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Initiation to Differential Calculus: Teaching Practices of Portuguese Secondary School Teachers

Adilson de Campos [a]
João Pedro da Ponte [b]

[a] Instituto Federal de Santa Catarina, Departamento Acadêmico de Linguagem, Tecnologia, Educação e Ciência, Florianópolis, SC, Brasil.
[b] Universidade de Lisboa, Instituto de Educação (IE-ULisboa), Lisboa, Portugal.

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ABSTRACT

Context: The study of teaching practice is an emerging topic of research and is directly linked to the importance of improving the teaching and learning of Mathematics. **Goals:** To understand the core aspects of how secondary school teachers teach Differential Calculus. **Design:** The study uses a qualitative methodology, based on an interpretative paradigm, and is carried out by means of three case studies. **Environment and participants:** Three Portuguese secondary school teachers who teach Mathematics A took part in the study. **Data collection and analysis:** The data were collected through direct observation of lessons and semi-structured interviews. The data analysis gave rise to two dimensions: (i) general aspects, and (ii) specific aspects of didactic exploration. The general aspects are split into three subcategories: (i) lesson structure; (ii) interactions in the lessons; and (iii) working with the tasks. The specific aspects were also split into three subcategories: (i) connecting concepts; (ii) time for the students to elaborate and present their reasoning; and (iii) interaction between the graphical and algebraic aspects. **Results:** The results of the study point to the almost exclusive use of textbooks without using other tasks. **Conclusions:** The conclusions point to a significant emphasis by the teachers on the connection between the different concepts involved. The approach adopted seems to be more focused on conceptual questions than procedural techniques.

Keywords: Teaching Practice; Mathematics; Secondary Education; Teaching Differential Calculus.

Corresponding author: Adilson de Campos. Email: adilsoncampos28@gmail.com

Iniciação ao Cálculo Diferencial: Práticas Letivas de Professores do Ensino Secundário Português

RESUMO

Contexto: O estudo da prática profissional do professor constitui um tema emergente de pesquisa e está diretamente associado à importância de se melhorar o ensino e a aprendizagem da Matemática. **Objetivos:** Conhecer os aspectos centrais da prática letiva de professores envolvendo o ensino de tópicos de Cálculo Diferencial no ensino secundário. **Design:** O estudo segue uma opção metodológica de natureza qualitativa, inscrita em um paradigma interpretativo, sendo concretizado por meio da realização de três estudos de caso. **Ambiente e participantes:** Participam do estudo três professores do ensino secundário português que lecionam Matemática A. **Coleta e análise de dados:** Os dados foram coletados por meio da observação direta de aulas e também através de entrevistas semi-estruturadas. A partir da análise dos dados, duas dimensões emergiram: (i) aspectos gerais e (ii) aspectos específicos da exploração didática. Os aspectos gerais apresentam três subcategorias: (i) estrutura das aulas; (ii) interações nas aulas; e (iii) trabalho com as tarefas. Os aspectos específicos também apresentam três subcategorias: (i) conectando conceitos; (ii) espaço para o aluno elaborar e apresentar o seu raciocínio; e (iii) interação entre os aspectos gráficos e algébricos. **Resultados:** Os resultados do estudo apontam para um uso quase exclusivo do exercício dos manuais sem que exista a utilização de outras tarefas. **Conclusões:** As conclusões evidenciam uma ênfase significativa por parte dos professores para a conexão entre os diferentes conceitos envolvidos. A abordagem adotada parece estar mais focada em questões conceituais do que em técnicas procedimentais.

Palavras-chave: Prática Letiva; Matemática; Ensino Secundário; Ensino do Cálculo Diferencial.

INTRODUCTION

The research community has focused a lot of attention on teachers in recent decades, leading to a notable evolution in how teachers are viewed. In the 1960s, teachers were studied from the point of view of their personality, whereby it was assumed that teachers with certain character traits (cordial, communicative) enabled better student performance (Good, Biddle & Goodson, 1997). In a later phase, interest shifted to what happened in the classroom, with special focus on the teacher's observable behaviour. The idea was that the students' learning was an immediate and direct consequence of the teacher's action in the classroom (Calderhead, 1987). This paradigm became known as the "process-product" but it proved extremely limited, as it reduced everything to the teacher's observable actions in the classroom. Taking into account the limitations of the "process-product" paradigm, interest moved onto the teachers' thinking processes or their mental life, on the assumption that what teachers do in the classroom is substantially influenced by what they think. This perspective, in turn, gave rise to a range of approaches. A first approach is linked to teachers' knowledge (Ball, Thames & Phelps, 2008; Shulman, 1986), recognising that teachers "possess a body of specialised knowledge" (Calderhead, 1987, p. 1). Another approach follows a more interpretative line, focusing on the teachers' beliefs and conceptions (Cooney, 1985; Pajares, 1992; Thompson, 1992). A third, more recent approach, focuses mainly on the teachers' practice (Boaler, 2003; Saxe, 1999).

This study follows the third approach, focusing on the practice of Mathematics teachers, specifically with regard to teaching Differential Calculus topics in secondary education. Its aim is to understand the core aspects of how secondary school teachers teach this topic. After this introduction explaining the backdrop to the study, the next section outlines the theoretical background, where we debate how Differential Calculus is taught, especially focusing on the teaching of this topic in secondary education; we then discuss the teaching practice, with special emphasis on the terminology and the teaching methods used. Afterwards, we outline and justify the methodological choices adopted and present the context and participants in the study. The following section is the analysis of the data, portraying the general and specific aspects of the didactic exploration carried out by the teachers. Finally, we discuss the three cases considered and we present the conclusions.

