Research and Study Paths in the Teaching of Mathematics at Secondary school relative to the Rational Functions

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ABSTRACT

This research proposes the teaching of mathematics in the secondary school by means of the Study and Research Paths (SRP). A longitudinal analysis has been carried out during two years with secondary school students (aged 15-17). The results obtained in the last part of the path are presented in the present work, during which the rational functions were studied. The aspects connected to the processes of mesogenesis, topogenesis and cronogenesis are analyzed, according to Chevallard in the Anthropologic Theory of Didactics (ATD).

Key words: Mathematical Education; Research and Study Paths (RSP); Rational functions, Secondary Level.

1. Introduction

In the secondary school in Argentina traditional teaching methods are mainly used. Students do not take any responsibility in this process. The main activity and major responsibility during the lesson is in the hands of the teacher. Therefore, the teacher is in charge of choosing, introducing and communicating the Mathematical Organizations (MOs), which constitute the curriculum. On the other hand, the students have a reproductive role in relation to the knowledge, which in turn is presented as finished, transparent, unquestionable and, in a way, dead. This phenomenon has been called *monumentation of knowledge* (Chevallard, 2004, 2007, 2009) and one of its main characteristics is that it presents the MOs as finished works, i.e. as already created and meaningless objects.

Therefore, the Research and Study Paths (RSPs) have been proposed as an answer to the problem. They substitute the "inventorying-of-knowledge" school paradigm for another of questioning the world paradigm. One of the objectives of the research is to introduce and develop these SRPs in real mathematic lessons in the secondary school level in Argentina. From a generative question Q_0*: How to operate with any curves knowing only its graphic representation and the unit of axes?* (Llanos, Otero, 2012, 2013a, 2013b) the whole path is developed.

In the present work there is a description of the results generated in the last part of the SRP, arising from a question that derivates from the original Q_0, referred to as Q_3*: How to do the quotient between polynomial functions, knowing only its graphic representations and the unit of its axes?* The construction of an answer to this question in this part of the SRP leads to the study of the MOs of the rational functions (Otero, Llanos, Gazzola, 2012).

2. Theoretical Framework

The Anthropologic Theory of Didactics (ATD) is adopted (Chevallard, 1999), and in particular, the notions of Research and Study Paths and the *mesogenesis, topogenesis* and *chronogenesis* constructs. Within this theory, knowledge is the result of an answer to a *Q* mathematical question, which is meaningful to the study community and whose answer needs of the elaboration of new Q_i derivative questions.

The RSPs are developed from a Q_0 generative question, and from a group of Q_i questions which derivate from it. Therefore, the RSPs allow redefining the curriculum based on a group of questions, instead of on the production of answers as is the case of ordinary teaching. The teaching by means of RSP needs of important changes in the dominant pedagogy, which affects the process of study and the ecology, the survival of the RSPs. These modifications impact on the *topogenesis*, *mesogenesis* and *chronogenesis* processes; which are central in the didactic analysis proposed by the ATD (Chevallard, 2011).

2.1. Didactic Functions: topogonesis, mesogenesis and chronogenesis

In a RSP, the *M* didactic means is constructed in and by the class, from both external productions (as the ones available in books, Internet, etc.) or internal (the ones elaborated by the class); process referred to by Chevallard (2009) as *mesogenesis*. The *M* means is constituted from what has been done in the class between the teacher and the students, and not only by the teacher. The place each one takes is known as *topogenesis*. The students´*topos* is broadened as they can not only contribute with their R_x answer, but also they can introduce in *M* any works they wish to include. This change produces another important one in the teacher´s *topos*, who takes over the role of *study director* of *Q*, or *research director*.

The changes in the *mesogenesis* and the *topogenesis* produce modifications in the *chronogenesis*, in the management and "control" of the didactic times. In a RSP, the constitution and the "working" of the *M* means requires a dilatation of the school times, i.e. an extension of the hours required by it. The *chronogenesis, mesogenesis* and *topogenesis* didactic functions (Chevallard, 1985, 2009) are used to describe the functioning of RSP in the selected courses.

