_{p(\cdot),\,\mathrm{loc}}(\overline{Q})$. From (5.2), (5.19) we deduce $u \in C([0,T]; L_{2,\mathrm{loc}}(\overline{\Omega}))$ (it means that $u \in \mathbb{U}^b_{p,\mathrm{loc}}(\overline{Q})$) and the initial condition (1.3) is true. Hence, the function u is a weak solution of problem (1.1) – (1.3).

Step 3 (the correctness of identity (5.30)). To verify the correctness of identity (5.30) we use the monotonicity method [33].

Let $v \in L_{2,\mathrm{loc}}(\overline{Q})$ be an arbitrary function such that $\partial_i v \in L_{p_i(\cdot),\,\mathrm{loc}}(\overline{Q})$, $i = \overline{1,n}$, let $\zeta(x')$, $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$, be a nonnegative continuously differentiable function with bounded support, and let $\theta \in C^1_c(0,T)$, $\theta \geqslant 0$. By virtue of condition $(\mathbf{A}_3)$ (see (3.4)), for every $l \in \mathbb{N}$ we have

$$\iint_Q \big[\sum_{i=0}^{n} (a_i(u_l) - a_i(v))(\partial_i u_l - \partial_i v) + \mu(u_l - v)^2\big]\zeta\theta e^{-2\mu t}\,dxdt \geqslant 0. \qquad (5.31)$$

We rewrite inequality (5.31) as

$$\iint_Q \big[\sum_{i=0}^{n} a_i(u_l)\partial_i u_l\big]\zeta\theta e^{-2\mu t}\,dxdt - \iint_Q \big[\sum_{i=0}^{n} \big(a_i(u_l)\partial_i v + a_i(v)(\partial_i u_l - \partial_i v)\big) + \mu(u_l - v)^2\big]\zeta\theta e^{-2\mu t}\,dxdt \geqslant 0 \quad \forall\, l \in \mathbb{N}. \qquad (5.32)$$

Assume $m \in \mathbb{N}$ such that $\operatorname{supp}\zeta \subset \{x' \,|\, |x'| \leqslant \tau(m)\}$. Using Lemma 1, we obtain from identity (5.28) as $l \geqslant m$

$$\iint_Q \big[\sum_{i=0}^{n} a_i(u_l)\partial_i u_l\big]\zeta\theta e^{-2\mu t}\,dxdt = \iint_Q |u_l|^2\zeta(\theta'/2 - \mu\theta)e^{-2\mu t}\,dxdt - \iint_Q \big[\sum_{i=1}^{k} a_i(u_l)u_l\partial_i\zeta - fu_l\zeta\big]\theta e^{-2\mu t}\,dxdt. \qquad (5.33)$$

From (5.32) and (5.33) we get

$$\iint_Q |u_l|^2\zeta(\theta'/2 - \mu\theta)e^{-2\mu t}\,dxdt - \iint_Q \big[\sum_{i=1}^{k} a_i(u_l)u_l\partial_i\zeta - fu_l\zeta\big]\theta e^{-2\mu t}\,dxdt - \iint_Q \big[\sum_{i=0}^{n} \big(a_i(u_l)\partial_i v + a_i(v)(\partial_i u_l - \partial_i v)\big) + \mu(u_l - v)^2\big]\zeta\theta e^{-2\mu t}\,dxdt \geqslant 0. \qquad (5.34)$$

In (5.34) we pass to the limit as $l \to \infty$, and by virtue of (5.14), (5.15), (5.25) – (5.27) we infer

$$\iint_Q |u|^2\zeta(\theta'/2 - \mu\theta)e^{-2\mu t}\,dxdt - \iint_Q \big[\sum_{i=1}^{k} \chi_i u\partial_i\zeta - fu\zeta\big]\theta e^{-2\mu t}\,dxdt - \iint_Q \big[\sum_{i=0}^{n} \big(\chi_i\partial_i v + a_i(v)(\partial_i u - \partial_i v)\big) + \mu(u - v)^2\big]\zeta\theta e^{-2\mu t}\,dxdt \geqslant 0. \qquad (5.35)$$

In view of Lemma 1 it follows from (5.29) next equality

$$\iint_Q \big[\sum_{i=0}^{n} \chi_i \partial_i u\big]\, \zeta\theta e^{-2\mu t}\, dxdt = \iint_Q |u|^2 \zeta(\theta'/2 - \mu\theta) e^{-2\mu t}\, dxdt$$
$$- \iint_Q \big[\sum_{i=1}^{k} \chi_i u \partial_i \zeta - f u \zeta\big] \theta e^{-2\mu t}\, dxdt. \quad (5.36)$$