THEORETICAL BACKGROUND

a) Teaching differential calculus

How the topic of Differential Calculus is tackled in the secondary education syllabus is very different depending on each country. In Brazil, for example, Differential Calculus has not been included in the secondary education syllabus since the early 1960s (Carvalho, 1996; Spina, 2002), and was definitively removed from the Brazilian secondary school syllabus as part of the Modern Mathematics Movement reform (Ávila, 1991). In Portugal, in contrast, Differential Calculus topics have remained on the secondary education syllabus since the start of the 20th century (Santos, 2010). Aires and Vásquez (2005) state that a reform – the Carneiro Pacheco reform, dated 14 October 1936 – removed the study of derivatives from the syllabus. However, as the same authors say, the concept of derivative was reintroduced in the following reform, the Pires de Lima reform of 17 September 1947. Today, Differential Calculus topics are included in the secondary education Mathematics A curriculum in the Science and Technology Scientific-Humanities course and the Socioeconomic Sciences courses[1].

The debate about the place and nature of Calculus education has been especially intense in the United States. A reformist movement took place in the country in the second half of the 1980s (Hughes-Hallet, 2006; Schoenfeld, 1995) based on the growing concern about the quality of students' learning, which led to a "vigorous debate" in the research community (Tall, Smith, & Piez, 2008). Schoenfeld (1995) states that "this discontent paved the way for change" (p. 1) and three basic principles were suggested as regards the approach to Differential and Integral Calculus: (i) encouraging the use of technology; (ii) the so-called "Rule of three", in other words, an equal emphasis on graphical, numerical and analytical aspects – as well as encouraging a better understanding, "this approach gives students with weak manipulative skills a chance to grasp the concepts behind Calculus" (p. 4); and (iii) emphasis on problem-solving, modelling and conceptual understanding – "the development of conceptual understanding, not algebraic technique, should be the guiding force" (p. 3).

The reformist movements were mainly directed at the Calculus taught in higher education (both Differential Calculus and Integral Calculus), but also ended up influencing the teaching of Calculus at secondary education level. Given the specificity of this study, it is important to better understand the relationship between the Calculus taught at secondary education level and at higher education level. On this matter, two major American institutions, the National Council of Teachers of Mathematics (NCTM) and the Mathematical Association of America (MAA) asked how secondary schools and universities should plan the Calculus course, given that the course is situated in the transition from secondary education Mathematics to post-secondary education Mathematics. With the intention of contributing to this question, the two

[1] The Portuguese education system has 12 school years before entering higher education. Of these twelve years, nine are considered basic education, and the three last years are secondary education. In the three years of secondary education, the students are channelled to a group of higher education courses, with different Mathematics curriculums depending on the course: Sciences, Humanities, Technologies or Arts.

institutions issued a joint declaration, expressed in the 2012 note[2], with the following recommendations:

> Although calculus can play an important role in secondary school, the ultimate goal of the K–12 mathematics curriculum should not be to get students into and through a course in calculus by twelfth grade but to have established the mathematical foundation that will enable students to pursue whatever course of study interests them when they get to college. The college curriculum should offer students an experience that is new and engaging, broadening their understanding of the world of mathematics while strengthening their mastery of tools that they will need if they choose to pursue a mathematically intensive discipline.

b) Teacher's professional practice

The first question to ask is what we understand by "practice". As Ponte (2014) states, for an in-depth study, it is important to start by discussing this concept, given that "professional practice has been presented in the research literature from a variety of perspectives, often very simplistic, sometimes identified as the teacher's actions and other times from the point of view of the overall curriculum perspective" (p. 6).

Saxe (1999) defines practices as "recurrent socially organized activities that permeate daily life" (p. 1-25). Boaler (2003), dwelling precisely on classroom practices, defines them as "the recurrent activities and norms that develop in classrooms over time, in which teachers and students engage" (p. 1-3). This author believes the teaching actions change the opportunities created for the students, and she lists three types of teaching according to the structure conferred by the teacher: (i) *very structured*, which reduces the cognitive demands of the task, (ii) *unstructured* and with a high degree of freedom, which causes a certain frustration both for the teacher and the students, and (iii) *intermediate level* of freedom and structure, without, however, reducing the cognitive demand of the task.

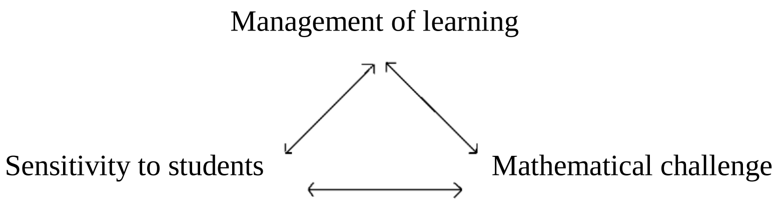
[2] The joint note can be read at: <https://www.nctm.org/Standards-and-Positions/Position-Statements/ Calculus/>