The present research attempts to answer the following questions: What characteristics of the rational functions MO allows developing the RSP? What modifications are identified in the first and second development of this part of the RSP?

3. Research Methodology

The research is a qualitative, ethnographic and exploratory one; and corresponds to a longitudinal analysis. The RSP is introduced in an experimentally controlled way in secondary school courses. Each implementation lasts for two years; i.e. that students begin the RSP in the 4^{th} year and continue on the following year, in the 5^{th} year. In total, four implementations were carried out: two of them simultaneously in each year, in two cohorts. In total, 121 students took part in the research.

During the implementations all the written protocols of the students were obtained, through the scanning technique; which enables to register the students´ daily work in the RSP. "General" audio recordings were taken, which were completed with the teacher´s field notes (before and after each meeting); apart from the observations carried out by the researchers. The obtained results are described from a triangulation of the recorded data.

4. The MOs of the rational functions in the longitudinal study

The RSP suggested to study the MOs of the mathematic program starts by a Q_0 generative question*: How to operate with any curves knowing only their graphic representation and the unit of the axes?* (Llanos, Otero, 2013a; Otero, Llanos, 2011). All throughout the longitudinal study, three questions deriving from the generative question Q_0 have been developed. They depend on the curves and operations considered in each case. The starting point is the multiplication of two straight lines, as they are the most familiar curves to the students when the RSP is started. Then, operations with parabolas are done. The elaboration of possible answers to these questions has allowed reconstructing different MOs:

- of polynomial functions of the second degree, developed from the Q_1 question: *How to multiply two straight lines knowing only its graphic representations and the union of the axes?*(Llanos, Otero, 2012, 2013a, 2013b);
- of the polynomial functions in general developed from the Q_2 question: *How to multiply several straight line or straight lines and parabolas knowing only its graphic representations and the union in the axes?* (Llanos, Otero, Bilbao, 2011);
- of the MOs of the rational functions (Otero, Llanos, Gazzola, 2012) whose results are described as follows.

4.1. The MOs of the rational functions

The last part of the path corresponding to the longitudinal study is introduced from the Q_3 question: *How to do the quotient between polynomial functions knowing only its graphic representations and the unit in the axes?* The introduction to the study of the rational functions is done from the quotient of curves. The analysis and construction of the reference knowledge (MOR) shows that from a Q_3 question other derivative questions can be generated which enable to reconstruct other MOs of the study program. The MOR characteristics are described in Diagram 1 below.

