The Concept of Exponential Function in the Context of Vergnaud Theory

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ABSTRACT

In order to teach exponential functions as well as to study the process of conceptualization throughVergnaudTheory of Conceptual Fields, adidactic sequence was designed. The sequence was tested with four groups of students [between15-16 years old] of a secondary school. The analysis suggested the existence of an escalation in the conceptualization of the exponential function linked to representational systems, ranging from linear schemes, to exponential. However, this escalation did not occur in all representational systems at the very same time.

1. Introduction

The conceptualization requires a dialectical relationship between schemes and situations. The situations require schemes, and more schemes allow to facemore complex situations (Vergnaud,1990; 2007a; 2007b; 2011; 2013).The scheme is not a conduct by itself, but has the function of generating the behavior within the situation. For this reason, it is possible to study the schemesthat underlie student responses, through their written resolutions. We are particularly interested in operational invariants [OI] that make the schemeoperatory, i.e. the concepts-in-action and theorems-in-action. The first are pertinentcategories, and the latter areimplicit intuitive knowledge.The concepts-in-action and theorems-in-action assumed themselves and therefore they are mutually built.

The concept consists of operational invariants (the significant), situations (the reference) and representational systems (thesignifier). Thus, to study the development of the concepts these three planes have to be analyzed at the very same time.

Below are the situations and the overall results of the implementation according to an analysis based on the Theory of Conceptual Fields (Vergnaud,1990).

2. Methodology And Analysis Categories

It was designed and implemented a set of situations and exercises about exponential function. There were considered five representationalsystems [RS]. **Numeric** [RSN]: tables and calculations with numbers. **First Order Algebraic** [RSA1]: algebraic procedureswith initialized parameters (e.g.$2 \,.\, 5^x = 3$).**Second Order Algebraic**[RSA2]: algebraic procedures with uninitialized parameters (e.g.$.\, b^x = c$). **Graphic – Analytic**-[RSG]: Cartesian graph axes. **Written Verbal** [RSWV]: written language forms. The study was conducted in two fourth year courses (between 15-16 years old) of high school, during two months and a half. In class, the students studied in reduced groups of four or five members each. All the classes were recorded and the responses of the 59 students were always collected;this produced 885 protocols. A protocol is a student's answer to a situation. The students'answers were analyzed protocol by protocol and classified into stages (Sureda& Otero,2013). The classification (Table 1) is the same for both groups.

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Stage	**Indicator**	**Acronym**
Linear	Linear answer in all representational systems.	[LR]
Partly Non Linear	Non Linear answer in at least one representational system.	[PNLR]
Non Linear	Non Linear answer in all of the representational systems.	[NLR]
Partly Exponentially	Exponential answer in at least one representational system.	[PER]
Exponential	Exponential answer in all of the representational systems.	[ER]

Table 1

The study was performed again the following year with 56 students. The analysis showed the student's process of conceptualization was developed by the five states or levels mentioned.

3. Situations And Analysis

Secondary students are interested in saving money for the senior trip. The work is framed in the insurance savings context. That is, to study how varying the amount of money set fixed term, at compound interest. The students had studied linear functions and simple interest. The implementation began with a discussion about the operation of compound interest. Then, the first situation was given to them.

3.1. First Situation

Each group of students has got 12000 pesos for their senior trip (that will be in thirty months from that moment on). Theinterestsrate of three banks and the amount of money after the first month of capitalization are given to the students. The students were asked to explain how they calculated the amount of money for the first month. They had to calculate the amount of money for three months randomly and plot the variation of the amount of money given through coordinate axes. In the last one, students should say what function is plotted. The answers to this situation were classified according to the categories listed in Table 1 as follows (Table 2). The column "absent " indicates the number of students of both courses that did not resolve the situation for missing the class. The "total" column indicates the number of students from both courses.

[LR]	[PNLR]	[NLR]	[PER]	[ER]	Absents	Total
12	33	4	-	-	10	59

Table 2

Almost all students respond in a linear or partially linear way. Here appeared the first non-linear responses.
Linear Response [LR]: In the numericalRS, the students calculated the money of the first, totaling 132 pesos per month. In this way, the increase of the amount of money is constant. In the RSA1 they wrote the equation:$Bank\ 1 = 12000 + 132.t$. Then, they drew three lines on the Cartesian axes system and afterwards they said that they had plotted the linear function. In this way, the actions are linear in the four RS involved (RSN, RSA1, RSG and RSWV).

These resolutions allow us to infer that the theorems-in-action involved in each RS are linear (Table 3).

