# Economics and Mathematical Theory of Games

Ajda Fošner

The theory of games is a branch of applied mathematics that is used in economics, management, and other social sciences. Moreover, it is used also in military science, political science, international relations, computer science, evolutionary biology, and ecology. It is a field of mathematics in which games are studied. The aim of this article is to present matrix games and the game theory. After the introduction, we will explain the methodology and give some examples. We will show applications of the game theory in economics. We will discuss about advantages and potential disadvantages that may occur in the described techniques. At the end, we will represent the results of our research and its interpretation.

*Key Words:* the theory of games, matrix games, economics
JEL *Classification:* C7

## Introduction

We all know a lot of different games that are played for relaxation or a financial gain. There are also a lot of people (young and old) that are playing games every day. Some of these games (roulette, for example) involve no skills and are purely games of chance. We will not be interested in this kind of games. However, on the other side there are games (chess, for example) that are entirely games of skills. Moreover, there are also games (football, for example) that involve both, chance and strategy. Finally, there are so called *games of strategy and conflict.* They involve choices of alternative strategies, conflicting interests of the players, and payoff to the players. Playing the stock market, developing real estate, conducting a business against competitors – these are examples of such games. Actually, these and other similar activities would not ordinarily be thought of as games but they are games in the sense that we have just described.

The game theory started to develop in 1944 with the book *Theory of Games and Economics Behavior,* written by John von Neumann and Oscar Morgenstern. This book was a major step in the use of mathematical analyses to solve some problems in the modern society. In 1970s the theory of games was applied to biology. Nowadays, the game theory is an

*Dr Ajda Fošner is an Associate Professor at the Faculty of Management Koper, University of Primorska, Slovenia.*

*Managing Global Transitions* 10 (3): 245–256

TABLE 1 Price decision

| | $C_2$: raise | $C_2$: no change | $C_2$: lower |
|---|---|---|---|
| $C_1$: raise | 0 | −10 | −50 |
| $C_1$: no change | 3 | 0 | −10 |
| $C_1$: lower | 50 | 3 | 0 |

important tool in many fields. Until now, eight game theorists have won the Nobel Memorial Prize in Economics Science. Let us also mention that John Maynard Smith was awarded the Crafoord Prize for applications of the game theory to biology.

Usually we speak of a game as a process of interaction that involves individuals (two or more). Game is a subject of some fixed rules and has a specified collection of payoffs according to every possible outcome. As we already mentioned, there are games that are played for relaxation. On the other hand, there are games that are studied by the scientists. These games may be far from amusing, as it is illustrated by the following examples.

The first example is actually an example of a game, which we can be found in economics.

### EXAMPLE 1

Suppose that a specific product is manufactured only by two companies, the company $C_1$ and the company $C_2$. Of course, they are in competition for the entire market. We also know that the first company $C_1$ is the larger company and it has a larger share of the market. Every January both companies have to decide whether to raise, lower, or not change the price for the product. In table 1, the gains and losses (in millions of euros) for the first company corresponding to the various possible pairs of decisions are represented.

The natural question here is: What decision should the first company make according to the table 1? This is the point, where we can use the theory of games and mathematical analyses to solve the problem.

The next example is so called *battle of the sexes.*

### EXAMPLE 2

Mary and George decided to go out on a date this evening. Of course, they have to decide where to meet and what to do together. They have two possibilities: they can meet in the center of the city and go to the cinema, or they can go dancing to the nearby dancing club. Mary likes

TABLE 2 Battle of the sexes

| | G: dancing | G: cinema |
|---|---|---|
| M: dancing | M: 3<br>G: 2 | M: 1<br>G: 1 |
| M: cinema | M: 0<br>G: 0 | M: 2<br>G: 3 |

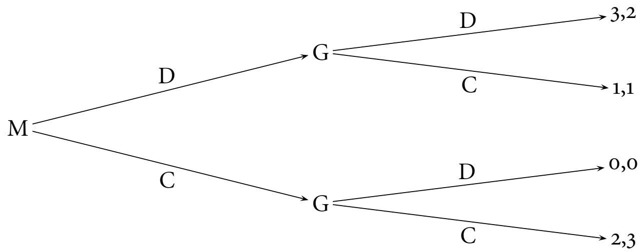


FIGURE 1 Battle of the sexes

to dance and she would prefer to go to the dancing club. On the other hand, George would like to go to the cinema. In any case, they both prefer doing something together than canceling the date.