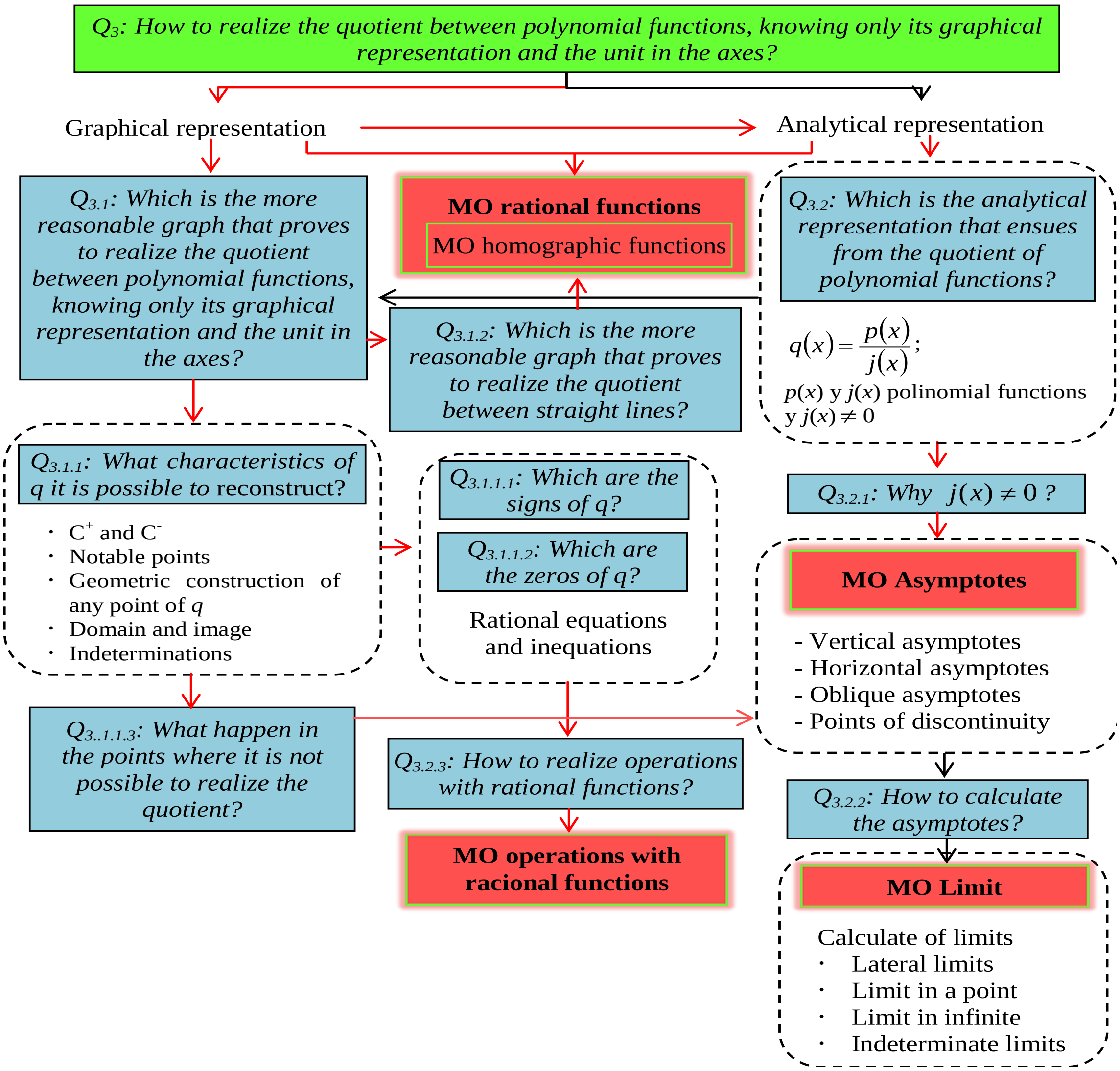
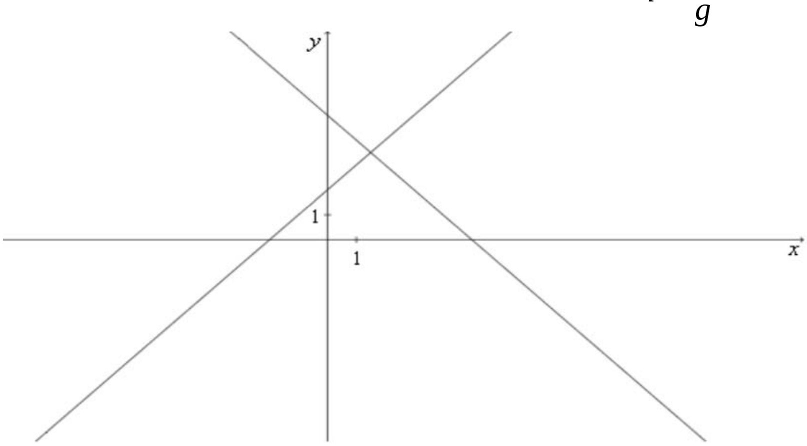


Diagram 1: MOR description for Q_3.