Aiming at conceptualising teacher’s professional practices, Ponte and Serrazina (2004) split them into three major groups: (i) teaching practices, (ii) professional practices in the institution, and (iii) training practices. With regard to the teaching practices, these authors set out five main aspects: (i) the tasks set, (ii) the materials used, (iii) communication in the classroom, (iv) curriculum management practices, and (v) assessment practices. In this study we consider the teaching practices, with special focus on the tasks set and the classroom communication, as we consider these two aspects bedrocks of the teaching practice. In relation to the tasks, Ponte (2005) highlights two essential dimensions: the degree of the mathematical challenge, related to the perception of the difficulty of a question, and the degree of structure, which varies between “open” and “closed”. Stein and Smith (1998) argue that the tasks set in the classroom are directly linked to the way they are presented by the teacher, the method used to organise the students’ work and the learning environment created. The authors propose a framework in relation to the mathematical tasks in the classroom, identifying three phases in the work on a task: (i) how it appears in the curricular material; (ii) how it is presented by the teacher; and (iii) how it is carried out by the students. They say the nature of the task may change when it goes from one phase to the next, emphasising factors linked to keeping high cognitive levels.

With respect to the teacher’s teaching practice, Potari and Jaworski (2002) present the theoretical construct called *Teaching Triads (TT).* The goals of this construct are both to encourage reflection by the teacher about teaching and to serve as an instrument for the analysis of practice. The model presents three domains: *management of learning, sensitivity to students* and *mathematical challenge.*

Figure 1

Teaching Triad (Potari & Jaworski, 2002, p. 353)



Potari and Jaworski (2002) say that the *management of learning* describes the teacher's role in constructing the learning environment in the classroom, which can include the grouping of students, planning of the tasks and the defining of norms. *Sensitivity to students* describes how the teacher attends to the needs of the students and the forms of interaction in the classroom. Finally, the *mathematical challenge* describes the difficulty of the challenge given to the students to incite mathematical thinking and activity.

METHODOLOGY AND CONTEXT OF THE STUDY

The present investigation took place within the scope of a PhD at the Institute of Education of the University of Lisbon – Portugal. Regarding ethical issues, the project was submitted to the Ethics Committee of the Institute of Education of the University of Lisbon, and was approved on February 19, 2019 (opinion number 609). Thus, respecting the Ethics Charter for Research in Education and Training[3] of the aforementioned Institute.

Aimed at identifying and understanding the core aspects of the teaching practice of secondary school teachers who teach Differential Calculus, we drew up the following research question: *How can one describe the didactic exploration of the Differential Calculus topics by the teacher in terms of the structure adopted, the approach used, the tasks set and the mathematical representations used?* This question, in turn, can be broken down into five sub-questions:

- How are the lessons structured?

- What is the nature of the interactions (communication) in the lessons?

- What does the teacher emphasise during the lessons?

- What kind of tasks are set and how does the teacher work on them in the lessons?

- How are the Differential Calculus topics explored from the point of view of the representations? Is a certain method of representing favoured?

The goal of the study and the need to appraise the meanings produced by the teachers led us to opt for a qualitative methodology underpinned by an interpretative paradigm (Denzin & Lincoln, 2006; Erickson, 2012). Therefore, we chose to carry out case studies (Stake, 2007; Yin, 2010) involving three

[3] Diário da República Portuguesa, 2nd series — N.º 52 — march 15, 2016.

teachers who teach Mathematics A in secondary schools in the Lisbon region. The choice of teachers who took part in the study (João, Maria José and Mariana) considered the fact the teachers have significant teaching experience in mathematics (over fifteen years) and are well integrated in their respective schools, each of them having worked at the school for over ten years when the data was collected.

The two female teachers, Mariana and Maria José, teach mathematics in the 11^{th} school year, while João teaches Mathematics to 12^{th} year students. Mariana has been a teacher for approximately eighteen years and teaches Mathematics to the 2^{nd} and 3^{rd} cycle basic education students (10-11 and 12-14 year olds). As for João, as well as his secondary school teaching activities, he is also the headmaster's assistant. Maria José has been a teacher for thirty-one years and currently teaches mathematics to 3^{rd} cycle students and is the coordinator of joint projects for the school she belongs to.

Focusing on the narrative dimension of the teachers' discourse, we attempted to interpret it (Merrian, 1988). Therefore, through direct observation of the lessons and by carrying out interviews, each teacher gave rise to a case study. The analytical part was split into three phases. The first phase occurred when the data was collected. The second phase occurred after the data collection when the aim was to write up the case. After each case was written, it was sent to the teachers for their appraisal, when they could ask for clarifications or suggest certain parts be rectified or erased. This was done with two goals in mind: (i) to obtain the teacher's validation as regards our interpretation, and (ii) to safeguard any ethical issues involving the teachers as agreed beforehand. After the validation, the third and final analytical phase comprised assessing the three case studies in conjunction.

The data analysis gave rise to two dimensions or categories, namely the general aspects and the specific aspects of the didactic exploration. Each of these two dimensions, in turn, was split into three subcategories. In the general aspects: (i) the structure of the lessons; (ii) the interactions in the lessons; and (iii) the work with the tasks. As for the specific aspects: (i) connecting concepts; (ii) time for the students to elaborate and present their reasoning; and (iii) interaction between the graphical and algebraic aspects.