RSN	*"The money set IC increases the same every month"*	[LR]
RSA1	*"It is possible to calculate the money saved in CI made by a linear algebraic expression"*	[LR]
RSG	*"The graphic representation of growth of money set in CI, is a straight line"*	[LR]
RSWV	*"Is a linear function because the graphic is a straight line"*	[LR]

Table 3 Shows a set of linear schemes, complex and complete, that are expressed in each RS.

Partly Non LinearResponse[PNLR]: In general, students answered in non-linear in the RSN and linearly in all the others. They calculated recursively the simple interest, as follows. Bank 1 - Month 1: $12000\,.\,0{,}011 + 12000 = 12132$. Month 2: $12132\,.\,0{,}011 + 12132 = 12265{,}452$. This action, genesis of an exponential action, allowed them to obtain the compoundcapitalization. To show how does the amount of money varied placed at interest, they drew three lines and then they claimed that they had plotted a linear function (Figure 1).

Situación 1

Un grupo de chicos tiene \$12000 para su viaje de egresados y los quieren poner en un plazo fijo a interés compuesto por 30 meses, que es el momento de viajar. Se averiguaron las tasas de algunos bancos y se sabe que:

poc c/peso. la taza es del 1,1% x c/ 100$ se gana 1,1 x mes

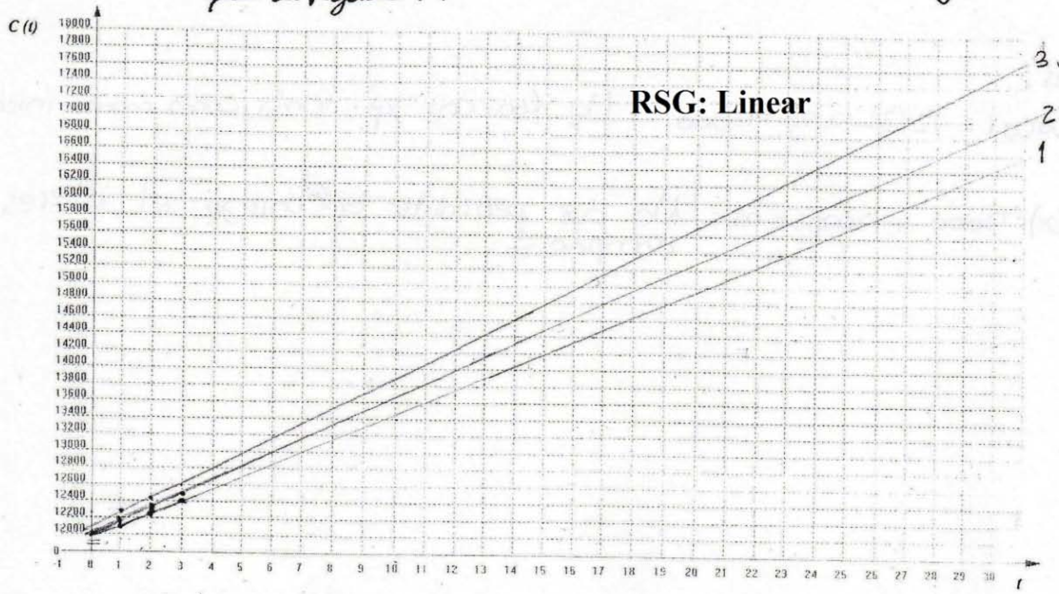
La tasa mensual del **Banco 1** es de 0,011 y les permite tener \$12132 cumplido el primer mes.

La tasa mensual del **Banco 2** es de 0,012 y les permite tener \$12144 cumplido el primer mes.

La tasa mensual del **Banco 3** es de 0,013 y les permite tener \$12156 cumplido el primer mes.

a) ¿Cómo calcularon los bancos ese primer mes?

b) Realiza un gráfico aproximado de la variación del dinero en cada banco; calculando al menos tres valores. ¿A qué función corresponde la Representación gráfica que dibujaste?*



Recordá que es muy importante dejar todas las cuentas que haces en la hoja, y no borrar nada de lo que escribas.

(A) BANCO 1: f(x) = 12000 + 12000 . 0,011

12132 = 12000 + 12000 . 0,011

BANCO 2: 12144 = 12000 + 12000 . 0,012

BANCO 3. 12156 = 12000 + 12000 . 0,013

(B) BANCO 1: 2do mes: 12132 + 12132 . 0,011 = 12265,45. **RSN: Non Linear**

3er mes: 12265,45 + 12265,45 . 0,011 = 12400,37.

BANCO 2: 2do mes: 12144 + 12144 . 0,012 = 12289,73.

3er mes: 12289,73 + 12289,73 . 0,012: 12437,3

BANCO 3: 2do mes: 12156 + 12156 . 0,013 = 12314

3er mes: 12314 + 12314 . 0,013 = 12474,

RSWV: Linear * Corresponde a una función lineal?