In the final stage of the path, the students benefit from the geometric and analytic calculation techniques previously developed. The questions initially proposed are:

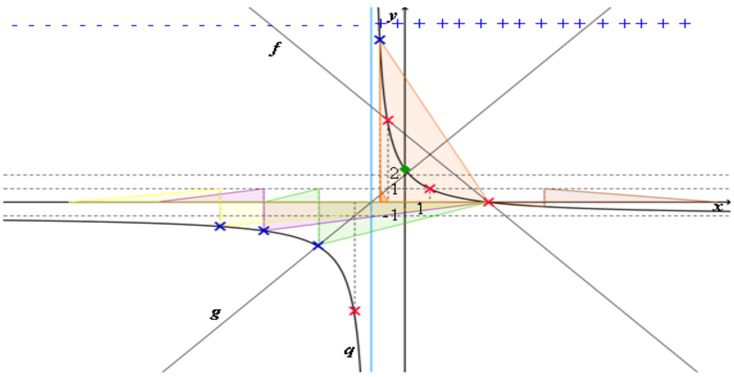
Functions f and g are given by the graphics of Picture 1 .Function $q = \dfrac{f}{g}$



Picture 1: Graphic representation of the straight lines *f* and *g*

(a) What could be the most reasonable graphic for q?
(b) What characteristics of the graphic of q could you justify?
(c) Is q a function?

In order to obtain the curve for *q* it is necessary to identify the signs (C^+ and C^-), the outstanding points (zeros, ones, minus ones, multiples of the union and the intersection between the curves, which is one); points represented in red in Picture 2. It is also possible to construct other points, by means of the geometric construction which has been previously obtained, out of the similar triangles using as data the union in the axes, identified in blue colour. The point represented in green colour also corresponds to *q*, given that in this case: $q = \dfrac{f}{2}$; i.e., half the ordinate *f* in the abscissa.



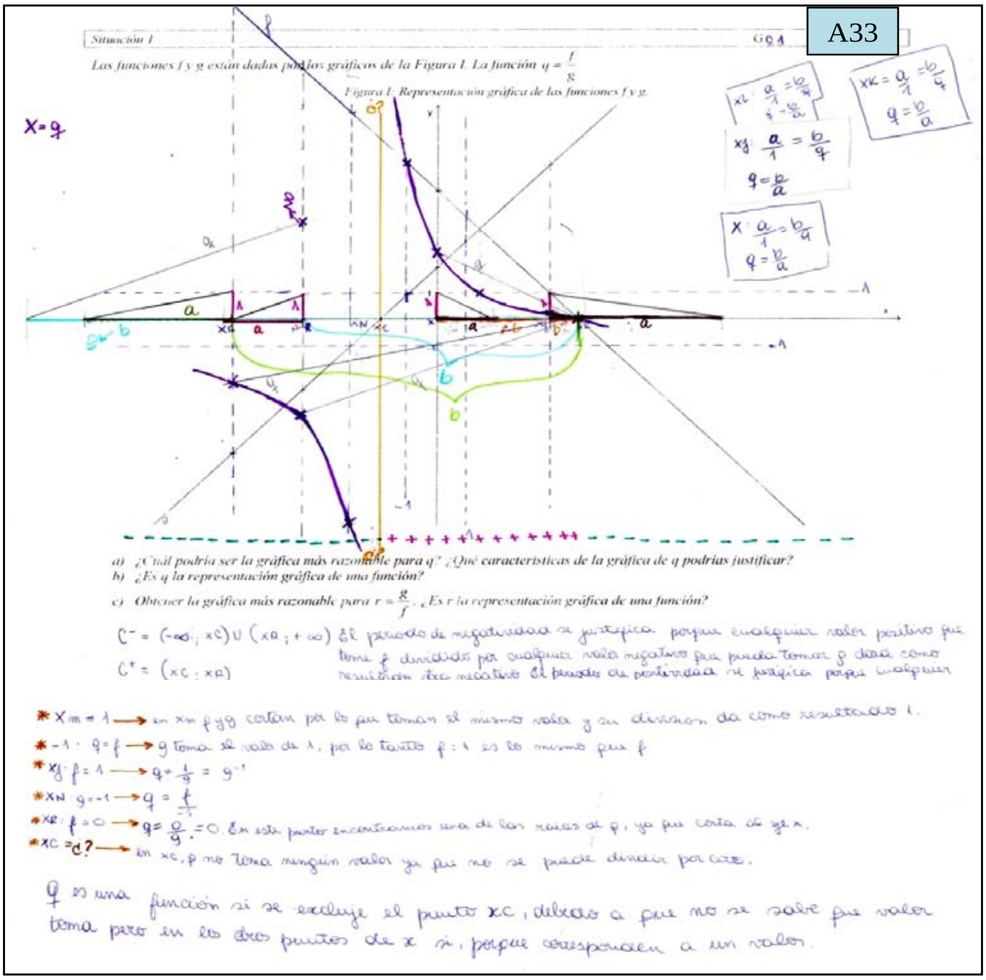
Picture 2: Cartesian representation of *q* out of the outstanding points and the ones obtained from construction.