GENERAL ASPECT OF THE DIDACTIC EXPLORATION

In this section, considering each of the three teachers who took part in the study, we present three general categories of didactic exploration (structure of the lessons, interactions in the lessons and the work with the tasks) that emerged from the data analysis.

a) Structure of the lessons

We can categorise João's lessons into two basic types: lessons doing exercises and presentation of a new concept. The lessons to present a new concept invariably involved four stages: (i) to start with, the teacher would *ask the students some initial questions* about the lesson topic. These questions served two basic purposes for the teacher: they provided a reason for why topics are selected and also trigger connections to other topics already studied; (ii) next, the teacher *shows a video* where the topic is presented (usually taken from the school textbook), pausing it to make clarifications; (iii) afterwards, *the teacher outlines a synthesis of the theory on the blackboard*; and finally (iv) *some examples are discussed and the students are asked to do a set of exercises*. With regard to João's exercise lessons, they are split into two parts: (i) initially *time is given for the students to work on the exercises*, when the teacher walks around the classroom assisting the students; and (ii) the second part involves *commented solving of the exercises*, whereby the solution is presented on the blackboard by the teacher or a student.

In Mariana's lessons a new concept is presented in three stages: (i) to start with, the teacher *presents the concept in question*; it is the chance for students to be intuitive, and as she made a point of emphasising, does not involve formalities; (ii) next, Mariana *discusses one or more examples* in which the concept is applied; and (iii) in a third stage, near the end of the lesson, a list of exercises is set. Not much time is set aside for the students to do the exercises, with the first two stages occupying almost the entirety of the lesson. Nearly always it is the teacher who solves the exercises on the blackboard.

Maria José's Differential Calculus lessons were given remotely (online lessons) given the state of emergency decreed in the country because of the pandemic caused by the Sars-Cov-2 virus. These lessons, as regards their structure, can be split into three stages: (i) the lesson begins with an *initial moment* that can take one of three forms: preliminary dialogue with the students about the topic to be studied; discussion of an exercise done in the previous lesson, or a review of previous concepts; (ii) the teacher then *shows a video where the concept is presented*, pausing from time to time to provide clarifications; finally (iii) *exercise solving* work is done. This final stage can take two forms: together, but with the teacher asking the students questions and then making a final summary, or with a chosen student sharing his/her solution, which is discussed in the class, at the end of which the teacher does a final summary on the blackboard.

b) Interactions in the lessons

Mariana seems to favour directly interacting with the class, by questioning the whole class in the three steps of the lesson. The second most common type of interaction is the teacher speaking one-to-one with the student, when the student raises a question or asks the teacher for clarification. Less frequent is the interaction among the students when they do the exercises. These last interactions are of a spontaneous nature, given that the teacher does not explicitly recommend that the work is carried out individually or collectively.

As for João, he seems to give preference to one-to-one interaction with a student and interaction among the students. This is closely related to the emphasis given by the teacher to the classroom exercises. With regard to the teacher's interaction with the student on an individual basis, this occurs when the exercises are being done and before the moment of discussion, and seems to allow the teacher to tackle a handful of the main questions, mistakes and alternative solutions that the students present, in order to prepare the ground for a discussion among the whole class, when these issues are taken up again by the teacher.

Maria José dedicates a lot of time to interaction with the whole class and also to interaction with a student in particular. It is very common, during the presentation phase of a concept or when the video is paused, for the teacher to ask a question to a specific student. During the solution-sharing phase by one student, an interaction takes place between that student and the teacher, when questions are asked and further clarifications are given. In this dialogue, it is not rare for the teacher to ask questions to the class to allow students to identify possible mistakes by their classmate or to make a contribution to the solution presented.

c) Working on the tasks set

With regard to the tasks set, João gave preference to working with the school textbooks at gradually increasing levels of difficulty, so the students could gain confidence. Meanwhile, even though he mentioned that the aspect of Differential Calculus the students most liked was its application that involves problem solving, this part was not worked on in the lessons, with strictly mathematical exercises. With regard to the exercises, João did not make them too structured, leaving a significant part of the work for the student, and therefore not reducing their degree of difficulty. The following excerpt illustrates both the type of task set, and how it was worked on: with the teacher encouraging interaction with the students, not simply giving a ready answer to their questions:

> The teacher set question 49 of the textbook to be done in the lesson. The question is as follows: "Consider the function *f*, of domain R^+, defined by $f(x) = 3x^2 + x + 2\ln(x)$. Calculate $f'(1)$, using the derivative at one point."
>
> The students worked on the question and swapped ideas. I noticed, right beside me, a student turn around and exchange some thoughts with two students behind him. They talked about the question. The teacher walks around the classroom and helps the students.
>
> I overhear a dialogue between the teacher and a student:
>
> Student: Teacher... but how can I do it?
>
> Teacher: Following the question... using the definition... what do we get?
>
> Student: [Remains quiet].
>
> The teacher then goes to the blackboard and writes: $f'(1) = \lim_{x\to 1} \dfrac{f(x) - f(1)}{x-1}$.
>
> Student: Ah... if it was 2... the *x* tends towards 2?
>
> Teacher: Yes... using the definition!

The teacher, still at the blackboard, writes: $\lim_{x \to 0} \frac{\ln(x+1)}{x} = 1$.

And corrects it straight away:

Teacher: This is going to be useful here [pointing to the recently written fundamental limit].

Student: Do I have to think about a change in variable?

Teacher: Yes... *x* tends towards 1... so *y*?