The payoffs of this game are represented in table 2. The numbers quantify the utilities obtained by Mary and George for each combination (the first number is for Mary and the second one for George). We can present the outcomes of the battle of the sexes also with a graph (figure 1).

Here, each vertex represents a point of choice for a player (M – Mary, G – George). The lines out of the vertex show a possible action for that player (D – dancing, C – cinema). The payoffs are specified at the end of the graph (Vega-Redondo 2003).

More examples can be found in (Mizrahi and Sullivan 1993), (Khoury and Parsons 1981), and (Brown and Brown 1977).

## Methodology and Examples

The mathematical theory of games deals with the situations in which two or more persons with conflict interests are involved. The outcome of such games depends on some chance, but primarily on skills and intelligence of the participants. In some certain areas of economics, politics, military science, and operations research there are many conflicting situations to which the theory of games can be applied.

In this section, we shall discuss only about two-person games (primarily zero-sum two person games) because of the difficulties that arise in the mathematical theory of $n$-person games. We will also assume that the players play as well as it is possible.

First, we will introduce the matrix game played by two players. Let $G$ be a $m \times n$ matrix of some real numbers (here are no restrictions: numbers can be positive, negative, or even zero). This matrix is also called the *payoff matrix* of a given matrix game which is played by two persons. One of the players is the row player $R$ and the other player is the column player $C$.

$$G = \begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ g_{31} & g_{32} & \cdots & g_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ g_{m1} & g_{m2} & \cdots & g_{mn} \end{bmatrix}. \tag{1}$$

Let us point out that the entries $g_{ij}$ of a matrix $G$ represent the payoff from $C$ to $R$: numbers $g_{ij}$ are 'wins' for a row player and 'looses' for a column player. Here, of course, a negative win is a loss and a zero win is a draw.

How do we play this game? The row player $R$ chooses a row (the natural number between 1 and $m$) and the column player $C$ chooses the column (the natural number between 1 and $n$) of a given matrix $G$. Then they simultaneously tell their choices. Of course, before that the players must not disclose their choices to each other. The most common way to play this game is that both, the row player and the column player, write their choices secretly on a slip of paper and then they simultaneously expose the written number.

Now, suppose that $R$ chooses row $i$ ($i \leq i \leq m$) and $C$chooses column $j$ ($1 \leq j \leq n$). Then the number $g_{ij}$ in the matrix $G$ is the chosen number. If $g_{ij} > o$, then $C$ pays $R$ exactly $g_{ij}$ EUR. On the contrary, if $g_{ij} < O$, then $R$ pays $C$ the amount of $-g_{ij}$ EUR. In the case of $g_{ij} = 0$ no payments are made.

Let us show one simple example of a matrix game.

EXAMPLE 3

$$G = \begin{bmatrix} -1 & 10 \\ 1 & 5 \end{bmatrix}. \tag{2}$$

Suppose that the above matrix represents the payoff matrix of a matrix game. If we look at the numbers in the given matrix, then we see that the row player can win 10 EUR by choosing the first row (this is the best outcome for him). On the other hand, the column player will probably not choose the second column since in that case he will definitely lose at least 5 EUR. However, if he chooses the first column, he can lose just 1 EUR or even win 1 EUR. Thus, if *C* is intelligent, he will choose the first column. And if the row player concludes this assumption, he will decide to play the second row and win 1 EUR.