There is a special emphasis on the analysis of the behaviour of the curve in the zero denominators.

4.2. The MO effectively reconstructed

Two protocols from the students participating in the RSP are selected in the different cohorts to describe the range and limitations in the different performances. The students obtain a graphic representation for *q* by adapting the techniques previously constructed. The results obtained in the last part of the longitudinal study, impact mainly in the *cronogenesis* level, providing that the students recover the techniques studied before and adapt them with no need to increase the expected school time. Also at the *mesogenesis* level, the students do not present any difficulties to obtain the representation of the curve; even when they need to develop other concepts which they have not dealt with in the previous year.

In Picture 3, the results of A33 are presented, student corresponding to the first cohort. This student obtains a well approximated graphic of *q*, from the signs and the construction of outstanding points. He identifies that *q* corresponds to a function, only if the value that makes zero the denominator, is excluded from the domain. He cannot formalize the adaptation he performs of the geometric technique in order to construct any point in the curve, and he does not analyze what is going on when *x* takes higher and higher values in absolute value. This student only points out that there is something going on in the point where the denominator function becomes zero.

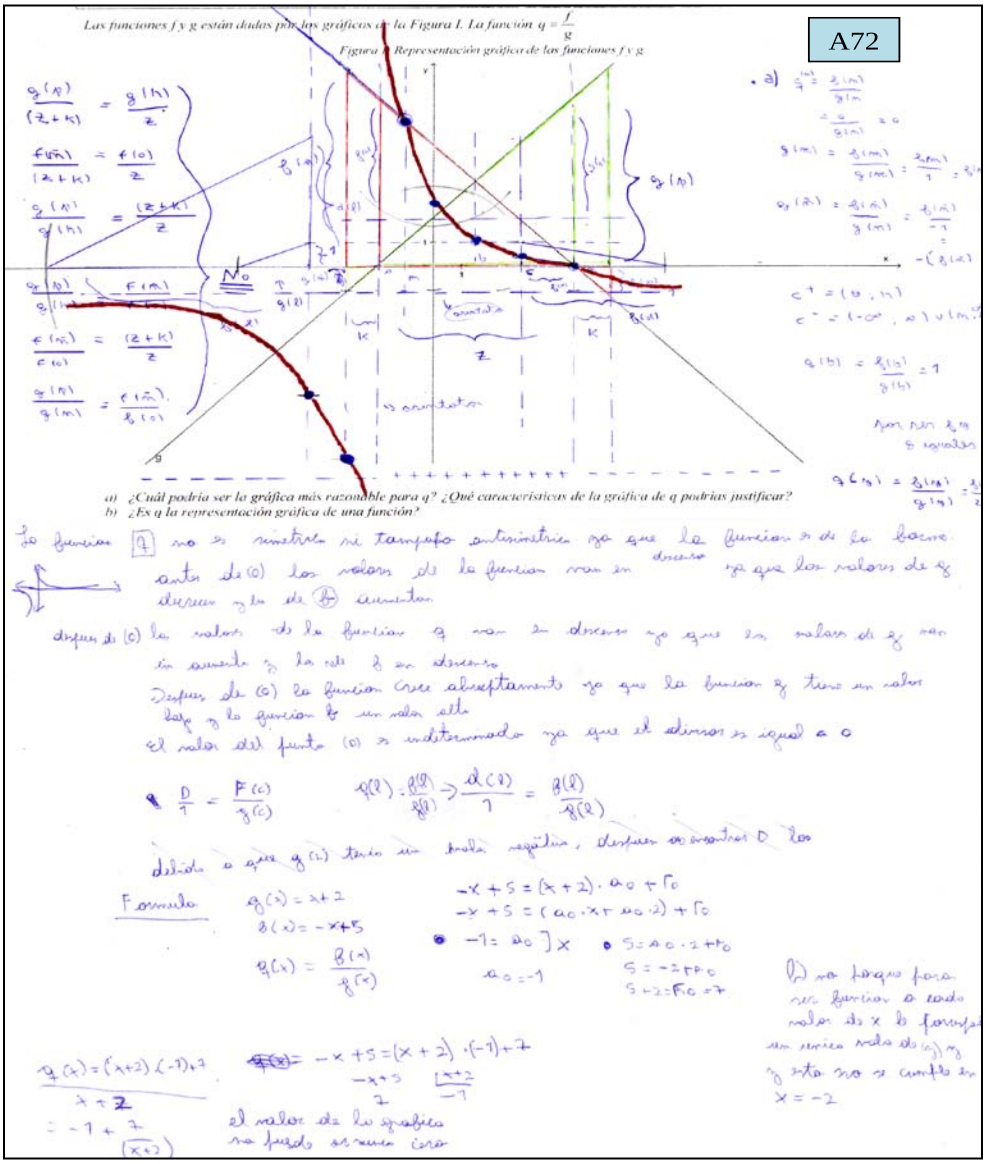


Picture 3: Protocol corresponding to student A33.

In contrast to the results obtained in the first cohort, in the second, the students introduce the problem of constructing vertical and horizontal asymptotes of *q*, apart from the analytic representation. These modifications in the *mesogenesis* level are justified by the decisions the teacher has considered in each case.

In the protocol of A72, of Picture 4 (second cohort), allows to interpret that this student constructs the characteristics of *q* and also the analytic representation. In order to deal with the graphic, the student analyses the signs, marks all the outstanding points and represents the asymptotes. Then, he justifies through a written verbal representation system that the curve takes very high values in absolute value for points next to the vertical asymptote. He obtains an approximate analytic representation, moving the units in the respective axes;