Student: Ah... I put *y* equal to *x* minus 1.

Teacher: So test it there... this question is for those who are not alert, because we have to carefully follow the question statement... if it asks you to use the definition and you don't do that... you're going to fail in a test or an exam... so read it carefully.

After around 15 minutes, the solution is presented on the blackboard by the teacher, who explains each of the steps:

$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1} \frac{3x^2 + x + 2\ln x - 4}{x-1}^{\frac{0}{0}} =$$

$$\underset{\substack{y=x-1 \\ x=y+1}}{=} \lim_{y \to 0} \frac{3(y+1)^2 + y + 1 + 2\ln(y+1) - 4}{y}$$

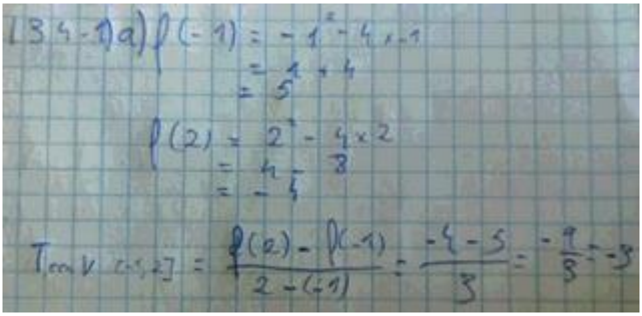
$$= \lim_{y \to 0} \frac{3(y^2 + 2y + 1) + y + 1 - 4}{y} + 2\lim_{y \to 0} \frac{\ln(y+1)}{y}$$

$$= \lim_{y \to 0} \frac{3y^2 + 6y + y + 3 - 3}{y} + 2x1 = \lim_{y \to 0} \frac{3y^2 + 7y}{y} + 2$$

$$= \lim_{y \to 0} \frac{y(3y+7)}{y} + 2 = 3x0 + 7 + 2 = 9$$

(João, *lesson record, 25 March*)

Similarly, Maria José states that a core issue when working with Differential Calculus involves the application of the concepts in problem solving, mentioning that this enables the students to visualise a meaning behind what they are learning. Meanwhile, this work was not done in the classroom, with the teacher explaining that time constraints prevented this. Therefore, the tasks set by the teacher were restricted to the exercises in the school textbook and also were limited to a strictly mathematical context, with the teacher attaching considerable importance to the proper use of the mathematical notation. Maria José seems to prefer working on the problems in the solutions of the students who were invited to explain their answers, as the following excerpt shows:

> The teacher starts by reading the exercise: “Given the real function defined by $f(x) = x^2 - 4x$. Calculate the average rate of variation between: a) -1 and 2.
>
> The student’s solution is shared on the screen.
>
> 
>
> Teacher: Minus 1 uhm... so the average rate of variation... in the interval minus one, two... is equal to the f of 2 minus the f of 1... over 2 minus -1... Can you spot any mistake?
>
> Student: The minus one there, shouldn’t that be between brackets?
>
> Teacher: The minus one has to be between brackets... right? [The teacher goes through the student’s answer speaking aloud]... so, as the variation rate gave minus 3, so it was negative, this means that the coefficient of the secant line is...
>
> Student: It’s negative too.
>
> Teacher: It’s negative too... Now can you write the equation of the secant line?
>
> Student: Yes.

Teacher: So, what is it?... *y* equals to minus 3...

Student: Minus 3*x*.

Teacher: Plus *b*... and we need another point on the line. Right?

Student: Yes.

(*Maria José, lesson record, 9 June*)

Mariana also gives preference to working with the textbook exercises and these are always related to a strictly mathematical context. The teacher did not explore the possibility of a real-world application of the Differential Calculus concepts, transmitting the idea that the work involving these concepts was more related to the acquisition of across-the-board skills, such as persistence, logical thinking and abstraction. The teacher sought to give a greater degree of structure to the solution, and proposed brief summaries (which she called "cheat sheets") that outlined the steps to be followed by the students, as the following excerpt shows. This option seems to considerably reduce the difficulty of the task. Mariana said this rather rigid structure was related to the difficulties experienced by this particular class, stating that in another class perhaps this more structured work would not be necessary.

The teacher went over the 4 steps adopted in studying the monotony of the function through the use of the derivative. The teacher again writes them on the blackboard:

1º) Derivative: f'(x);

2º) The derivative zeros: f'(x) = 0;

3º) The construction of the table;

4º) Writing the answer.

According to the teacher, most of the exercises can be done following these four steps. After discussing an example with the students, the next exercise from the textbook is proposed: "*g* is a function whose derivative is defined, in IR, with $g'(x)=(x-1).(x-3)^2$. The function *g* has relative extreme(s) in: A) only in x = 1; B) only in x = 1; C) only in x = 1 and x = 3; D) in x = 1, x = 2 and x = 3".

The teacher then writes on the blackboard:

$$g'(x) = (x-1).(x-3)^2$$

Teacher: Do you see how in this exercise the derivative is given... which we studied in the "g" line?

Students: The zeros.

The teacher then writes the answer on the blackboard to the equation g'(x) = 0.

$$\boxed{\begin{aligned} & g'(x) = 0 \Leftrightarrow (x-1).(x-3)^2 = 0 \\ & \Leftrightarrow x-1 = 0 \vee (x-3)^2 = 0 \\ & \Leftrightarrow x = 1 \vee x = 3 \end{aligned}}$$

Teacher: Now we have to do the last...