Therefore, the result that we conclude is: the row player will play the second row and the column player will choose the first column. Therefore, *R* will win 1 EUR and *C* will lose 1 EUR.

From the matrix *G*, we can easily see that the row player will always win if he chooses the second row. On the other hand, he can lose, if he tries with the first row. Hence, the row player can guarantee that he will always win at least 1 EUR by choosing the second row and the column player can guarantee by playing the first column that he will get no more than that.

We can say that there is a sort of equilibrium in this game and the value 1 EUR represents and illustrates the so called *rationality assumption.* We assume that we should choose our strategy in such a way that the worst thing that could happen to us is as good as possible. Therefore, in this kind of games the rational assumption is that we expect to win only what we can guarantee for ourselves.

## STRATEGIES IN MATRIX GAMES

If the row player of a matrix game *G* chooses row $i$, $1 \leq i \leq m$, the worst thing that can happen is that he would win (or lose, if the entry is negative) the least entry in the chosen row. Thus, he would choose the row whose least entry is as big as possible. In other words, he would play any row whose least entry has the value

$$\max_{1\leq i\leq m}(\min_{1\leq j\leq n} g_{ij}). \quad (3)$$

This is called *the maximin strategy.*

Similarly, if the column player of a matrix game *G* chooses row $j$, $1 \leq j \leq n$, the worst thing that can happen is that he loses the largest entry in the chosen column. Therefore, if the column player tries to make his outcome as good as possible, he will choose the column whose largest entry is as small as possible. That entry has the value

TABLE 3 Maximin strategy and minimax strategy

| | | | | | Row minima |
|---|---|---|---|---|---|
| | 0 | 3 | 1 | 4 | 0 |
| | 3 | −2 | 4 | −1 | −2 |
| | 3 | −3 | 2 | −2 | −3 |
| Column maxima | 3 | 3 | 4 | 4 | |

$$\min_{1\leq j\leq n}(\max_{1\leq i\leq m} g_{ij}). \tag{4}$$

This is *the minimax strategy.*

In the Example 3, the maximum of the row minima is 1 and also the minimum of the column maxima is 1. This value is in the second row and first column ($g_{21} = 1$). Thus, maximin strategy for the row player is to play the second row and the minimax strategy for the column player is to play the first column.

In the above example, the maximin value is equal to the minimax value. The following example of a matrix game will show that this is not always true.

EXAMPLE 4

Let us consider a matrix game with the payoff matrix.

$$G = \begin{bmatrix} 0 & 3 & 1 & 4 \\ 3 & -2 & 4 & -1 \\ 3 & -3 & 2 & -2 \end{bmatrix}. \tag{5}$$

Table 3 will help us to calculate maximin and minimax values.

Now we can easily calculate that the maximin value is equal to 0 and the minimax value is 3. Therefore, the maximin strategy for the row player is to play the first row and the minimax strategy for the column player is to play the first or the second column. This means that the column player has two possibilities: he can choose either column 1 or column 2. Nevertheless, the most important here is that the maximin value is not equal to the minimax value.

If the row player plays the first row, he will not do worse than gain nothing. On the other hand, if the column player plays either the first or the second column, he will not do worse than lose 3 EUR. This means that if the row player chooses the first row and the column player plays the first column, then both players gain nothing. On the other hand, if the row player chooses the first row and the column player plays the second column, then the row player win 3 EUR.

Let us also mention that one of the players has to do better than lose on any given play. This points out that the maximin and minimax strategies are generally too cautious. Moreover, this game is unstable. In other words, if one of the players insists on always playing a particular strategy, then the other player can take the advantage of this fact. For example, if the row player decides to play the first row all the time, then (after a few plays) the column player will always choose the first column. Thus, both players will gain nothing. However, if the player $R$ notices that $C$ is always playing the first column, then the row player will switch to the second row. Thus, $R$ will win 3 EUR. Nevertheless, after a few plays $C$ will switch to column 2 and start taking 2 EUR from the row player. Furthermore, when the row player notices that $C$always plays column 2, he will start to choose row 1. Therefore, we are back at the beginning. This shows that no one strategy is good for either player if this game is played many times.