Figure 1

In this way, theorems-in-action that guide the action on the RSN are non-linear, but are linear in the RSG and RSWV (Table 4).

RSN	*"Money set CI does not increase the same every month"*	**[NLR]**
RSA1	*"The increment of money set CI is not calculated in the same way that the simple interest"*	**[NLR]**
RSG	*"The graphic representation of growth of money set IC is a straight line"*	**[LR]**
RSWV	*"The increment of money set CI is is a straight line"*	**[LR]**

Table 4

This shows that the schemes that direct the action in eachRS are different and do not evolve together. Even if the student'sactions in the differentRS are contradictory,the students do not warn this.

Non-linear Response [NLR]: They calculated recursively the simple interest for the first three months. Then they made the algebraic expression $x + (1{,}1\% \, . \, x)$. This equation is the attempt of make an algebraic calculation procedure, and therefore is not linear. Agreed with the group of algebraic expression class, students construct a non-linear graph, by uniting previously calculated points. Then they said that the graph does not correspond to a linear function. In this way, strategies are guided by theorems in non-linear act all SRs (Table 5).

RSN	*"Money does set in IC does not increase the same every month"*	**[NLR]**
RSA1	*"Increasing the money put at compound interest is not calculated in the same way that the simple interest"*	**[NLR]**
RSG	*"The graph of growth of money set IC is not a straight line"*	**[NLR]**
RSWV	*"Increasing the money put at compound interest is not a linear function"*	**[NLR]**

Table 5

The resolution of these stages show that as stated in theTCF, the explicitly of the operational invariants used in-action, appear lateron in the conceptualization.

3.2. Second Situation

The interest rate of the three banks are proposed, the money earned after the first month of capitalization, and a table showing the variation of the amount of money in the first bank during the first thirty months. The table shows that the amount of money does not increase the same way every month. The table has some empty boxes that students must complete. In the first task the student must complete the empty boxes and propose a formula. In the second task they had to build similar tables for the other two banks, and give the formulas. In the third task they had to determine the domain and image as to be functions. Then, they had to graph them on a given Cartesian axis system and explain the difference between this model and the previous one. The answers to this situation were classified according to Table 6.

[LR]	**[PNLR]**	**[NLR]**	**[PER]**	**[ER]**	**Absents**	**Total**
-	36	18	2	-	3	59

Table 6

The linear responses disappeared. The amount of partial responses and fully non-linear increased. The first exponential answers appeared partially.

Non-linear PartlyResponse[NLPR]: They calculated recursively the simple interest, and formulate an algebraic expression that aims the algebrization of the calculation procedure. But to show how varying the amount of money placed at interest,they draw three straight lines. In this way, the actions are guided by non-lineartheorems-in-action in the RSN and RSA1, but linear in the RSG (Table 4).

Non-linear Response [NLR]: They calculated recursively simple interest for the first three months, and try to make an algebraic calculation procedure. Then they constructed a non-linear graph. In this way, the actions were guided by theorems in non-linear in all RSs (Table 5).

Partly ExponentialResponse[PER]: They computed and completed the boxes using a non-linear procedure.

Mes (t)	Monto final del periodo C (t)	Monto al inicio del periodo C (t - 1)	Interés en cada periodo I ((t-1); t)	Tasa i = R/100
0	C(0) = 12000	--	--	--
1	C(1) = 12132	C(0) = 12000	I (0; 1) = 132	0,011
2	C(2) = 12265,452	C(1) = 12132	I (1; 2) = 133,452	0,011
3	C(3) = 12400,37197	C(2) = 12265,452	I (2; 3) = 134,919972	0,011
4	C(4) = 12536,77606	C(3) = 12400,37197	I (3; 4) = 136,4040917	0,011
5	C(5) = 12674,6806	C(4) = 12536,77606	I (4; 5) = 137,9045367	0,011
6	C(6) = 12814,10209	C(5) = 12674,6806	I (5; 6) = 139,4214866	0,011
7	C(7) = 12955,05721	C(6) = 12814,10209	I (6; 7) = 140,955123	0,011
8	C(8) = 13097,56284	C(7) = 12955,05721	I (7; 8) = 142,5056293	0,011
9	C(9) = 13241,63603	C(8) = 13097,56284	I (8; 9) = 144,0731912	0,011
10	C(10) = 13387,29403	C(9) = 13241,63603	I (9; 10) = 145,6579963	0,011
11	C(11)=13534,55426	C(10)=13387,29403		
12	C(12)=13683,43136	C(11)=13534,55426	**RSN: Non Linear**	
13	C(13)=13883,95214	C(12)=13683,43136		
14	C(14)=13986,12561	C(13)=13833,95214		

Figure 2

They wrote the algebraic expression $M_i\ (1+i)^t = Mf$. In the RS graph, they drew three lines. In this way, students resolved non-linearly in RSN, exponentially in RSA1 and linearly in the RSG.