Teacher: What is the degree?

Student: Second... no... third.

Teacher: Now we have to do it by steps... building a table...

(*Mariana, lesson record, 24 May*)

SPECIFIC ASPECTS OF THE DIDACTIC EXPLORATION

In this section we point out three specific categories of the didactic exploration that emerged from the data analysis (connecting concepts; time set aside for the students to elaborate and present their reasoning; and interaction between the graphical and algebraic aspects). Taking into account the emphasis conferred by the narrative dimension of the discourse (in the language of the participants) we insert excerpts from the lessons and the interviews.

a) Connecting concepts

João makes the connection between the concepts mainly in the phase immediately prior to the presentation of a new concept. This was observed, for example, when the logarithmic function was introduced, when the teacher looked again at the bijection concept and the inverse function, and based on the inverse function of the exponential function, discussed previously in a lesson, the logarithmic function was introduced. The teacher made a point of mentioning that in this introductory step it is always desirable to present the subject in question in a sequence alongside other subjects that the students are familiar with. He argued this was to show that there was a reason why each subject is studied, and that they are not selected at random.

Mariana also gives considerable emphasis to the connection and sequence of the concepts worked on in the lesson. She says that in Mathematics, one always ends up studying the same things. However, in the attempt to broaden out more, and in this particular area, the connection between the concepts is essential. According to the teacher, the connection between the concepts worked on is needed and a more intuitive initial approach is more suitable. This emphasis on fitting the work into a sequence is also, according to Mariana, related to the possibility of guiding the student towards an understanding of the concepts (and not simply memorising them). For this to happen, she stresses that the concepts must not be taught in isolation. The two following interview excerpts attempt to demonstrate the emphasis conferred by the teacher:

> It was exactly the moment that you saw, when they understood how you go from the average variation rate, which is a very basic concept for them... and straight away associated it to average velocity; they all understand what average velocity is and they all understand that really, that only gives us one piece of information in relation to the initial point and the final point and not what happens in between, and... they understand why you go from there to, to an instant and the application of the limit, which they themselves arrived at, didn't they?... if x has to tend to a.
>
> *(interview, topic 1, p. 2)*
>
> I think that they even... I still think it went relatively very well... I think they managed to understand... ah the reason for concluding the relationship between the sign of the derivative and the sign of the monotony of the function... and... and that is the part that I focused on with the graph... so they would understand... that this is, is... basically we are always seeing the same thing, but broadening it... so the question of monotony

was studied last year... but they only managed to study, it wasn't... they only saw the function graphically... and it's for them to understand that now we are going to do this without the graph... it's important that they understand that previously they did things through visualisation alone, and now they are going to justify these things analytically.

(interview, reflection 2, pp. 6-7)

Maria José, in turn, categorically stated in her lesson that the mathematical knowledge would be part of a sequence. She commented on the need for the students to study the limits, as she put it, "in a proper manner," to understand what would be taught next, namely the derivatives. The teacher even created a term for the sequencing of these mathematical subjects, calling it "parallelism". As an example, she stated that this "parallelism" occurred when she defined the derivative of a function at a point as the slope of the tangent line according to that point, and immediately afterwards deducing the equation of the line (a subject the students knew) using the derivative concept. In another situation, while she dealt with the differentiability and the discontinuity of functions, she implemented the language of logic (a subject studied earlier by the students) to organise a summary, as shown in the following excerpt:

> Teacher: So, I'm saying if it is discontinuous this means it has no derivative, right? I'm making the connection that one affirmation implies the other [and she writes the following]:
>
> $$\sim q \Rightarrow \sim p$$
>
> Teacher: This is the same thing as saying what? That *p* implies *q*...
>
> $$\sim q \Rightarrow \sim p$$
> $$p \Rightarrow q$$
>
> Teacher: We've seen that these two were equivalent... so how can I put this another way... a function with a finite derivative at a point is continuous, right? So, as I said… yesterday we saw that if *f* is discontinuous... if the function is discontinuous then it has no derivative... but this affirmation, as we have seen through logic, is equivalent to saying that *p* implies *q*... so I can

> say the following: That if the function has a finite derivative at a point it is continuous or it is differentiable at that point, OK?
>
> Teacher: So, a real function of real variable is said to be differentiable at a point when it has a finite derivative at this point... uhm... now you can turn to page 107.
>
> *(Maria José, lesson record, 18 June)*

b) Time for the students to elaborate and present their reasoning

Mariana said in one of the first interviews that she liked calling a student up to the blackboard to present and justify his/her lines of thinking about a given question. She would use this technique to show the students that different lines of thinking can lead to the same result. However, these situations did not take place in the lessons observed, and it was almost invariably left to the teacher to present the solution of the exercises on the blackboard. Hence, the lessons observed did not have time allotted for the presentation and justification of the students' thought processes. Even if the students worked on the exercises, they did so in a way more centred on the teacher, who gave a considerable structure to this task.

As for João, he allotted a considerable amount of lesson time for the students to elaborate, present and justify their reasoning, because he believes that the students worked very little (almost not at all) outside the classroom. The teacher called this "experimentation" time, a time where the students could try trial and error, then try again, where they could always ask for help from the teacher or classmates. In this experimentation phase the teacher seems to map out (while he walks around the classroom providing help) the main difficulties and reasoning adopted before taking up the general discussion again. The presentation and justification of a student's reasoning are included in the general discussion of the exercises. On these occasions, a student will be invited or will volunteer to present his/her ideas and the teacher, as well as questioning the student about the reasoning presented, will also try to involve the class in this discussion.