In the last few decades, a lot of mathematicians have been studied matrix games and the game theory. They have done a tremendous work at this field. Let us just mention the recent result of Akain, Gaubert, and Guterman (2011). They showed that several decision problems originating from max-plus or tropical convexity are equivalent to mean payoff (zero-sum, two players) game problems.

The following problems are basic in max-plus or tropical algebra. The problems are taken from Akain, Gaubert, and Guterman (2011). See also Akain, Gaubert, and Guterman (2010).

### PROBLEM 1

Given $m \times n$ matrices $A = (A_{ij})$ and $B = (B_{ij})$ with entries in $\mathbb{R} \cup \{-\infty\}$, does there exist a vector $x \in (\mathbb{R} \cup \{-\infty\})^n$ non-identically $-\infty$ such that the inequality $Ax \leq Bx$ holds in the tropical sense, i. e.,

$$\max_{1 \leq j \leq n}(A_{ij} + x_j) \leq \max_{1 \leq j \leq n}(B_{ij} + x_j) \tag{6}$$

for every $i \cup \{1, 2, \ldots, m\}$?

### PROBLEM 2

Given $m \times n$ matrices $A = (A_{ij})$ and $B = (B_{ij})$ with entries in $\mathbb{R} \cup \{-\infty\}$, and two vectors c, d of dimension $n$ with entries in $\mathbb{R} \cup \{-\infty\}$, does there exist a vector $x \in (\mathbb{R} \cup \{-\infty\})^n$ such that the inequality $Ax + c \leq Bx + d$ holds in the tropical sense, i. e.,

$$\max(\max_{1 \leq j \leq n}(A_{ij} + x_j), c_i) \leq \max(\max_{1 \leq j \leq n}(B_{ij} + x_j), d_i) \quad (7)$$

for every $i \cup \{1, 2, \ldots, m\}$?

PROBLEM 3

Given $m \geq n$ and an $m \times n$ matrix $A = (A_{ij})$ with entries in $\mathbb{R} \cup \{-\infty\}$, are the columns of $A$ tropically linearly dependent? For example, can we find scalars $x_1, x_2, \ldots, x_n \in \mathbb{R} \cup \{-\infty\}$, not all equal to $-\infty$, such that the equation $Ax = 0$ holds in the tropical sense, meaning that for every value of $i \cup \{1, 2, \ldots, m\}$, when evaluating the expression

$$\max_{1 \leq j \leq n}(A_{ij} + x_j) \quad (8)$$

the maximum is attained by at least two values of $j$?

Akain, Gaubert, and (Guterman 2011) proved that the first problem is equivalent to the existence of a winning initial state for a mean payoff game problem. The second problem can be transformed in linear time to the problem of knowing whether a prescribed initial state of a mean payoff game is winning, and vice versa. Moreover, the third problem can be transformed in quadratic time to the problem of the existence of a winning initial state in a mean payoff game.

## Using the Game Theory in Economics

As we already mentioned, economics is one of the major fields where the game theory is used. Bargaining, fair division, social network formation, and voting systems auctions are just some of the economics phenomena, which are analyzed with the use of a game theory. Note also that the payoff of the game represents the utility of individual players and in modeling situations this is money. In other words, money corresponds to the individual's utility. However, here we have to be careful because this assumption can be faulty (Vega-Redondo 2003).