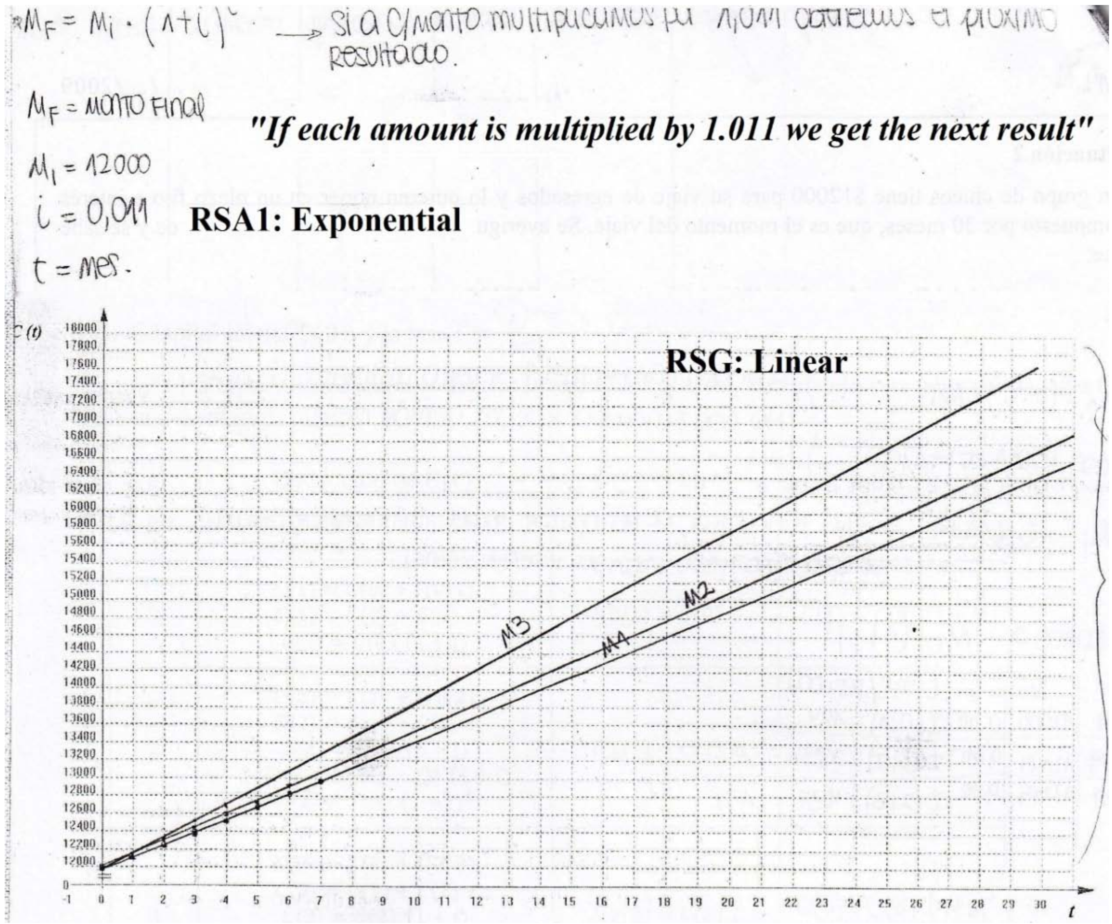


Figure 3

The theorems-in-action that seem to guide their actions aredescribed in Table 7.

RSN	*"Money set in IC does not increase in the same way every month"*	[NLR]
RSA1	*"If each amount is multiplied by 1.011, the next result is shown"*	[ER]
RSG	*"The graph of growth of money set IC is a straight line"*	[LR]

Table 7

On the one hand, this shows that to understanda problem within a RS and to solve it, does not implies the understanding of the problem in other RSs. On the other hand, it shows that when the knowledge about conceptual field is nascent, the arrangements established in aRS, arenot immediately reinterpreted in another.

At the end of this situation,the students invented the compound interest formula. In the third situation theystudied the differences between simple interest and compound interest. Then,theysolve a set of exercises.

3.3. Fourth Situation

Three people have SIV (swine influenza viruses). Each of them infect five people in the first hour. Then each of those five people infects another five the second hour. And so on. A formula is required, as well as a table with the number of infected people each hour during the first day and the respective graph. The aim is to generalize the formula for exponential functions. The answers to this situation were classified according to Table 8.