Assertions (5.35) and (5.36) imply

$$\iint_Q \big[\sum_{i=0}^{n} \chi_i\, \partial_i u\big] w\theta e^{-2\mu t}\, dxdt - \iint_Q \big[\sum_{i=0}^{n} \big(\chi_i \partial_i v + a_i(v)(\partial_i u - \partial_i v)\big)$$
$$+ \mu(u-v)^2\big]\zeta\theta e^{-2\mu t}\, dxdt \geqslant 0,$$

that is,

$$\iint_Q \big[\sum_{i=0}^{n} (\chi_i - a_i(v))(\partial_i u - \partial_i v) + \mu(u-v)^2\big]\zeta\theta e^{-2\mu t}\, dxdt \geqslant 0. \quad (5.37)$$

In (5.37) we put $v = u - \lambda\psi\varphi$, where λ is an arbitrary number, and $\psi \in \widetilde{W}^1_{p(\cdot),c}(\Omega)$, $\varphi \in C^1_c(0,T)$ are arbitrary functions. So, taking into account the arbitrariness of λ, we obtain the equality

$$\iint_Q \big[\sum_{i=0}^{n} \big(\chi_i - a_i(u - \lambda\psi\varphi)\big)\partial_i\psi\varphi + \lambda\mu(\psi\varphi)^2\big]\zeta\theta e^{-2\mu t}\, dxdt = 0.$$

Here we tend λ to 0, using conditions $(\mathbf{A}_1)$, $(\mathbf{A}_2)$, and Lebesgue dominated convergence theorem. Thus, taking into account the arbitrariness of ζ and θ, we deduce

$$\iint_Q \Big[\sum_{i=0}^{n} \big(\chi_i - a_i(u)\big)\partial_i\psi\Big]\varphi\, dxdt = 0, \quad \psi \in \widetilde{W}^1_{p(\cdot),c}(\Omega),\ \varphi \in C^1_c(0,T). \quad (5.38)$$

From (5.38) it follows (5.30).

Step 4 (the solution's estimate). Estimate (3.19) is obtained from (5.8), (5.12) and (5.13) by this way: $\langle u\rangle_m \leqslant \langle u - u_m\rangle_m + \langle u_m\rangle_m = \lim_{l\to\infty}\langle u_l - u_m\rangle_m + \langle u_m\rangle_m \leqslant C_2 e^{(1-\varkappa)m/2}$, where $C_2 := \sqrt{C_1} + C_5 = \sqrt{C_1}(2 + e^{1/2} - e^{-\varkappa/2})/(1 - e^{-\varkappa/2})$.

Now it is easy to see that the function u satisfies (3.16). Indeed, let $R > 0$ be an arbitrary number, and m be a natural number such that $m - 1 < R \leqslant m$. Using (3.19), we get

$$\langle u\rangle_R \leqslant \langle u\rangle_m \leqslant C_2 e^{(1-\varkappa)m/2} = C_2 e^{(1-\varkappa)(m-R)/2} e^{(1-\varkappa)R/2}$$
$$\leqslant C_2 e^{(1-\varkappa)/2} e^{-\varkappa R/2} e^{R/2} = \beta(R) e^{R/2}, \quad R \geqslant 1,$$

where $\beta(R) := C_2 e^{(1-\varkappa)/2} e^{-\varkappa R/2}$. Since $\beta(R) \to 0$ as $R \to +\infty$, then we have (3.16).

So, we have shown that u is a weak solution of problem (1.1) – (1.3) that satisfies (3.16) and (3.19). Theorem 2 is proved. □

References

1. A.N. Tikhonov, *Théoremes d'unicité pour l'équation de la chaleur*, *Mat. Sb.*, **42** (2) (1935), 199–216.
2. S. Täcklind, *Sur les classes quasianalytiques des solutions des équations aux dérivés partielles du type parabolique*, Nova Acta Regiae Soc. Sci. Upsal. Series 4, **10** (3) (1936), 3–55.
3. O.A. Oleinik, G.A. Iosifyan, *An analog of Saint-Venant principle and uniqueness of the solutions of the boundary-value problems in unbounded domains for parabolic equations*, *Usp. Mat. Nauk*, **31** (6) (1976), 142–166.
4. O.A. Oleinik, E.V. Radkevich, *The method of introducing a parameter for the investigation of evolution equations*, *Usp. Mat. Nauk*, **33** (5) (1978), 7–72.
5. Ph. Benilan, M.G. Grandall, M. Pierre, *Solutions of the porous medium equations in R^n under optimal conditions on initial values*, Indiana Univ. Math. J., **33** (1) (1984), 51–87.
6. A.E. Shishkov, *The solvability of the boundary-value problems for quasilinear elliptic and parabolic equations in unbounded domains in the classes of functions growing at the infinity*, Ukr. Math. J., **47** (2) (1985), 277–289.
7. A.E. Shishkov, V.F. Akulov, *Analogs of Teklind uniqueness classes for solutions of initial-boundary problems for some quasilinear degenerate parabolic equations*, Reports of the Academy of Sciences of UkrSSR. Ser. A, **5** (1989), 23–25.
8. E. Di Benedetto, M.A. Herrero, *On the Cauchy problem and initial traces for a degenerate parabolic equation*, Trans. Amer. Math. Soc., **314** (1) (1989), 187–224.
9. A.L. Gladkov, *The Cauchy problem in classes of increasing functions for the equation of filtration with convection*, Math. Sbornik, **186** (6) (1995), 35–56.
10. H. Brézis, *Semilinear equations in $\mathbb{R}^N$ without conditions at infinity*, Appl. Math. Optim., **12** (3) (1984), 271–282.
11. M.A. Herrero, M. Pierre, *The Cauchy problem for $u_t - \Delta u^m = 0$ when $0 < m < 1$*, Trans. Am. Math. Soc., **291** (1) (1985), 145–158.
12. F. Bernis, *Elliptic and parabolic semilinear parabolic problems without conditions at infinity*, Arch. Rational Mech. Anal., **106** (3) (1989), 217–241.
13. E. Di Benedetto, M.A. Herrero, *Non-negative solutions of the evolution p-Laplacian equation. Initial traces and Cauchy problem when $1 < p < 2$*, Arch. Rational Mech. Anal., **111** (3) (1990), 225–290.
14. M.M. Bokalo, *On unique solvability of boundary value problems for semilinear parabolic equations in unbounded domains without conditions at infinity*, Siberian Math. J., **34** (4) (1993), 620–627.
15. M.M. Bokalo, *Boundary value problems for semilinear parabolic equations in unbounded domains without conditions at infinity*, Siberian Math. J., **37** (5) (1996), 860–867.
16. L. Boccardo, Th. Gallouët, J.L. Vazquez, *Solutions of nonlinear parabolic equations without growth restrictions on the data*, Electronic J. Diff. Eq., **60** (2001), 1–20.