Scientists usually focus in their research on the sets of strategies, which are known as equilibrium in games. The most famous is the *Nash equilibrium.* A set of strategies is the Nash equilibrium if no player can do better by unilaterally changing his strategy. In other words, the Nash equilibrium is a concept of a game in which each player is assumed to know the strategies of the other players and no player has anything to gain by changing only his own strategy unilaterally. For example, David and John are in the Nash equilibrium if David is making the best decision taking into account John's decision and John is making the best decision taking into account David's decision. Let us point out that this does not necessary mean the best payoff for David and John. Usually players might improve their payoff, if they choose another strategy.

When analyzing some specific game, scientists usually start by presenting a game. This game is an abstraction of an economic situation. Then some solution concepts are chosen. At the end, the researchers illustrate which strategy sets are the equilibrium of the appropriate type. Here we can use either a *descriptive analyses* or a *prescriptive, normative analyses.* Some researchers believe that by finding the equilibrium of a studied game they can predict how actually human population will behave when they are in the situations analogous to the game. This is a descriptive way of analyses. Nevertheless, in normative analyses scientists see a game theory as a suggestion for how people ought to behave in a specific situation and not as a predictive tool for the behavior of human beings.

In the following, we will present a fundamental problem in the game theory, the so called *prisoner's dilemma,* which demonstrates why two people might not cooperate even if it is in both their best interests to do so.

In the prisoner's dilemma, two players can choose either cooperative or defective move. If both players cooperate then they both gain, but if one of the players defects, then he will gain more and the other, who cooperates, will gain less. And finally, if both players defect, both gain little.

### Example 5

A classical example of the prisoner's dilemma is presented as follows: Two criminals (say,$P_1$ and $P_2$) are arrested under the suspicion that they have committed a crime. However, the police do not have enough proofs to convict the suspects. Thus, the police separate both prisoners and visit each of them to offer the same deal: the one who offers evidence against the other one will be freed.

If both prisoners remain silent, then both are sentenced to only four months in jail. In fact, they are cooperating against the police and they both gain. However, if one of them betrays the other one and the other remains silent, the betrayer goes free (he gains more) and the other one will be punished with six-year sentence. If both prisoners betray, both will be punished (two years in jail), but less than if one of them refused

TABLE 4 The prisoner's dilemma

| | $P_1$: silent | $P_2$: betrays |
|---|---|---|
| $P_1$: silent | $P_1$: 4 months<br>$P_2$: 4 months | $P_1$: 6 years<br>$P_2$: goes free |
| $P_1$: betrays | $P_1$: goes free<br>$P_2$: 6 years | $P_1$: 2 years<br>$P_2$: 2 years |

TABLE 5 An example of the prisoner's dilemma

| | B: cooperates | B: defects |
|---|---|---|
| A: cooperates | A: 5 points<br>B: 5 points | A: –10 points<br>B: 10 points |
| A: defects | A: 10 points<br>B: –10 points | A: 0 points<br>B: 0 points |

to talk. The dilemma resides in the fact that each prisoner has a choice between only two options, but cannot make a good decision without knowing what the other one will do (table 4).

Therefore, the question is: how should the prisoners act?

In the next example, we will present the prisoner's dilemma between two players which can be applied to everyday life.

### EXAMPLE 6

In many situations the kind of distribution of losses and gains, that we have described in the above example, seems natural. For simplicity, we might consider the prisoner's dilemma as follows: if both players defect, then they both get 0 points. If only one defects, he will get 10 points and the cooperator 10 points. Finally, if both players cooperate, then each of them gets 5 points. Of course, there would always be a temptation to defect, since the gain for mutual cooperation is only 5 points and the gain for one-sided defection is 10 points (table 5).