[LR]	[PNLR]	[NLR]	[PER]	[ER]	Absents	Total
-	-	-	17	35	7	59

Table 8

Partial responses and completely non-linear responses disappear. The exponential partially answers increase. The first fully exponential responses appear.

Partly Exponential Response[PER]: they wrote various linear equations $f(x) = 3 + 5x$; $f(t) = (1 + 5)^3.t$; $f(x) = (3.5).t + 3$, etc.

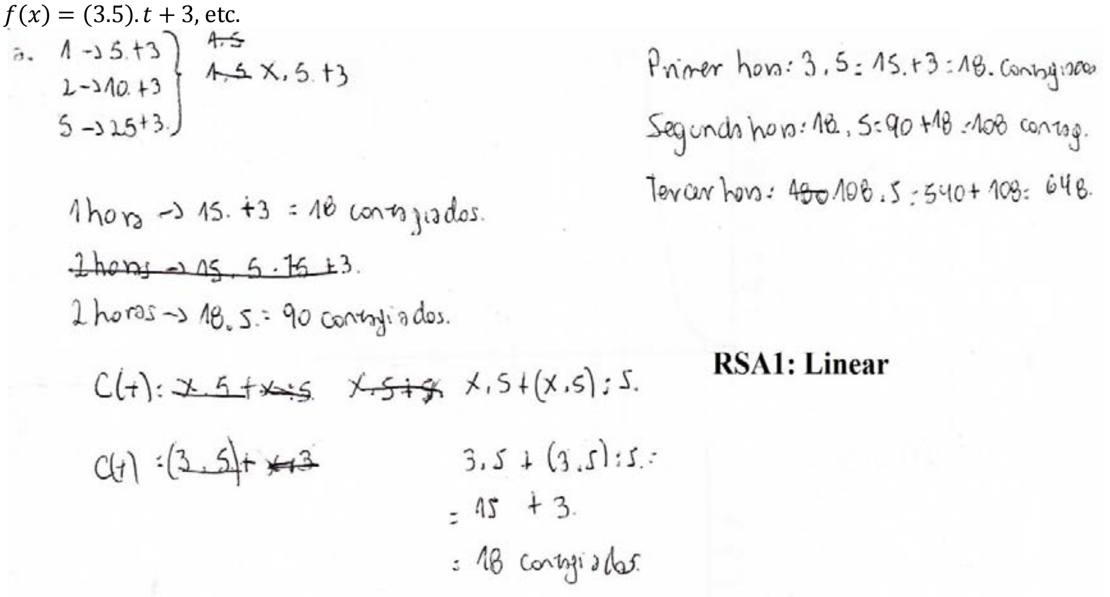


Figure 4

Once agreed the algebraic expression, they calculate some values and they built an exponential graph representation (Table 7).