17. A. Gladkov, M. Guedda, *Diffusion-absorption equation without growth restrictions on the data at infinity*, J. Math. Anal. Appl., **274** (1) (2002), 16–37.
18. C. Marchi, A. Tesei, *Higher-order parabolic equations without conditions at infinity*, J. Math. Anal. Appl., **269** (2002), 352–368.
19. N.M. Bokalo, *The well-posedness of the first boundary value problem and the Cauchy problem for some quasilinear parabolic systems without conditions at infinity*, J. Math. Sci., **135** (1) (2006), 2625–2636.
20. M. M. Bokalo, O. M. Buhrii, N. Hryadil, *Initial-boundary value problems for nonlinear elliptic-parabolic equations with variable exponents of nonlinearity in unbounded domains without conditions at infinity*, Nonlinear Analysis. Elsevier. USA, **192** (2020), 1–17.
21. M. Růžička, *Electroreological fluids: modeling and mathematical theory*, Springer-Verl., Berlin, 2000.
22. V. N. Samokhin, *On a class of equations that generalize equations of polytropic filtration*, Diff. Equat., **32** (5) (1996), 648–657.
23. O. Buhrii, S. Lavrenyuk, *Initial boundary-value problem for parabolic equation of polytropic filtration type*, Visn. Lviv Univ (Herald of Lviv University). Ser. Mech.-Math., **56** (2000), 33–43.
24. M.M. Bokalo, I.B. Pauchok, *On the well-posedness of a Fourier problem for nonlinear parabolic equations of higher order with variable exponents of nonlinearity*, Matematychni Studii, **26** (1) (2006), 25–48.
25. M. Bokalo, O. Domanska, *On well-posedness of boundary problems for elliptic equations in general anisotropic Lebesgue-Sobolev spaces,* Matematychni Studii, **28** (1) (2007), 77–91.
26. S. Antontsev, S. Shmarev, *Evolution PDEs with nonstandard growth conditions. Existence, uniqueness, localization, blow-up,* Atlantis Studies in Diff. Eq., Vol. 4, Paris: Atlantis Press, 2015.
27. V. Rădulescu, D. Repovš, *Partial differential equations with variable exponents: variational methods and qualitative analysis*, CRC Press, Boca Raton, London, New York, 2015.
28. M.M. Bokalo, O.M. Buhrii, R.A. Mashiyev, *Unique solvability of initial-boundary-value problems for anisotropic elliptic-parabolic equations with variable exponents of nonlinearity*, J. Nonl. Evol. Eq. Appl., **6** (2013), 67–87.
29. O. Buhrii, N. Buhrii, *Nonlocal in time problem for anisotropic parabolic equations with variable exponents of nonlinearities*, J. Math. Anal. Appl., **473** (2019), 695–711.
30. L. Diening, P. Harjulehto, P. Hästö, M. Růžička, *Lebesgue and Sobolev spaces with variable exponents*, Springer, Heidelberg, 2011.
31. N.M. Bokalo, *Energy estimates for solutions and unique solvability of the Fourier problem for linear and quasilinear parabolic equations*, Diff. Equat., **30** (8) (1994), 1226–1234.
32. V.A. Galactionov, A.E. Shishkov, *Saint-Venant's principle in blow-up for higher-order quasilinear parabolic equations,* Proc. R. Soc. Edinb. Sect. A, Math, **133** (2003), no. 5, 1075–1119.
33. J.-L. Lions *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Paris: Dunod, 1969.

Received 12.04.2022