The problem in the prisoner's dilemma is that if both players were purely rational, they would never cooperate, since the rational decision means that you make the decision, which is best for you whatever the other actor chooses. Hence, if both decision makers are rational, both will decide to defect and none of them will gain anything. However, if both would cooperate, both would gain. This paradox is formulated more explicitly through so-called *principle of suboptimization:* optimizing the outcome for a subsystem will in general not optimize the outcome for the system as a whole (the whole is more than the sum of its

TABLE 6 The prisoner's dilemma in the advertising

| | $F_2$: advertising | $F_2$: no advertising |
|---|---|---|
| $F_1$: advertising | $F_1$: loses<br>$F_2$: loses | $F_1$: benefits greatly<br>$F_2$: loses much |
| $F_1$: no advertising | $F_1$: loses much<br>$F_2$: benefits greatly | $F_1$: benefits<br>$F_2$: benefits |

parts). We will continue with the example of the prisoner's dilemma in the advertising.

EXAMPLE 7

At the beginning, let us mention that the advertising is sometimes called a real life example of the prisoner's dilemma.

Suppose that two competitive firms in Slovenia (we will denote them by $F_1$ and $F_2$) have to decide how much money they will spend on the advertising. The profit derived from the advertising for the firm $F_1$ is, of course, partially determined by the advertising made by the firm $F_2$. Similarly, the effectiveness of the firm $F_2$'s advertising depends on the advertising conducted by the firm $F_1$. If both, the firm $F_1$ and the firm $F_2$, choose to advertise receipts remain constant and the expenses increase due to the cost of the advertising. Both firms would benefit if they both decide not to advertise. However, if one of the firms chooses not to advertise, the others could benefit greatly by the advertising (table 6). More about the prisoner's dilemma you can read in Heylighen (1992; 1995).

## Conclusion

The game theory is a branch of mathematics that deals with the analyses of situations involving parties with conflicting interests. When dealing with simple games we can represent a complete mathematical solution. On the other hand, we can find principles of the game theory also in complicated games such as chess, cards, and checkers, as well as in real-world problems (in economics, property division, politics, and warfare). The theory of games is actually the theory of social situations. Namely, most researchers in the game theory focus on how groups of people interact: how intelligent individuals interact with one another in a specific situation to achieve their own goals.

In the classical game theory, players move, bet, or strategize simultaneously. In this branch of the game theory both hidden information and chance elements are frequent features.

The theory of games is one of the tools that have been used to study a wide variety of not just human but also animal behavior. It was first developed in economics to explore and understand the behavior of markets, firms, consumers and other economic behaviors. Nowadays the game theory has been applied to political, psychological, and sociological behaviors as well.

As we have already mentioned, the game theory is also used in the study of animal behavior. Note that even Charles Darwin made a few informal game theoretic statements. Later, the development of the game theory in economics was applied to biology.

Furthermore, the game theory has also been used to develop theories of ethical or normative behavior. In schools, the game theory helps to understand good and proper behavior.

Let us also mention that applications of the theory of games have not been so far reaching as was hoped. Actually considerable difficulties arise in the mathematical theory of $n$-person games for $n > 2$.

## References

Akian, M., S. Gaubert, and A. Guterman. 2010. 'The Correspondence Between Tropical Convexity and Mean Payoff Games.' Paper presented at the 19th International Symposium on Mathematical Theory of Networks and Systems – MTNS 2010, 5–9 July, Budapest.

———. 2011. 'Tropical Polyhedra are Equivalent to Mean Payoff Games.' *International Journal on Algebra and Computations* 33 (3): 325–48.

Brown, R. F., and B. W. Brown. 1977. *Applied Finite Mathematics.* Belmont, CA: Wadsworth Publishing Company.

Heylighen, F. 1992. 'Evolution, Selfishness and Cooperation.' *Journal of Ideas* 2 (4): 70–76.

———. 1995. *The Prisoner's Dilemma.* Brussels: Principia Cybernetica.

Khoury, S. J., and T. D. Parsons. 1981. *Mathematical Methods in Finance and Economics.* New York: Elsevier.

Mizrahi, A., and M. Sullivan. 1993. *Mathematics for Business, Life Sciences and Social Sciences.* New York: Wiley.

Vega-Redondo, F. 2003. *Economics and the Theory of Games.* Cambridge: Cambridge University Press.