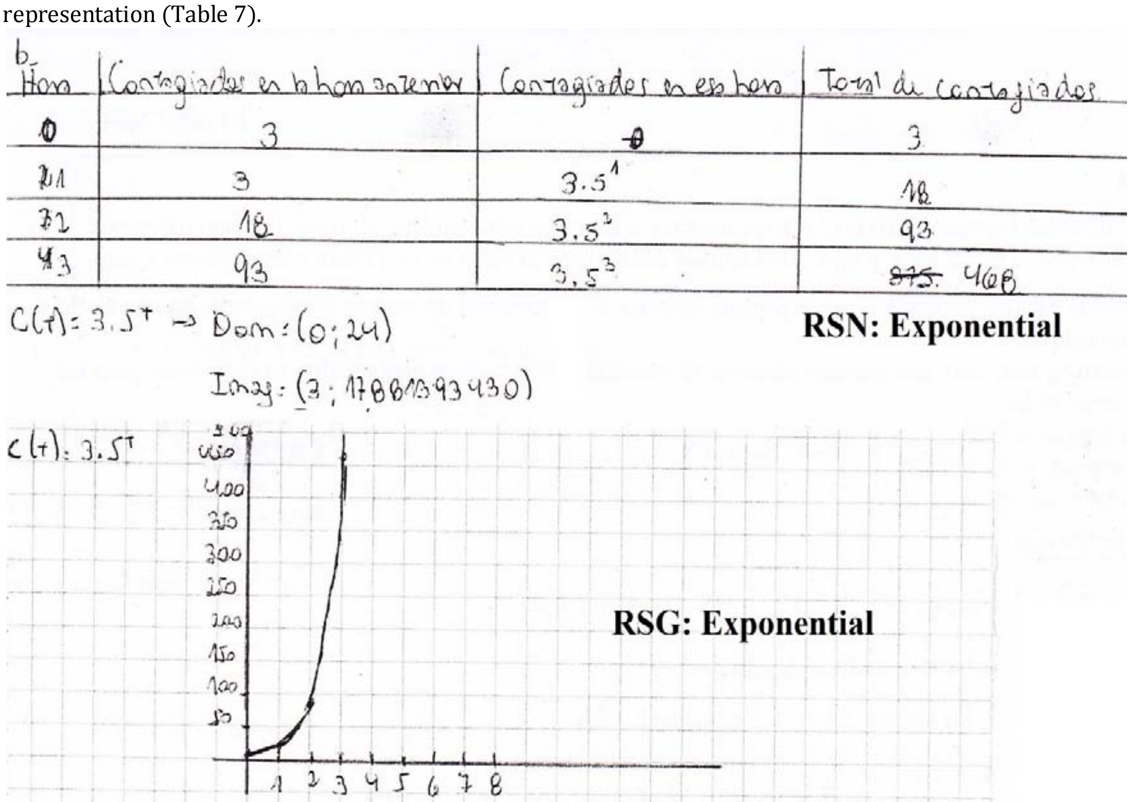


Figure 5

Exponential Response [ER]: Students constructed various exponential equations: $f(x) = 3\,.\,(5)^t$; $f(x) = 3\,.\left(1 + {}^{3}/_{5}\right)^t$; $f(t) = Mi\,.\,(1 + i)^t$; $f(t) = 3\,.\,(1 + 5)^t$; etc. Once agreed the algebraic expression, they calculated some values and built the graphic representation. In allRS responses are exponential. These decisions appear to be driven by exponential theorems-in-action (Table 9).

RSN	*"The increased is calculated on the immediately preceding amount"*	**[ER]**
RSA1	*"The algebraic expression is $f(t) = k\,.\,a^t$. Where t is the independent variable , a is the growth rate and k the initial amount"*	**[ER]**
RSG	*"The graphical representation of the variation is a rising curve"*	**[ER]**

Table 9

3.4. Fifth Situation

There are 50 amoebae in a laboratory that get duplicated by bipartition day by day. Anequationis require, a table with the number of the amoebas for 31-day and its graphics. The answers to this situation were classified according to Table 10.

[LR]	[PNLR]	[NLR]	[PER]	[ER]	Absents	Total
-	-	-	-	54	5	59

Table 10

The fully exponential responseis generalized.

Exponential Response [ER]: Students wrote the expression: $f(x) = 50\,.\,2^t$, they calculated some values, and built the graphic. AllRSresponses are exponential. By this, these resolutions appear to be driven by exponential operational invariants.

In the sixth situation they agreed the definition of exponential function. Then they resolveda set of exercises.