In the case of Maria José, as mentioned above, the lessons took place online. This ended up limiting both the lesson time and the individual support that could be given to the students as they elaborated and presented their reasoning. Given the circumstances, according to the teacher, the students should, ideally, work more independently. However, she added that this did not always happen. Owing to the circumstances, she made an effort to make herself more available to clarify students' questions and set aside time in the virtual environment for the students to ask her about their questions and present their solutions. Therefore, the most intense work ended up being the presentation and justification of the students' reasoning. This work was carried out during the students' problem-solving session, when they were invited to present and explain their solutions. In these moments, the teacher tries to involve the whole class, emphasising the aspects relative to the clarity of the presentation and the care that should be taken with the mathematical language adopted. Finally, a summary about the question was written by the teacher on the interactive blackboard, when the screen was shared with the students.

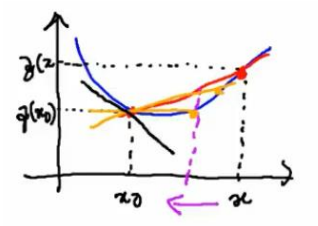
c) Interaction between the graphical and algebraic aspect

Mariana strongly emphasised the interaction between the graphical and the algebraic aspects in the work involving Differential Calculus, saying that she preferred, whenever possible, to present the topics graphically. As an example, when she presented the concept of the derivative as an instantaneous variation rate, there was a strong graphical element to her explanation, and only afterwards, she introduced the algebraic definition. In another situation, after discussing an exercise that dealt with solving a derivative algebraically at a point of a function defined by branches, the teacher made a point of showing the problem in graph form, highlighting the point at which the function was not derivable.

Maria José also puts a big emphasis on the interaction between these two aspects. Like Mariana, she introduced the concept of the derivative of a function with a graphical explanation, and even planned a GeoGebra animation to do so. However, a problem with the software meant the teacher had to improvise and draw graphical sketches by hand. The following excerpt, obtained through the transcription of an online recorded lesson (graphs produced by the teacher on the interactive board), aims to illustrate this process. The teacher made a point of saying it was very important for the students to be able to understand both forms and that the teacher should not overly focus on one to the detriment of the other. Maria José's liking for technology and her belief in using it through the GeoGebra dynamic geometric software seemed to help her work as regards reinforcing the interaction between the graphical and algebraic aspects.

> Teacher: Imagine the *x* here, with *x* nearing zero... I'm going to shorten this interval, so... imagine that the *x* is here; what are you going to sketch? So, what would the secant line be like? It would be something like... if the *x* was here, the secant line

would always be like this... so I'm going to bring the x closer and closer to the x_0, in other words, I want to see the limit at which x tends towards x_0... and then, we will get to a point when we will be right here at the x_0 and so I have a line of that form which is the tangent line... when it gets closer it's going to be more or less like this.



Teacher: So the slope of the tangent line that is equal to the limit when the x nears the x_0... we are considering the interval so we are going to calculate this variation equal to the derivative of the function at a point.

$$m_t = f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

The teacher then presents the second definition of the derivative by the limit:

Teacher: So, let's write this limit another way, replacing the x with x_0 plus h... I mean that definition can be written this way, which will be equal to having what? limit when h tends to zero... here what is equal to x? x_0 plus h... minus the f of x_0... over x minus the x_0 is the h... this is also the derivative at a x_0 abscissa point... which is equal to the slope of the tangent line.

$$m_t = f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Se $h = x - x_0$ então $x = x_0 + h$

$x \to x_0$ então $h \to 0$

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

(*lesson record, 17 June – Maria José*)

As for João, in some situations the interaction between the graphical and the algebraic aspects was more obvious than in others. For example, in the lessons about the exponential and logarithmic functions, this interaction was more obvious, with the teacher emphasising either the graphical constructions of the respective functions, or their analytical expressions. However, in the work that involved calculating the limits, the algebraic representation took precedence.

DISCUSSION

In this section, by grouping the three teachers who took part in the study together, we discuss what has been presented above. We highlight that, as a qualitative study of a heavily explorative nature, the discussion and conclusions presented here aim to broaden the understanding and knowledge about the phenomenon under analysis. Hence, the ideas outlined here do not intend to establish any kind of comparison among the teachers, assess them or suggest any type of generalisation that can be extrapolated to all Mathematics teachers. We begin by discussing the general aspects, then move onto the specific aspects of the didactic exploration.

a) General aspects of the didactic exploration

With regard to the structure of the lessons, the three teachers showed points of convergence and also significant differences. A common point between Maria José and João, for example, is the use of videos to introduce a concept, where both of them paused the video to ask the students questions, clarify questions or clear up any details not understood. Mariana did not use videos in her lessons. However, Maria José and João used video in significantly different ways. João gave a summary straight after the video, which was not done by Maria José. These two teachers also asked initial questions to prepare for the video that would be shown and set aside significant lesson time for the students to work independently, which did not happen in Mariana's lesson.