and from this result the problem of analyzing analytically the quotient is triggered. This allows identifying the vertical and horizontal asymptotes; which is a problem the class took over several times along the path.



Picture 4: Protocol corresponding to student A72.

In the implementations carried out, the study of *Q*3 has enabled to: construct the characteristics of the curve and the analytic representations of the rational functions; study the vertical, horizontal and oblique asymptotes and operate with rational functions. There was no further progress in relation to the limit MO although it is expected to be achieved in future implementations.

5. Some results

In the *topogenesis*, the achievements are relative to a "partial" distribution of the responsibilities between the teacher and the students. It has been referred to as "partial" as the students take over their role in the construction of the answers to any question, including the new ones. The limitation is detected in the generation of other questions by the students. The management of questions lies almost exclusively on the teacher.

The main differences in the *mesogenetic* level are closely connected to the decisions taken in the *topogenesis*. In the first cohort the teacher did not allow the students to move ahead beyond the graphic representation of the curve; whereas in the second, the study of the graphic and analytic representations is introduced from the beginning; and the justification of the vertical and horizontal asymptotes.

The time problem relative to the decisions in the *chronogenesis* level, which meant a big obstacle at the beginning, is recovered in the last part of the path; as the students support their results in the adaptation of other techniques studied before.

The four implementations performed allow justifying that the proposed RSP enables to construct the MO of the rational functions. Although the results obtained have been significant, a need to provide the RSP with a bigger openness has been considered; as it is the class the one that needs to decide what to study and with which depth it is done; and not only the teacher. The main differences between the two cohorts lie on this problem.

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