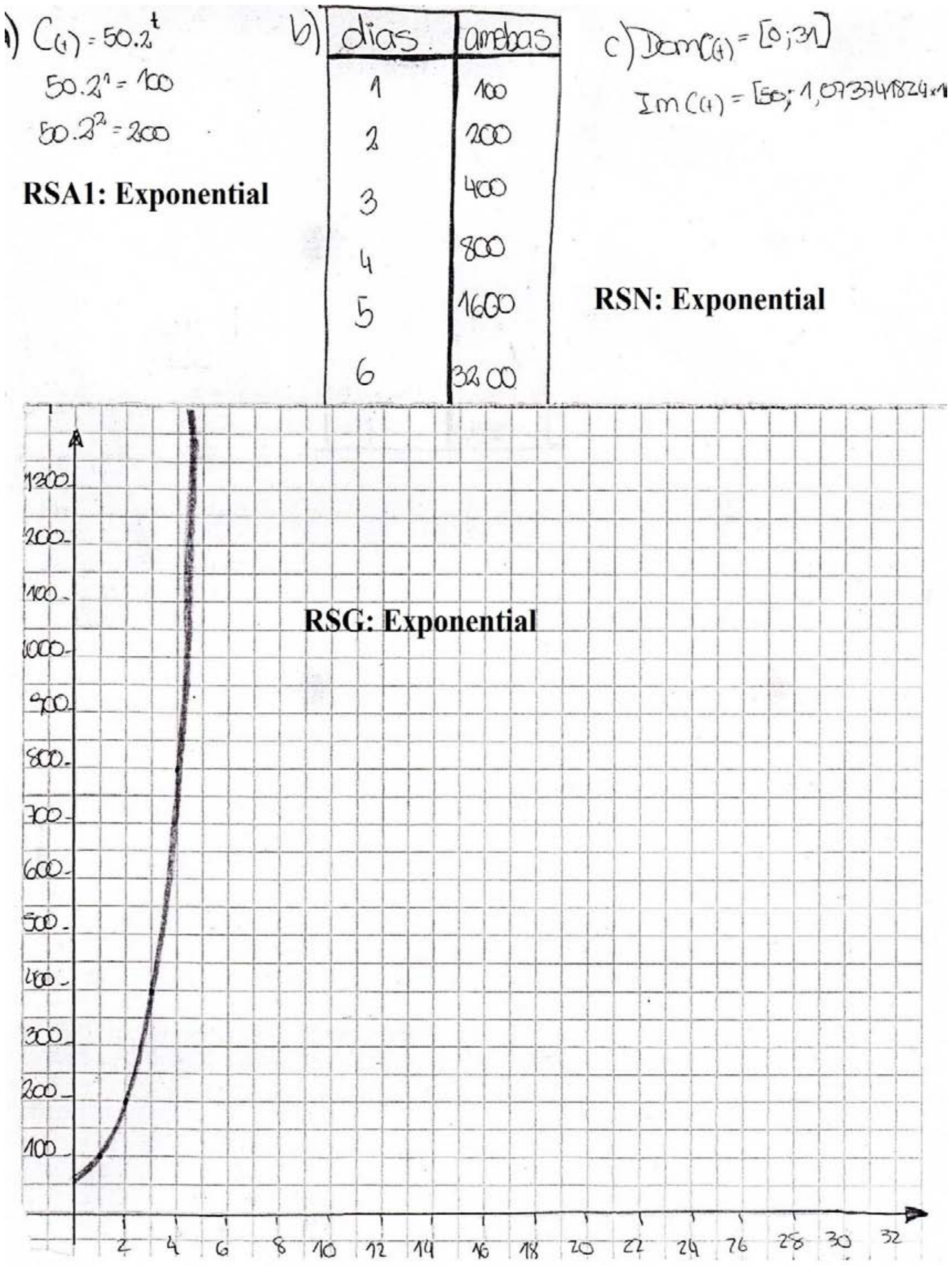


Figure 6

3.5. Seventh Situation

Each group of students has 5000 pesos for their senior trip that will be set on a fixed interest, to thirty months. The interest rate of the bank is known. They also have 2000 pesos that will not be settled on interest. A formula is require to calculate the money each month, including the money that was not put at interest; the amount of money for thirty months; the formula for calculating the amount of money without donation, and by a grant of 3200; all graphs;the domain and image of each one of them. The answers to this situation were classified according to Table 11.

[LR]	[PNLR]	[NLR]	[PER]	[ER]	Absents	Total
-	-	-	41	13	5	59

Table 11color in

Partially exponential responses reappear. Some answers are kept completely exponential.

Partly Exponential Response[PER]: Students write three equations exponential: $f_1(x) = 5000\,.\,1{,}013^t + 2000$; $f_2(x) = 5000\,.\,1{,}013^t + 3200andf_3(x) = 5000\,.\,1{,}013^t$.They usedequations to calculate the amount of money in a few months. With those values theyplotted points on a coordinate axis system, and built linear and non-linear graphs. That is, instead of drawing curves strictly increasing, they connected the points up and down. Or directly they drew straight lines.