As regards the lesson interactions, they seem to be closely associated to the structure and the focus each teacher wants to bring to the lesson. Mariana appears to prefer more direct contact with the students, asking questions to the class as a whole. This method is aligned to the more "teacher-centred" approach described by Boaler (2003). As for João, he puts a bigger focus on the students, favouring a more individualised contact between the teacher and the student, which he believes serves two purposes: (i) it helps the students, and (ii) identifies situations that can be exploited later. Maria José's interaction with one particular student seems to be linked to the opportunity to work on the reasoning of the students and involve the class in the discussion, and learn from this based on the solution presented (both as regards the structure and the content). Therefore, the interactions favoured by João and Maria José are linked to the intended focus, more centred on the teacher's questions (Boaler, 2003) and also on the discourse shared between the teacher and the students (Ponte & Serrazina, 2004).

In relation to the tasks worked on in the lesson, these are basically exercises taken from the school textbook (Ponte, 2005). However, each teacher works on the exercises in the lessons in a very different way, in direct correlation to maintaining (or not) the level of mathematical difficulty and the cognitive demands of the task (Stein & Smith, 1998). João and Maria José sought to leave a considerable part of the work to the students, providing sporadic support and not simply giving the answers, and above all setting time aside for the students to work on the task in their own time. These approaches are compatible with the intermediate level of freedom proposed by Boaler (2003). In contrast, Mariana opts for a more structured approach when it comes to solving the exercises (Boaler, 2003), which significantly reduces the mathematical challenge of the task.

Another point of convergence is the fact that these exercises are confined to strictly mathematical contexts. Nevertheless, even in sticking to a strictly mathematical context, the three teachers repeatedly emphasised the importance of the connection between the different concepts involved, stressing the understanding of the concepts more than the procedural steps to be carried out. As for applying the concepts to real-life situations, João and Maria José were more in favour, which was not an approach shared by Mariana.

b) Specific aspects of the didactic exploration

The three teachers put a big emphasis on following a logical sequence in tackling the subjects in the lessons. João did so mainly upon the introduction of a new concept. Mariana emphasised the connection and the sequence of subjects through the lesson, underlining that this connection is both important and necessary. Maria José seems to adopt a similar approach, saying that the mathematical knowledge will be imparted in a sequence, labelling the method "parallelism" whereby one subject that is known by the students is used to introduce a new subject.

As for the time given to the students to elaborate and present their reasoning, this was very different depending on each teacher. This time was less forthcoming in Mariana's lessons, while in João's case, the students are given time to elaborate their reasoning, mainly during the "experimentation" phase. The justification and presentation of the students' reasoning comes during the discussion phase, when one student presents his/her solution to the class. Maria José seems to give more emphasis to the students' justification and presentation of their reasoning, encouraging these moments by sharing the students' solutions, when she invites one student to present his/her reasoning and answer questions about the presentation.

Hence, taking into account the model of Potari and Jaworski (2002), João and Maria José seem to present a considerable level of *sensitivity to students*, which is translated into more time set aside for the students to elaborate, present and justify their reasoning in the lesson, However, these two teachers also gave significant emphasis to the domains of *management of learning* and sticking to the *mathematical challenge* in the tasks set. Mariana, in turn, seems to give more emphasis to the domain of *management of the learning*, with the lesson more focused on the teacher, in contrast to the *sensitivity to students* domain. The level of the *mathematical challenge* of the tasks set seems to be considerably low, given the highly structured approach adopted by the teacher.

The interaction between the graphical and the algebraic aspects seems to be a crucial concern of Mariana and Maria José's in their teaching of this topic. This was obvious in the way the two teachers introduced the concept of the derivative. Moreover, the recurrent use of GeoGebra by Maria José appeared to be an asset as regards consolidating this interaction. As for João, it was not possible to detect any emphasis given to the interaction between the graphical and algebraic aspects, with the algebraic component given significant preponderance in the work involving the calculation of limits.

CONCLUSION

In relation to the teaching of Differential Calculus, the conclusions point to almost exclusive use of the textbook exercises by the teacher without using other tasks. Another point to be highlighted is the Differential Calculus work in strictly mathematical contexts, with the absence of any interaction with other areas of knowledge or the possibility of applying such subjects to real-life situations (modelling). Meanwhile, even in strictly mathematical contexts, a very obvious emphasis was given by the three teachers to the connection between the different concepts worked on, whereby they sought to point out a close relationship between the concept of the topic being studied and the knowledge learned previously by the students. With regard to the approach taken by the teachers, this seems to be more focused on conceptual questions than only on repetitive calculation procedures and techniques (procedural techniques).

Another issue to point out is related to the emphasis given specifically in the case of the two female teachers of this study to the interaction between the graphical and algebraic aspects of working with Differential Calculus, specifically in the lessons involving the introduction of the concept of the derivative of a function at a point. Furthermore, one of the teachers seemed to integrate these two aspects more completely, despite the constraints caused by the fact the lessons where online, as she was the teacher who made most intensive and recurrent use of the technological resources, specifically the GeoGebra software.

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DECLARATION OF AUTHORS' CONTRIBUTION

J. P. P. supervised the PhD study of A. C., the research in which this article is based. Data collection and preliminary analysis was made by A. C.. Both authors discussed the planning and contributed to the final version of this article.

DECLARATION OF DATA AVAILABILITY

The data produced and which backs up the results of this study can be supplied by the authors upon reasonable request.

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