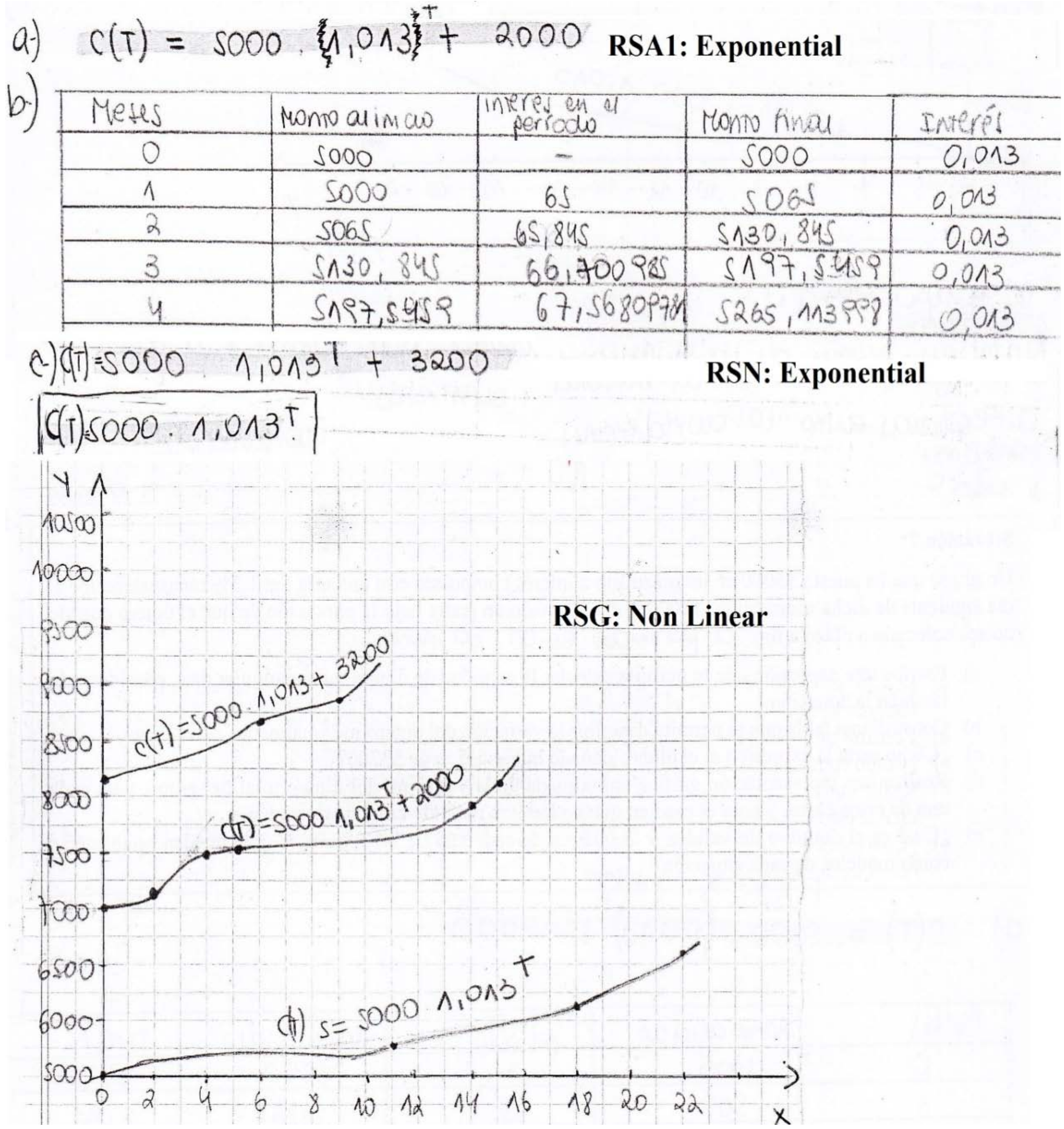


Figure 7

Responses were exponential in the numericRS as well as in the RSA1, but linear or non-linear in RSG (Table 7).

Exponential Response [ER]: Students wrote the same three exponential formulas. They usedthese equations to calculate the amount of money in a few months. With these values they plotted points on a coordinate axis system, and they constructed exponential graphs. Responses were exponential in all RS (Table 9).

In the eighth situation they studied asymptotes, and in the ninth situation they agreed the definition of exponential function for the form: $f(x) = k \cdot a^x + b$. Then they resolved the exercises. In the tenth situation they studied exponential equations, and in the eleventh situation they studied the variation of this family of functions. Finally a synthesiswas made and later on they gettested on the subject.

The tests showed that the students could differentiate an exponential function of another that is not, in all systems of representation [RSN, RSA1, RSG and RSWV]. It also shown that the students understood that the function in the system of numeric representation [RSN], does not increase the same between$x$and$x + 1$. Students could also figure out that the function in the system of first-order algebraic representation [RSA1] is expressedinthe form: $f(x) = k \cdot a^x + b$. They also noticed that in the written verbal representation system [RSWV] the function has the independent variablein the exponent. Finally, the graphic representation system [RSG] was drawn as a graph with a horizontal asymptote with a curve that wasnot straight, growing (or decreasing) in the whole domain.

4. Conclusions

The analysis shows the progressivity of the conceptualization of the exponential function, linked to systems of representation. This can be appreciated in the five types of responses identified. However, this progression does not occur in all systems of representation simultaneously. It shows that while students can solve a problem in a system of representation, they fail at the moment to do so in another system of representation, even within the same situation. This means that the results obtain from aRS, are not necessary reinterpreted in one another, at least when the conceptual domain knowledge is incipient. This could be because initially they would use different operational invariants according to the system of representation that the situation demanded. That is, the conceptualization of the exponential function requires new invariants, which are associated to different ways variations, and also new invariants associated on how to represent these invariants predicatively. On the other hand, the only schemes of functional variation that the student possesses are linear, which are largely consolidated by the use, the students used them consistently in all systems of representation.

Finally, the conceptualization of the exponential function, assumes a complex activity that emerges in rich and varied situations, that does not involves simultaneously all the systems of representation. The construction and consolidation of schemes linked to the exponential variation largely exceeds the duration of this implementation and possibly extends beyond the school years.

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